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HEEGAARD FLOER INVARIANTS IN CODIMENSION ONE

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Abstract. Using Heegaard Floer homology, we construct a numerical invariant for any smooth, oriented 4-manifold X with the homology of $S^1 \times S^3$. Specifically, we show that for any smoothly embedded 3-manifold Y representing a generator of $H_3(X)$, a suitable version of the Heegaard Floer d invariant of Y, defined using twisted coefficients, is a diffeomorphism invariant of X. We show how this invariant can be used to obstruct embeddings of certain types of 3-manifolds, including those obtained as a connected sum of a rational homology 3-sphere and any number of copies of $S^1 \times S^2$. We also give similar obstructions to embeddings in certain open 4-manifolds, including exotic $\mathbb{R}^4$s.

1. Introduction

A powerful way to study a nonsimply connected manifold X is to look at invariants of a codimension one submanifold Y dual to an element of $H^1(X)$. This idea goes back to work of Pontrjagin, Rohlin, and Novikov in the 1950s and 1960s exploring "codimension 1 (and higher) signatures" (see [24, 31]). In dimension 4, if X has the homology of $S^1 \times S^3$, then the Rohlin invariant of a submanifold Y representing a generator of $H_3(X)$, with spin structure induced from X, is a diffeomorphism invariant of X [33]. (We call Y a *cross-section* of X.) This invariant has interpretations in terms of Seiberg–Witten theory [21] and conjecturally in terms of Yang–Mills theory [34–37]. More recently, Frøyshov [5] observed that if X has a cross-section Y that is a rational homology 3-sphere, the invariant $h(Y, \mathfrak{s}_X) \in \mathbb{Q}$ associated to the unique Spin^c structure $\mathfrak{s}_X$ on Y induced from X, which is defined using monopole Floer homology, is also a smooth invariant of X. Frøyshov's argument uses only the rational homology cobordism invariance property of $h(Y, \mathfrak{s}_X)$, so it applies verbatim to the version of $h(Y, \mathfrak{s}_X)$ defined by Kronheimer and Mrowka [13, §39.1] (presumed, but not known, to be equal to Frøyshov's) and the similarly defined Heegaard Floer correction term $d(Y, \mathfrak{s}_X)$ [25]. In this paper, we extend the range of the Heegaard Floer invariant to an arbitrary smooth 4-manifold X with the homology of $S^1 \times S^3$, without the requirement that X admit a rational homology sphere cross-section. Note that this is a nontrivial restriction; for instance, the Alexander polynomial obstructs the existence of such cross-sections.

The definition of the correction term $d(Y, \mathfrak{s})$ for a rational homology sphere Y relies on the fact that $\mathrm{HF}^\infty(Y, \mathfrak{s})$ is isomorphic to $\mathbb{F}[U, U^{-1}]$. (Here $\mathbb{F}$ denotes the field of two elements.) In our earlier work [15, 16], we showed how to extend the definition of the correction terms for manifolds with $b_1(Y) > 0$ for which $\mathrm{HF}^\infty(Y, \mathfrak{s})$

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is "standard". (Here, $\mathfrak{s}$ is assumed to be a torsion spin^c structure.) Work of Lidman [17] shows that this condition holds whenever the triple cup product on $H^1(Y;\mathbb{Z})$ vanishes identically. However, an arbitrary 4-manifold X with the homology of $S^1 \times S^3$ need not have any cross-section with standard HF^∞.

In the present paper, we use a further generalization of the correction terms. For any subspace A of $H^1(Y)$ on which the triple cup product vanishes, we show in Theorem 3.1 that the *twisted* Heegaard Floer homology group $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ with coefficients in $M_A = \mathbb{F}[H^1(Y)/A]$ is standard in a suitable sense, allowing us to define a twisted correction term $d(Y,\mathfrak{s};M_A)$. (The case where $A = 0$ has been studied by Behrens and Golla [1].) When Y is a cross-section of X, we identify a particular such subspace by studying the cohomology of the infinite cyclic cover $\tilde{X}$. Our main result, which is stated more precisely below as Theorem 4.10, is as follows.

Theorem 1.1. *Let X be a homology $S^1 \times S^3$, and let Y be any cross-section of X representing a fixed generator y of $H_3(X)$. Then the correction term of $(Y,\mathfrak{s}_X)$, suitably normalized, is independent of the choice of Y. Thus, we obtain an invariant $\tilde{d}(X,y)$, which depends only on the diffeomorphism type of X and the choice of generator $y \in H_3(X)$.*

In principle, the invariant $\tilde{d}(X,y)$ could be used to detect exotic smooth structures on $S^1 \times S^3$, but we do not know of any candidates.

A more tractable application comes from the behavior of $\tilde{d}(X,y)$ under reversing either the orientation of X or the choice of generator of $H_3(X)$. In general, the four numbers $\tilde{d}(\pm X,\pm y)$ are *a priori* unrelated to each other, so they can obstruct the existence of symmetries that reverse the orientations of X or Y. Moreover, in Section 3.3 we describe a class of 3-manifolds which are called *d-symmetric*; this includes any manifold of the form $Q \# n(S^1 \times S^2)$, where Q is a rational homology sphere and $n \geq 0$. The following proposition describes some further symmetries of the invariants $\tilde{d}(\pm X,\pm y)$.

Proposition 1.2. *Let X be a homology $S^1 \times S^3$.*

- *If X has a cross-section that is a rational homology sphere, then*

$$\tilde{d}(X,y) = \tilde{d}(-X,y) = -\tilde{d}(X,-y) = -\tilde{d}(-X,-y). \tag{1.1}$$

- *If X has a cross-section that is d-symmetric, then*

$$\tilde{d}(X,y) = -\tilde{d}(-X,-y) \quad \text{and} \quad \tilde{d}(-X,y) = -\tilde{d}(X,-y). \tag{1.2}$$

- *If X is the mapping torus of a diffeomorphism $\phi\colon Y \to Y$, then*

$$\begin{aligned} \tilde{d}(X,y) = \tilde{d}(-X,y) &= \underline{d}(Y,\mathfrak{s}_X) + \frac{b_1(Y)}{2}, \\ \tilde{d}(X,-y) = \tilde{d}(-X,-y) &= \underline{d}(-Y,\mathfrak{s}_X) + \frac{b_1(Y)}{2}, \end{aligned} \tag{1.3}$$

where $\underline{d}$ denotes the twisted correction term defined by Behrens and Golla [1].

In particular, the failure of (1.1) or (1.2) for a given 4-manifold X enables us to obstruct the existence of particular types of cross-sections in X. In Section 5, we apply this obstruction to the study of 3-dimensional Seifert surfaces for knotted 2-spheres in S^4.

The proof of Theorem 1.1 relies on examining the lift of a cross-section of X to the infinite cyclic cover $\tilde{X}$. In fact, our techniques are more general; we consider a d invariant associated to any open 4-manifold $\tilde{X}$ satisfying certain homological properties similar to those of an infinite cyclic cover and any embedded 3-manifold Y representing a generator of $H_3(\tilde{X})$, which we also call a cross-section. While this quantity depends on the choice of Y and not just its homology class, we prove an inequality relating the invariants of disjoint cross-sections, which implies Theorem 1.1 in the case where $\tilde{X}$ is actually the $\mathbb{Z}$ cover of a homology $S^1 \times S^3$. In the general case, the inequality still gives interesting restrictions on the types of cross-sections that can occur. As one application (Example 4.12), we construct an exotic $\mathbb{R}^4$ that has no d-symmetric 3-manifold sufficiently far out in its end. In his forthcoming Ph.D. thesis, Mckee Krumpak has shown that results analogous to those of this paper hold in the setting of monopole homology [13].

2. The surgery formula

In this section, we state a twisted version of the mapping cone formula for the Heegaard Floer homology of surgeries on knots. This formula is known to experts but does not appear in the literature; the proof is a straightforward generalization of Ozsváth and Szabó's original integer surgery formula [29]. We will use this formula in Section 3 in order to prove that HF^∞ with appropriately twisted coefficients has a standard form.

2.1. **Heegaard Floer preliminaries.** Throughout the paper, all Heegaard Floer homology groups are taken over the ground field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Singular and simplicial homology and cohomology groups are taken with coefficients in $\mathbb{Z}$ unless otherwise specified.

We first provide a brief overview of Heegaard Floer homology with twisted coefficients. See Ozsváth–Szabó [27] for the original definition and Jabuka–Mark [11] for an excellent exposition. Here we emphasize two aspects of the theory that will be needed later: passing from HF^+ to HF^∞ via the U–completed version $\mathbf{HF}^\infty$ and the behavior of the coefficient modules under cobordism maps.

Let Y be a closed, connected, oriented 3-manifold, and let $\mathfrak{s}$ be a spin^c structure on Y. Let $\mathcal{H}_Y = \mathbb{F}[H^1(Y)]$; this can be identified with the ring of Laurent polynomials in $b_1(Y)$ variables.[1] Associated to $(Y, \mathfrak{s})$, there are chain complexes $\mathrm{CF}^-(Y, \mathfrak{s}; \mathcal{H}_Y)$, $\mathrm{CF}^\infty(Y, \mathfrak{s}; \mathcal{H}_Y)$, and $\mathrm{CF}^+(Y, \mathfrak{s}; \mathcal{H}_Y)$ over $\mathcal{H}_Y[U]$, well-defined up to chain homotopy equivalence, which fit into a short exact sequence

$$0 \to \mathrm{CF}^-(Y, \mathfrak{s}; \mathcal{H}_Y) \xrightarrow{\iota} \mathrm{CF}^\infty(Y, \mathfrak{s}; \mathcal{H}_Y) \xrightarrow{\pi} \mathrm{CF}^+(Y, \mathfrak{s}; \mathcal{H}_Y) \to 0. \tag{2.1}$$

We use $\mathrm{CF}^\circ(Y, \mathfrak{s}; \mathcal{H}_Y)$ to refer to any of the three complexes (or, by abuse of notation, the exact sequence relating them). Note that $\mathrm{CF}^\circ(Y, \mathfrak{s}; \mathcal{H}_Y)$ always has a relative $\mathbb{Z}$–grading, which multiplication by U drops by 2. If $\mathfrak{s}$ is torsion, multiplication by any element of $\mathcal{H}_Y$ preserves this grading, and the grading lifts to an absolute $\mathbb{Q}$–grading. (When $\mathfrak{s}$ is nontorsion, one must put a nontrivial grading on $\mathcal{H}_Y$ to define the relative $\mathbb{Z}$–grading, but we shall focus on torsion spin^c structures throughout the paper.) If M is any $\mathcal{H}_Y$–module, then let $\mathrm{CF}^\circ(Y, \mathfrak{s}; M) = \mathrm{CF}^\circ(Y, \mathfrak{s}; \mathcal{H}_Y) \otimes_{\mathcal{H}_Y} M$;

[1] The ring $\mathcal{H}_Y$ is called R_Y in [11]; we use the notation $\mathcal{H}_Y$ to avoid confusion with the manifold R_Y in Section 4.

these groups fit into a short exact sequence just like (2.1).[2] The homology groups are denoted by $\mathrm{HF}^\circ(Y,\mathfrak{s};M)$ and fit into a long exact sequence

$$\cdots \to \mathrm{HF}^-(Y,\mathfrak{s};M) \xrightarrow{\iota_M} \mathrm{HF}^\infty(Y,\mathfrak{s};M) \xrightarrow{\pi_M} \mathrm{HF}^+(Y,\mathfrak{s};M) \to \dots . \tag{2.2}$$

(We will frequently omit the subscripts from ι_M and π_M unless they are needed for clarity.)

Any element $\zeta \in H_1(Y)$ induces a degree -1, $\mathcal{H}_Y[U]$–linear chain map

$$\mathcal{A}_\zeta\colon\ \mathrm{CF}^\circ(Y,\mathfrak{s};\mathcal{H}_Y) \to \mathrm{CF}^\circ(Y,\mathfrak{s};\mathcal{H}_Y),$$

which is well-defined up to chain homotopy. Let $\mathcal{A}_\zeta^M$ denote the induced map on $\mathrm{CF}^\circ(Y,\mathfrak{s};M)$. These induce an action of $\Lambda^*(H_1(Y)/\operatorname{Tors})\otimes\mathcal{H}_Y[U]$ on $\mathrm{HF}^\circ(Y,\mathfrak{s};M)$. Moreover, following [11, Remark 5.2], define

$$Z_M = \{\alpha \in H^1(Y) \mid \alpha m = m\ \forall m \in M\}, \tag{2.3}$$

$$Z_M^\perp = \{\zeta \in H_1(Y) \mid \langle \alpha,\zeta\rangle = 0\ \forall \alpha \in Z_M\}. \tag{2.4}$$

For any $\zeta \in Z_M^\perp$, $\mathcal{A}_\zeta^M$ is chain-homotopic to 0. Thus, the H_1 action descends to an action of $\Lambda^*(H_1(Y)/Z_M^\perp)\otimes\mathcal{H}_Y[U]$.

If A is a subspace of $H^1(Y)$ such that $H^1(Y)/A$ is torsion-free (i.e., a direct summand of $H^1(Y)$), let $M_A = \mathbb{F}[H^1(Y)/A]$, viewed as an $\mathcal{H}_Y$–module via the quotient map. Concretely, if $\alpha_1,\dots,\alpha_n$ are a basis for $H^1(Y)$ such that $A = \operatorname{Span}(\alpha_1,\dots,\alpha_k)$ and $t_i \in \mathcal{H}_Y$ corresponds to α_i, then

$$M_A = \mathcal{H}_Y/(t_1-1,\dots,t_k-1).$$

Moreover, $Z_{M_A} = A$ and $Z_M^\perp = A^\perp$, and $H_1(Y)/A^\perp$ is naturally isomorphic to the dual of A.

Ignoring the $\mathcal{H}_Y$–module structure and the H_1 action, we note the following basic fact.

Lemma 2.1. *Let Y be a closed, oriented 3-manifold, $\mathfrak{s}$ a torsion spin^c structure on Y, and $A \subset H^1(Y)$ a direct summand of rank k. Then $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ is a free, finitely generated $\mathbb{F}[U,U^{-1}]$–module with rank at most 2^k.*

Proof. Ozsváth and Szabó [27, Theorem 10.12] proved the $k=0$ case:

$$\mathrm{HF}^\infty(Y,\mathfrak{s};\mathcal{H}_Y) \cong \mathbb{F}[U,U^{-1}],$$

where every element of $H^1(Y)$ acts by the identity. Thus, assume $k>0$. Up to an overall shift, assume that $\mathrm{HF}^\infty(Y,\mathfrak{s};\mathcal{H}_Y)$ is supported in even integer gradings. Consider the universal coefficient spectral sequence for changing coefficients from $\mathcal{H}_Y$ to M_A. The E^2 page satisfies

$$E^2_{p,q} = \operatorname{Tor}_p^{\mathcal{H}_Y}(\mathrm{HF}^\infty(Y,\mathfrak{s};\mathcal{H}_Y), M_A) = \begin{cases} \mathbb{F}^{\binom{k}{p}} & q \text{ even},\\ 0 & q \text{ odd}.\end{cases}$$

For each $s \in \mathbb{Z}$, by summing over all p,q with $p+q=s$, we thus deduce that $\dim_{\mathbb{F}} \mathrm{HF}^\infty_s(Y,\mathfrak{s};M_A) \le 2^{k-1}$. Choose bases (over $\mathbb{F}$) for the summands in grading

[2] We will not be focusing on nontorsion spin^c structures in this paper, but the fact that $\mathrm{CF}^\circ(Y,\mathfrak{s};\mathcal{H}_Y)$ is relatively $\mathbb{Z}$–graded (and not just $\mathbb{Z}/2d\mathbb{Z}$–graded for some $d>0$) is one of the advantages of twisted coefficients in other settings. Moreover, depending on the choice of M, the $\mathbb{Z}$–grading sometimes descends to $\mathrm{CF}^\circ(Y,\mathfrak{s};M)$ even when $\mathfrak{s}$ is nontorsion. See [11, §3] for a nice discussion.

0 and 1; since $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ is relatively $\mathbb{Z}$-graded, these combine to give a basis for $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ over $\mathbb{F}[U,U^{-1}]$. □

In the proof of the surgery formula (Theorem 2.3) below, we will need to pass from a result about HF^+ to a result about HF^∞. This is best done by first considering the U-completed version, introduced in [18]. Define

$$\mathbf{CF}^\infty(Y,\mathfrak{s};M)=\mathrm{CF}^\infty(Y,\mathfrak{s};M)\otimes_{\mathbb{F}[U,U^{-1}]}\mathbb{F}[[U,U^{-1}].$$

Denote the homology of this complex by $\mathbf{HF}^\infty(Y,\mathfrak{s};M)$. Because $\mathbb{F}[[U,U^{-1}]$ is flat over $\mathbb{F}[U,U^{-1}]$, we have

$$\mathbf{HF}^\infty(Y,\mathfrak{s};M)\cong\mathrm{HF}^\infty(Y,\mathfrak{s};M)\otimes_{\mathbb{F}[U,U^{-1}]}\mathbb{F}[[U,U^{-1}]. \tag{2.5}$$

Because multiplication by U drops grading by 2, it can also be understood as a grading-preserving map $\mathrm{CF}^+(Y,\mathfrak{s};M)\to\mathrm{CF}^+(Y,\mathfrak{s};M)[2]$.[3] It is easy to see that $\mathbf{CF}^\infty(Y,\mathfrak{s};M)$ is isomorphic to the inverse limit of the directed system

$$\dots\xrightarrow{U}\mathrm{CF}^+(Y,\mathfrak{s};M)[-2]\xrightarrow{U}\mathrm{CF}^+(Y,\mathfrak{s};M)\xrightarrow{U}\mathrm{CF}^+(Y,\mathfrak{s};M)[2]\xrightarrow{U}\dots.$$

For conciseness, we write

$$\mathbf{CF}^\infty(Y,\mathfrak{s};M)\cong\varprojlim(\mathrm{CF}^+(Y,\mathfrak{s};M),U).$$

There is therefore a short exact sequence

$$0\to\varprojlim{}^1(\mathrm{HF}^+(Y,\mathfrak{s};M),U)_{*-1}\to\mathbf{HF}^\infty_*(Y,\mathfrak{s};M)\to\varprojlim(\mathrm{HF}^+(Y,\mathfrak{s};M),U)_*\to 0$$

(where the $*$ denotes the homological grading). The system $(\mathrm{HF}^+(Y,\mathfrak{s};M),U)$ satisfies the Mittag-Leffler condition, since for all n sufficiently large, the image of U^n is equal to the image of $\pi\colon\ \mathrm{HF}^\infty(Y,\mathfrak{s};M)\to\mathrm{HF}^+(Y,\mathfrak{s};M)$. (See, e.g., [38, Proposition 3.5.7].) Thus, the derived functor $\varprojlim{}^1(\mathrm{HF}^+(Y,\mathfrak{s};M),U)$ vanishes, and we deduce that

$$\mathbf{HF}^\infty(Y,\mathfrak{s};M)\cong\varprojlim(\mathrm{HF}^+(Y,\mathfrak{s};M),U). \tag{2.6}$$

For a nontorsion spin^c structure $\mathfrak{s}$, $\mathbf{HF}^\infty(Y,\mathfrak{s};M)$ does not generally determine $\mathrm{HF}^\infty(Y,\mathfrak{s};M)$; see [18, Section 2]. However, when $\mathfrak{s}$ is torsion and $M=M_A$ for some summand $A\subset H^1(Y)$, the two theories are essentially interchangeable. Specifically, by Lemma 2.1 and (2.5), $\mathbf{HF}^\infty(Y,\mathfrak{s};M_A)$ is a finitely generated, free $\mathbb{F}[[U,U^{-1}]$–module whose rank (over $\mathbb{F}[[U,U^{-1}]$) is the same as the rank of $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ (over $\mathbb{F}[U,U^{-1}]$). It is thus clear how to recover $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ from $\mathbf{HF}^\infty(Y,\mathfrak{s};M_A)$. Moreover, the action of $\mathcal{H}_Y$ is grading-preserving, and the action of $H_1(Y)$ drops grading by 1, so these actions on $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ and $\mathbf{HF}^\infty(Y,\mathfrak{s};M_A)$ are readily identified.

Next, we discuss the cobordism maps on twisted Heegaard Floer homology. If $W\colon Y\to Y'$ is a cobordism between closed, connected, oriented 3-manifolds, consider the exact sequence

$$H^1(\partial W)\xrightarrow{\delta_W}H^2(W,\partial W)\xrightarrow{j_W}H^2(W), \tag{2.7}$$

and let $K(W)=\mathrm{im}(\delta_W)=\ker(j_W)$. The map δ_W makes $\mathbb{F}[K(W)]$ into an $\mathcal{H}_Y$–$\mathcal{H}_{Y'}$ bimodule. Given an $\mathcal{H}_Y$–module M, let $M(W)=M\otimes_{\mathcal{H}_Y}\mathbb{F}[K(W)]$. Note that the map j_W is given by the intersection form on W; if this form vanishes (meaning there

[3]We use the following convention: If C is a graded vector space and $n\in\mathbb{Q}$, $C[n]$ denotes the graded vector space with $C[n]_k=C_{k-n}$.

are no classes in $H_2(W)$ of nonzero square), then $K(W) = H^2(W, \partial W)$. According to [28, §2.7], for any spin^c structure $\mathfrak{t}$ on W, there is an induced map

$$F^\circ_{W,\mathfrak{t}}\colon\ \mathrm{HF}^\circ(Y, \mathfrak{t}_Y; M) \to \mathrm{HF}^\circ(Y', \mathfrak{t}|_{Y'}; M(W)), \tag{2.8}$$

which is an invariant of $(W, \mathfrak{t})$ up to multiplication by units in $\mathcal{H}_Y$ and $\mathcal{H}_{Y'}$.

For future reference, let us describe $\mathbb{F}[K(W)]$ in the case where W is given by a single handle attachment. To be precise, we assume that

$$W = Y \times [0,1] \cup k\text{-handle},$$

where the handle is attached along a $(k-1)$–sphere in $Y \times \{1\}$ and $k \in \{1,2,3\}$. (Note that any connected cobordism between connected 3-manifolds has a handle decomposition with only 1-, 2-, and 3-handles.) In each of these cases, it is easy to describe $K(W)$ as an $\mathcal{H}_Y$–module. It is easier to work in terms of homology, identifying the sequence (2.7) with

$$H_2(\partial W) \to H_2(W) \to H_2(W, \partial W)$$

via Poincaré duality.

- When $k = 1$, the inclusion $Y \to W$ induces an isomorphism $H_2(Y) \to H_2(W)$, and hence $K(W) = H^2(W, \partial W) \cong H^1(Y)$. Hence, for any $\mathcal{H}_Y$–module M, we have $M(W) \cong M$. Similarly, when $k = 3$, if we let $t \in \mathcal{H}_{Y_1}$ denote the Poincaré dual of the attaching sphere (which is assumed to be nonseparating and therefore a primitive class), we see that $\mathbb{F}[K(W)] \cong \mathcal{H}_{Y_1}/(t-1)$, and therefore $M(W) \cong M/(t-1)M$.
- When $k = 2$, let $K \subset Y$ denote the attaching circle for the 2-handle. The exact sequence on homology for the pair (W, Y) gives

$$0 \to H_2(Y) \to H_2(W) \to \mathbb{Z} \to H_1(Y), \tag{2.9}$$

  where $1 \in \mathbb{Z}$ maps to $[K] \in H_1(Y)$.

  If K is rationally null-homologous, let $d > 0$ denote its order in $H_1(Y)$. A capped-off rational Seifert surface for K produces a class $[\hat{S}] \in H_2(W)$ that maps to $d \in \mathbb{Z}$ in (2.9). Therefore, $H_2(W) \cong H_2(Y) \oplus \mathbb{Z}$; the $H_2(Y)$ summand is canonical, while the $\mathbb{Z}$ is generated by $[\hat{S}]$. If the 2-handle is attached along a multiple of the rational longitude for K (meaning that the self-intersection of $[\hat{S}]$ is zero), then the map $H_2(Y') \to H_2(W)$ is surjective, so $K(W) \cong H^2(W, \partial W) \cong H^1(Y) \oplus \mathbb{Z}$. We thus have $\mathbb{F}[K(W)] \cong \mathcal{H}_Y[t, t^{-1}]$, and for any $\mathcal{H}_Y$–module M, $M(W) \cong M[t, t^{-1}]$. On the other hand, if the self-intersection of $[\hat{S}]$ is nonzero, then the image of $H_2(Y')$ in W agrees with the image of $H_2(Y)$. Therefore, $K(W) \cong H^1(Y) \cong H^1(Y')$, and $M(W) \cong M$ for any $\mathcal{H}_Y$–module M.

  If K represents a nontorsion element of $H_1(Y)$, then the map $H_2(Y) \to H_2(W)$ is an isomorphism, so $K(W) \cong H^1(Y)$. In this case, however, $H^1(Y')$ is smaller than $H^1(Y)$; indeed, we may find an identification of $\mathcal{H}_{Y_1}$ with $\mathcal{H}_{Y_2}[t, t^{-1}]$.

2.2. **The exact triangle with twisted coefficients.** Throughout this section, let Y be a closed, oriented 3-manifold, and let $\mathfrak{s}$ be a torsion spin^c structure on Y. Let $K \subset Y$ be a null-homologous knot, and let S be a Seifert surface for K.

For any integer m, let W_m be the m-framed 2-handle cobordism from Y to $Y_m = Y_m(K)$. Let $S_m \subset W_m$ denote the capped-off Seifert surface; when $m = 0$,

we may view this as lying in Y_0. For any integer k, let $\mathfrak{t}_{m,k}$ denote the unique spin^c structure on W_m with

$$\mathfrak{t}_{m,k}|_Y = \mathfrak{s}, \quad \langle c_1(\mathfrak{t}_{m,k}), [S_m]\rangle + m = 2k, \tag{2.10}$$

and let $\mathfrak{s}_{m,k} = \mathfrak{t}_{m,k}|_{Y_m}$. Additionally, let W'_m denote W_m with reversed orientation, viewed as a cobordism from Y_m to Y.

As seen in the previous section, when $m \neq 0$, we have a natural identification $\mathcal{H}_Y \cong \mathcal{H}_{Y_m}$, so we may view any $\mathcal{H}_Y$–module M as an $\mathcal{H}_{Y_m}$–module, and vice versa, and $M(W_m) \cong M$. Likewise, if we consider M as an $\mathcal{H}_{Y_m}$ module, then the module induced on Y by W'_m is again isomorphic to M. It follows that there are maps

$$F^\circ_{W_m,\mathfrak{t}_{m,k}}\colon\ \mathrm{HF}^\circ(Y,\mathfrak{s};M) \to \mathrm{HF}^\circ(Y_m,\mathfrak{s}_{m,k};M),$$
$$F^\circ_{W'_m,\mathfrak{t}_{m,k}}\colon\ \mathrm{HF}^\circ(Y_m,\mathfrak{s}_{m,k};M) \to \mathrm{HF}^\circ(Y,\mathfrak{s};M).$$

On the other hand, when $m = 0$, we have $\mathcal{H}_{Y_0} \cong \mathcal{H}_Y[t^{\pm 1}]$, and for any $\mathcal{H}_Y$–module M, $M(W_0) \cong M[t^{\pm 1}]$. Hence, for each k, we have a map

$$F^\circ_{W_0,\mathfrak{s}_{0,k}}\colon\ \mathrm{HF}^\circ(Y,\mathfrak{s};M) \to \mathrm{HF}^\circ(Y_0,\mathfrak{t}_{0,k};M[t^{\pm 1}]),$$

which can then be extended to a map

$$\underline{F}^\circ_{W_0,\mathfrak{s}_{0,k}}\colon\ \mathrm{HF}^\circ(Y,\mathfrak{s};M)[t^{\pm 1}] \to \mathrm{HF}^\circ(Y_0,\mathfrak{t}_{0,k};M[t^{\pm 1}])$$

by the formula

$$\underline{F}^\circ_{W_0,\mathfrak{s}_{0,k}}(x \otimes t^i) = t^i F^\circ_{W_0,\mathfrak{s}_{0,k}}(x).$$

When $m \neq 0$, for each $[k] \in \mathbb{Z}/m$, we define

$$\mathrm{HF}^\circ(Y_0,[\mathfrak{s}_{m,k}];M) = \bigoplus_{l \equiv k \pmod m} \mathrm{HF}^\circ(Y_0,\mathfrak{s}_{0,l};M).$$

A key property is that for m sufficiently large and $|k| \leq \frac{m}{2}$, the only non-zero summand in this decomposition of $\mathrm{HF}^+(Y_0,[\mathfrak{s}_{m,k}];M)$ is $\mathrm{HF}^+(Y_0,\mathfrak{s}_{0,k};M)$. The following theorem is a slight generalization of [11, Theorem 9.1 and Proposition 9.3].

Theorem 2.2. *Let Y be a closed, oriented 3-manifold, and let $\mathfrak{s}$ be a torsion spin^c structure on Y. Let $K \subset Y$ be a null-homologous knot, and let S be a Seifert surface for K. Let M be a module for $\mathcal{H}_Y$. For any integer $m > 0$ and any $[k] \in \mathbb{Z}/m$, there is a sequence of $\mathcal{H}_Y[t^{\pm 1}] \otimes \mathbb{F}[U]$–linear maps:*

$$\begin{array}{ccc} \mathrm{HF}^+(Y_m(K),\mathfrak{s}_{m,k};M)[t^{\pm 1}] & \xrightarrow{\quad\underline{F}\quad} & \mathrm{HF}^+(Y,\mathfrak{s};M)[t^{\pm 1}] \\ & {\scriptstyle \underline{H}}\nwarrow \quad \swarrow{\scriptstyle \underline{G}} & \\ & \mathrm{HF}^+(Y_0(K),[\mathfrak{s}_{m,k}];M[t^{\pm 1}]) & \end{array} \tag{2.11}$$

Moreover, up to an overall power of t, the map $\underline{F}$ appearing in (2.11) *is given by*

$$\underline{F} = \sum_{l \equiv k \pmod m} F^+_{W'_m,\mathfrak{t}_{m,l}} \otimes t^{\lfloor l/m \rfloor}. \tag{2.12}$$

2.3. **The mapping cone.** Let $\mathrm{CFK}^\infty(Y,K;\mathcal{H}_Y)$ denote the totally twisted knot Floer complex of (Y,K) with coefficients, coming from a doubly pointed Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$. This is generated by tuples $[\mathbf{x},i,j]$ with $j-i=A(\mathbf{x})$, where $A(\mathbf{x})$ is the Alexander grading of $\mathbf{x}$, with differential given by

$$\partial[\mathbf{x},i,j]=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}}\#\widehat{\mathcal{M}}(\Phi)e^{\mathcal{A}(\phi)}[\mathbf{y},i-n_w(\phi),j-n_z(\phi)],$$

where $\mathcal{A}$ is the additive assignment used for defining twisted coefficients. Define an action of $\mathbb{F}[U,U^{-1}]$ on C by $U\cdot[\mathbf{x},i,j]=[\mathbf{x},i-1,j-1]$. Fix an $\mathcal{H}_Y$–module M, and let $C=\mathrm{CFK}^\infty(Y,K;M)=\mathrm{CFK}^\infty(Y,K;\mathcal{H}_Y)\otimes_{\mathcal{H}_Y}M$. Let $\mathbf{C}=C\otimes_{\mathbb{F}[U,U^{-1}]}\mathbb{F}[[U,U^{-1}]$.

Note that C can be identified as either $\mathrm{CF}^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w;M)$ or $\mathrm{CF}^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z;M)$ by ignoring either j or i respectively. There is thus a chain homotopy equivalence $\Phi\colon C\to C$ which takes $C\{j<s\}$ into $C\{i<s\}$ for any s and therefore descends to a homotopy equivalence $C\{j\geq s\}\to C\{i\geq s\}$. (Note that there is no control on how Φ interacts with the second filtration on each complex.) The map Φ also extends naturally to $\mathbf{C}$.

For each $s\in\mathbb{Z}$, let $A_s^+=C\{\max(i,j-s)\geq 0\}$, and let $B^+=C\{i\geq 0\}=\mathrm{CF}^+(Y,\mathfrak{s};M)$. Define maps

$$v_s^\infty,h_s^\infty\colon C\to C$$

as follows: v_s^∞ is the identity, and h_s^∞ is multiplication by U^s followed by Φ. It is easy to verify that these maps descend to

$$v_s^+,h_s^+\colon A_s^+\to B^+,$$

defined just as in [29]. (That is, v_s^+ is the projection onto $C\{i\geq 0\}$, and h_s^+ is the projection onto $C\{j\geq s\}$, followed by multiplication by U^s to identify this with $C\{j\geq 0\}$, followed by Φ.)

Let

$$D_{0,s}^\infty\colon C[t^{\pm1}]\to C[t^{\pm1}]$$

be the map of $\mathcal{H}_Y[t^{\pm1}]\otimes\mathbb{F}[U,U^{-1}]$–modules given by

$$D_{0,s}^\infty=v_s^\infty+t\cdot h_s^\infty=1+tU^s\Phi.$$

This descends to a map

$$D_{0,s}^+\colon A_s^+[t^{\pm1}]\to B^+[t^{\pm1}],$$

given by

$$D_{0,s}^+=v_s^++t\cdot h_s^+.$$

Let $\underline{X}_{0,s}^+$ (resp. $\underline{X}_{0,s}^\infty$) denote the mapping cone of $D_{0,s}^+$ (resp. $D_{0,s}^\infty$). Let $\underline{\mathbf{X}}_{0,s}^\infty$ be the U-completion of $\underline{X}_{0,s}^\infty$, which can be viewed as the mapping cone of the extension of $D_{0,s}^\infty$ to $\mathbf{C}[t^{\pm1}]$. Clearly, $\varprojlim(\underline{X}_{0,s}^+,U)=\underline{\mathbf{X}}_{0,s}^\infty$.

The surgery formula then states the following.

Theorem 2.3. *For any $s\in\mathbb{Z}$, there are isomorphisms of relatively graded $\mathcal{H}_Y\otimes\mathbb{F}[U]$–modules*

$$H_*(\underline{X}_{0,s}^+)\cong\mathrm{HF}^+(Y_0,\mathfrak{s}_{0,s};M[t^{\pm1}]),\tag{2.13}$$

$$H_*(\underline{\mathbf{X}}_{0,s}^\infty)\cong\mathbf{HF}^\infty(Y_0,\mathfrak{s}_{0,s};M[t^{\pm1}]).\tag{2.14}$$

If $M = M_A$ for some summand $A \subset H^1(Y)$ and $s = 0$, we also have

$$H_*(\underline{X}^\infty_{0,0}) \cong \mathrm{HF}^\infty(Y_0, \mathfrak{s}_{0,0}; M[t^{\pm 1}]). \tag{2.15}$$

Moreover, under each isomorphism, the map

$$\underline{F}^\circ_{W_0,\mathfrak{t}_{0,s}}\colon\ \mathrm{HF}^\circ(Y, \mathfrak{s}; M)[t^{\pm 1}] \to \mathrm{HF}^\circ(Y_0, \mathfrak{s}_{0,s}; M[t^{\pm 1}])$$

is given (up to a power of t) by the inclusion of the subcomplex $B^\circ \subset \underline{X}^\circ_{0,s}$.

Proof. We begin with (2.13). Just as in the untwisted case, the large surgery formula, states that for m sufficiently large and $|s| \le m/2$, there is an identification of A^+_s with $\mathrm{CF}^+(Y_m(K), \mathfrak{s}_{m,s}; M)$ such that the maps v^+_s and h^+_s compute $F^+_{W'_m,\mathfrak{t}_{m,s}}$ and $F^+_{W'_m,\mathfrak{t}_{m,s+m}}$, respectively. (See [26, Theorem 4.4] or [29, Theorem 2.3]; the proof goes through identically with coefficients in M.) Just as in [29], we use the procedure of "truncation" applied to the surgery exact sequence of Theorem 2.2 to obtain (2.13). Taking inverse limits and using (2.6) yields (2.14).

The proof of (2.15) follows just as in [17, Lemma 4.10], using Lemma 2.1 to observe that HF^∞ is finitely generated and free over $\mathbb{F}[U, U^{-1}]$. □

We also describe the H_1 action. For any $\zeta \in H_1(Y)$, the induced chain map $\mathcal{A}_\zeta\colon C \to C$ commutes with the differential on C and commutes up to homotopy with Φ: say $\mathcal{A}_\zeta\Phi + \Phi\mathcal{A}_\zeta = \partial H_\zeta + H_\zeta\partial$. We may then extend $\mathcal{A}_\zeta$ to $\underline{X}^\infty_{0,s}$ by the formula

$$\tilde{\mathcal{A}}_\zeta(a, b) = (\mathcal{A}_\zeta(a), tU^s H_\zeta(a) + A_\zeta(b)),$$

which descends to $\underline{X}^+_{0,s}$. These maps give an action of $\Lambda_*(H_1(Y)/Z^\perp_M)$ on $H_*(\underline{X}^\circ_{0,s})$. Moreover, there is an easy identification

$$H_1(Y)/Z^\perp_M \cong H_1(Y_0)/Z^\perp_{M[t^{\pm 1}]}.$$

Following through the proof of Theorem 2.3, it is not hard to see that these chain maps $\tilde{\mathcal{A}}_\zeta$ agree with the H_1 action on $\mathrm{HF}^+(Y_0, \mathfrak{s}_{0,s}; M[t^{\pm 1}])$. (See [9, Section 4.2].) We do not need to worry about defining a chain map associated to the homology class of the meridian of K in $H_1(Y_0)$, since its action on $\mathrm{HF}^+(Y_0, \mathfrak{s}_{0,s}; M[t^{\pm 1}])$ is 0.

For the purposes of this paper, the most important consequence of the preceding discussion is the following.

Proposition 2.4. *Let Y be a closed, oriented 3-manifold, let $\mathfrak{s}$ be a torsion spin^c structure on Y, and let M be a finitely generated $\mathcal{H}_Y$–module. Let K be a null homologous knot in Y, let W be the 2-handle cobordism from Y to $Y_0(K)$, let $\mathfrak{t}_0$ be the torsion extension of $\mathfrak{s}$ to W, and let $\mathfrak{s}_0 = \mathfrak{t}_0|_{Y_0(K)}$. Then the map*

$$F^\infty_{W,\mathfrak{t}_0}\colon\ \mathrm{HF}^\infty(Y, \mathfrak{s}; M) \to \mathrm{HF}^\infty(Y_0(K), \mathfrak{s}_0; M[t^{\pm 1}])$$

is an isomorphism.

Proof. We apply Theorem 2.3. Let $C = \mathrm{CFK}^\infty(Y, K; M)$ denote the doubly filtered knot Floer complex of (Y, K) with coefficients in M, and let $\Phi\colon C \to C$ denote the homotopy equivalence discussed above. The surgery formula then says that $\mathrm{HF}^\infty(Y_0, \mathfrak{s}_0; M[t^{\pm 1}])$ can be computed as the mapping cone of

$$(1 + t\Phi)\colon\ C[t^{\pm 1}] \to C[t^{\pm 1}],$$

and the map $F^\infty_{W,\mathfrak{t}}$ is given (up to a power of t) by the inclusion of C into the second copy of $C[t^{\pm 1}]$. From this description, it is easy to see that $\mathrm{HF}^\infty(Y_0, \mathfrak{s}_0; M[t^{\pm 1}]) \cong$

$\mathrm{HF}^\infty(Y,\mathfrak{s};M)$, where the action of t is given by Φ_*^{-1}, and that $F^\infty_{W,\mathfrak{t}}$ is an isomorphism. □

Remark 2.5. Following [30], we may adapt the results of this section (specifically Proposition 2.4) to the case where K is merely a rationally null homologous knot, representing a class of order $d>1$ in $H_1(Y)$. Assume that K has trivial self-linking, so that it has a well-defined 0-framing. We briefly sketch the necessary modifications to the surgery formula, leaving details to the reader. The set of relative spin^c structures for K, $\mathrm{Spin}^c(Y,K)$, forms an affine set for $H^2(Y,K)$. Spin^c structures on $Y_0(K)$ then correspond to the orbits of the action of $\mathrm{PD}[K_\lambda]$, the Poincaré dual of the 0-framed pushoff of K, each of which has d elements. The relative spin^c structures also correspond naturally with spin^c structures on the 2-handle cobordism $W_0(K)$.

Associated to each $\xi\in\mathrm{Spin}^c(Y,K)$, there is a doubly filtered complex $C_\xi=\mathrm{CFK}^\infty(Y,K,\xi;M)$ and quotients A^+_ξ and B^+_ξ (see [30] for all definitions). We also have maps

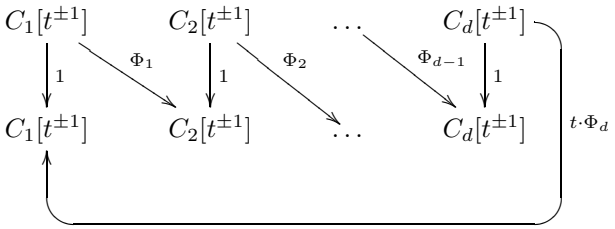
$$v^\infty_\xi\colon C_\xi\to C_\xi,\quad h^\infty_\xi\colon C_\xi\to C_{\xi+\mathrm{PD}[K_\lambda]},$$

which induce

$$v^+_\xi\colon A^+_\xi\to B^+_\xi,\quad h^\infty_\xi\colon A^+_\xi\to B^+_{\xi+\mathrm{PD}[K_\lambda]},$$

defined similarly to the above. Specifically, v^∞_ξ is the identity, and h^∞_ξ is a homotopy equivalence induced by Heegaard moves.

Suppose $\{\xi_1,\dots,\xi_d\}$ is the orbit corresponding to a torsion spin^c structure $\mathfrak{s}_0$ on Y_0, where $\xi_{i+1}=\xi+\mathrm{PD}[K_\lambda]$ (indices modulo d). Let $\mathfrak{s}_i$ be the (absolute) spin^c structure on Y extending ξ_i, and let $\mathfrak{t}_i$ be the spin^c structure on W_0 corresponding to ξ_i. Write C_i for C_{ξ_i} and Φ_i for $h^\infty_{\xi_i}$. The twisted mapping cone that computes $\mathrm{HF}^\infty(Y_0,\mathfrak{s}_0;M[t^{\pm1}])$ has the form



Up to isomorphism, it doesn't matter which of the Φ_i arrows comes with a power of t; the important point is that exactly one of them does. The map

$$F^\infty_{W_0,\mathfrak{t}_i}\colon \mathrm{HF}^\infty(Y,\mathfrak{s}_i;M)\to\mathrm{HF}^\infty(Y_0,\mathfrak{s}_0;M[t^{\pm1}])$$

is given (up to a power of t) by the inclusion of C_i in the bottom row. Just as in the proof of Proposition 2.4, we deduce that this map is an isomorphism. We will make use of this generalization in Section 4.

3. Twisted correction terms

In this section, we define the twisted correction terms. Throughout, let Y be a closed, oriented 3-manifold, and let $A\subset H^1(Y)$ be a direct summand on which the triple cup product vanishes. As above, let $\mathcal{H}_Y=\mathbb{F}[H^1(Y)]$, and let $M_A=\mathbb{F}[H^1(Y)/A]$, viewed as an $\mathcal{H}_Y$–module.

3.1. **Construction of the invariants.** To begin, we show that $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ is standard, in the following sense.

Theorem 3.1. *Let Y be a closed, oriented 3-manifold, and let $\mathfrak{s}$ be a torsion spin^c structure on Y. Let $A \subset H^1(Y)$ be a direct summand on which the triple cup product vanishes, and let $M_A = \mathbb{F}[H^1(Y)/A]$. Then*

$$\mathrm{HF}^\infty(Y,\mathfrak{s};M_A) \cong \Lambda^*(A)\otimes\mathbb{F}[U,U^{-1}]$$

as a $\Lambda^*(H_1(Y)/A^\perp)\otimes\mathbb{F}[U,U^{-1}]$–module (where the action of $\Lambda^*(H_1(Y)/A^\perp)$ on $\Lambda^*(A)$ is induced from the natural action of $\Lambda^*(H_1(Y)/\mathrm{Tors})$ on $\Lambda^*(H^1(Y))$).

Proof. We induct on the rank of $H^1(Y)/A$, starting with the extremal case when $A = H^1(Y)$ and the triple cup product on $H^1(Y)$ vanishes identically. The statement in this case follows from [17], as explained in [15, Theorem 3.2].

For the induction, assume that $H^1(Y)/A \neq 0$. Let $J \subset Y$ be a knot representing a primitive homology class in $A^\perp \subset H_1(Y)$ such that $\langle \beta, [J]\rangle = 1$ for some $\beta \in H^1(Y) \smallsetminus A$. Let Z be obtained by surgery on J with some arbitrary framing, and let $K \subset Z$ denote the core of the surgery solid torus, so that $Y = Z_0(K)$. Let W be the 2-handle cobordism from Z to Y, and let $\iota_Y\colon Y \to W$ and $\iota_Z\colon Z \to W$ be the inclusions.

The map $\iota_Z^*\colon H^1(W) \to H^1(Z)$ is an isomorphism, and $\iota_Y^* \circ (\iota_Z^*)^{-1}$ restricts to an injection on A. Let $A' \subset H^1(Z)$ be the image of this restriction, and let $M_{A'} = \mathbb{F}[H^1(Z)/A']$. Then $M_A \cong M_{A'}(W) \cong M_{A'}[t^{\pm 1}]$. Also, let $\mathfrak{s}'$ be the restriction to Z of the unique spin^c structure on W that extends $\mathfrak{s}$.

By the induction hypothesis,

$$\mathrm{HF}^\infty(Z,\mathfrak{s}';M_{A'}) \cong \Lambda^*(A')\otimes\mathbb{F}[U,U^{-1}]$$

as a $\Lambda^*(H_1(Z)/A'^\perp)\otimes\mathbb{F}[U,U^{-1}]$–module. The result then follows from Proposition 2.4. □

Remark 3.2. As noted in the proof of Lemma 2.1 above, the other extremal case of Theorem 3.1 was proven by Ozsváth and Szabó [27, Theorem 10.12]: when $A = 0$, the totally twisted homology satisfies

$$\mathrm{HF}^\infty(Y,\mathfrak{s};\mathcal{H}_Y) \cong \mathbb{F}[U,U^{-1}].$$

Remark 3.3. Note that Theorem 3.1 does *not* describe the structure of $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ as an $\mathcal{H}_Y$–module. The action of any element of $H^1(Y)$ is a grading-preserving automorphism of $\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)$ that commutes with the action of $\Lambda^*(H_1(Z)/A^\perp)\otimes\mathbb{F}[U,U^{-1}]$, but in principle this map need not be the identity.

We may now define the d invariant that we use below, which is analogous to the d_{top} invariant defined in [16, Definition 3.3]. We make use of notation from [15]. First, given any finitely generated, free abelian group V and any $\Lambda^*(V)$–module N, define $\mathcal{Q}^V(N) = N/(V\cdot N)$ and $\mathcal{K}^V(N) = \{n \in N \mid v\cdot n = 0\ \forall v \in V\}$ (i.e., the quotient and kernel of the action of V, respectively). We sometimes omit the superscripts if they are understood from context.

Definition 3.4. Let Y be a closed, oriented 3-manifold, $\mathfrak{s}$ a torsion spin^c structure, and $A \subset H^1(Y)$ a subspace on which the triple cup product vanishes. Let

$$\pi\colon\ \mathrm{HF}^\infty(Y,\mathfrak{s};M_A) \to \mathrm{HF}^+(Y,\mathfrak{s};M_A)$$

denote the canonical map. Then there are isomorphisms

$$\mathcal{Q}^{H_1(Y)/A^\perp}(\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)) \cong \mathbb{F}[U,U^{-1}], \tag{3.1}$$

$$\mathcal{Q}^{H_1(Y)/A^\perp}(\mathrm{im}(\pi)) \cong \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U], \tag{3.2}$$

such that the induced map

$$\bar{\pi}\colon\ \mathcal{Q}^{H_1(Y)/A^\perp}(\mathrm{HF}^\infty(Y,\mathfrak{s};M_A)) \to \mathcal{Q}^{H_1(Y)/A^\perp}(\mathrm{im}(\pi))$$

is the natural projection. The *correction term* $d(Y,\mathfrak{s};M_A) \in \mathbb{Q}$ is defined as minimal grading in which $\bar{\pi}$ is nontrivial or, equivalently, as the grading of $1 \in \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$ under the identification (3.2). The *shifted correction term* is defined as

$$\tilde{d}(Y,\mathfrak{s};M_A) = d(Y,\mathfrak{s};M_A) - \mathrm{rank}(A) + \frac{b_1(Y)}{2}. \tag{3.3}$$

If $H^2(Y)$ is torsion-free, so that Y has a unique torsion spin^c structure, we sometimes omit $\mathfrak{s}$ from the notation.

When $A = H^1(Y)$ (so that $M_A = \mathbb{F}$) and the triple cup product vanishes identically, $d(Y,\mathfrak{s};\mathbb{F}) = d_{\mathrm{top}}(Y,\mathfrak{s})$. When $A = 0$ (so that $M_A = \mathcal{H}_Y$), $d(Y,\mathfrak{s};\mathcal{H}_Y)$ is precisely the invariant $\underline{d}$ defined by Behrens and Golla [1].

Example 3.5. When $Y = S^1 \times S^2$, it is easy to compute directly from a Heegaard diagram that

$$d(Y;\mathbb{F}) = \frac{1}{2}, \qquad \tilde{d}(Y;\mathbb{F}) = 0,$$
$$d(Y;\mathcal{H}_Y) = -\frac{1}{2}, \qquad \tilde{d}(Y;\mathcal{H}_Y) = 0.$$

Example 3.6. If Y is obtained by 0-surgery on a knot $K \subset S^3$, then $\tilde{d}(Y;\mathbb{F})$ and $\tilde{d}(Y;\mathcal{H}_Y)$ are both determined by the knot Floer complex of K. Specifically, combining [1, Example 3.9], [25, Proposition 4.12], and [23, Proposition 1.6], we have

$$\tilde{d}(Y;\mathbb{F}) = d(Y;\mathbb{F}) - \frac{1}{2} = d_{\mathrm{top}}(Y) - \frac{1}{2} = d(S^3_1(K)) = -2V_0(K), \tag{3.4}$$

$$\tilde{d}(Y;\mathcal{H}_Y) = d(Y;\mathcal{H}_Y) + \frac{1}{2} = d_{\mathrm{bot}}(Y) + \frac{1}{2} = d(S^3_{-1}(K)) = 2V_0(\bar{K}), \tag{3.5}$$

where V_0 is a nonnegative integer invariant defined by Ni and Wu [23, Section 2.2], and $\bar{K}$ denotes the mirror of K. (The second inequality in (3.5), proven by Behrens and Golla, is special to the case of 0-surgery on knots in S^3.)

In particular, if K is either the right-handed trefoil $T_{2,3}$ or its positive, untwisted Whitehead double $D(T_{2,3})$, then $V_0(K) = 1$ and $V_0(\bar{K}) = 0$. The statement for $T_{2,3}$ is a straightforward computation; the statement for $D(T_{2,3})$ follows from the fact that $\mathrm{CFK}^\infty(D(T_{2,3}))$ is isomorphic to $\mathrm{CFK}^\infty(T_{2,3})$ plus an acyclic summand that does not affect V_0 [8, Proposition 6.1]. Hence, if Y is obtained by 0-surgery

on either of these knots, we have

$$d(Y;\mathbb{F}) = -\frac{3}{2}, \qquad \tilde{d}(Y;\mathbb{F}) = -2,$$
$$d(Y;\mathcal{H}_Y) = -\frac{1}{2}, \qquad \tilde{d}(Y;\mathcal{H}_Y) = 0,$$
$$d(-Y;\mathbb{F}) = \frac{1}{2}, \qquad \tilde{d}(-Y;\mathbb{F}) = 0,$$
$$d(-Y;\mathcal{H}_{-Y}) = \frac{3}{2}, \qquad \tilde{d}(-Y;\mathcal{H}_{-Y}) = 2.$$

(The results for 0-surgery on the trefoil were also proven earlier by Ozsváth and Szabó [25].)

Example 3.7. Let T^3 denote the 3-torus. Because the triple cup product on $H^1(T^3)$ is nonvanishing, the invariant $d(T^3; M_A)$ is only defined when $\operatorname{rank} A = 0$, 1, or 2. When $A = 0$, [25, Proposition 8.5] shows that

$$d(T^3;\mathcal{H}_{T^3}) = \frac{1}{2}, \qquad \tilde{d}(T^3;\mathcal{H}_{T^3}) = 2.$$

On the other hand, we will see below in Example 4.14 that when $\operatorname{rank} A = 1$ or 2, $\tilde{d}(T^3; M_A) = 0$. Since any automorphism of $H^1(T^3)$ can be realized by a self-diffeomorphism of T^3, it suffices to compute these invariants for a single subspace A of either rank. Note also that T^3 admits orientation-reversing diffeomorphisms, so the the same statements hold with either orientation on T^3.

3.2. **Relation with untwisted invariants.** We now describe the relationship between Definition 3.4 and the invariants defined in [15]. Suppose the triple cup product on $H^1(Y)$ vanishes identically, so that the untwisted homology group $\mathrm{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$ is standard:

$$\mathrm{HF}^\infty(Y,\mathfrak{s};\mathbb{F}) \cong \Lambda^* H^1(Y) \otimes \mathbb{F}[U,U^{-1}]$$

as a $\Lambda^*(H_1(Y)/\operatorname{Tors}) \otimes \mathbb{F}[U,U^{-1}]$–module. In [15], we defined an "intermediate correction term" $d(Y,\mathfrak{s},V)$ associated to each subspace $V \subset H_1(Y)$. In particular, $d_{\mathrm{top}}(Y,\mathfrak{s}) = d(Y,\mathfrak{s},\{0\})$ and $d_{\mathrm{bot}}(Y,\mathfrak{s}) = d(Y,\mathfrak{s},H_1(Y))$. The two constructions are related as follows.

Proposition 3.8. *If the triple cup product on $H^1(Y)$ vanishes identically, then for each summand $A \subset H^1(Y)$, we have*

$$d(Y,\mathfrak{s};M_A) \le d(Y,\mathfrak{s},A^\perp), \tag{3.6}$$

and therefore

$$d(Y,\mathfrak{s};M_A) \le d_{\mathrm{bot}}(Y,\mathfrak{s}) + \operatorname{rank}(A), \tag{3.7}$$

$$\tilde{d}(Y,\mathfrak{s};M_A) \le d_{\mathrm{bot}}(Y,\mathfrak{s}) + \frac{b_1(Y)}{2}. \tag{3.8}$$

(The case where $A = 0$ was proven by Behrens and Golla [1, Proposition 3.8].)

Proof. To begin, note that $M_A = \mathbb{F}[H^1(Y)/A]$ is a commutative ring with unit, not just a module over $\mathcal{H}_Y$, and the projection $\mathcal{H}_Y \to M_A$ is a ring homomorphism. Let $n = b_1(Y)$ and $k = \operatorname{rank} A$; we assume $n > 0$. For concreteness, let $a_1,\dots,a_n$ be a basis for $H^1(Y)$ such that $a_1,\dots,a_k$ are a basis for A. Let $\zeta_1,\dots,\zeta_n$ be the dual basis for $H_1(Y)/\operatorname{Tors}$, so that $A^\perp = \operatorname{Span}(\zeta_{k+1},\dots,\zeta_n)$.

Let $C_* = \mathrm{CF}^\infty(Y, \mathfrak{s}; M_A)$, with differential denoted by ∂. As a simplification, let us shift the homological grading on C_* so that it lies in $\mathbb{Z}$ (rather than $\mathbb{Z}+q$ for some rational number q). Furthermore, if $A = 0$, so that $\mathrm{HF}^\infty(Y, \mathfrak{s}; M_A) \cong \mathbb{F}[U, U^{-1}]$, we assume that the nonzero groups are in even grading. If we consider $\mathbb{F}$ as an M_A–module, where each element of $H^1(Y)/A$ acts as the identity, then by definition, $H_q(C_*) = \mathrm{HF}^\infty_q(Y, \mathfrak{s}; M_A)$, while $H_q(C_* \otimes_{M_A} \mathbb{F}) = \mathrm{HF}^\infty_q(Y, \mathfrak{s})$.

Since the untwisted $\mathrm{HF}^\infty(Y, \mathfrak{s})$ is standard, we have

$$H_q(C \otimes_{M_A} \mathbb{F}) \cong \mathbb{F}^{2^{n-1}}.$$

By Theorem 3.1, if $k = 0$, we have

$$H_q(C_*) \cong \begin{cases} \mathbb{F} & q \text{ even}, \\ 0 & q \text{ odd}, \end{cases} \tag{3.9}$$

while if $k > 0$, we have

$$H_q(C_*) \cong \mathbb{F}^{2^{k-1}} \tag{3.10}$$

for all q. As in Remark 3.3, right now we only know that the isomorphisms (3.9) and (3.10) hold on the level of groups; we shall see shortly that they hold on the level of M_A–modules as well.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{HF}^\infty(Y, \mathfrak{s}; M_A) & \xrightarrow{\pi_{M_A}} & \mathrm{HF}^+(Y, \mathfrak{s}; M_A) \\ \downarrow{\scriptstyle \otimes 1} & & \downarrow{\scriptstyle \otimes 1} \\ \mathrm{HF}^\infty(Y, \mathfrak{s}; M_A) \otimes_{M_A} \mathbb{F} & \xrightarrow{\pi_{M_A} \otimes 1} & \mathrm{HF}^+(Y, \mathfrak{s}; M_A) \otimes_{M_A} \mathbb{F} \\ \downarrow{\scriptstyle g^\infty} & & \downarrow{\scriptstyle g^+} \\ \mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F}) & \xrightarrow{\pi_{\mathbb{F}}} & \mathrm{HF}^+(Y, \mathfrak{s}; \mathbb{F}) \end{array} \tag{3.11}$$

Here π_{M_A} and $\pi_{\mathbb{F}}$ are the usual maps $\mathrm{HF}^\infty \to \mathrm{HF}^+$, and g^∞ and g^+ are the natural change-of-coefficient maps. As above, let $\mathcal{K}^{A^\perp}\,\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$ denote the subspace of $\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$ consisting of all $x \in \mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$ for which $\zeta \cdot x = 0$ for all $\zeta \in A^\perp$, and let $J^+(Y, \mathfrak{s}, A^\perp)$ denote the image of the restriction of $\pi_{\mathbb{F}}$ to $\mathcal{K}^{A^\perp}\,\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$. The invariant $d(Y, \mathfrak{s}, A^\perp)$ is defined to be the minimal grading in which the induced map

$$\bar{\pi}_{\mathbb{F}}\colon\ \mathcal{Q}^{H_1(Y)/A^\perp}\mathcal{K}^{A^\perp}\,\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F}) \to \mathcal{Q}^{H_1(Y)/A^\perp} J^+(Y, \mathfrak{s}, A^\perp)$$

is nontrivial.

Claim 1. The upper-left vertical map in (3.11) is an isomorphism; equivalently, the action of M_A on $\mathrm{HF}^\infty(Y, \mathfrak{s}; M_A)$ is trivial.

Claim 2. The map g^∞ is injective with image equal to $\mathcal{K}^{A^\perp}\,\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$.

Assuming these two claims, it follows that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{HF}^\infty(Y, \mathfrak{s}; M_A) & \xrightarrow{\pi_{M_A}} & \mathrm{HF}^+(Y, \mathfrak{s}; M_A) \\ \downarrow{\scriptstyle \cong} & & \downarrow \\ \mathcal{K}^{A^\perp}\,\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F}) & \xrightarrow{\pi_{\mathbb{F}}} & J^+(Y, \mathfrak{s}, A^\perp) \end{array}$$

which then descends to

$$\begin{array}{ccc} \mathcal{Q}\,\mathrm{HF}^\infty(Y,\mathfrak{s};M_A) & \xrightarrow{\pi_{M_A}} & \mathcal{Q}\,\mathrm{HF}^+(Y,\mathfrak{s};M_A) \\ \cong\downarrow & & \downarrow \\ \mathcal{QK}^{A^\perp}\,\mathrm{HF}^\infty(Y,\mathfrak{s};\mathbb{F}) & \xrightarrow{\pi_{\mathbb{F}}} & \mathcal{Q}J^+(Y,\mathfrak{s},A^\perp). \end{array}$$

By comparing the definitions of the two d invariants, it is then easy to see that (3.6) holds. Equations (3.7) and (3.8) then follow from [15, Proposition 3.4].

To prove Claims 1 and 2, we use the universal coefficients spectral sequence, which we explain in some detail because morphisms of spectral sequences can be confusing. To begin, take a free resolution of $\mathbb{F}$ as an M_A–module:

$$0 \leftarrow \mathbb{F} \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \dots \xleftarrow{d_{n-k}} F_{n-k} \leftarrow 0,$$

where $F_p = M_A^{\binom{n-k}{p}}$. Consider the complex

$$\mathcal{C}_s = \bigoplus_{p+q=s} C_q \otimes F_p$$

(nonzero only when $0 \le p \le n-k$) with differential $D_s\colon \mathcal{C}_s \to \mathcal{C}_{s-1}$ given by

$$D_s = \sum_{p+q=s} (-1)^s \partial_q \otimes d_p.$$

Observe that $H_s(\mathcal{C}_*) \cong H_s(C_* \otimes_{M_A} \mathbb{F}) \cong \mathrm{HF}^\infty_s(Y,\mathfrak{s};\mathbb{F})$.

The spectral sequence comes from considering the p filtration on $\mathcal{C}_*$, so that the differential on the E^r has (p,q)–bigrading $(-r, r-1)$. The E^1 page is given by

$$E^1_{p,q} \cong H_q(C_*) \otimes_{M_A} F_p \cong H_q(C_*)^{\binom{n-k}{p}}, \tag{3.12}$$

and the E^2 page is given by

$$E^2_{p,q} \cong \mathrm{Tor}_p^{M_A}(H_q(C_*),\mathbb{F}). \tag{3.13}$$

In particular, in the $p=0$ column, we have

$$E^1_{0,*} \cong \mathrm{HF}^\infty(Y,\mathfrak{s};M_A) \quad \text{and} \quad E^2_{0,*} \cong \mathrm{HF}^\infty(Y,\mathfrak{s};M_A) \otimes_{M_A} \mathbb{F}, \tag{3.14}$$

and the upper-left vertical map in (3.11) is the natural quotient map. Furthermore, there is a filtration

$$0 = G^{-1}_s \subset G^0_s \subset G^1_s \subset \cdots \subset G^{n-k}_s = H_s(\mathcal{C}_*) \tag{3.15}$$

so that

$$E^\infty_{p,q} \cong G^p_{p+q}/G^{p-1}_{p+q}; \tag{3.16}$$

in particular, the $p=0$ column $E^\infty_{0,*}$ is identified with the subspace G^0_*. The map g^∞ from (3.11) is given by the identification (3.14), followed by the successive quotients taking $E^2_{0,*} \to E^\infty_{0,*}$, followed by the inclusion of G^0_* into $H_*(\mathcal{C}_*) \cong \mathrm{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$.

By (3.12), we have

$$\dim_{\mathbb{F}} E^1_{p,q} = \binom{n-k}{p} \dim_{\mathbb{F}} H_q(C_*) = \begin{cases} \binom{n-k}{p} & k=0,\ q \text{ even} \\ 0 & k=0,\ q \text{ odd} \\ \binom{n-k}{p} 2^{k-1} & k>0. \end{cases} \tag{3.17}$$

Summing over $p+q=s$, we see that

$$\sum_{p+q=s} \dim_{\mathbb{F}} E^1_{p,q} = 2^{n-1} = \dim_{\mathbb{F}} H_s(\mathcal{C}_*) = \sum_{p+q=s} \dim_{\mathbb{F}} E^\infty_{p,q},$$

which implies that the spectral sequence collapses at the E^1 page. Looking in the $p=0$ column, we see that the successive quotient maps

$$E^1_{0,q} \to E^2_{0,q} \to \cdots \to E^\infty_{0,q}$$

are all isomorphisms, which proves Claim 1 and the injectivity statement of Claim 2.

It remains to identify G^0_* with $\mathcal{K}^{A^\perp}\operatorname{HF}^\infty(Y,\mathfrak{s};M)$. For each $i=1,\dots,n$, there is a chain map $\mathcal{A}_{\zeta_i}\colon C_* \to C_{*-1}$; these give rise to the action of H_1. As noted above, the maps $\mathcal{A}_{\zeta_{k+1}},\dots,\mathcal{A}_{\zeta_n}$ are null-homotopic (see [11, Remark 5.2]), but they are still defined on the chain level. Indeed, we extend each A_{ζ_i} to a map on $\mathcal{C}_*$ by tensoring with the identity map on F_*. The maps $\mathcal{A}_{\zeta_i *}$ induced on the homology of $\mathcal{C}_*$ (which, as noted above, is isomorphic to $\operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$) then generate the action of $\Lambda^*(H_1(Y)/\operatorname{Tors})$ on $\operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$; that is, $\zeta_i \cdot x = \mathcal{A}_{\zeta_i *}(x)$.

Moreover, the restriction of $A_{\zeta_i *}$ to G^0_* agrees with the action of ζ_i on $\operatorname{HF}^\infty(Y,\mathfrak{s};M_A)$. For $i=k+1,\dots,n$, this action vanishes, so

$$G^0_q \subset \mathcal{K}^{A^\perp}_q \operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$$

for each grading q. Because $\operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$ is standard, we can see that $\mathcal{K}^{A^\perp}\operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$ is a free $\mathbb{F}[U,U^{-1}]$–module with

$$\operatorname{rank}_{\mathbb{F}[U,U^{-1}]} \mathcal{K}^{A^\perp}\operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F}) = \frac{1}{2^{n-k}} \operatorname{rank}_{\mathbb{F}[U,U^{-1}]} \operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F}) = 2^k$$

since taking the kernel of each ζ_i for $i=k+1,\dots,n$ cuts down the rank by a factor of 2. If $k=0$, then $\mathcal{K}^{A^\perp}\operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$ is a single tower, with 0 and $\mathbb{F}$ in alternating gradings; otherwise, $\mathcal{K}^{A^\perp}\operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$ has dimension 2^{k-1} in each grading. By (3.17), we see that

$$\dim_{\mathbb{F}} \mathcal{K}^{A^\perp}_q \operatorname{HF}^\infty(Y,\mathfrak{s};\mathbb{F}) = \dim_{\mathbb{F}} E^1_{0,q} = \dim_{\mathbb{F}} G^0_q,$$

so $G^0_* = \mathcal{K}^{A^\perp}\operatorname{HF}^\infty(Y,\mathfrak{s};M)$ as required. □

Example 3.9. Although we do not know of an actual manifold Y for which equality fails to hold in (3.6), this seems unlikely to be true in general. Figure 1 represents the totally twisted complexes $\operatorname{CF}^\infty(Y,\mathfrak{s};\mathcal{H}_Y)$ and $\operatorname{CF}^+(Y,\mathfrak{s};\mathcal{H}_Y)$ for a hypothetical $(Y,\mathfrak{s})$ with $b_1(Y)=1$. Writing $\mathcal{H}_Y=\mathbb{F}[t^{\pm1}]$, we view $\operatorname{CF}^\infty(Y,\mathfrak{s};\mathcal{H}_Y)$ as a complex over $\mathbb{F}[t^{\pm1},U^{\pm1}]$ generated by a,b,c,d. A solid arrow represents a coefficient of 1 in the differential, a dashed arrow represents $1-t$, and a dotted arrow represents $1-t^2=(1-t)^2$. The pattern repeats infinitely in both directions in CF^∞ and infinitely upward in CF^+. The numbers at the left represent the Maslov gradings, which we have chosen in analogy with $S^1\times S^2$.

Clearly, $\operatorname{HF}^\infty(Y,\mathfrak{s};\mathcal{H}_Y)$ is isomorphic to $\mathbb{F}[U,U^{-1}]$, generated as an $\mathbb{F}[t^{\pm1},U^{\pm1}]$–module by $(1-t)b+Uc$, with the relation $(1-t)((1-t)b+Uc)=0$. Also, $\operatorname{HF}^+(Y,\mathfrak{s};\mathcal{H}_Y)$ is generated as an $\mathbb{F}[t^{\pm1}]$–module by $\{U^n((1-t)b+Uc) \mid n\le -1\}$ along with b, with the relations that $(1-t)U^n((1-t)b+Uc)=0$ and $(1-t)^2b=0$.

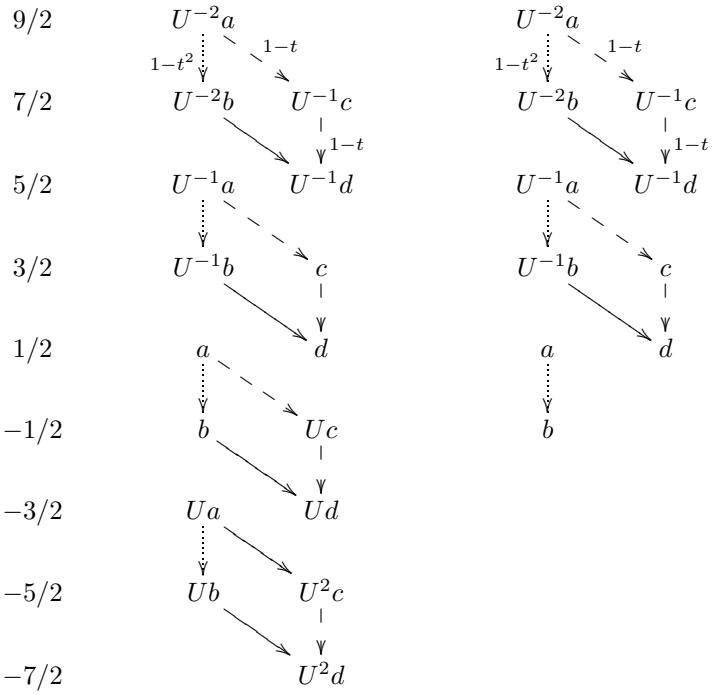


FIGURE 1. CF^∞ (left) and CF^+ (right) for a hypothetical manifold $(Y,\mathfrak{s})$ with $d(Y,\mathfrak{s};\mathcal{H}_Y) < d_{\mathrm{bot}}(Y,\mathfrak{s})$, as in Example 3.9.

We thus see that $d(Y,\mathfrak{s};\mathcal{H}_Y) = -1/2$. (Notice that the short exact sequence

$$0 \to \mathrm{im}(\pi) \to \mathrm{HF}^+(Y,\mathfrak{s};\mathcal{H}_Y) \to \mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s};\mathcal{H}_Y) \to 0$$

does not split over $\mathcal{H}_Y$, although it does split over $\mathbb{F}$.)

On the other hand, we can also view the same figure as representing the untwisted complexes $\mathrm{CF}^\infty(Y,\mathfrak{s};\mathbb{F})$ and $\mathrm{CF}^+(Y,\mathfrak{s};\mathbb{F})$. Now the solid arrows represent the differential, the dashed arrows represent the chain map $\mathcal{A}_\zeta$ associated to a generator of $H_1(Y)$, and the dotted arrows represent a chain null-homotopy of $\mathcal{A}_\zeta^2$. Here, $\mathrm{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$ is generated over $\mathbb{F}[U,U^{-1}]$ by a and c, with c generating the "bottom tower" $\mathcal{K}^{\mathcal{H}_Y}\,\mathrm{HF}^\infty(Y,\mathfrak{s};\mathbb{F})$. Also, $\mathrm{HF}^+(Y,\mathfrak{s};\mathbb{F})$ is generated as an $\mathbb{F}$–module by $\{U^n a, U^n c \mid n \le 0\} \cup \{b\}$. We therefore deduce that $d_{\mathrm{bot}}(Y,\mathfrak{s}) = 3/2$.

Moreover, it is not hard to modify the construction to make the difference $d_{\mathrm{bot}}(Y,\mathfrak{s}) - d(Y,\mathfrak{s};\mathcal{H}_Y)$ arbitrarily large.

As noted above in Example 3.6, Behrens and Golla [1, Example 3.9] proved that for any knot $K \subset S^3$, $\underline{d}(S^3_0(K)) = d_{\mathrm{bot}}(S^3_0(K))$. Thus, a manifold for which (3.6) is a strict inequality, as in the putative example just discussed, would not be homology cobordant to 0-surgery on any knot in S^3.

Remark 3.10. Proposition 3.8 implies that the twisted d invariants can in principle give stronger constraints on intersection forms of 4-manifolds bounded by Y than the untwisted d invariants from [15]. For instance, if Z is a negative semi-definite 4-manifold bounded by Y such that the restriction map $H^1(Z) \to H^1(Y)$ is trivial

and $\mathfrak{t}$ is a spin^c structure on Z whose restriction to Y is torsion, Ozsváth and Szabó [25, Theorem 9.15] showed that

$$c_1(\mathfrak{t})^2 + b_2^-(Z) \leq 4d_{\mathrm{bot}}(Y, \mathfrak{t}|_Y) + 2b_1(Y),$$

and Behrens and Golla [1, Theorem 1.1] proved an analogous statement with $d(Y, \mathfrak{t}|_Y; \mathcal{H}_Y)$ in place of $d_{\mathrm{bot}}(Y, \mathfrak{t}|_Y)$. The latter result is thus a potentially stronger bound. Likewise, for any summand $A \subset H^1(Y)$, it is not hard to prove a stronger analogue of [15, Theorem 4.7] using $d(Y, \mathfrak{s}; M_A)$ in place of $d(Y, \mathfrak{s}, A^\perp)$ (where A is chosen such that $A^\perp = V$).

Definition 3.11. For a 3-manifold Y and a torsion spin^c structure $\mathfrak{s}$ on Y, we say that $(Y, \mathfrak{s})$ is *d-simple* if the triple cup product on $H^1(Y)$ vanishes identically (so that $\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$ is standard) and for every subspace $A \subset H^1(Y)$, equality holds in (3.8), i.e.,

$$\tilde{d}(Y, \mathfrak{s}; A) = d_{\mathrm{bot}}(Y, \mathfrak{s}) + \frac{b_1(Y)}{2}. \tag{3.18}$$

We say that Y is d-simple if $(Y, \mathfrak{s})$ is d-simple for each torsion spin^c structure $\mathfrak{s}$ on Y.

If $(Y, \mathfrak{s})$ is d-simple, then for each $A \subset H^1(Y)$, (3.6) is an equality, meaning that

$$d(Y, \mathfrak{s}, A^\perp) = d_{\mathrm{bot}}(Y, \mathfrak{s}) + \mathrm{rank}(A).$$

In other words, the untwisted d invariants of $(Y, \mathfrak{s})$ are simple in the sense of [15, Corollary 3.5]. In particular, we have

$$d_{\mathrm{top}}(Y, \mathfrak{s}) = d_{\mathrm{bot}}(Y, \mathfrak{s}) + b_1(Y),$$

and hence

$$\tilde{d}(Y, \mathfrak{s}; M_A) = d_{\mathrm{top}}(Y, \mathfrak{s}) - \frac{b_1(Y)}{2}. \tag{3.19}$$

3.3. **Orientation reversal.** Unlike with the original d invariant for rational homology spheres, Examples 3.5 and 3.6 show that $\tilde{d}(Y, \mathfrak{s}; M_A)$ does not determine $\tilde{d}(-Y, \mathfrak{s}; M_A)$. The only relation between these quantities occurs in the extremal cases where $A = 0$ or $A = H^1(Y)$.

Proposition 3.12. *Let Y be a closed, oriented 3-manifold and let $\mathfrak{s}$ be a torsion spin^c structure on Y. Then*

$$\tilde{d}(Y, \mathfrak{s}; \mathcal{H}_Y) + \tilde{d}(-Y, \mathfrak{s}; \mathcal{H}_{-Y}) \geq 0. \tag{3.20}$$

If the triple cup product on $H^1(Y)$ vanishes so that $\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$ is standard, then

$$\tilde{d}(Y, \mathfrak{s}; \mathbb{F}) + \tilde{d}(-Y, \mathfrak{s}; \mathbb{F}) \leq 0. \tag{3.21}$$

Proof. Behrens and Golla [1, Proposition 3.7] showed that the totally twisted d invariant is additive under connected sums; thus,

$$d(Y, \mathfrak{s}; \mathcal{H}_Y) + d(-Y, \mathfrak{s}; \mathcal{H}_{-Y}) = d(Y \# -Y, \mathfrak{s} \# \mathfrak{s}; \mathcal{H}_{Y\#-Y}).$$

Note that $Y \# -Y$ is the boundary of the 4-manifold $Z = (Y \smallsetminus B^3) \times [0, 1]$, whose intersection form vanishes identically. Applying [1, Theorem 1.1] (see Remark 3.10 above), we deduce that

$$0 \leq d(Y \# -Y, \mathfrak{s} \# \mathfrak{s}; \mathcal{H}_{Y\#-Y}) + b_1(Y).$$

This implies (3.20).

For the second statement, we have $d(Y,\mathfrak{s};\mathbb{F}) = d_{\mathrm{top}}(Y,\mathfrak{s})$ by definition and $d(-Y,\mathfrak{s};\mathbb{F}) = d_{\mathrm{top}}(-Y,\mathfrak{s}) = -d_{\mathrm{bot}}(Y,\mathfrak{s})$ by [16, Proposition 3.7]. Moreover, by [16, Lemma 3.5], we have $d_{\mathrm{top}}(Y,\mathfrak{s}) \le d_{\mathrm{bot}}(Y,\mathfrak{s}) + b_1(Y)$. Thus

$$d(Y,\mathfrak{s};\mathbb{F}) + d(-Y,\mathfrak{s};\mathbb{F}) \le b_1(Y),$$

which implies (3.21). □

Motivated by Proposition 3.12, we make the following definition.

Definition 3.13. Given a closed, oriented 3-manifold Y and a torsion spin^c structure $\mathfrak{s}$ on Y, we say that $(Y,\mathfrak{s})$ is *d-symmetric* if for every summand $A \subset H^1(Y)$ on which the triple cup product vanishes, we have

$$d(-Y,\mathfrak{s};M_A) = -d(Y,\mathfrak{s};M_A). \tag{3.22}$$

We say Y is d-symmetric if $(Y,\mathfrak{s})$ is d-symmetric for every torsion spin^c structure $\mathfrak{s}$ on Y.

Combining (3.18) and (3.19) with [16, Proposition 3.7], we immediately deduce that if both $(Y,\mathfrak{s})$ and $(-Y,\mathfrak{s})$ are d-simple, then they are both d-symmetric. (We do not know of an example where $(Y,\mathfrak{s})$ is d-simple while $(-Y,\mathfrak{s})$ is not.)

3.4. **Connected sums.** The behavior of twisted d-invariants under connected sums is also potentially more complicated than in the untwisted setting. Given summands $A_1 \subset H^1(Y_1)$ and $A_2 \subset H^1(Y_2)$, $A_1 \oplus A_2$ is naturally a summand of $H^1(Y_1 \# Y_2)$. Evidently, if the triple cup product vanishes on each A_i, then it vanishes on $A_1 \oplus A_2$ as well. Adapting the usual proof of additivity of d invariants (see [25, Theorem 4.3], [16, Proposition 3.8], [15, Proposition 4.3]), it is straightforward to see that

$$d(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2, M_{A_1\oplus A_2}) \ge d(Y_1,\mathfrak{s}_1,M_{A_1}) + d(Y_2,\mathfrak{s}_2,M_{A_2}). \tag{3.23}$$

Proving the reverse inequality requires orientation reversal, which is not available. However, if Y_1 and Y_2 are d-simple, then we have

$$\begin{aligned} d(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2, M_{A_1\oplus A_2}) &\le d(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2, (A_1 \oplus A_2)^\perp) \\ &= d(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2, A_1^\perp \oplus A_2^\perp) \\ &= d(Y_1,\mathfrak{s}_1,A_1^\perp) + d(Y_2,\mathfrak{s}_2,A_2^\perp) \\ &= d(Y_1,\mathfrak{s}_1,M_{A_1}) + d(Y_2,\mathfrak{s}_2,M_{A_2}), \end{aligned}$$

so equality holds.

Proposition 3.14. *If Y is of the form $Q \# n(S^1 \times S^2)$, where Q is a rational homology sphere and $n \ge 0$, then Y is d-simple and therefore d-symmetric. Indeed, if $\mathfrak{s} = \mathfrak{t} \# \mathfrak{t}_0 \# \cdots \# \mathfrak{t}_0$, where $\mathfrak{t}$ is a spin^c structure on Q and $\mathfrak{t}_0$ is the unique torsion spin^c structure on $S^1 \times S_2$, then for any subspace $A \subset H^1(Y)$, we have*

$$\tilde{d}(Y,\mathfrak{s};M_A) = d(Q,\mathfrak{t}).$$

Proof. Clearly, any rational homology sphere is d-simple, as is $S^1 \times S^2$. Given a subspace $A \subset \#n(S^1 \times S^2)$, there is a self-diffeomorphism of $\#n(S^1 \times S^2)$ such that the pullback of A can be viewed as $A_1 \oplus \cdots \oplus A_n$, where each A_i is a subspace of H^1 of the ith $S^1 \times S^2$ summand. (That is, any change of basis on $H^1(\#n(S^1 \times S^2))$ can be realized geometrically by handleslides.) The result then follows. □

3.5. **Congruence condition.** Given a closed, oriented 3-manifold Y and a torsion spin^c structure $\mathfrak{s}$, there is an invariant $\rho(Y,\mathfrak{s}) \in \mathbb{Q}/2\mathbb{Z}$, defined to be the congruence class of

$$\frac{c_1(\mathfrak{t})^2 - \sigma(W)}{4}, \tag{3.24}$$

where $(W,\mathfrak{t})$ is any spin^c 4–manifold with boundary $(Y,\mathfrak{s})$. (Here $\sigma(W)$ denotes the signature of W.) In this section we prove the following.

Proposition 3.15. *For any closed, oriented 3-manifold Y, any torsion spin^c structure $\mathfrak{s}$, and any subspace $A \subset H^1(Y)$ on which the triple cup product vanishes, we have*

$$\tilde{d}(Y,\mathfrak{s};M_A) \equiv \rho(Y,\mathfrak{s}) \pmod{2\mathbb{Z}}.$$

The case where Y is a rational homology sphere was proven by Ozsváth and Szabó [25, Theorem 1.2].

Proof. Suppose $b_1(Y) = n$ and $\operatorname{rank}(A) = k$. In the proof of Theorem 3.1, we inductively produced a spin^c cobordism $(W_1,\mathfrak{t}_1)\colon (Y_1,\mathfrak{s}_1) \to (Y,\mathfrak{s})$ by successively attaching $n-k$ 2-handles along 0-framed knots. The untwisted homology $\mathrm{HF}^\infty(Y_1,\mathfrak{s}_1;\mathbb{F})$ is standard, and the cobordism induces an isomorphism

$$F^\infty_{W_1,\mathfrak{t}_1}\colon \mathrm{HF}^\infty(Y_1,\mathfrak{s}_1;\mathbb{F}) \to \mathrm{HF}^\infty(Y,\mathfrak{s};M_A).$$

Since $c_1(\mathfrak{t}_1)$ is torsion by construction, the grading shift of $F^\infty_{W_1,\mathfrak{t}_1}$ is equal to $\frac{k-n}{2}$. It follows that

$$d(Y,\mathfrak{s};M_A) \equiv d_{\mathrm{top}}(Y_1,\mathfrak{s}_1) - \frac{n-k}{2} \pmod{2\mathbb{Z}}.$$

By [16, Lemma 3.5],

$$d_{\mathrm{top}}(Y_1,\mathfrak{s}_1) \equiv d_{\mathrm{bot}}(Y_1,\mathfrak{s}_1) + k \pmod{2\mathbb{Z}}.$$

Next, we find a cobordism $(W_2,\mathfrak{t}_2)\colon (Y_2,\mathfrak{s}_2) \to (Y_1,\mathfrak{s}_1)$, where Y_2 is a rational homology sphere, again obtained by successively attaching k 2-handles along 0-framed knots. By [25, Proposition 9.3], the map

$$F^\infty_{W_2,\mathfrak{t}_2}\colon \mathrm{HF}^\infty(Y_2,\mathfrak{s}_2) \to \mathrm{HF}^\infty(Y_1,\mathfrak{s}_1)$$

is injective and takes $\mathrm{HF}^\infty(Y_2,\mathfrak{s}_2) \cong \mathbb{Z}[U,U^{-1}]$ to the bottom tower in $\mathrm{HF}^\infty(Y_1,\mathfrak{s}_1)$. Hence,

$$d_{\mathrm{bot}}(Y_1,\mathfrak{s}_1) \equiv d(Y_2,\mathfrak{s}_2) - \frac{k}{2} \pmod{2\mathbb{Z}}.$$

Combining these congruences, we see that

$$\begin{aligned} d(Y,\mathfrak{s};M_A) &\equiv d(Y_2,\mathfrak{s}_2) + k - \frac{n}{2} \pmod{2\mathbb{Z}}, \\ \tilde{d}(Y,\mathfrak{s};M_A) &\equiv d(Y_2,\mathfrak{s}_2) \equiv \rho(Y_2,\mathfrak{s}_2) \pmod{2\mathbb{Z}}. \end{aligned}$$

Finally, the spin^c cobordisms $(W_1,\mathfrak{t}_1)$ and $(W_2,\mathfrak{t}_2)$, each of which has signature 0, give us $\rho(Y,\mathfrak{s}) = \rho(Y_1,\mathfrak{s}_1) = \rho(Y_2,\mathfrak{s}_2)$, which concludes the proof. □

4. Cross-sections of open 4-manifolds

4.1. **Topological preliminaries.** Throughout this section, we will be working with open 4-manifolds obtained as the infinite cyclic covers of homology $S^1 \times S^3$s. For greater generality, we begin by stating the salient algebraic-topology properties of such manifolds and then work throughout with open manifolds satisfying those properties. (The terminology is motivated by Hughes and Ranicki [10], who have a stronger, homotopy-theoretic notion that they call a *ribbon*.)

Definition 4.1. A *homology ribbon* is a smooth, connected, orientable, open 4-manifold $\tilde{X}$ with two ends that satisfies the following properties:

(1) $H_3(\tilde{X}) \cong \mathbb{Z}$.
(2) The intersection form on $H_2(\tilde{X}) \cong H_c^2(\tilde{X})$ vanishes.
(3) For each end ϵ of $\tilde{X}$ and any field $\mathbf{k}$, we have $H_1(\tilde{X}, \epsilon; \mathbf{k}) \cong H_2(\tilde{X}, \epsilon; \mathbf{k}) \cong 0$.

We call $\tilde{X}$ a homology $S^3 \times \mathbb{R}$ if (in addition to the above properties) $H_1(\tilde{X}) = H_2(\tilde{X}) = 0$ and a rational homology $S^3 \times \mathbb{R}$ if $H_1(\tilde{X}; \mathbb{Q}) = H_2(\tilde{X}; \mathbb{Q}) = 0$.

Proposition 4.2. *Let X be a smooth, closed, oriented 4-manifold such that $H_*(X) \cong H_*(S^1 \times S^3)$, and let $p\colon \tilde{X} \to X$ denote the universal abelian cover of X, with deck transformation group $\mathbb{Z}$. Then $\tilde{X}$ is a homology ribbon, and $p_*\colon H_3(\tilde{X}) \to H_3(X) \cong \mathbb{Z}$ is an isomorphism.*

Proof. For property (1), Milnor [19, Remark 1] shows that $H_3(\tilde{X}) \cong H^0(\tilde{X}) \cong \mathbb{Z}$.

Let $\tau\colon \tilde{X} \to \tilde{X}$ denote a generator of the deck transformation group. Note that $H_*(\tilde{X})$ is a $\mathbb{Z}[t, t^{-1}]$–module, where t acts by τ_*. The Milnor exact sequence

$$\cdots \to H_i(\tilde{X}) \xrightarrow{1-t} H_i(\tilde{X}) \xrightarrow{p_*} H_i(X) \to H_{i-1}(\tilde{X}) \to \cdots \tag{4.1}$$

implies that $1 - t$ is an isomorphism on $H_1(\tilde{X})$ and $H_2(\tilde{X})$ and zero on $H_3(\tilde{X})$. It follows that $p_*\colon H_3(\tilde{X}) \to H_3(X)$ is an isomorphism.

For each integer $m \geq 1$, let $\tilde{X} \xrightarrow{q_m} X_m \xrightarrow{p_m} X$ denote the intermediate m-fold cover of X with deck group $\mathbb{Z}/m$. A standard argument shows that when m is a prime power p^k, $H_*(X_m; \mathbb{Z}_p) \cong H_*(S^1 \times S^3; \mathbb{Z}_p)$, and therefore $H_2(X_m; \mathbb{Q}) = 0$ since $H_*(X_m; \mathbb{Z})$ is finitely generated. In particular, the intersection form on $H_2(X_m; \mathbb{Z})$ is trivial. Now, given any classes $a, b \in H_2(\tilde{X})$, let Σ_a, Σ_b be closed, oriented, embedded surface representatives that intersect transversally. For m sufficiently large, the restriction of q_m to $\Sigma_a \cup \Sigma_b$ is a diffeomorphism onto its image, so $a \cdot b = q_{m*}(a) \cdot q_{m*}(b) = 0$. This proves property (2).

For property (3), it is easiest to work with simplicial homology. Choose a finite triangulation of X, and lift it to a locally finite triangulation of $\tilde{X}$. After possibly replacing τ by τ^{-1}, we may assume that τ shifts in the direction of the end ϵ. Then

$$C_*(\tilde{X}, \epsilon; \mathbf{k}) \cong C_*(\tilde{X}; \mathbf{k}) \otimes_{\mathbf{k}[t,t^{-1}]} \mathbf{k}[[t, t^{-1}].$$

Since $\mathbf{k}[[t, t^{-1}]$ is as flat as a $\mathbf{k}[t, t^{-1}]$–module, we have

$$H_*(\tilde{X}, \epsilon; \mathbf{k}) \cong H_*(\tilde{X}; \mathbf{k}) \otimes_{\mathbf{k}[t,t^{-1}]} \mathbf{k}[[t, t^{-1}].$$

Note that $H_*(\tilde{X}; \mathbf{k})$ is finitely generated as a $\mathbf{k}[t, t^{-1}]$–module. Since $1 - t$ acts as an isomorphism on $H_j(\tilde{X}; \mathbf{k})$ for $j = 1, 2$, we have

$$H_j(\tilde{X}; \mathbf{k}) \cong \bigoplus_{l=1}^{n} \mathbf{k}[t, t^{-1}]/(p_l),$$

where each p_l is a nonzero, monic polynomial. Since p_l is invertible in $\mathbf{k}[[t,t^{-1}]$, we deduce that

$$H_j(\tilde{X};\mathbf{k}) \otimes_{\mathbf{k}[t,t^{-1}]} \mathbf{k}[[t,t^{-1}] = 0,$$

as required. □

For the rest of this section, unless otherwise specified, $\tilde{X}$ will denote an arbitrary homology ribbon without the requirement that it is the cover of a homology $S^1 \times S^3$.

A *cross-section* of $\tilde{X}$ is a connected, smoothly embedded, oriented 3-manifold Y representing a generator of $H_3(\tilde{X})$. To find such a cross-section, one can proceed as in Example 3 of the introduction to [10], which treats the case of a manifold with a single end. Using a proper exhaustion of $\tilde{X}$, one finds a smooth proper map $f\colon \tilde{X} \to \mathbb{R}$ with the ends going to $\pm\infty$. (The choice of a generator of $H_3(\tilde{X}) \cong H_c^1(\tilde{X})$ determines which end goes to $+\infty$.) Then there is a component of the preimage of a regular value that is a cross-section. Denote the closures of the components of $\tilde{X} \smallsetminus Y$ by L_Y and R_Y so that $Y = \partial L_Y = -\partial R_Y$ as an oriented manifold. Note that reversing the orientation of Y (and hence the class in $H_3(\tilde{X})$ that Y represents) interchanges the roles of L_Y and R_Y: $L_{-Y} = R_Y$ and $R_{-Y} = L_Y$.

If disjoint cross-sections Y_1 and Y_2 represent the same homology class, we say that Y_2 is to the right of Y_1, denoted $Y_1 \prec Y_2$, if $Y_2 \subset R_{Y_1}$. This notion depends on which homology class Y_1 and Y_2 represent; if $Y_1 \prec Y_2$, then $-Y_2 \prec -Y_1$. In what follows, whenever we write $Y_1 \prec Y_2$, we implicitly assume that Y_1 and Y_2 represent the same homology class. If $Y_1 \prec Y_2$, let $W(Y_1,Y_2)$ be the closure of $\tilde{X} \smallsetminus (L_{Y_1} \cup R_{Y_2})$; this is an oriented cobordism from Y_1 to Y_2.

Fix a torsion spin^c structure $\mathfrak{s}$ on $\tilde{X}$. By abuse of notation, the restriction of $\mathfrak{s}$ to any cross-section Y or any cobordism $W(Y_1,Y_2)$ will also be denoted by $\mathfrak{s}$. If $\tilde{X}$ is in fact the $\mathbb{Z}$–cover of X, a rational homology $S^1 \times S^3$, then let $\mathfrak{s}_X$ denote the pullback of the unique spin^c structure on X.

We begin with a few basic facts concerning the algebraic topology of cross-sections. First, note that the Mayer–Vietoris sequence for $\tilde{X} = L_Y \cup R_Y$ shows that $H_3(L_Y) \cong H_3(R_Y) \cong \mathbb{Z}$. Next, consider the long exact sequence on cohomology (both ordinary and compactly supported) for the pair (L_Y, Y):

(4.2)

$$\begin{array}{ccccccccccc}
H_c^0(L_Y) & \longrightarrow & H^0(Y) & \longrightarrow & H_c^1(L_Y,Y) & \longrightarrow & H_c^1(L_Y) & \xrightarrow{j_Y^c} & H^1(Y) & \xrightarrow{\delta_Y^c} & H_c^2(L_Y,Y) \\
\downarrow & & \downarrow{=} & & \downarrow & & \downarrow{\kappa_L} & & \downarrow{=} & & \downarrow \\
H^0(L_Y) & \xrightarrow{\cong} & H^0(Y) & \xrightarrow{0} & H^1(L_Y,Y) & \longrightarrow & H^1(L_Y) & \xrightarrow{j_Y} & H^1(Y) & \xrightarrow{\delta_Y} & H^2(L_Y,Y)
\end{array}$$

Note that $H_c^0(L_Y) = 0$ since L_Y is noncompact. By Poincaré duality, $H_c^1(L_Y,Y) \cong H_3(L_Y) \cong \mathbb{Z}$. By looking at the same diagram with coefficients in $\mathbb{Z}/p$ for each prime p, we deduce that the coboundary $H^0(Y) \to H_c^1(L_Y,Y)$ is an isomorphism, the map j_Y^c is injective, and the quotient $H^1(Y)/H_c^1(L_Y) \cong \mathrm{im}(\delta_Y^c)$ is torsion-free. (That is, $H_c^1(L_Y)$ is a direct summand of $H^1(Y)$.) In particular, $H_c^1(L_Y)$ is a finitely generated, free abelian group; let $b_1^c(L_Y)$ denote its rank. Moreover, the map $\kappa_L\colon H_c^1(L_Y) \to H^1(L_Y)$ is injective. (Analogous statements hold with R_Y in place of L_Y.)

Next, consider the Mayer–Vietoris sequences (in both ordinary and compactly supported cohomology) for the decomposition $\tilde{X} = L_Y \cup_Y R_Y$, and the natural

maps between them:

$$\begin{array}{ccccccccc} H^0(Y) & \longrightarrow & H^1_c(\tilde{X}) & \longrightarrow & H^1_c(L_Y)\oplus H^1_c(R_Y) & \longrightarrow & H^1(Y) & \longrightarrow & H^2_c(\tilde{X}) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \kappa_X} & & \downarrow{\scriptstyle \kappa_L\oplus\kappa_R} & & \downarrow{\scriptstyle =} & & \downarrow \\ H^0(Y) & \longrightarrow & H^1(\tilde{X}) & \longrightarrow & H^1(L_Y)\oplus H^1(R_Y) & \longrightarrow & H^1(Y) & \longrightarrow & H^2(\tilde{X}) \end{array} \tag{4.3}$$

(The construction of the upper sequence is most readily carried out if one uses the simplicial version of cohomology with compact supports, as described in [7, §3.3]; we remark that exactness uses the fact that Y is compact.) Just as before, we deduce that the map $H^1_c(L_Y)\oplus H^1_c(R_Y)\to H^1(Y)$ is injective, meaning that the images of $H^1_c(L_Y)$ and $H^1_c(R_Y)$ in $H^1(Y)$ intersect trivially. However, this map need not be surjective, as in the following example.

Example 4.3. Suppose X is a homology $S^1\times S^3$ obtained as the mapping torus of a self-diffeomorphism of some 3-manifold Y. Then $\tilde{X}\cong Y\times\mathbb{R}$. If we consider $Y=Y\times\{0\}$ as a cross-section of $\tilde{X}$, it is easy to check that $H^*_c(L_Y)\cong H^*_c(R_Y)\cong 0$. In particular, the coboundary $H^1(Y)\to H^2(\tilde{X})$ in (4.3) is an isomorphism.

In the case where $\tilde{X}$ is a rational homology $S^3\times\mathbb{R}$, the situation simplifies considerably, so that we can use ordinary rather than compactly supported cohomology throughout.

Lemma 4.4. *If $\tilde{X}$ was a rational homology $S^3\times\mathbb{R}$, then:*

1. *The maps $\kappa_L\colon H^1_c(L_Y)\to H^1(L_Y)$ and $\kappa_R\colon H^1_c(R_Y)\to H^1(R_Y)$ are isomorphisms, so j_Y and j^c_Y are identified.*
2. *We have $H^1(Y)\cong H^1(L_Y)\oplus H^1(R_Y)\cong H^1_c(L_Y)\oplus H^1_c(R_Y)$.*
3. *The sequence*

$$0\longrightarrow H^1(L_Y)\xrightarrow{j_Y} H^1(Y)\xrightarrow{\delta_Y} H^2(L_Y,Y)\longrightarrow 0$$

is short exact and splits, and $H^2(L_Y,Y)\cong H^1(R_Y)$.

Proof. If $\tilde{X}$ is a rational homology $S^3\times\mathbb{R}$, then $H^1(\tilde{X})=0$, and $H^2(\tilde{X})$ and $H^2_c(\tilde{X})$ are both torsion groups. It follows that the maps $H^1(L_Y)\oplus H^1(R_Y)\to H^1(Y)$ and $H^1_c(L_Y)\oplus H^1_c(R_Y)\to H^1(Y)$ in (4.3) are both isomorphisms, so $\kappa_L\oplus\kappa_R$ is as well. Moreover, by the exact sequence for $(\tilde{X},R_Y)$ and excision, $H^1(R_Y)\cong H^2(\tilde{X},R_Y)\cong H^2(L_Y,Y)$. The restriction map $H^1(R_Y)\to H^1(Y)$ provides a splitting for the short exact sequence. □

Returning to the general case, it is useful to consider one more version of the Mayer–Vietoris sequence, which again is most easily proved using simplicial cohomology as in [22, §25]. If ϵ denotes the left end of X corresponding to L_Y, we have an exact sequence

$$H^1(\tilde{X},\epsilon)\to H^1_c(L_Y)\oplus H^1(R_Y)\to H^1(Y)\to H^2(\tilde{X},\epsilon). \tag{4.4}$$

In particular, if we take coefficients in $\mathbb{Q}$ and apply property (3) from Definition 4.1 together with universal coefficients, we see that

$$H^1(Y;\mathbb{Q})\cong H^1_c(L_Y;\mathbb{Q})\oplus H^1(R_Y;\mathbb{Q}). \tag{4.5}$$

Finally, we recall the locally finite homology groups of a (noncompact) polyhedral space Z, $H^{lf}_*(Z)$. These can be defined in greatest generality using an inverse

limit; see Laitinen [14, §2]. When Z has a locally finite triangulation, it is easiest to use the simplicial version: the chain group $C_i^{lf}(Z)$ consists of possibly infinite sums of i-simplices, and the differential is defined in the usual way. The universal coefficient theorem [14, Proposition 2.8] relating locally finite homology and compactly supported cohomology takes a slightly unusual form: for any principal ideal domain R, there is an exact sequence

$$0 \to \mathrm{Ext}(H_c^{n+1}(Z), R) \to H_n^{lf}(Z; R) \to \mathrm{Hom}(H_c^n(Z), R) \to 0. \tag{4.6}$$

Additionally, if Z is an n-dimensional manifold, then there is a Poincaré duality isomorphism $H_k^{lf}(Z) \cong H^{n-k}(Z)$ [14, Theorem 3.1].

4.2. **Correction terms.** We will be considering Heegaard Floer homology with coefficients in the module

$$\mathcal{L}_Y := M_{H_c^1(L_Y)} = \mathbb{F}[H^1(Y)/H_c^1(L_Y)] = \mathbb{F}[\mathrm{im}(\delta_Y^c)]. \tag{4.7}$$

The key observation is the following.

Proposition 4.5. *Let $\tilde{X}$ be a homology ribbon and let $\mathfrak{s}$ be a torsion spin^c structure on $\tilde{X}$. For any cross-section Y, the restriction of the triple cup product form on $H^1(Y)$ to the image of $H_c^1(L_Y)$ vanishes identically. Therefore,*

$$\mathrm{HF}^\infty(Y, \mathfrak{s}; \mathcal{L}_Y) \cong \Lambda^*(H_c^1(L_Y)) \otimes \mathbb{F}[U, U^{-1}]$$

as a $\Lambda^*(H_1(Y)/(H_c^1(L_Y)^\perp)) \otimes \mathbb{F}[U, U^{-1}]$–module. Moreover, we may naturally identify $H_1(Y)/(H_c^1(L_Y))^\perp$ with $H_1^{lf}(L_Y)/\mathrm{Tors}$. Analogous statements hold with R_Y in place of L_Y.

Proof. First, note that there is a fundamental class $[L_Y, Y] \in H_4^{lf}(L_Y, Y)$ which maps to the fundamental class $[Y] \in H_3(Y)$ under the boundary map. (If we are given a locally finite triangulation of L_Y with Y as a subcomplex, the fundamental class is given as the sum of all the 4-simplices, with signs determined by the orientation, and the boundary map $H_4^{lf}(L_Y, Y) \to H_3(Y)$ is just the usual simplicial boundary.) We then argue just as we would if L_Y were a compact manifold: For any $\alpha_1, \alpha_2, \alpha_3 \in H_c^1(L_Y)$, we have

$$\begin{aligned}
\langle i^*(\alpha_1) \cup i^*(\alpha_2) \cup i^*(\alpha_3), [Y] \rangle &= \langle i^*(\alpha_1 \cup \alpha_2 \cup \alpha_3), [Y] \rangle \\
&= \langle \alpha_1 \cup \alpha_2 \cup \alpha_3, i_*([Y]) \rangle \\
&= \langle \alpha_1 \cup \alpha_2 \cup \alpha_3, i_*(\partial([L_Y, Y])) \rangle \\
&= 0.
\end{aligned}$$

The second and third lines make use of the pairing between $H_c^3(L_Y)$ and $H_3^{lf}(L_Y)$ given in (4.6) (taking $R = \mathbb{Z}$). The same exact sequence also yields the identification

$$H_1(Y)/(H_c^1(L_Y))^\perp \cong H_1^{lf}(L_Y)/\mathrm{Tors}\,.$$

Finally, the statement about HF^∞ follows directly from Theorem 3.1. □

Proposition 4.5 allows us to consider the twisted correction term $d(Y, \mathfrak{s}; \mathcal{L}_Y)$ and the shifted version $\tilde{d}(Y, \mathfrak{s}; \mathcal{L}_Y)$. By definition, we have

$$\tilde{d}(Y, \mathfrak{s}; \mathcal{L}_Y) = d(Y, \mathfrak{s}; \mathcal{L}_Y) + \frac{b_1(Y) - 2b_1^c(L_Y)}{2}. \tag{4.8}$$

Observe that when $\tilde{X}$ is a homology $S^3 \times \mathbb{R}$, Lemma 4.4 implies that $b_1^c(L_Y) = b_1(L_Y)$ and $\chi(L_Y) = b_1(Y) - 2b_1(L_Y)$, and hence

$$\tilde{d}(Y, \mathfrak{s}; \mathcal{L}_Y) = d(Y, \mathfrak{s}; \mathcal{L}_Y) + \frac{\chi(L_Y)}{2}. \tag{4.9}$$

Our choice of coefficients in $\mathcal{L}_Y$ (as opposed to the analogous construction using the cohomology of R_Y) is justified by the following lemma.

Lemma 4.6. *Let $\tilde{X}$ be a homology ribbon. Suppose Y_1, Y_2 are disjoint, homologous cross-sections of $\tilde{X}$ with $Y_1 \prec Y_2$, and let $W = W(Y_1, Y_2)$ be the cobordism between them. Then $\mathcal{L}_{Y_1}(W) \cong \mathcal{L}_{Y_2}$.*

Proof. According to (4.7), we are trying to show that

$$\mathbb{F}[\mathrm{im}(\delta^c_{Y_1})] \otimes_{\mathbb{F}[H^1(Y_1)]} \mathbb{F}[K(W)] \cong \mathbb{F}[\mathrm{im}(\delta^c_{Y_2})].$$

We prove this by constructing an exact sequence

$$H^1(Y_1) \to \mathrm{im}(\delta^c_{Y_1}) \oplus K(W) \to \mathrm{im}(\delta^c_{Y_2}) \to 0$$

as follows.

First, by property (2) in Definition 4.1, the intersection form on $H_2(W)$ vanishes identically. Therefore, the map $j_W \colon H^2(W, \partial W) \to H^2(W)$ vanishes identically, meaning that $K(W) = H^2(W, \partial W)$.

Consider the following commutative diagram, whose rows and columns are exact:

$$\begin{array}{ccccccc}
H^1(Y_1 \cup Y_2, Y_2) & \hookrightarrow & H^1(Y_1 \cup Y_2) & \twoheadrightarrow & H^1(Y_2) & \xrightarrow{0} & H^2(Y_1 \cup Y_2, Y_2) \\
\downarrow{=} & & \downarrow{\gamma} & & \downarrow{\delta^c_{Y_2}} & & \downarrow{=} \\
H^1(Y_1 \cup Y_2, Y_2) & \xrightarrow{f} & H^2_c(L_{Y_2}, Y_1 \cup Y_2) & \xrightarrow{g} & H^2_c(L_{Y_2}, Y_2) & \xrightarrow{h} & H^2(Y_1 \cup Y_2, Y_2) \\
 & & \downarrow & & \downarrow & & \\
 & & H^2_c(L_{Y_2}) & \xrightarrow{=} & H^2_c(L_{Y_2}) & &
\end{array}$$

We easily deduce that $\mathrm{im}(\delta^c_{Y_2}) \subset \mathrm{im}(g)$ and that $g^{-1}(\mathrm{im}(\delta^c_{Y_2})) = \mathrm{im}(\gamma)$; thus, the middle row gives rise to an exact sequence

$$H^1(Y_1 \cup Y_2, Y_2) \xrightarrow{f} \mathrm{im}(\gamma) \xrightarrow{g} \mathrm{im}(\delta^c_{Y_2}) \longrightarrow 0.$$

Of course, $H^1(Y_1 \cup Y_2, Y_2) \cong H^1(Y_1)$.

Next, the Mayer–Vietoris sequence for the decomposition

$$(L_{Y_2}, Y_1 \cup Y_2) = (L_{Y_1}, Y_1) \cup (W, \partial W)$$

shows that

$$H^2_c(L_{Y_2}, Y_1 \cup Y_2) \cong H^2_c(L_{Y_1}, Y_1) \oplus H^2(W, \partial W).$$

Under this identification, it is easy to see that the image of γ is identified with $\mathrm{im}(\delta^c_{Y_1}) \oplus H^2(W, \partial W)$, as required. □

Proposition 4.7. *Let $\tilde{X}$ be a homology ribbon and let $\mathfrak{s}$ be a torsion spin^c structure on X. Suppose Y_1, Y_2 are disjoint, homologous cross-sections of $\tilde{X}$ with $Y_1 \prec Y_2$. Then*

$$\tilde{d}(Y_1, \mathfrak{s}; \mathcal{L}_{Y_1}) \leq \tilde{d}(Y_2, \mathfrak{s}; \mathcal{L}_{Y_2}). \tag{4.10}$$

TABLE 1. Summary of handle additions in the proof of Proposition 4.7. The convention is that $\Delta b_1(Y) = b_1(Y_2) - b_1(Y_1)$, etc. It is easy to see that $\Delta b_1(Y) - 2\Delta b_1^c(L_Y) = \chi(W)$ and that $\Delta b_1(Y) = \Delta b_1^c(L_Y) + \Delta b_1^c(R_Y)$.

Handle type	$\Delta b_1(Y)$	$\Delta b_1^c(L_Y)$	$\Delta b_1^c(R_Y)$	$\chi(W)$	$F^\infty_{W,\mathfrak{s}}$
1	$+1$	$+1$	0	-1	injective
2, $[K]$ torsion	$+1$	0	$+1$	$+1$	isomorphism
2, $[K]$ nontorsion	-1	-1	0	$+1$	surjective
3	-1	0	-1	-1	isomorphism

Proof. It suffices to assume that the cobordism $W = W(Y_1, Y_2)$ is given by a single-handle attachment. To be precise, let L'_{Y_1} denote the union of L_{Y_1} with the closure of a product neighborhood of Y, and assume that

$$L_{Y_2} = L'_{Y_1} \cup k\text{-handle},$$

where $k \in \{1, 2, 3\}$. When $k = 2$, because the intersection form on $\tilde{X}$ vanishes, L_{Y_2} cannot be obtained by attaching a 2-handle to a rationally null-homologous curve with nonzero framing.

The proof proceeds as follows. By Lemma 4.6, the cobordism W induces maps

$$F^\circ_{W,\mathfrak{s}}\colon\ \mathrm{HF}^\circ(Y_1, \mathfrak{s}; \mathcal{L}_{Y_1}) \to \mathrm{HF}^\circ(Y_2, \mathfrak{s}; \mathcal{L}_{Y_2}).$$

Since $c_1(\mathfrak{s})$ is torsion, the grading shift of $F^\circ_{W,\mathfrak{s}}$ is equal to $-\frac{\chi(W)}{2}$. Note that $\chi(W) = 1$ when $k = 2$ and -1 otherwise.

We will show in each case that $F^\infty_{W,\mathfrak{s}}$ descends to an isomorphism

$$\mathcal{Q}\,\mathrm{HF}^\infty(Y_1, \mathfrak{s}; \mathcal{L}_{Y_1}) \to \mathcal{Q}\,\mathrm{HF}^\infty(Y_2, \mathfrak{s}; \mathcal{L}_{Y_2}).$$

Thus, by the usual argument,

$$d(Y_1, \mathfrak{s}; \mathcal{L}_{Y_1}) \le d(Y_2, \mathfrak{s}; \mathcal{L}_{Y_2}) + \frac{\chi(W)}{2}. \tag{4.11}$$

At the same time, we will see in each case that

$$\chi(W) = (b_1(Y_2) - 2b_1^c(L_{Y_2})) - (b_1(Y_1) - 2b_1^c(L_{Y_1}))\,, \tag{4.12}$$

from which (4.10) follows.

We now consider the different values for k. A summary can be found in Table 1.

(1) If $L_{Y_2} \cong L'_{Y_1} \cup 1$-handle, then $Y_2 \cong Y_1 \# S^1 \times S^2$, so $b_1(Y_2) = b_1(Y_1) + 1$. By looking at the exact sequence on cohomology for the pair (L_{Y_2}, L_{Y_1}), we see that $H^1_c(L_{Y_2}) \cong H^1_c(L_{Y_1}) \oplus \mathbb{Z}$, where a generator of the $\mathbb{Z}$ factor maps to the Poincaré dual of $[\{\mathrm{pt}\} \times S^2]$ in $H^1(Y_2)$. Hence, $b_1^c(L_{Y_2}) = b_1^c(L_{Y_1}) + 1$, so (4.12) holds.

As in [28, §4.3], we have

$$\mathrm{HF}^\infty(Y_2, \mathfrak{s}_2; \mathcal{L}_{Y_2}) \cong \mathrm{HF}^\infty(Y_1, \mathfrak{s}_1; \mathcal{L}_{Y_1})[\tfrac{1}{2}] \oplus \mathrm{HF}^\infty(Y_1, \mathfrak{s}_1; \mathcal{L}_{Y_1})[-\tfrac{1}{2}],$$

where the action of the generator of $H_1(S^1 \times S^2)$ takes the first summand to the second, and the map $F^\infty_{W,\mathfrak{s}}$ is given by the inclusion of the first summand.

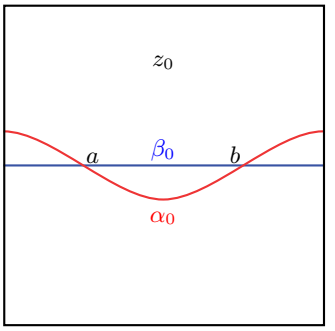


FIGURE 2. Standard Heegaard diagram $(T^2, \alpha_0, \beta_0, z_0)$ for $S^1 \times S^2$.

(2) If L_{Y_2} is obtained by attaching a 2-handle to L'_{Y_1} along a curve $K \subset Y_1$, there are two possibilities.

If K represents a torsion class in $H_1(Y)$, then the 2-handle must be attached along the rational longitude of K; otherwise, there would be a closed surface in $\tilde{X}$ with nontrivial self-intersection, which violates our assumptions. Thus, Y_2 is obtained by 0-surgery on K, and $b_1(Y_2) = b_1(Y_1) + 1$. Moreover, the inclusion $L_{Y_1} \to L_{Y_2}$ induces an isomorphism $H^1_c(L_{Y_2}) \to H^1_c(L_{Y_1})$. By Proposition 2.4 (and its extension to the rationally null-homologous case in Remark 2.5), $F^\infty_{W,\mathfrak{s}}$ is an isomorphism that respects the H_1 actions.

If K represents a nontorsion class in $H_1(Y_1)$, then $b_1(Y_2) = b_1(Y_1) - 1$. Equation (4.5) implies there is a class $\alpha \in H^1_c(L_{Y_1}; \mathbb{Q})$ such that $\langle \alpha|_{Y_1}, [K] \rangle = 1$, and therefore $[K]$ is also nontorsion in $H_1^{lf}(L_{Y_1})$. The restriction map $H^1_c(L_{Y_2}) \to H^1_c(L_{Y_1})$ is injective, with image equal to the set of elements that evaluate to 0 on $[K]$, so $b^c_1(L_{Y_2}) = b^c_1(L_{Y_1}) - 1$. Just as in the untwisted case (see [25, Proposition 9.3]), $F^\infty_{W,\mathfrak{s}}$ descends to an isomorphism

$$\mathrm{HF}^\infty(Y_1, \mathfrak{s}; \mathcal{L}_{Y_1})/([K] \cdot \mathrm{HF}^\infty(Y_1, \mathfrak{s}; \mathcal{L}_{Y_1})) \xrightarrow{\cong} \mathrm{HF}^\infty(Y_2, \mathfrak{s}; \mathcal{L}_{Y_2}),$$

which respects the H_1 actions.

(3) Suppose L_{Y_2} is obtained by attaching a 3-handle to L'_{Y_1} along an embedded, nonseparating sphere $S \subset Y_2$, which necessarily represents a primitive homology class. Then $Y_1 \cong Y_2 \# S^1 \times S^2$, so $b_1(Y_2) = b_1(Y_1) - 1$, and $H^1_c(L_{Y_2}) \cong H^1_c(L_{Y_1})$. As seen in Section 2.1, we have $\mathcal{L}_{Y_2} \cong \mathcal{L}_{Y_1}/(1-t) \cong \mathcal{L}_{Y_1}(W)$, where t is the class in $H^1(Y_1)$ Poincaré dual to $[S]$.

By [28, Lemma 4.11], we may represent Y_1 by a split Heegaard diagram

$$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z') = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \# (T^2, \alpha_0, \beta_0, z_0),$$

where $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ represents Y_2, $(T^2, \alpha_0, \beta_0, z_0)$ is a standard diagram for $S^1 \times S^2$ as shown in Figure 2, and the connected sum is taken in the regions containing the basepoints. The curves α_0, β_0 meet in two points a, b. For any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there is a pair of holomorphic bigons $\phi^\pm_{\mathbf{x}} \in \pi_2(\mathbf{x} \times \{a\}, \mathbf{x} \times \{b\})$. We may choose the additive assignment such that for each $\mathbf{x}$, the disks $\phi^+_{\mathbf{x}}$ and $\phi^-_{\mathbf{x}}$ contribute 1 and t, respectively, in the differential. Just as in Ozsváth and Szabó's proof of the Künneth formula for

connected sums [27, Theorem 6.2], with respect to a sufficiently stretched complex structure, $\mathrm{CF}^\infty(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z'; \mathcal{L}_{Y_1})$ is then isomorphic to the mapping cone

$$\mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z; \mathcal{L}_{Y_2})[t^{\pm 1}] \xrightarrow{1-t} \mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z; \mathcal{L}_{Y_2})[t^{\pm 1}],$$

where the two copies correspond to a and b respectively. The map $F^\infty_{W,\mathfrak{s}}$ is given on the chain level by setting $t = 1$ and projecting onto the second factor. It follows that

$$F^\infty_{W,\mathfrak{s}} \colon \mathrm{HF}^\infty(Y_1, \mathfrak{s}; \mathcal{L}_{Y_1}) \to \mathrm{HF}^\infty(Y_2, \mathfrak{s}; \mathcal{L}_{Y_2})$$

is an isomorphism.

In each of the three cases, it is easy to see that the induced maps on $\mathcal{Q}\,\mathrm{HF}^\infty$ are isomorphisms and that (4.12) holds, as required. □

Proposition 4.8. *If $\tilde{X}$ is a homology $S^3 \times \mathbb{R}$, then $\tilde{d}(Y, \mathfrak{s}; \mathcal{L}_Y)$ is an* even *integer.*

Proof. By Proposition 3.15, we know that $\tilde{d}(Y, \mathfrak{s}; \mathcal{L}_Y) \equiv \rho(Y, \mathfrak{s}) \pmod{2\mathbb{Z}}$, where $\rho(Y, \mathfrak{s})$ is defined by equation (3.24). Since the spin^c structure $\mathfrak{s}$ is a spin structure, we may take the manifold W in (3.24) to be a spin manifold, and so the term $c_1(\mathfrak{t})^2 = 0$. As in the proof of [37, Theorem 3.4], the signature of W is the same as the signature of the open manifold

$$W_\infty = W \cup_Y R_Y.$$

The vanishing of the homology of $\tilde{X}$ together with property (3) from Definition 4.1 implies that the intersection form on the spin manifold W_∞ is unimodular, and hence van der Blij's theorem [20, §5] says that its signature is divisible by 8. □

Remark 4.9. By turning the cobordism W around, it is also easy to see how the quantity $b_1^c(R_Y)$ behaves: we see that $b_1^c(R_{Y_2}) - b_1^c(R_{Y_1})$ equals 0 in the case of a 1-handle addition or a 2-handle addition along a nontorsion curve, 1 in the case of a 2-handle addition along a torsion curve, and -1 in the case of a 3-handle addition. In particular, we see from Table 1 that the quantity

$$b_1(Y) - b_1^c(L_Y) - b_1^c(R_Y)$$

is independent of the choice of cross-section Y. The Mayer–Vietoris sequence (top row of (4.3)) shows that this quantity equals the rank of the coboundary map $H^1(Y) \to H^2_c(\tilde{X})$.

4.3. **Invariants for homology $S^1 \times S^3$s.** We are now finally able to prove the main theorem from the introduction, which we restate as follows.

Theorem 4.10. *Let X be an oriented homology $S^1 \times S^3$, let $\tilde{X}$ be its infinite cyclic cover, and let $\mathfrak{s}_X$ be the spin^c structure on $\tilde{X}$ pulled back from X. Then for any cross-section Y of $\tilde{X}$, the shifted correction term $\tilde{d}(Y, \mathfrak{s}_X; \mathcal{L}_Y)$ depends only on the homology class of Y in $H_3(\tilde{X})$ or equivalently on its image $y \in H_3(X)$. We denote this number by $\tilde{d}(X, y)$; it is an invariant of X under orientation-preserving diffeomorphisms that preserve the choice of homology class.*

Proof. Fix a generator for $H_3(\tilde{X})$. Let τ be a generator of the deck transformation group such that for any two cross-sections Y, Y' representing the fixed generator, $\tau^{-n}(Y) \prec Y' \prec \tau^n(Y)$ for all n sufficiently large. For any $n \in \mathbb{Z}$, note that

$$\tilde{d}(\tau^n(Y), \mathfrak{s}_X; \mathcal{L}_{\tau^n(Y)}) = \tilde{d}(Y, \mathfrak{s}_X; \mathcal{L}_Y),$$

since the spin^c structure $\mathfrak{s}_X$ on $\tilde{X}$ is τ-invariant and the deck transformation τ^n takes L_Y to $L_{\tau^n(Y)}$. Thus, by Proposition 4.7, we have

$$\tilde{d}(Y, \mathfrak{s}_X; \mathcal{L}_Y) \leq \tilde{d}(Y', \mathfrak{s}; \mathcal{L}_{Y'}) \leq \tilde{d}(Y, \mathfrak{s}_X; \mathcal{L}_Y),$$

and hence equality holds. □

Next, we prove the symmetries stated in Proposition 1.2. It is more convenient to work in the more general setting of open manifolds. Given a homology ribbon $\tilde{X}$ equipped with a spin^c structure $\mathfrak{s}$ and any cross-section Y of $\tilde{X}$, define

$$\tilde{d}(\tilde{X}, Y) = \tilde{d}(Y, \mathfrak{s}; \mathcal{L}_Y). \tag{4.13}$$

(For convenience, we suppress the spin^c structure $\mathfrak{s}$ from the notation.) When $\tilde{X}$ is the $\mathbb{Z}$ cover of a homology $S^1 \times S^3$ X and $\mathfrak{s} = \mathfrak{s}_X$, then by definition $\tilde{d}(\tilde{X}, Y) = \tilde{d}(X, y)$.

There are two possible orientation changes to consider.

- If we leave the orientation on $\tilde{X}$ fixed but change the orientation of Y, the roles of L_Y and R_Y are interchanged: $L_{-Y} = R_Y$ and $R_{-Y} = L_Y$. According to our definition, we have

$$\tilde{d}(\tilde{X}, -Y) = \tilde{d}(-Y, \mathfrak{s}; \mathcal{R}_Y), \tag{4.14}$$

  where $\mathcal{R}_Y = \mathbb{F}[H^1(Y)/H^1(R_Y)]$.
- If we reverse both the orientations of both $\tilde{X}$ and Y, then the roles of L_Y and R_Y do not change, since $\partial(-L_Y) = -Y$. Thus, we may write

$$\tilde{d}(-\tilde{X}, -Y) = \tilde{d}(-Y, \mathfrak{s}; \mathcal{L}_Y). \tag{4.15}$$

  Combining this argument with the previous one, we deduce that

$$\tilde{d}(-\tilde{X}, Y) = \tilde{d}(Y, \mathfrak{s}; \mathcal{R}_Y). \tag{4.16}$$

Proof of Proposition 1.2. Suppose that Y is a cross-section of a homology ribbon $\tilde{X}$. If Y is a rational homology sphere, then $\tilde{d}(X, Y) = d(Y, \mathfrak{s})$. By inspecting equations (4.13) through (4.16), it is immediate that

$$\tilde{d}(\tilde{X}, Y) = \tilde{d}(-\tilde{X}, Y) = -\tilde{d}(\tilde{X}, -Y) = -\tilde{d}(-\tilde{X}, -Y). \tag{4.17}$$

Likewise, when Y is merely d-symmetric, we obtain

$$\tilde{d}(-\tilde{X}, -Y) = -\tilde{d}(\tilde{X}, Y) \quad \text{and} \quad \tilde{d}(-\tilde{X}, Y) = -\tilde{d}(\tilde{X}, -Y). \tag{4.18}$$

These translate to (1.1) and (1.2), respectively.

Finally, if X is the mapping torus of a diffeomorphism $\phi\colon Y \to Y$ and $\tilde{X}$ is its universal cover, then $\tilde{X} \cong Y \times \mathbb{R}$. As seen in Example 4.3, we have $H^1_c(L_Y) = H^1_c(R_Y) = 0$, and therefore the coefficient modules $\mathcal{L}_Y$ and $\mathcal{R}_Y$ are both simply $\mathcal{H}_Y$. Equations (4.13) through (4.16) yield (1.3). □

The following proposition is an immediate consequence of Proposition 4.7 and equation (4.18).

Proposition 4.11. *Let $\tilde{X}$ be a homology ribbon and let $\mathfrak{s}$ be a torsion spin^c structure on $\tilde{X}$. Suppose $Y_1 \prec Y_2$ are disjoint cross-sections of $\tilde{X}$ and that $(Y_1, \mathfrak{s})$ and $(Y_2, \mathfrak{s})$ are both d-symmetric. Then $\tilde{d}(\tilde{X}, Y_1) = \tilde{d}(\tilde{X}, Y_2)$. Moreover, if Y' is any other cross-section with $Y_1 \prec Y' \prec Y_2$, then*

$$\tilde{d}(-\tilde{X}, -Y') = -\tilde{d}(\tilde{X}, Y') = -\tilde{d}(\tilde{X}, Y_1).$$

This result is useful for obstructing the presence of d-symmetric cross-sections (e.g., rational homology spheres) in the ends of exotic $\mathbb{R}^4$s, as in the following example.

Example 4.12. Let K denote the positive, untwisted Whitehead double of the right-handed trefoil, let $Y = S^3_0(K)$, and let W be obtained by attaching a 0-framed 2-handle to D^4 along K, so that $\partial W = Y$. Let Z denote the complement of a topological slice disk for K, with $\pi_1(Z) = \mathbb{Z}$. We may choose a smooth structure on $Z_0 = Z \smallsetminus \{\text{pt}\}$; then Z_0 is an open, smooth 4-manifold with $\partial Z_0 = Y$. Then $R = W \cup_Y -Z_0$ is an exotic $\mathbb{R}^4$, and $\tilde{X} = (W \smallsetminus B^4) \cup_Y -Z_0$ is an exotic $S^3 \times \mathbb{R}$ with one end smoothly modeled on $S^3 \times (-\infty, 0]$. Since $b_1(L_Y) = 0$, we have $\mathcal{L}_Y = \mathbb{F}[H^1(Y)]$. As seen in Example 3.6, $\tilde{d}(\tilde{X}, Y) = 0$ and $\tilde{d}(-\tilde{X}, -Y) = 2$. If the generator of $H_3(Z_0)$ were represented by any d-symmetric manifold, this would contradict Proposition 4.11.

Remark 4.13. The existence of an exotic $\mathbb{R}^4$ not containing a homology sphere arbitrarily far out in its end seems to be 4-manifold folklore; compare [12, Remark 1, p. 96]. The proof depends on Donaldson's diagonalization theorem. Bob Gompf pointed out to us that the extension of Donaldson's theorem to nonsimply connected manifolds [4] can be used to show that there is no rational homology sphere arbitrarily far out in the end.

Example 4.14. The three-torus T^3 embeds in $\mathbb{R}^4$, so it occurs as a cross-section of $S^3 \times \mathbb{R}$. By Lemma 4.4, we have $H^1(T^3) = H^1(L_{T^3}) \oplus H^1(R_{T^3}) = H^1_c(L_{T^3}) \oplus H^1_c(R_{T^3})$. Because the triple cup product vanishes on each summand, one summand must have rank 1 and the other rank 2; by varying the orientations, we may interchange them. Because $S^3 \prec T^3 \prec S^3$, we deduce in either case that $\tilde{d}(T^3, \mathfrak{s}; \mathcal{L}_{T^3}) = 0$. As we saw in Example 3.7, this means that for any subspace $A \subset H^1(T^3)$ of rank 1 or 2, we have $\tilde{d}(T^3, \mathfrak{s}; M_A) = 0$.

On the other hand, let X be a homology $S^1 \times S^3$ obtained as the mapping torus of a self-diffeomorphism of $Y = T^3$. (Such manifolds play a key role in the construction of the Cappell–Shaneson homotopy spheres [2,3].) Then $H^1_c(L_{T^3}) = H^1_c(R_{T^3}) = 0$. From Example 3.7, we deduce that $\tilde{d}(X, y) = \tilde{d}(-X, -y) = 2$. Hence, X does not admit any d-symmetric cross-section. (Of course, because any cross-section of X admits a degree 1 map to T^3 and therefore has nonvanishing triple cup product, cross-sections of the form $Q \# n(S^1 \times S^2)$ are automatically excluded.)

5. Applications to knotted spheres

In this section, we use the invariants defined above to study Seifert surfaces for 2-knots in S^4. Given a smoothly embedded, oriented 2-sphere Σ in S^4, a Seifert surface is a smoothly embedded, compact, connected, oriented 3-manifold with boundary Σ. Let $X(\Sigma)$ denote the surgered manifold $S^4 \smallsetminus \text{nbd}(\Sigma) \cup (D^3 \times S^1)$, which is a homology $S^1 \times S^3$. Any Seifert surface of Σ can be capped off to be a cross-section of $X(\Sigma)$. (In a slight abuse of notation, if $Y \smallsetminus B^3$ occurs as a Seifert surface of Σ, we will sometimes say that Σ has Y as a Seifert surface.) The homology class y of a capped-off Seifert surface Y in $H_3(X)$ is determined by the orientation of Σ; therefore, we define $\tilde{d}(\Sigma) = \tilde{d}(X(\Sigma), y)$, which is an invariant of the smooth isotopy class of Σ.

Let Σ^r denote Σ with reversed orientation, and let $\bar{\Sigma}$ denote the image of Σ under a reflection of S^4. If $Y \subset X(\Sigma)$ is a capped-off Seifert surface for Σ, then

$$\tilde{d}(\Sigma^r) = \tilde{d}(X(\Sigma), -y),$$
$$\tilde{d}(\bar{\Sigma}) = \tilde{d}(-X(\Sigma), y),$$
$$\tilde{d}(\bar{\Sigma}^r) = \tilde{d}(-X(\Sigma), -y).$$

The four numbers $\tilde{d}(\Sigma)$, $\tilde{d}(\Sigma^r)$, $\tilde{d}(\bar{\Sigma})$, and $\tilde{d}(\bar{\Sigma}^r)$ may *a priori* all be different. We say that Σ is *invertible*, *positive amphicheiral*, or *negative amphicheiral* if Σ is smoothly isotopic to Σ^r, $\bar{\Sigma}$, or $\bar{\Sigma}^r$, respectively; the $\tilde{d}$ invariant can thus be used to obstruct such symmetries. Moreover, the symmetries from Proposition 1.2 each translate to a symmetry of the 2-knot invariants. For instance, if Σ has a Seifert surface Y that is d-symmetric, then

$$\tilde{d}(\Sigma) = -\tilde{d}(\bar{\Sigma}^r) \quad \text{and} \quad \tilde{d}(\Sigma^r) = -\tilde{d}(\bar{\Sigma}). \tag{5.1}$$

In particular, if Σ is a *ribbon knot* (i.e., bounds an immersed 3-ball with ribbon singularities), a theorem of Yanagawa [39] states that Σ has a Seifert surface diffeomorphic to $\#n(S^1 \times S^2) \smallsetminus B^3$ for some n; it follows that

$$\tilde{d}(\Sigma) = \tilde{d}(\Sigma^r) = \tilde{d}(\bar{\Sigma}) = \tilde{d}(\bar{\Sigma}^r) = 0.$$

Likewise, if Σ is a fibered 2-knot with capped-off fiber Y, then

$$\tilde{d}(\Sigma) = \tilde{d}(\bar{\Sigma}) = \tilde{d}(Y, \mathfrak{s}_X; \mathcal{H}_Y) \quad \text{and} \quad \tilde{d}(\Sigma^r) = \tilde{d}(\bar{\Sigma}^r) = \tilde{d}(-Y, \mathfrak{s}_X; \mathcal{H}_Y).$$

Example 5.1. If Σ is the 5-twist-spin of the right-handed trefoil, then Σ has the Poincaré homology sphere as a fiber [32, p. 306]. Hence, $\tilde{d}(\Sigma) = d(\bar{\Sigma}) = 2$ and $\tilde{d}(\Sigma^r) = d(\bar{\Sigma}^r) = -2$. We deduce that Σ is neither reversible (which was also proven by Gordon [6]) nor negative amphicheiral.

Example 5.2. Let Σ be the 6-twist-spin of the right-handed trefoil K and let Y be the fiber which is the 6-fold cyclic branched cover of K. As explained in [32, p. 307], Y can be obtained by $(0,0)$ surgery on the positive Whitehead link or, equivalently, by $(0,0,-1)$ surgery on the Borromean rings. (Hence, Y has an alternate description as the circle bundle of Euler number -1 over the torus.) Likewise, $-Y$ can be obtained by $(0,0,1)$ surgery on the Borromean rings. Let $\mathfrak{s}$ denote the unique torsion spin^c structure on Y. Ozsváth and Szabó [25, Lemma 8.7] proved that $d(-Y, \mathfrak{s}; \mathcal{H}_{-Y}) = -1$, and therefore $\tilde{d}(\Sigma^r) = -2$. A very similar computation (following the proofs of [25, Lemmas 8.6 and 8.7]) shows that $d(Y, \mathfrak{s}; \mathcal{H}_Y) = 1$, so $\tilde{d}(\Sigma) = 0$. By (1.2), it follows that Σ does not have any Seifert surface that is d-symmetric, such as any manifold of the form $Q \# n(S^1 \times S^2)$.

Remark 5.3. Just as with classical knots, the degree of the Alexander polynomial $\Delta(\Sigma)$ provides a lower bound on b_1 of any Seifert surface for Σ. In particular, if Σ admits a Seifert surface that is a rational homology sphere, then $\Delta(\Sigma) = 1$. It would thus be interesting to find a 2-knot Σ with $\Delta(\Sigma) = 1$ that fails to satisfy (5.1) and therefore does not admit a rational homology sphere Seifert surface. (We do not know of any other such obstruction.)

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