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RESEARCH

Regularity of CR-mappings between Fuchsian type hypersurfaces in $\mathbb{C}^2$

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Abstract
We investigate regularity of CR-mappings between real-analytic infinite type hypersurfaces in $\mathbb{C}^2$. We show that, under the *Fuchsian type* condition, all (respectively formal or smooth) CR-diffeomorphisms between them are automatically analytic. The Fuchsian condition appears to be in a certain sense optimal for the regularity problem.

1 Introduction

The problem of regularity of CR-maps between CR-submanifolds in complex space is of fundamental importance in the field of Several Complex Variables. Starting from the classical work of Cartan [4], Chern and Moser [5], Pinchuk [23], and Lewy [20], a large amount of publications is dedicated to various positive results in this well-developed direction. In particular, when both the source and the target are real-analytic, the expected regularity of smooth CR-maps is C^ω, i.e., they are *analytic* (this property implies that the CR-maps extend holomorphically to a neighborhood of the source manifold). We refer the reader to the book of Baouendi–Ebenfelt–Rothschild [2], the survey of Forstnerič [9], the book of Berhano, Cordaro, and Hounie [3], and the introduction in [14] for the set-up of the theory of CR-maps, a historical outline of the analyticity problem, its connections with the boundary regularity of holomorphic maps/the reflection principle, and the connections of the problem to the theory of linear PDEs.

In the particularly well-studied case of real-analytic hypersurfaces in $\mathbb{C}^2$, it has been known for some time that CR-diffeomorphisms of finite D'Angelo type hypersurface are automatically analytic (see, e.g., Baouendi–Jacobowitz–Treves [1]). (Note that in the $\mathbb{C}^2$-case finite D'Angelo type is equivalent to the Hörmander–Kohn bracket-generating condition and Tumanov non-minimality). In the case of infinite type but Levi-non-flat hypersurfaces, when there exists a complex variety $X \subset M$ passing through the reference point p in the source hypersurface M, some partial analyticity results are available. For instance, analyticity has been established by Ebenfelt [8] for so-called 1-*non-minimal hypersurfaces* (see the notion of *non-minimality order* below), and by Ebenfelt–Huang [6] for the case of maps admitting a one-sided holomorphic extension.

On the other hand, in the recent paper [14], Kossovskiy and Lamel discovered the existence of real-analytic hypersurfaces in $\mathbb{C}^N$, $N \geq 2$ which are C^∞ CR-equivalent, but are inequivalent analytically. In particular, it follows that C^∞ CR-diffeomorphisms between real-analytic Levi-non-flat hypersurfaces in $\mathbb{C}^2$ are *not* analytic in general. Moreover, it shows that the equivalence problem for non-minimal real-analytic CR-structures is of a more *intrinsic* nature, as a map realizing an equivalence does not necessarily arise from the biholomorphic equivalence of the CR-manifolds as submanifolds in complex space.

A natural question immediately raised by the results in [14] is to identify an optimal class of "regular"

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real-analytic hypersurfaces, for which CR-diffeomorphism are still analytic. The goal of the current paper is to address this question in the $\mathbb{C}^2$-case. We consider the class of *Fuchsian type hypersurfaces* introduced by the authors in [7] (this condition is described explicitly in terms of the defining function of a hypersurface), and prove that CR-diffeomorphisms of Fuchsian type hypersurfaces are automatically analytic. We also show the invariance and optimality of the Fuchsian type condition.

Another result of us concerns the problem of convergence of *formal* CR-maps. Similarly to the analyticity issue, this problem has attracted a lot of attention of experts in complex analysis in the last few decades (see, e.g., the survey [19] of Lamel–Mir). Theorem 1.5 establishes a convergence result for formal CR-maps in the Fuchsian type case.

We now formulate the results below in detail. We start with describing the precise class of hypersurfaces considered in this paper. In light of the above, we deal with germs of Levi-non-flat real-analytic hypersurfaces $M \subset \mathbb{C}^2$ considered near a point of *infinite* type $p \in M$. If M is such a hypersurface, there is a unique germ of a complex hypersurface (complex curve) $X \subset M$ passing through p. The complex hypersurface X consists of all infinite type points in M near p; it is non-singular and we will also refer to it as the *infinite type locus of* M. We say that (M, p) is of *generic infinite type* if the canonical extension of the Levi form

$$\mathcal{L}_p : T_p^{1,0} \times T_p^{1,0} \longrightarrow \mathbb{C}T_pM/\mathbb{C}T_p^{\mathbb{C}}M$$

from M to its *complexification* $M^{\mathbb{C}} \subset \mathbb{C}^2 \times \overline{\mathbb{C}^2}$ locally vanishes only on the complexification $X^{\mathbb{C}} \subset \mathbb{C}^2 \times \overline{\mathbb{C}^2}$ of X. (We refer the reader to Section 2 for details). If M is a Levi-non-flat real-analytic hypersurface with infinite type locus X, then M must be of generic infinite type at points p lying outside of a proper real-analytic subset of X.

We say that local holomorphic coordinates (z, w), where $w = u + iv$, near p are *admissible* (for M) if in these coordinates, p becomes the origin and M is given by

$$v = \frac{1}{2}u^m\left(\epsilon|z|^2 + \sum_{k,l\geq 2} h_{kl}(u)z^k\bar{z}^l\right) =: h(z, \bar{z}, u), \quad \epsilon = \pm 1 \tag{1.1}$$

(such admissible coordinates always exist under the generic infinite type assumption, see [16]); in particular, in these coordinates $X = \{w = 0\}$. The integer $m \geq 1$ is an important invariant of an infinite type hypersurface called *the non-minimality order*, and M with such non-minimality order is called *m-non-minimal*. For an even m, we can further normalize ϵ to be equal to 1, while for an odd m, ϵ is a biholomorphic invariant. Note that the form (1.1) is stable under the group of dilations

$$\begin{aligned} &z \mapsto \lambda z,\\ &w \mapsto \mu w,\\ &\mu^{1-m} = \epsilon|\lambda|^2,\\ &\lambda \in \mathbb{C} \setminus \{0\},\\ &\mu \in \mathbb{R}. \end{aligned} \tag{1.2}$$

We are now able to describe the Fuchsian condition.

Definition 1.1 An infinite type hypersurface (1.1) is called *a hypersurface of Fuchsian type*, if its defining function $h(z, \bar{z}, u)$ satisfies

$$\begin{aligned} &\text{ord } h_{22}(w) \geq m-1; \ \text{ord } h_{23}(w) \geq 2m-2; \ \text{ord } h_{33}(w) \geq 2m-2;\\ &\text{ord } h_{2l}(w) \geq 2m-l+2, \ \ 4 \leq l \leq 2m+1;\\ &\text{ord } h_{kl}(w) \geq 2m-k-l+5, \ \ k \geq 3, \ l \geq 3, \ 7 \leq k+l \leq 2m+4. \end{aligned} \tag{1.3}$$

We point out that

- The Fuchsian condition requires vanishing of an appropriate part of the $(2m+4)$-jet of the defining function h at 0;
- It is easy to see from (1.3) that for $m = 1$ the Fuchsian type condition holds automatically, while for $m > 1$ it fails to hold in general;
- As will be shown in Section 3, the Fuchsian type property is holomorphically invariant.s

Remark 1.2 The property of being Fuchsian extends earlier versions of this property given, respectively, in the work [16] of Kossovskiy–Shafikov and the work [13] of Kossovskiy–Lamel. In the paper [16], a Fuchsian property of *generically spherical* hypersurfaces (1.1) was introduced. It is possible to check that for a generically spherical hypersurface the two notions of being Fuchsian coincide. In the paper [13], general hypersurfaces (1.1) were considered, but the notion of Fuchsian type considered there is weaker than that given in [7] and in the present paper; it serves to guarantee the regularity of infinitesimal CR-automorphisms, while the property (1.3) guarantees regularity of *arbitrary* CR-maps. The property introduced in [13] is more appropriately addressed as *weak Fuchsian type*, while the property (1.3) as the (actual) Fuchsian type.

Now our main analyticity results are as follows.

Theorem 1.3 *Let $M, M^* \subset \mathbb{C}^2$ be real-analytic hypersurfaces, and let M be of Fuchsian type at a point $p \in M$. Let*

U be an open neighborhood of p in $\mathbb{C}^2$ *. Then any* C^∞ *CR-diffeomorphism* $H : M \cap U \longrightarrow M^*$ *is analytic.*

By applying the Hanges–Treves propagation principle [10], we are able to address the regularity at an arbitrary infinite type point.

Theorem 1.4 *Let* $M, M^* \subset \mathbb{C}^2$ *be real-analytic Levi-non-flat hypersurfaces, and U an open neighborhood of p in* $\mathbb{C}^2$*. Assume that* $U \cap M$ *contains a Fuchsian type point q. Then any* C^∞ *CR-diffeomorphism* $H : M \cap U \longrightarrow M^*$ *is analytic.*

We further obtain a result on the convergence of formal power series maps between Fuchsian type hypersurfaces.

Theorem 1.5 *Let* $M, M^* \subset \mathbb{C}^2$ *be real-analytic hypersurfaces, and let M be of Fuchsian type at a point* $p \in M$*. Then any formal invertible power series map* $H : (M, p) \longrightarrow (M^*, p^*)$, $p^* \in M^*$ *is convergent.*

Remark 1.6 As follows from the invariance of the Fuchsian type property under formal power series transformations (see Theorem 4.3), the target hypersurface M^* is also of Fuchsian type at the respective point $p^* = H(p)$.

Theorem 1.5 extends earlier results in this direction obtained in [12] in the case $m = 1$. It also extends, in a certain sense, the result in [13] on the regularity of infinitesimal CR-automorphisms of Fuchsian type hypersurfaces to the case of general maps (not necessarily appearing as flows of infinitesimal CR-automorphisms). However, as discussed above, the Fuchsian type condition in [13] is more mild and involves only vanishing conditions on the coefficient functions h_{kl}, $k + l \leq 7$ (unlike the conditions in (1.3)). As arguments in Section 3 show, the case of a general CR-mapping requires considering *all* the coefficients h_{kl} in (1.3), as they appear in the complete (singular) system of ODEs determining a CR-map.

2 Preliminaries

2.1 Infinite type real hypersurfaces

We recall that if $M \subset \mathbb{C}^2$ is a real-analytic hypersurface, then for any $p \in M$ there exist so-called *normal coordinates* (z, w) centered at p for M. The coordinates being normal means that (z, w) is a local holomorphic coordinate system near p in which $p = 0$ and for which near 0, M is defined by an equation of the form

$$v = F(z, \bar{z}, u)$$

for some germ F of a holomorphic function on $\mathbb{C}^3$ which satisfies the normality condition

$$F(z, 0, u) = F(0, \bar{z}, u) = 0$$

and the reality condition $F(z, \bar{z}, u) \in \mathbb{R}$ for $(z, u) \in \mathbb{C} \times \mathbb{R}$ close to 0 (see, e.g., [2]). Equivalently, $v = F(z, \bar{z}, u)$ defines a real hypersurface, and in the coordinates (z, w), we have $Q_{(0,u)} = \{(0, w) \in U : w = u\}$.

We also recall that M is *of infinite type* at p if there exists a germ of a non-trivial complex curve $X \subset M$ through p. It turns out that in normal coordinates, such a curve X is necessarily defined by $w = 0$ (because $X = Q_0 = \{w = 0\}$); in particular, any such X is non-singular. It also turns out that M is Levi-flat if and only if in normal coordinates, it is defined by $v = 0$. Thus a Levi-non-flat real-analytic hypersurface M is of infinite type at p if and only if in normal coordinates (z, w) as above, the defining function F satisfies $F(z, \bar{z}, 0) = 0$. In other words, M is of infinite type if and only if it can defined by an equation of the form

$$v = u^m \psi(z, \bar{z}, u), \text{ with } \psi(z, 0, u) = \psi(0, \bar{z}, u) = 0 \text{ and } \psi(z, \bar{z}, 0) \not\equiv 0, \tag{2.1}$$

where $m \geq 1$. It turns out that the integer $m \geq 1$ is independent of both the choice of $p \in X$ and also of the choice of normal coordinates for M at p (see [21]), and we say that M is *m-infinite type* along X (or at p).

We are going to utilize a number of different ways to write a defining function. Throughout this paper, we use the *complex defining function* Θ in which M is defined by

$$w = \Theta(z, \bar{z}, \bar{w});$$

it is obtained from F by solving the equation

$$\frac{w - \bar{w}}{2i} = F\left(z, \bar{z}, \frac{w + \bar{w}}{2}\right)$$

for w, and it agrees with the function defining the Segre varieties in those coordinates, that is, $Q_Z = \{(z, \Theta(z, \bar{Z})) : z \in U^z\}$. We are going to make extensive use of the Segre varieties and refer the reader to [2] for a discussion of their properties in the general case, and to [14] for specific properties in the infinite type setting.

The complex defining function (in normal coordinates) satisfies the conditions

$$\Theta(z, 0, \tau) = \Theta(0, \chi, \tau) = \tau, \quad \Theta(z, \chi, \bar{\Theta}(\chi, z, w)) = w.$$

If M is of m-infinite type at p, then $\Theta(z, \chi, \tau) = \tau\theta(z, \chi, \tau)$ and thus M is defined by the equation $w = \bar{w}\theta(z, \bar{z}, \bar{w}) = \bar{w} + \bar{w}^m \tilde{\theta}(z, \bar{z}, \bar{w})$, where $\tilde{\theta}$ satisfies $\tilde{\theta}(z, 0, \tau) = \tilde{\theta}(0, \chi, \tau) = 0$ and $\tilde{\theta}(z, \chi, 0) \neq 0$.

We also note that the external complexification $M^{\mathbb{C}}$ of M, which is the hypersurface in $\mathbb{C}^2 \times \overline{\mathbb{C}^2}$ defined by

$M^{\mathbb{C}} = \{(Z, \zeta) \in U \times \bar{U} : Z \in Q_{\bar{\zeta}}\}$, is conveniently defined as the graph of the complex defining function Θ, i.e.,

$$w = \Theta(z, \chi, \tau).$$

We also introduce the real line

$$\Gamma = \{(z, w) \in M : z = 0\} = \{(0, u) \in M : u \in \mathbb{R}\} \subset M, \tag{2.2}$$

and recall that

$$Q_{(0,u)} = \{w = u\}, \quad (0, u) \in \Gamma$$

for $u \in \mathbb{R}$. This property, as already mentioned, is actually equivalent to the normality of the coordinates (z, w). More precisely, for any real-analytic curve γ through p one can find normal coordinates (z, w) in which a small piece of γ corresponds to Γ in (2.2).

We finally notice that a real-analytic Levi-non-flat hypersurface $M \subset \mathbb{C}^2$ has infinite type points of two kinds, which we will refer to as *generic* and *exceptional* infinite type points, respectively. A generic point $p \in M$ is characterized by the condition that the complexified Levi form of M only degenerates on the complexified infinite type locus $w = \tau = 0$ near p. (The complexified Levi form is defined similarly to the classical Levi form, but instead the (1, 0) and the (0, 1) vector fields are considered on the complexification $M^{\mathbb{C}}$, see, e.g., [2]). We refer to a non-generic point p as *exceptional*. We note that the set of exceptional points is a proper real-analytic subvariety of X and that $p \in X$ is generic if and only if the Levi determinant of M vanishes to order m along *any* real curve γ passing through p which is transverse to X at p.

A generic infinite type point is characterized in normal coordinates by requiring in addition to (2.1) the condition $\psi_{z\bar{z}}(0, 0, 0,) \neq 0$. If p is a generic infinite type point, we can further simplify M to the form (1.1), or alternatively to the *exponential form*

$$w = \bar{w}e^{i\bar{w}^{m-1}\varphi(z,\bar{z},\bar{w})}, \quad \text{where} \quad \varphi(z, \bar{z}, \bar{w}) = \pm z\bar{z} + \sum_{k,l\geq 2} \varphi_{kl}(\bar{w})z^k\bar{z}^l \tag{2.3}$$

(see, e.g., [16]).

2.2 Real hypersurfaces and second-order differential equations.

There is a natural way to associate to a Levi non-degenerate real hypersurface $M \subset \mathbb{C}^N$ a system of second-order holomorphic PDEs with 1 dependent and $N-1$ independent variables by using the Segre family of the hypersurface M. This remarkable construction goes back to E. Cartan [4] and Segre [24] (see also a remark by Webster [28]), and was recently revisited in the work of Sukhov [25, 26] in the non-degenerate setting, and in the work of Kossovskiy, Lamel, and Shafikov in the degenerate setting (see [13, 14, 16, 17]). For the convenience of the reader, we recall this procedure in the case $N = 2$, but refer to the above references for more details.

So assume that $M \subset \mathbb{C}^2$ is a smooth real-analytic hypersurface passing through the origin and $U = U^z \times U^w$ is chosen small enough. The second-order holomorphic ODE associated to M is uniquely determined by the condition that for every $\zeta \in U$, the function $h(z, \zeta) = w(z)$ defining the Segre variety Q_ζ as a graph is a solution of this ODE. To be more precise, one can show that the Levi-non-degeneracy of M (at 0) implies that near the origin, the Segre map $\zeta \mapsto Q_\zeta$ is injective and the Segre family has the so-called transversality property: if two distinct Segre varieties intersect at a point $q \in U$, then their intersection at q is transverse (actually it turns out that, again due to the Levi-non-degeneracy of M, the Segre varieties passing through a point p are uniquely determined by their tangent spaces T_pQ_ζ). Thus, $\{Q_\zeta\}_{\zeta\in U}$ is a 2-parameter family of holomorphic curves in U with the transversality property, depending holomorphically on $\bar{\zeta}$. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [11]) that there exists a unique second-order holomorphic ODE $w'' = \Phi(z, w, w')$ such that for each $\zeta \in U$, $w(z) = h(z, \bar{\zeta})$ is one of its solutions.

We can carry out the construction of this ODE concretely by utilizing the complex defining equation $w = \Theta(z, \chi, \tau)$ introduced above. Recall that the Segre variety Q_ζ of a point $\zeta = (a, b) \in U$ is now given as the graph

$$w(z) = \rho(z, \bar{a}, \bar{b}). \tag{2.4}$$

Differentiating (2.4) once, we obtain

$$w' = \rho_z(z, \bar{a}, \bar{b}). \tag{2.5}$$

The system of equations (2.4) and (2.5) can be solved, using the implicit function theorem, for $\bar{a}$ and $\bar{b}$. This gives us holomorphic functions A and B such that

$$\bar{a} = A(z, w, w'), \ \bar{b} = B(z, w, w').$$

The application of the implicit function theorem is possible since the Jacobian of the system consisting of (2.4) and (2.5) with respect to $\bar{a}$ and $\bar{b}$ is just the Levi determinant of M for $(z, w) \in M$ ([2]). Differentiating (2.5) once more, we can substitute $\bar{a} = A(z, w, w')$ and $\bar{b} = B(z, w, w')$ to obtain

$$w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: \Phi(z, w, w'). \tag{2.6}$$

Now (2.6) is a holomorphic second-order ODE, for which all of the functions $w(z) = h(z, \zeta)$ are solutions by construction. We will denote this associated second-order ODE by $\mathcal{E} = \mathcal{E}(M)$.

More generally it is possible to associate a completely integrable PDE to any of a wide range of CR-submanifolds (see [25, 26]) such that the correspondence $M \to \mathcal{E}(M)$ has the following fundamental properties:

(1) Every local holomorphic equivalence $F : (M, 0) \to (M', 0)$ between CR-submanifolds is an equivalence between the corresponding PDE systems $\mathcal{E}(M), \mathcal{E}(M')$;
(2) The complexification of the infinitesimal automorphism algebra $\mathfrak{hol}^{\omega}(M, 0)$ of M at the origin coincides with the Lie symmetry algebra of the associated PDE system $\mathcal{E}(M)$ (see, e.g., [22] for the details of the concept).

In contrast to the case of a finite type real hypersurface described above, if $M \subset \mathbb{C}^2$ is of infinite type at the origin one, cannot associate to M a regular second-order ODE or even a more general PDE system near the origin such that the Segre varieties are graphs of solutions. However, in [16] and [13], Kossovskiy, Lamel, and Shafikov found an injective correspondence associating to a hypersurface $M \subset \mathbb{C}^2$ at a generic infinite type point a certain *singular* complex ODE $\mathcal{E}(M)$ with an isolated singularity at the origin. We are going to base our normal form construction on this construction, which is therefore extensively used in the paper (more details are given in Section 3).

We finally point out that at *exceptional* infinite type points, one can still associate a system of singular complex ODEs to a real-analytic hypersurface $M \subset \mathbb{C}^2$ (although possibly of higher order $k \geq 2$) as in the paper [15] Kossovskiy–Lamel–Stolovitch.

2.3 Complex differential equations with an isolated singularity

We will again just gather the facts from the classical theory of singular (complex) differential equations, and refer the reader to, e.g., [11, 18, 27] for any details.

A linear system $\mathcal{L}$ of (holomorphic) first-order ODEs on a domain $G \subset \mathbb{C}$ (or simply a *linear system* in a domain G) is an equation of the form $y'(x) = A(x)y(x)$, where $A : G \to \mathbb{C}^{n \times n}$ is a matrix-valued holomorphic map on G and $y(x) = (y_1(x), ..., y_n(x))$ is an n-tuple of (unknown) functions. The set of solutions of $\mathcal{L}$ near a point $p \in G$ is isomorphic to $\mathbb{C}^n$ by $y \mapsto y(p)$. Because every germ y of a solution of $\mathcal{L}$ at $p \in G$ extends analytically along any path $\gamma \subset G$ starting at p, any solution $y(x)$ of $\mathcal{L}$ is defined in all of G as a (possibly multi-valued) analytic function. If G is a punctured disc, centered at 0, we say that $\mathcal{L}$ has an *isolated singularity* (at $x = 0$). If $A(x)$ has a pole at the isolated singularity $x = 0$, we say that the system has a *meromorphic singularity*. As the solutions of $\mathcal{L}$ are holomorphic in any proper sector $S \subset G$ of a sufficiently small radius with vertex at $x = 0$, it is important to study the behavior of the solutions as $x \to 0$. If for every sector $S = \{x \in G : |x| < \delta, \alpha < \arg x < \beta\}$ there exist constants $C > 0$ and $a \in \mathbb{R}$ such that for every solution y of $\mathcal{L}$ defined in S we have that $||y(x)|| \leq C|x|^a$ holds for $x \in S$, then we say that $x = 0$ is a *regular singularity*, otherwise we say it is an *irregular singularity*.

An important condition ensuring regularity of a singularity is due to L. Fuchs: We say that the singular point $x = 0$ is *Fuchsian* if $A(x)$ has a pole of order at most 1 at $x = 0$. If 0 is a Fuchsian singularity, then $x = 0$ is a regular singular point. Another important property of Fuchsian singularities is that every formal power series solution (at $x = 0$) of the equation is actually *convergent*. The dynamical system associated to a Fuchsian singularity corresponds to the dynamical system of the vector field

$$x\frac{\partial}{\partial x} + A(x)y\frac{\partial}{\partial y},$$

which is "almost" non-resonant in the sense of Poincaré-Dulac.

However, in the *non*-Fuchsian case we encounter very different behaviors, both of solutions and of mappings between linear systems with such a singularity. A generic solution of a non-Fuchsian system

$$y' = \frac{1}{x^m}B(x)y, \quad m \geq 2$$

does not have polynomial growth in sectors, and generic formal power series solutions of such a system (as well as formal equivalences between generic non-Fuchsian systems) are divergent. The dynamics associated to a non-Fuchsian singularity correspond to the dynamics of the vector field

$$x^m\frac{\partial}{\partial x} + A(x)y\frac{\partial}{\partial y},$$

which is always resonant, in the sense of Poincaré-Dulac.

Further information on the classification of isolated singularities can be found in, e.g., [11] or [27].

Fuchsianity admits a certain extension to the non-linear case as well, giving rise to the notion of *Briot–Bouquet type ODEs*, that is, ODEs of the form

$$xy' = F(x, y), \tag{2.7}$$

where x lies in a neighborhood of 0 in $\mathbb{C}$, y is n-dimensional and F is holomorphic in a neighborhood of 0 in $\mathbb{C}^{n+1}$. Briot–Bouquet ODEs are similar to linear systems of ODEs with a Fuchsian singularity in many respects; for example, their formal power series solutions are necessarily *convergent* (see, e.g., [18]). Dynamics associated to a Briot–Bouquet type ODE corresponds to the dynamics of the vector field

$$x\frac{\partial}{\partial x} + F(x, y)\frac{\partial}{\partial y}.$$

We also note that a Briot–Bouquet type ODE whose principal matrix $F_y(0,0)$ has no positive integer eigenvalues has at least one holomorphic solution (see [18]).

3 The associated ODE approach to the mapping problem

We consider a real-analytic hypersurface with defining equation as in (1.1). The complex defining function of such a hypersurface is given by

$$w = \bar{w} + i\bar{w}^m\left(\epsilon|z|^2 + \sum_{k,\ell\geq 2}\Theta_{k\ell}(\bar{w})z^k\bar{z}^\ell\right). \tag{3.1}$$

We recall from subsection 2.1 that this means that the Segre family $\mathcal{S} = \{Q_{(\xi,\eta)}\}$ of M is given by

$$w = \bar{\eta}e^{i\bar{\eta}^{m-1}\varphi(z,\bar{\xi},\bar{\eta})}, \quad \text{where} \quad \varphi(z,\bar{\xi},\bar{\eta}) = \epsilon z\bar{\xi} + \sum_{k,\ell\geq 2}\varphi_{k\ell}(\bar{\eta})z^k\bar{\xi}^\ell \tag{3.2}$$

We will need the following fact proved in [7]:

Lemma 3.1 *(see [7]). Let $H(z,w) = \big(F(z,w),G(z,w)\big)$ be a formal transformation vanishing at the origin, with invertible Jacobian $H'(0)$, which maps a hypersurface defined by (1.1) or equivalently (3.2) into another such hypersurface. Then H satisfies*

$$F_z(0,0) = \lambda, \quad G_w(0,0) = \mu, \quad G = O(w),$$
$$G_z = O(w^{m+1}), \quad \mu^{1-m} = |\lambda|^2, \quad \lambda \in \mathbb{C}\setminus\{0\},\ \mu \in \mathbb{R}. \tag{3.3}$$

In addition, we have

$$G_{w^\ell}(0,0) \in \mathbb{R}, \quad \text{for } \ell \leq m. \tag{3.4}$$

Lemma 3.1 implies in particular that any transformation H between hypersurfaces defined by equations of the form (1.1) can be factored as

$$H = H_0 \circ \psi,$$

for some dilation ψ of the form (1.2) and where H_0 is a transformation of the form:

$$z \mapsto z + f(z,w), \quad w \mapsto w + wg_0(w) + w^m g(z,w)$$

with

$$\begin{aligned} &f_z(0,0) = 0,\\ &g_0(0) = 0,\\ &g(z,w) = O(zw),\\ &g_0^{(\ell)}(0) \in \mathbb{R},\\ &\ell \leq m-1. \end{aligned} \tag{3.5}$$

(for $m = 1$ the last condition is void). In fact, one can also represent H as $H = \psi \circ H_0$ (with a different H_0). We therefore consider the classification problem only under transformations (3.5).

We now recall that [13, 16] showed that we can associate to a hypersurface in the form (1.1) a second-order singular holomorphic ODE $\mathcal{E}(M)$ given by

$$w'' = w^m\Phi\left(z, w, \frac{w'}{w^m}\right), \tag{3.6}$$

where $\Phi(z,w,\zeta)$ is holomorphic near the origin in $\mathbb{C}^3$, and satisfies $\Phi = O(\zeta^2)$. This ODE is characterized by the condition that any of the functions $w(z) = \Theta(z,\xi,\eta)$, for $(\xi,\eta) \in \bar{U}$, is a solution of the ODE (3.6)). We will decompose Φ as

$$\Phi(z,w,\zeta) = \sum_{j,k\geq 0,\ell\geq 2}\Phi_{jk\ell}z^k w^j\zeta^\ell \tag{3.7}$$

or

$$\Phi(z,w,\zeta) = \sum_{k\geq 0,l\geq 2}\Phi_{kl}(w)z^k\zeta^l. \tag{3.8}$$

We now recall the approach used in [15] and [7]. Considering the transformation rule for second-order ODEs and adapting it to ODEs (3.6) and maps (3.5) expanded as $\tilde{f}(z,w) = z + f(z,w)$, and $\tilde{g}(z,w) = w + wg_0(w) + w^m g(z,w)$, we get (see [7, 15]):

$$\begin{aligned}\Phi(z,w,\zeta) &= \frac{1}{J}\Big[\big(1 + f_z + w^m f_w\cdot\zeta\big)^3\big(1 + g_0(w) + w^{m-1}g\big)^m\\ &\quad\cdot\Phi^*\Big(z + f, w + wg_0(w) + w^m g\\ &\quad, \frac{g_z + \zeta(1 + wg_0' + g_0 + mw^{m-1}g + w^m g_w)}{(1 + g_0(w) + w^{m-1}g)^m(1 + f_z + w^m\zeta f_w)}\Big)\\ &\quad + I_0(z,w) + I_1(z,w)\zeta + I_2(z,w)w^m\zeta^2\\ &\quad + I_3(z,w)w^{2m}\zeta^3\Big],\end{aligned} \tag{3.9}$$

where $\zeta := \frac{w'}{w^m}$ and

$$
\begin{aligned}
J &= (1+f_z)(1+wg_0'+g_0+mw^{m-1}g+w^mg_w)\\
&\quad - w^mf_wg_z,\\
I_0 &= g_zf_{zz}-(1+f_z)g_{zz},\\
I_1 &= \big(1+wg_0'+g_0+mw^{m-1}g+w^mg_w\big)f_{zz}-w^mf_wg_{zz}\\
&\quad -2(1+f_z)(mw^{m-1}g_z+w^mg_{zw})+2w^mg_zf_{zw},\\
I_2 &= w^mg_zf_{ww}-(1+f_z)(wg_0''+2g_0'+m(m-1)w^{m-2}g\\
&\quad +2mw^{m-1}g_w+w^mg_{ww})-2f_w(mw^{m-1}g_z+w^mg_{zw})\\
&\quad +2(1+wg_0'+g_0+mw^{m-1}g+w^mg_w)f_{zw},\\
I_3 &= (1+wg_0'+g_0+mw^{m-1}g+w^mg_w)f_{ww}\\
&\quad -f_w(wg_0''+2g_0'+m(m-1)w^{m-2}g+2mw^{m-1}g_w\\
&\quad +w^mg_{ww}).
\end{aligned} \tag{3.10}
$$

Importantly, (3.9) is an identity in the *free variables* z, w, ζ, where the latter triple runs a suitable open neighborhood of the origin in $\mathbb{C}^3$.

We recall then that, by collecting in (3.9) terms with $z^kw^j\zeta^l$, $l=0,1$, we obtain a system of PDEs of the kind:

$$
\begin{aligned}
f_{zz} &= U(z,w,g_0,g_0',f,g,f_z,g_z,f_w,g_w,f_{zw},g_{zw}),\\
g_{zz} &= V(z,w,g_0,g_0',f,g,f_z,g_z,f_w,g_w,f_{zw},g_{zw})
\end{aligned} \tag{3.11}
$$

for some germs of holomorphic functions U, V at the origin. Given a choice of (respectively holomorphic or formal) data

$$
\begin{aligned}
&f(0,w)=f_0(w),\\
&f_z(0,w)=f_1(w),\\
&g(0,w)=0,\\
&g_1(0,w)=g_1(w),
\end{aligned}
$$

the Cauchy–Kowalevskaya theorem guarantees the existence of a unique (respectively holomorphic or formal) solution to (3.11) with this data.

The associated functions $\tilde f(z,w)=z+f(z,w)$, $\tilde g(z,w)=w+wg_0(w)+g(z,w)$ transform $\mathcal{E}^*$ to the (up to the initial data unique) $\mathcal{E}$. The initial conditions also imply that $(\tilde f,\tilde g)$ is of the form required in (3.5). To determine then the Cauchy data

$$
Y(w) := \big(f_0(w), f_1(w), g_0(w), g_1(w)\big), \tag{3.12}
$$

we collect in (3.9) terms with $z^kw^j\zeta^l$, $j=0,1$, $l=2,3$. This gives us a system of *singular* second-order ODEs:

$$
\begin{aligned}
w^{m+1}g_0'' &= T_1(w,g_0,g_1,f_0,f_1,wg_0',w^mg_1',w^mf_0',w^mf_1'),\\
w^{2m}g_1'' &= T_2(w,g_0,g_1,f_0,f_1,wg_0',w^mg_1',w^mf_0',w^mf_1'),\\
w^{2m}f_0'' &= T_3(w,g_0,g_1,f_0,f_1,wg_0',w^mg_1',w^mf_0',w^mf_1'),\\
w^{2m}f_1'' &= T_4(w,g_0,g_1,f_0,f_1,wg_0',w^mg_1',w^mf_0',w^mf_1')
\end{aligned} \tag{3.13}
$$

(we again refer to [7, 15] for details).

Our Fuchsian type condition is obtained by requiring that, roughly speaking, the arising system of ODEs (3.13) is Fuchsian (Briot–Bouquet). This is explained in the next section.

4 Fuchsian type ODEs and regularity of formal mappings

4.1 The normal form problem for Fuchsian type hypersurfaces

First, we translate the Fuchsian type condition for hypersurfaces (1.1) described in the Introduction onto the language of associated ODEs. For the functions Φ, Φ^*, we make use of the expansion (3.8). We now introduce

Definition 4.1 An ODE $\mathcal{E}$, defined by (3.6), is called *Fuchsian* (or *a Fuchsian type ODE*), if Φ satisfies the conditions:

$$
\begin{aligned}
&\operatorname{ord}\Phi_{02}(w)\ge m-1;\ \operatorname{ord}\Phi_{03}(w)\ge 2m-2;\ \operatorname{ord}\Phi_{12}(w)\\
&\quad\ge m-1;\ \operatorname{ord}\Phi_{13}(w)\ge 2m-2;\\
&\operatorname{ord}\Phi_{0l}(w)\ge 2m-l+2,\ 4\le l\le 2m+1;\ \operatorname{ord}\Phi_{k2}(w)\\
&\quad\ge 2m-k,\ 2\le k\le 2m+1;\\
&\operatorname{ord}\Phi_{kl}(w)\ge 2m-k-l+3,\ k\ge 1,\ l\ge 3,\ 5\le k+l\\
&\quad\le 2m+2.
\end{aligned} \tag{4.1}
$$

We make use of the following:

Proposition 4.2 *(See [7]). For a Fuchsian type hypersurface $M\subset\mathbb{C}^2$, its associated ODE $\mathcal{E}(M)$ is of Fuchsian type as well.*

We next prove the invariance of the Fuchsian type condition.

Theorem 4.3 *The property of being Fuchsian for a hypersurface (1.1) does not depend on the choice of (formal or holomophic) coordinates of the kind (1.1).*

Proof In view of Definition 4.1, we can switch to associated ODEs and it is enough to prove the invariance of the Fuchsianity for them. As discussed above, we can restrict to transformations (3.5). Let us consider then the transformation rule (3.9) (with a fixed transformation within it), when the source ODE (with the defining function Φ^*) is of Fuchsian type. We then claim the following: for all the coefficient functions Φ_{kl}, $k\ge 0$, $l\ge 2$ involved in the Fuchsianity conditions (4.1), *with the exception of the coefficients functions* Φ_{k2}, Φ^*_{k2}, $k\ge 2$, the Fuchsian conditions (4.1) are satisfied. Indeed, we fix any (k,l) relevant to (4.1), and from the transformation rule (3.9) we can see that the target coefficient function Φ_{kl} is a sum of three groups of terms: (i) terms $\Phi^*_{\alpha\beta}$ with $\alpha+\beta \geq k+l$ which are multiplied by a power series in w with order at 0 at least $k+l-\alpha-\beta$; (ii) terms $\Phi^*_{\alpha\beta}$ with $\alpha+\beta<k+l$; (iii) terms arising from the expressions $I_j,\ 0\leq j\leq 3$ (relevant for $l=2,3$ only). In view of the linearity of the Fuchsianity conditions in k,l, it is not difficult to see that terms of the first kind all have order at 0 at least as the one required for the Fuchsianity. Terms of the second kind already all have order bigger than the one required for Fuchsianity. Finally, terms of the third kind automatically provide order at least $2m$ sufficient for the Fuchsianity, except for the case $l=2$. For $k=0,1$ and $l=2$ though even the automatically provided order m suffices, and this proves the claim.

It remains to deal with terms Φ_{k2} with $k\geq 2,\ 2\leq k\leq 2m+1$. We note, however, that the ODEs under consideration have a real structure, which is why (in view of the reality condition) we have

$$\operatorname{ord} h_{kl}(w) = \operatorname{ord} h_{lk}(w) \tag{4.2}$$

for all k,l. This, in view of the transfer relations between Φ and h, gives, in particular:

$$\begin{aligned}\operatorname{ord} \Phi_{k2}(w) &= \operatorname{ord} h_{k+2,2}(w) = \operatorname{ord} h_{2,k+2}(w)\\ &= \operatorname{ord} \Phi_{0,k+2}(w) \geq 2m-k\end{aligned}$$

(the last inequality follows from the Fuchsianity condition for $\Phi_{0,k+2}$ being already proved). This finally proves the theorem.

□

We now proceed with the proof of Theorem 1.5. We follow the scheme in Section 3, and obtain a system of singular ODEs of the kind (3.13) for the Cauchy data $Y(w)$, as in(3.12), assuming the source ODE (with the defining function Φ^*) is of Fuchsian type. For the purposes of this section, we prefer to write down the obtained system in the form

$$w^{m+1}g_0'' = S\big(w,Y(w),wY'(w)\big),\ \ w^{2m}X'' = T\big(w,Y(w),wY'(w)\big), \tag{4.3}$$

where

$$X(w) := (g_1(w),f_0(w),f_1(w)),\qquad Y(w) := (g_0(w),X(w)),$$

and S,T are holomorphic near the origin.

For the functions T,S we will use the expansion

$$T(w,Y,\tilde{Y}) = \sum_{\alpha,\beta\geq 0} T_{\alpha,\beta}(w)Y^\alpha\tilde{Y}^\beta, \tag{4.4}$$

where α,β are multiindices, and similarly for S. We now shall prove the following key:

Proposition 4.4 *Under the Fuchsian type condition, the coefficient functions $T_{\alpha,\beta}(w),S_{\alpha,\beta}(w)$ satisfy*

$$\begin{aligned}&\operatorname{ord} T_{\alpha,\beta} \geq 2m-1-|\alpha|-|\beta|,\\ &\operatorname{ord} S_{\alpha,\beta} \geq m-|\alpha|-|\beta|,\ \ |\alpha|+|\beta|>0.\end{aligned} \tag{4.5}$$

Proof For the proof, we make use of (4.1) (applied for the source defining function Φ^*), and then study carefully the contribution of terms Φ^*_{kl} into the basic identity (3.9). Let us fix for the moment some positive value of $|\alpha|+|\beta|$. Then it is straightforward to check, by considering (3.9), that $T_{\alpha,\beta}$ as above can arise only from Φ^*_{kl} with $k+l\leq|\alpha|+|\beta|+4$, while $S_{\alpha,\beta}$ as above can arise only from Φ^*_{kl} with $k+l\leq|\alpha|+|\beta|+2$. (And in the latter cases a respective Φ^*_{kl} is a factor for $Y^\alpha(wY')^\beta$). Now it is not difficult to verify that (4.1) implies (4.4). □

Corollary 4.5 *For the* $(0,0)$ *coefficient functions in (4.3) we have*

$$\operatorname{ord} S_{0,0}\geq m;\quad \operatorname{ord} T_{0,0}\geq 2m-1. \tag{4.6}$$

As a consequence, for the target ODE defining function Φ we have

$$\begin{aligned}&\Phi_{0j2}=0,\ \ 0\leq j\leq m-2;\\ &\Phi_{1j2}=\Phi_{0j3}=\Phi_{1j3}=0,\ \ 0\leq j\leq 2m-3;\\ &\Phi_{0,m-1,2}=\Phi^*_{0,m-1,2};\ \Phi_{0,2m-2,3}=\Phi^*_{0,2m-2,3};\\ &\Phi_{1,2m-2,2}=\Phi^*_{1,2m-2,2};\ \Phi_{1,2m-2,3}=\Phi^*_{1,2m-2,3}.\end{aligned} \tag{4.7}$$

Proof As follows from the definition of $S_{\alpha,\beta},T_{\alpha,\beta}$ and the Fuchsianity, all terms in the first equation in (4.3) have order at least m in w with possibly the exception of terms arising from $S_{0,0}$, while all terms in the second equation in (4.3) have order at least $2m-1$ in w with possibly the exception of terms arising from $T_{0,0}$. This proves (4.6). To prove (4.7), we note that the $(m-1)$-jet of $S_{0,0}$ and the $(2m-2)$-jet of $T_{0,0}$, respectively, are formed from differences between coefficients Φ_{kjl} and Φ^*_{kjl} aparent in (4.7), and this proves (4.7).

□

We shall now prove that any solution of the system of singular ODEs (4.3). In view of the discussion in Section 3, this would imply the convergence of the formal map between the given ODEs (3.6) and the given real hypersurfaces, and hence the assertion of Theorem 1.5.

Let $H(w)$ be such a formal solution of (4.3). We decompose it as

$$H(w)=P(w)+Z(w), \tag{4.8}$$

where $P(w)$ is a polynomial without constant term of degree $\leq 2m-1$, while where $Z(w)$ is a formal series of the kind $O(w^{2m})$. The substitution (4.8) (for a fixed $(P(w))$ turns (4.3) into a similar system of ODEs for the unknown function $Z(w)$. We shall now prove

Lemma 4.6 *The transformed system (in the same way as the initial system) satisfies*

$$\begin{aligned} &\mathrm{ord}\ \tilde{S}_{01} \geq m-1, \\ &\mathrm{ord}\ \tilde{S}_{10} \geq m-1, \\ &\mathrm{ord}\ \tilde{T}_{01} \geq 2m-2, \\ &\mathrm{ord}\ \tilde{T}_{10} \geq 2m-2 \end{aligned} \tag{4.9}$$

(the tilde here stands for coefficients of the transformed system).

Proof The proof of the lemma is obtained by putting together the expansion (4.4), the conditions (4.4), and the fact that $P(w)$ is vanishing at the origin. □

Now, based on Lemma 4.6, we perform the substitution

$$Z := w^{2m} U, \tag{4.10}$$

which turns the "tilde" system into a new system of four meromorphic ODEs for the unknown function U, which, according to (4.8), has a formal solution $U(w)$ vanishing at the origin. It is straightforward to check then, by combining (4.10) and (4.9), that the new system can be written in the form

$$w^2 U' = R(w, U, wU'), \tag{4.11}$$

where R is a holomorphic function defined near the origin. Performing finally in the standard fashion the substitution

$$V := wU'$$

and introducing the extended vector function $\mathbf{U} := (U, V)$, we obtain a first-order ODE

$$w\mathbf{U}' = Q(w, \mathbf{U}'), \tag{4.12}$$

where Q is a holomorphic near the origin function. The ODE (4.12) is a Briot–Bouquet type ODE (see Section 2), and hence its formal solutions are convergent, as required.

This completes the proof of Theorem 1.5. □

5 Regularity of smooth mappings between Fuchsian type hypersurfaces

In this section we shall prove Theorem 1.3. Compared to the proof of Theorem 1.5, we need an additional argument, which is the following regularity result for Fuchsian (Briot–Bouquet) systems of meromorphic ODEs.

Proposition 5.1 *Consider a first-order real ODE*

$$xy' = F(x, y), \quad x \in [0, a], \tag{5.1}$$

with y being n-dimensional, $n \geq 1$, and F analytic. Assume it has a solution $y(x)$ which is C^∞ on $[0, a]$. Then $y(x)$ is analytic everywhere on $[0, a]$.

Remark 5.2 A singular ODE (5.1) belongs to the classical class of Briot–Bouquet type ODEs discussed in Section 2. Their formal solutions at the singular point $x = 0$ are *convergent*, which, however does *not* say anything about the regularity of smooth solutions, which is why Proposition 5.1 requires a separate proof.

Proof of Proposition 5.1 The analyticity of $y(x)$ everywhere outside $x = 0$ follows from the analyticity of the given ODE, which is why we consider only the analyticity at the singularity $x = 0$. First, consider the Taylor series $\hat{y}(x)$ of $y(x)$. Since, again, (5.1) is a Briot–Bouquet ODE, $\hat{y}(x)$ is convergent. Hence, taking $y - \hat{y}(x)$ as a new unknown function, we get an ODE again of the kind (5.1) which has now a *flat* at $x = 0$ solution on $[0, a]$. We assume, by contradiction, that this solution is not identical zero near $x = 0$. Substituting the flat solution into the new ODE (5.1) and equalizing the Taylor series in both sides, we conclude that $F(x, 0) = 0$. Hence we conclude that the (again analytic) right hand side expands as

$$F(x, y) = A(x)y + \cdots,$$

where $A(x)$ is an analytic at the origin matrix, and dots stand for terms of degree at least 2 in y.

Second, let us use the notation $|y(t)|$ for the Euclidean norm, and $||y||$ for the sup norm of y on $[0, a]$. Since y is flat at 0, we may shrink the interval to make $||y||$ small. Using the analyticity of F, we then have the bound

$$|F(x, y(x))| \leq C|y(x)|, \tag{5.2}$$

where C is a constant depending on $||y||$.

Third, we make a simple observation that $|y|$ cannot vanish for $x > 0$. Indeed, any solution with $y(x_0) = 0$, $x_0 \neq 0$ would need to be identical zero by uniqueness near x_0, and hence identical zero by the analyticity of the ODE.

Fourth, we do the following: we "resolve the singularity" of (5.1) by making the substitution

$$x := e^t, \quad t \in (-\infty, \ln a].$$

Now the ODE (5.1) reads as

$$\frac{dy}{dt} = F(e^t, y) =: \tilde{F}(t, y). \tag{5.3}$$

We denote the new solution by $y(t)$ and still have

$$|\tilde{F}(t, y(t))| \leq C|y(t)|. \tag{5.4}$$

Now we need to obtain certain bounds. Taking the limit in the triangle inequality, we have

$$\frac{d}{dt}|y(t)| \leq \left|\frac{dy}{dt}\right|.$$

In view of this and the inequality (5.4),

$$\frac{d}{dt}\ln|y(t)| = \frac{1}{|y(t)|}\frac{d}{dt}|y(t)| \leq \frac{1}{|y(t)|}\left|\frac{dy}{dt}\right| \leq C,$$

and by integrating over $[t, \ln a]$ we obtain

$$\ln|y(\ln a)| - \ln|y(t)| \leq C(\ln a - t). \tag{5.5}$$

Simplifying (5.5) and applying exp, we finally get for the initial function $y(x)$:

$$|y(x)| \geq \tilde{C} \cdot x^C \tag{5.6}$$

($\tilde{C}$ is some other constant, which is non-zero since $|y(a)|$ is non-zero!). But (5.6) is a contradiction with the fact that $y(x)$ is flat near 0, and this proves the desired analyticity statement. □

Remark 5.3 The assertion of Proposition 5.1 holds also for a *complex* Briot–Bouquet ODE, i.e., when $y(x)$ is complex-valued and F is complex analytic (one just has to split the real and imaginary parts, and this immediately gives an already *real* ODE (5.1) for the vector function formed from the real and imaginary parts of y).

We are now in the position to prove Theorem 1.3.

Proof of Theorem 1.3 We come back to the proof of Theorem 1.3. Note that a hypersurface (1.1) necessarily contains (the germ at the origin of) the real line $L = \{z = 0, \ \mathrm{Im}\, w = 0\}$. This means, in particular, that for the given map $H(z, w)$, the vector functions $H(0, w), H_z(0, w)$ are well defined on L and are holomorphic in its open neighborhood. Arguing now identically to the above proof of Theorem 1.5, we reduce the analyticity problem for the given CR-map to the analyticity of C^∞ smooth solutions of an ODE identical to (4.12). The only difference is that, instead of substituting a formal power series map into the basic identity (3.9), we substitute into (3.9) a holomorphic map in a domain Ω, containing 0 in its closure and coming from the analyticity of the map in a neighborhood of the Levi-non-degenerate part of M. In view of the above, the Cauchy data (3.12) of the map H is C^∞ on the real line, and so is a solution of (4.12) under discussion. We then apply Proposition 5.1 (together with Remark 5.3) and conclude that the desired solution of (4.12) is analytic, and so is the Cauchy data (3.12) and hence the map H. This completely proves the theorem. □

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