

# Elliptic Bubbles in Moser’s 4D Quadratic Map: the Quadfurcation \*

Arnd Bäcker<sup>†</sup> and James D. Meiss<sup>‡</sup>

**Abstract.** Moser derived a normal form for the family of four-dimensional, quadratic, symplectic maps in 1994. This six-parameter family generalizes Hénon’s ubiquitous 2D map and provides a local approximation for the dynamics of more general 4D maps. We show that the bounded dynamics of Moser’s family is organized by a codimension-three bifurcation that creates four fixed points—a bifurcation analogous to a doubled, saddle-center—which we call a *quadfurcation*. In some sectors of parameter space a quadfurcation creates four fixed points from none, and in others it is the collision of a pair of fixed points that re-emerge as two or possibly four. In the simplest case the dynamics is similar to the cross product of a pair of Hénon maps, but more typically the stability of the created fixed points does not have this simple form. Up to two of the fixed points can be doubly-elliptic and be surrounded by bubbles of invariant two-tori; these dominate the set of bounded orbits. The quadfurcation can also create one or two complex-unstable (Krein) fixed points. Special cases of the quadfurcation correspond to a pair of weakly coupled Hénon maps near their saddle-center bifurcations.

**Key words.** Hénon map, symplectic maps, saddle-center bifurcation, Krein bifurcation, invariant tori

**AMS subject classifications.** 37J10 37J20 37J25 70K43

**1. Introduction.** Multi-dimensional Hamiltonian systems model dynamics on scales ranging from zettameters, for the dynamics of stars in galaxies [1, 2], to nanometers, in atoms and molecules [3, 4]. Hamiltonian flows generate symplectic maps on Poincaré sections [5, §9.14], and numerical algorithms for these flows can be symplectic [6, 7]. Symplectic maps also arise directly in discrete-time models of such phenomena as molecular vibrations [8, 9], stability of particle storage rings [10, 11], heating of particles in plasmas [12], microwave ionization of hydrogen [13] and chaos in celestial mechanics [14].

A map  $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is canonically symplectic for coordinates  $x \in \mathbb{R}^n$  and momenta  $y \in \mathbb{R}^n$  if its Jacobian matrix,  $Df(x, y)$ , satisfies

$$(1.1) \quad Df^T J Df = J, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where  $J$  is the Poisson matrix. In particular this implies that the map is volume preserving:  $\det(Df) = 1$ .

Perhaps the most famous symplectic map is the area-preserving map introduced by Hénon in 1969 as an elemental model to inform his studies of celestial mechanics [15]. This map is also the simplest nonlinear symplectic map, since it contains a single quadratic term, and yet—as Hénon showed—every quadratic area-preserving map can be reduced to his form [16].

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\*Submitted to the editors June 14, 2019.

**Funding:** This work was funded by the U.S. National Science Foundation under grants DMS-1440140 and DMS-181248, and by the Deutsche Forschungsgemeinschaft under grant KE 537/6–1

<sup>†</sup>Technische Universität Dresden, Institut für Theoretische Physik and Center for Dynamics, 01062 Dresden, Germany and Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, 01187 Dresden, Germany [arnd.baecker@tu-dresden.de](mailto:arnd.baecker@tu-dresden.de)

<sup>‡</sup>University of Colorado, Department of Applied Mathematics, Boulder, CO 80309-0526, USA [jdm@colorado.edu](mailto:jdm@colorado.edu)

Quadratic maps are useful because they model the dynamics of smooth maps in the neighborhood of a fixed point or of a flow near a periodic orbit. For example, quadratic terms in the power series give a local description of the dynamics near an accelerator mode of Chirikov's standard map [17]. More generally, any symplectic diffeomorphism can be  $C^\infty$  approximated by a polynomial map on a compact set [18].

Higher-dimensional analogues of Hénon's map were proposed in [19], and similar maps were used to study the stickiness of regions near an elliptic fixed point [20], the resonant formation of periodic orbits and invariant circles [21, 22, 23, 24], bifurcations due to twist singularities [25], and the dynamics near a homoclinic orbit to a saddle-center fixed point [26]. Such maps model a focusing-defocusing (FODO) magnet cell in a particle accelerator and have been used to study the structure of bounded orbits, the dynamic aperture, and robustness of invariant tori [27, 28, 29, 30].

In 1994, Moser [31] showed that every quadratic symplectic map on  $\mathbb{R}^{2n}$  is conjugate to the form

$$(1.2) \quad f = \alpha \circ \sigma \circ \beta.$$

Here  $\alpha, \beta : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  are symplectic maps,  $\beta$  is linear and  $\alpha$  is affine, and  $\sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  is a symplectic shear:

$$(1.3) \quad \sigma(x, y) = (x, y - \nabla V(x)),$$

where  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a cubic potential. There are several immediate consequences of this representation. Firstly, if the quadratic map  $f$  has finitely many fixed points, as it generically will, then there are at most  $2^n$  [31]. Note that more generally a quadratic non-symplectic map on a  $2n$ -dimensional space could have as many as  $2^{2n}$  isolated fixed points. Secondly, since the inverse of  $\sigma$  is also a quadratic shear of the same form (replace  $V$  by  $-V$ ), the inverse of any quadratic, symplectic map is also quadratic. More generally, the inverse of a quadratic diffeomorphism could be a polynomial map of higher degree [32, Thm 1.5]; for example, the inverse of the volume-preserving map  $(x, y, z, w) \mapsto (x, y + x^2, z + y^2, w + z^2)$  has degree eight. The form (1.2) also applies to cubic maps, but not to higher degree polynomial maps [33].

In this paper we study the dynamics of Moser's map in four dimensions. The normal form for the 4D case is reviewed and slightly transformed for convenience in §2. We argue in §3 that its fixed points are most properly viewed as arising from a bifurcation in which they emerge from a single fixed point as parameters are varied away from a codimension-three surface. Since this bifurcation often results in the creation of four fixed points, we call it a *quadfurcation*, with thanks to Strogatz who, "with tongue in cheek," proposed the term in an exercise for 1D ODEs in his well-known textbook [34, Ex. 3.4.12].

As we will see in §3.1, the unfolding of the quadfurcation in Moser's map can lead to (i) the creation of four fixed points from none, or (ii) a collision and re-emergence of two pairs of fixed points, or (iii) even the collision of a pair leading to four fixed points. The stability of these fixed points is investigated in §3.2. The unfolding of the quadfurcation along paths in parameter space is studied in §3.3-§3.4. When the map is reversible, §3.5, additional cases occur including the simplest one: the Cartesian product of a pair of area-preserving maps.

We numerically investigate the creation of families of invariant two-tori, as expected from KAM theory, around doubly elliptic fixed points in §4. In §5 we observe that these bubbles of elliptic orbits strongly correlate with the regions of bounded dynamics.

Since Moser's map is affinely conjugate to the general quadratic, symplectic map, it must have a limit in which it reduces to a pair of uncoupled Hénon maps—we show this in §6.

## 2. Moser's Quadratic, Symplectic Map.

**2.1. Four-Dimensional Normal Form.** For the two-dimensional case, the map (1.2) can be generically transformed by an affine coordinate change to the Hénon map  $H$ ,

$$(2.1) \quad H(x, y) = (-y + a_h + x^2, x),$$

with a single parameter  $a_h$ . When this map has an elliptic fixed point (for  $-3 < a_h < 1$ ), it is conjugate to the map whose dynamics were first studied by Hénon [15]. By a similar transformation Moser showed [31] that in four dimensions, (1.2) can generically be written as

$$(2.2) \quad (x', y') = f(x, y) = (C^{-T}(-y + \nabla V(x)), Cx),$$

where

$$(2.3) \quad C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

$$V = A_1 x_1 + A_2 x_2 + \frac{1}{2} A_3 x_1^2 + \varepsilon_2 x_1^3 + x_1 x_2^2,$$

and  $x \equiv (x_1, x_2) \in \mathbb{R}^2$ ,  $y \equiv (y_1, y_2) \in \mathbb{R}^2$ . Here there are two discrete parameters,  $\varepsilon_1 \equiv \det(C) = \alpha\delta - \beta\gamma = \pm 1$ , and  $\varepsilon_2 \equiv \pm 1$  or 0. The remaining six parameters are free. We will refer to (2.2) as *Moser's map* below. It is convenient to think of the six real parameters of  $f$  as  $(A_1, A_2, A_3)$  and  $(\alpha, \delta, \mu)$ , where

$$(2.4) \quad \mu = \beta + \gamma.$$

Indeed, given these six, and the sign  $\varepsilon_1$ , we can determine the off-diagonal elements of  $C$  from

$$(2.5) \quad \beta, \gamma = \frac{1}{2}\mu \pm \sqrt{\varepsilon_1 - \alpha\delta + \mu^2/4}.$$

The choice of the sign here is unimportant since this simply replaces  $C$  with  $C^T$ , and the resulting map is conjugate to the inverse of (2.2); see §2.3. Note that (2.5) has real solutions only when  $\mu^2 \geq 4(\alpha\delta - \varepsilon_1)$ , and that  $C$  is symmetric only at the lower bound of this inequality.

The map (2.2) is easily seen to be symplectic (1.1) as it is the composition of the symplectic shear (1.3), the Poisson map,  $J(x, y) = (-y, x)$ , and the linear symplectic map

$$(x, y) \mapsto (C^{-T}x, Cy).$$

**2.2. Shifted Coordinates.** As a first step in the analysis of the dynamics of (2.2), we will study its fixed points. To do this it is convenient to define shifted variables and parameters. There is a codimension-three set of parameters where the map has exactly one fixed point, and focusing on this set simplifies the calculations more generally.

For any matrix  $C$  and when  $\varepsilon_2 \neq 0$ , the map (2.2) has exactly one fixed point at

$$(2.6) \quad \begin{aligned} x^Q &= (\delta, \tfrac{1}{2}\mu), \\ y^Q &= Cx^Q, \end{aligned}$$

when the parameters  $(A_1, A_2, A_3)$  of the potential (2.3) take the values

$$(2.7) \quad \begin{aligned} A_1^Q &= 3\delta^2\varepsilon_2 + \tfrac{1}{4}\mu^2, \\ A_2^Q &= \delta\mu, \\ A_3^Q &= 2\alpha - 6\delta\varepsilon_2. \end{aligned}$$

To see this, and to simplify the computations it is convenient to shift coordinates so that the origin is at the point  $(x^Q, y^Q)$  and to define new shifted parameters:

$$(2.8) \quad \begin{aligned} (\xi, \eta) &= (x - x^Q, y - y^Q), \\ a &= A_1 - A_1^Q + \delta(A_3 - A_3^Q), \\ b &= A_2 - A_2^Q, \\ c &= A_3 - A_3^Q. \end{aligned}$$

In these new coordinates, (2.2) becomes

$$(2.9) \quad (\xi', \eta') = M(\xi, \eta) = (\xi + C^{-T}(-\eta + C\xi + \nabla U(\xi)), C\xi),$$

where the new potential,

$$(2.10) \quad U = a\xi_1 + b\xi_2 + \tfrac{1}{2}c\xi_1^2 + \varepsilon_2\xi_1^3 + \xi_1\xi_2^2,$$

is the same as  $V$  from (2.3) upon replacing  $(A_1, A_2, A_3)$  by  $(a, b, c)$ . This shifted form of Moser's quadratic, symplectic map is convenient because several computations can be carried out more easily and many of the expressions we obtain below will be more compact.

The map (2.9) is generated by the discrete Lagrangian

$$(2.11) \quad \mathcal{L}(\xi, \xi') = (\xi' - \xi)^T C\xi - U(\xi),$$

through the equation

$$\eta' d\xi' - \eta d\xi = d\mathcal{L}(\xi, \xi').$$

In other words the map is implicitly defined by  $\eta = -\partial_\xi \mathcal{L}(\xi, \xi')$  and  $\eta' = \partial_{\xi'} \mathcal{L}(\xi, \xi')$ . This means that  $M$  is *exact* symplectic [35], and of course, that it preserves the symplectic form  $d\xi \wedge d\eta$ . Note also that if we denote an orbit of (2.9) as a sequence

$$(2.12) \quad \{(\xi_t, \eta_t) \in \mathbb{R}^4 \mid (\xi_t, \eta_t) = M(\xi_{t-1}, \eta_{t-1}), t \in \mathbb{Z}\}$$

and define the action of a finite portion by

$$\mathcal{A} = \sum_{t=j}^{k-1} \mathcal{L}(\xi_t, \xi_{t+1}),$$

then each stationary point of  $\mathcal{A}$ , for fixed endpoints, is a segment of an orbit with the momentum determined by  $\eta_{t+1} = C\xi_t$ .

**2.3. Second Difference Form and ODE Limit.** The shifted form (2.9) of Moser's map (2.2) can be written as a second difference equation. Denoting an orbit as (2.12), then  $\eta_t = C\xi_{t-1}$  and the map (2.9) is equivalent to

$$(2.13) \quad C^T(\xi_{t+1} - \xi_t) - C(\xi_t - \xi_{t-1}) = \nabla U(\xi_t).$$

One immediate consequence of (2.13) is that the replacement  $C \rightarrow C^T$  is clearly equivalent to inverting the map. Therefore the invariant sets of the Moser map with  $C \rightarrow C^T$  are the same as those of the original map. Similarly, note that the replacement  $C \rightarrow -C$  together with  $\xi \rightarrow -\xi$  and  $c \rightarrow -c$  leaves the Moser map invariant. We will also use the form (2.13) in §3.2 and §5.

To emphasize the different roles of the symmetric and antisymmetric parts of  $C$ , let

$$(2.14) \quad \begin{aligned} C &= C^s + C^a, \\ C^s &\equiv \frac{1}{2}(C + C^T) = \begin{pmatrix} \alpha & \mu/2 \\ \mu/2 & \delta \end{pmatrix}, \\ C^a &\equiv \frac{1}{2}(C - C^T) = \begin{pmatrix} 0 & \nu/2 \\ -\nu/2 & 0 \end{pmatrix}, \end{aligned}$$

where  $\nu = \beta - \gamma$  and, as before  $\mu = \beta + \gamma$ . Then (2.13) becomes

$$(2.15) \quad C^s(\xi_{t+1} - 2\xi_t + \xi_{t-1}) - C^a(\xi_{t+1} - \xi_{t-1}) = \nabla U(\xi_t).$$

In this form the map closely resembles a pair of second-order differential equations.

Indeed in a neighborhood of the origin in the phase space,  $(\xi, \eta)$ , and in the space of the new parameters,  $(a, b, c, \nu)$ , (2.15) approaches a Lagrangian system of ODEs. To see this, formally introduce a parameter  $h$ , and scale

$$(a, b, c, \nu) \rightarrow (h^4 a, h^4 b, h^2 c, h\nu).$$

Here  $h\nu$  represents the deviation from symmetry, so that  $C \rightarrow C^s + hC^a$ . Then in the limit  $h \rightarrow 0$ , the second difference equation (2.13) limits on a system of ODEs in a scaled time  $\tau = ht$ , and a new variable

$$\xi_t \rightarrow h^2 q(\tau).$$

This scaling implies that for the potential (2.10),  $\nabla_\xi U(\xi) \rightarrow h^4 \nabla_q U(q)$ . Moreover, as  $h \rightarrow 0$ , the second difference  $\xi_{t+1} - 2\xi_t + \xi_{t-1} \rightarrow h^4 \ddot{q}(\tau) + \mathcal{O}(h^5)$  and the first difference  $\xi_{t+1} - \xi_{t-1} \rightarrow 2h^3 \dot{q}(\tau) + \mathcal{O}(h^4)$ . Substituting these into (2.15), gives the limiting system

$$(2.16) \quad C^s \ddot{q} - 2C^a \dot{q} = \nabla U(q),$$

as  $h \rightarrow 0$ . Thus the symmetric part of  $C$  corresponds to a mass matrix, multiplying the acceleration. By contrast,  $C^a$  corresponds to a Coriolis-like force, which is proportional to the velocity.

The system (2.16) is obtained from the Lagrangian

$$(2.17) \quad L(q, \dot{q}) = \frac{1}{2} \dot{q}^T C^s \dot{q} + q^T C^a \dot{q} + U(q).$$

To convert (2.16) into a Hamiltonian system define the canonical momenta

$$p(\tau) = \frac{\partial L}{\partial \dot{q}} = C^s \dot{q} - C^a q,$$

giving a Hamiltonian,  $H = p\dot{q} - L$ , that has Coriolis and centripetal terms:

$$(2.18) \quad H(q, p) = \frac{1}{2} p^T C^s p + p^T C^s C^a q - \frac{1}{2} q^T (C^a C^s C^a) q - U(q).$$

Thus  $C^s$  is the mass matrix, and  $U$  is the *negative* of the potential energy. The antisymmetric matrix  $C^a$  contributes both a Coriolis-like term, bilinear in  $q$  and  $p$ , and a centripetal-like term, quadratic in  $q$ . We will use this interpretation, for the case of symmetric  $C$ , in §3.5.

**3. Quadfurcation.** From the general theory [31] we know that Moser's map (2.2) and hence (2.9) has at most four (isolated) fixed points. On the codimension-three surface  $a = b = c = 0$  in parameter space there is a single fixed point (unless  $\varepsilon_2 = 0$ ). As we will see below, there are sectors in parameter space near this surface for which there are no fixed points, and sectors for which there are four. It seems appropriate to call the creation of four fixed points from none a *quadfurcation*. In some cases a quadfurcation can be analogous to a simultaneous pair of co-located saddle-center bifurcations; however, the stabilities of the resulting fixed points are usually not those of a pair of decoupled area-preserving maps, namely the Cartesian product of 2D saddles and centers.

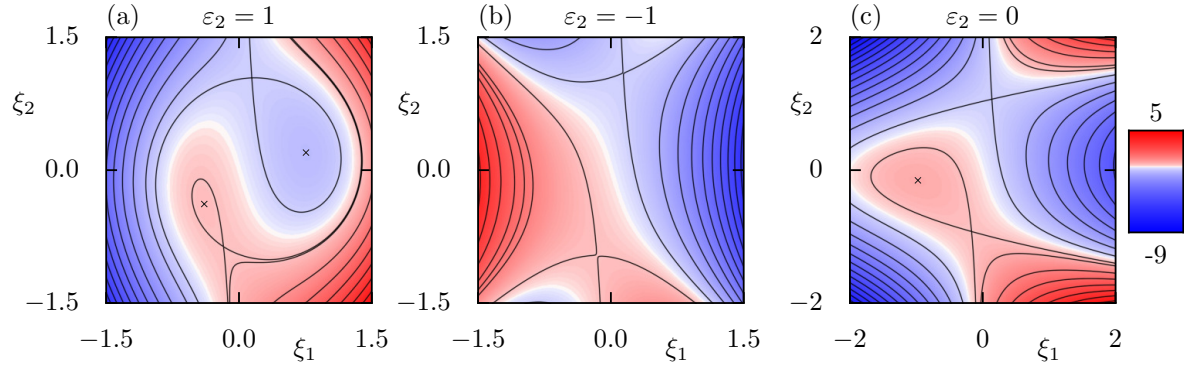
In the following subsections we study the fixed points, their stability, and the structure of the region of phase space around the elliptic fixed points that contains bounded orbits.

**3.1. Fixed Points.** The coordinates,  $\xi^*$ , of the fixed points of (2.9) are critical points of the cubic polynomial (2.10). Several contour plots of  $U(\xi)$  are shown in Fig. 1. Critical points satisfy the equations

$$(3.1) \quad 0 = \nabla U(\xi^*) = \begin{pmatrix} a + c\xi_1^* + 3\varepsilon_2\xi_1^{*2} + \xi_2^{*2} \\ b + 2\xi_1^*\xi_2^* \end{pmatrix}.$$

Note that the positions are independent of the matrix  $C$ , though the momenta, determined by  $\eta^* = C\xi^*$ , depend on the full matrix. Note that if  $\nabla U(\xi^*) = 0$  for parameters  $(a, b, c)$ , then it is also zero at the point  $-\xi^*$  for parameters  $(a, b, -c)$  and at the point  $(\xi_1^*, -\xi_2^*)$  for  $(a, -b, c)$ . Thus we can restrict attention to  $b, c \geq 0$ .

The case  $a = b = c = 0$  is an organizing center for the solutions of (3.1). In this case the second component immediately implies that either  $\xi_1^* = 0$  or  $\xi_2^* = 0$ . Then, whenever  $\varepsilon_2 \neq 0$ , the first implies that both  $\xi_1^* = \xi_2^* = 0$ . We call this the *quadfurcation point*. Since the matrix elements  $(\alpha, \delta, \mu)$  are still free parameters, it occurs on a codimension-three surface



**Figure 1.** Contour plots of the potential  $U$  (2.10) for  $(a, b, c) = (-1, -0.3, -1)$ . (a)  $\varepsilon_2 = 1$ . Here there are four critical points, implying four fixed points of (2.9). (b)  $\varepsilon_2 = -1$ , two critical points. (c)  $\varepsilon_2 = 0$ , three critical points.

in the six-dimensional parameter space. The off-diagonal elements of the matrix,  $\beta$  and  $\gamma$ , are then fixed up to exchange by the condition  $\det(C) = \varepsilon_1$ , (2.5). In the parameterization (2.9) the quadfurcation surface is just the three-plane  $a = b = c = 0$ . In Moser's original parameterization, this surface is determined by (2.7).

More generally if  $b \neq 0$  then (3.1) implies that  $\xi_1^* \neq 0$ , so

$$(3.2) \quad \xi_2^* = -\frac{b}{2\xi_1^*}.$$

Substituting into the first component of (3.1) then shows that  $\xi_1^*$  must be a root of the scalar polynomial

$$(3.3) \quad P(v; a, b, c, \varepsilon_2) = 3\varepsilon_2 v^4 + cv^3 + av^2 + \frac{1}{4}b^2.$$

When  $\varepsilon_2 \neq 0$  this polynomial is quartic, and so has at most four roots. Since  $P$  has no linear term, it has exactly one root in  $\mathbb{C}$  only when  $a = b = c = 0$ , on the quadfurcation set. When  $\varepsilon_2 = 0$ ,  $P(v)$  is at most cubic, and there are at most three isolated roots. Several examples are shown in Fig. 2.

There are various regions in the parameter space  $(a, b, c)$  that have different numbers of fixed points. We now determine the bifurcation sets, which separate these regions: to find these sets when  $\varepsilon_2 \neq 0$ , it is easiest to solve for the surfaces on which there are double roots, i.e.,  $P(v) = 0$  and  $P'(v) = 0$ . First,  $P$  always has a critical point,  $P'(v) = 0$ , at  $v = 0$ , and it has two more critical points if

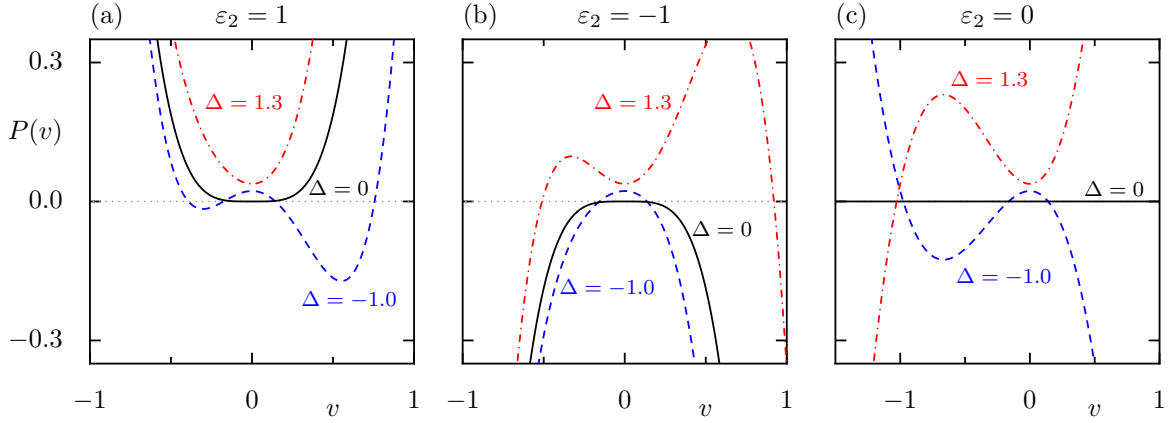
$$(3.4) \quad \varepsilon_2 a < a_+(c) \equiv \frac{3}{32}c^2.$$

Eliminating  $v$  from the two equations  $P(v) = P'(v) = 0$  gives the discriminant

$$b^2 [1728\varepsilon_2 b^4 + 9(-3c^4 + 48\varepsilon_2 ac^2 - 128a^2)b^2 - 16a^3c^2 + 192\varepsilon_2 a^4] = 0.$$

Thus there are double roots at  $b = 0$ , and on the surfaces  $b = \pm\sqrt{D_{\pm}(a, c)}$ , where

$$(3.5) \quad D_{\pm}(a, c) = \frac{\varepsilon_2}{384} \left( 3(c^2 - 8\varepsilon_2 a)^2 - 64a^2 \pm \frac{|c|}{\sqrt{3}}(3c^2 - 32\varepsilon_2 a)^{3/2} \right).$$



**Figure 2.** The polynomial  $P(v)$  (3.3) along the curve  $(a, b, c) = \Delta(1, 0.3, 1)$  as  $\Delta$  varies. (a)  $\varepsilon_2 = 1$  where four roots are created when  $\Delta$  decreases through zero; (b)  $\varepsilon_2 = -1$  where there are two roots for any  $\Delta \neq 0$ ; and (c)  $\varepsilon_2 = 0$  with one root for  $\Delta > 0$ , infinitely many at  $\Delta = 0$ , and three for  $\Delta < 0$ .

For these surfaces to be real, the radical in (3.5) must be real, i.e., (3.4) must be satisfied. Moreover letting

$$(3.6) \quad a_-(c) \equiv \frac{1}{12}c^2,$$

then  $D_+(\varepsilon_2 a_-, c) = 0$  and  $D_-(0, c) = 0$ . To define real-valued functions let

$$(3.7) \quad \begin{aligned} b_+(a, c) &= \begin{cases} \sqrt{D_+(a, c)}, & a < \varepsilon_2 a_-, \text{ and } \varepsilon_2 a < a_+ \\ 0, & \text{otherwise} \end{cases}, \\ b_-(a, c) &= \begin{cases} \sqrt{D_-(a, c)}, & a < 0, \text{ and } \varepsilon_2 a < a_+ \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

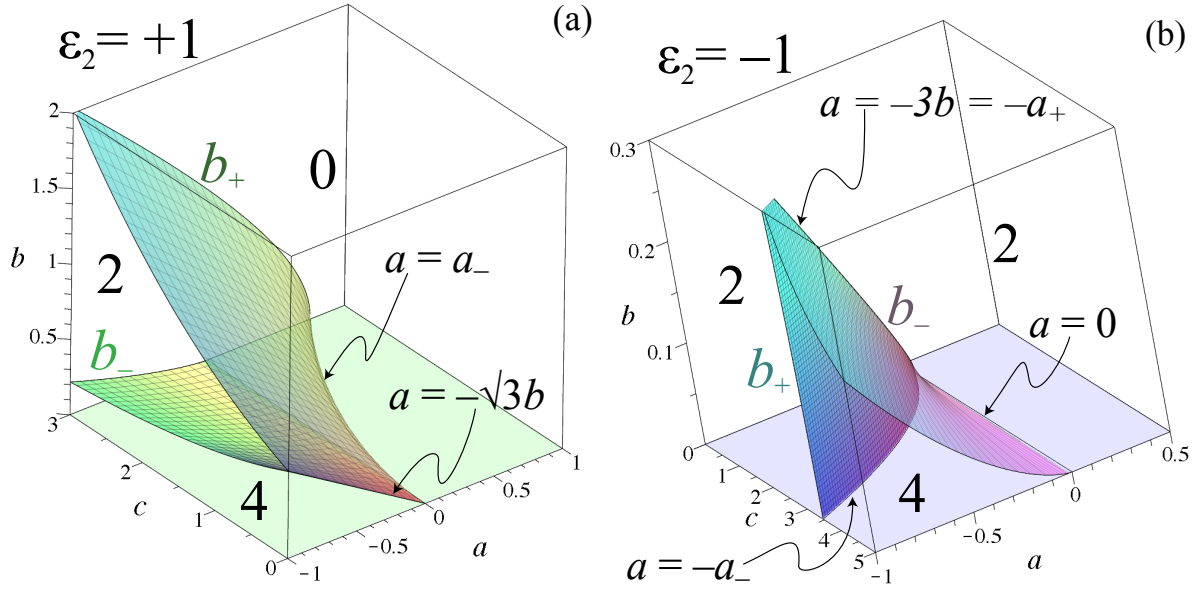
The resulting surfaces are shown in Fig. 3.

If  $\varepsilon_2 = 1$ , then when  $b > b_+$  there are no real roots. At the upper surface,  $b = b_+(a, c)$ , which is nonzero for  $a < a_-$ , two roots are created. Two additional roots are created upon crossing  $b_-(a, c)$ , which is nonzero for  $a < 0$ , see Table 1. The two surfaces  $b_{\pm}(a, c)$  intersect at  $c = 0$  on the line  $a = -\sqrt{3}|b|$ . Crossing this codimension-two set  $b_- = b_+$  at  $c = 0$  and moving into the region  $b < b_-$  thus corresponds to the simultaneous creation of four fixed points at two different locations, i.e., to a pair of simultaneous saddle-center bifurcations, as we will see in §3.2.

If  $\varepsilon_2 = -1$ , then (3.3) has four real roots only if  $-a_+ < a < 0$  and  $b_+ < b < b_-$ . Note that  $b_+$  is nonzero only when  $-a_- < a < 0$ . Inside cusp-like shape formed from the  $b_{\pm}$  surfaces, as shown in Fig. 3(b), there are four roots. Going outwards from this region, by either crossing  $b_-$  or  $b_+$ , two solutions disappear in a saddle-center bifurcation. Thus on the surfaces  $b = b_+$  or  $b = b_-$  there are three roots (one of them with multiplicity 2). When these surfaces merge, on the curve  $a = -a_+$  and  $b = b_{\pm}(-a_+, c) = c^2/32$ , there are two fixed points:

$$(\xi_1^*, \xi_2^*) = \begin{cases} \frac{c}{8}(1, -1), & (\text{multiplicity } 3) \\ \frac{c}{24}(-1, 9), & (\text{multiplicity } 1) \end{cases} \quad (\varepsilon_2 = -1, a = -3b = -a_+).$$





**Figure 3.** Surfaces in  $(a, b, c)$  for which there are double roots of  $P$  (3.3) (a) when  $\varepsilon_2 = 1$  and (b) when  $\varepsilon_2 = -1$ . For  $\varepsilon_2 = 1$ , when  $b$  is sufficiently positive there are no real solutions, as  $b$  decreases through the  $b_+$  surface (3.7), two solutions are created, and finally when it passes through the  $b_-$  surface there are four solutions. For  $\varepsilon_2 = -1$ , there are two solutions if  $b$  is sufficiently large. An additional two solutions are created upon moving through either of the  $b_{\pm}$  surfaces. Curves where bifurcation surfaces intersect are also shown in each panel: in (a)  $a = a_-(c)$  on the  $b = 0$  plane and  $a = -\sqrt{3}b$  on the  $c = 0$  plane, and in (b)  $a = 0$  and  $-a_-(c)$  at  $b = 0$  and  $a = -3b = -a_+(c)$ .

The cases  $b = 0$  and  $c = 0$  require special treatment. When  $c = 0$  but  $b \neq 0$ , the polynomial (3.3) has no cubic term and the fixed points can be solved for explicitly:

$$(3.8) \quad \xi_1^* = \pm \sqrt{-\frac{1}{6}\varepsilon_2 \left( a \pm \sqrt{a^2 - 3\varepsilon_2 b^2} \right)} \quad (c = 0),$$

where  $\xi_2^*$  is then obtained from (3.2). Note that there are four possible points here, with choices for the outer  $\pm$  and the inner  $\pm$ . This equation gives real solutions only when both square roots are real. When  $\varepsilon_2 = 1$ , (3.8) gives four real solutions if  $a < -\sqrt{3}|b|$ . On the boundary  $a = -\sqrt{3}|b|$ , these four solutions are created in two pairs, at

$$(\xi_1^*, \xi_2^*) = \pm \sqrt{-\frac{a}{6}} \left( 1, -\sqrt{3} \operatorname{sgn}(b) \right) \quad (\varepsilon_2 = 1, a = -\sqrt{3}|b|, c = 0).$$

These pairs merge at the quadfurcation point  $a = b = c = 0$ . If  $\varepsilon_2 = -1$  then only the inner  $+$  sign choice is valid and (3.8) gives two real solutions whenever  $a$  or  $b \neq 0$ . Table 1 delineates the possibilities.

If  $b = 0$ , then (3.1) implies that either  $\xi_1^* = 0$  or  $\xi_2^* = 0$ . The first component of (3.1) is then trivially a quadratic function. In this case the four solutions are

$$(3.9) \quad (\xi_1^*, \xi_2^*) = \begin{cases} (0, \pm\sqrt{-a}) \\ \left( \frac{\varepsilon_2}{6}(-c \pm \sqrt{c^2 - 12\varepsilon_2 a}), 0 \right) \end{cases} \quad (b = 0).$$

| $\varepsilon_2$ | $b$ | $c$ | Number of Fixed Points |            |                                                    |                             |                                               |
|-----------------|-----|-----|------------------------|------------|----------------------------------------------------|-----------------------------|-----------------------------------------------|
|                 |     |     | 0                      | 1          | 2                                                  | 3                           | 4                                             |
| 1               |     |     | $b > b_+$              | $b = b_+$  | $b \in (b_-, b_+)$                                 | $b = b_-$                   | $b < b_-$                                     |
|                 | 0   |     | $a > a_-$              | $a = a_-$  | $0 < a < a_-$                                      | $a = 0$                     | $a < 0$                                       |
|                 |     | 0   | $a > -\sqrt{3} b $     |            | $a = -\sqrt{3} b $                                 |                             | $a < -\sqrt{3} b $                            |
|                 | 0   | 0   | $a > 0$                | $a = 0$    |                                                    |                             | $a < 0$                                       |
| -1              |     |     |                        |            | $a \notin (-a_+, 0]$ ,<br>or $b \notin [b_+, b_-]$ | $b = b_+$ ,<br>or $b = b_-$ | $a \in (-a_+, 0)$ ,<br>and $b \in (b_+, b_-)$ |
|                 | 0   |     |                        |            | $a \notin [-a_-, 0]$                               | $a = -a_-, 0$               | $a \in (-a_-, 0)$                             |
|                 |     | 0   |                        |            | always                                             |                             |                                               |
|                 | 0   | 0   |                        | $a = 0$    | $a \neq 0$                                         |                             |                                               |
| 0               |     |     |                        | $a > -a_0$ | $a = -a_0$                                         | $a < -a_0$                  |                                               |
|                 | 0   |     |                        | $a > 0$    | $a = 0$                                            | $a < 0$                     |                                               |
|                 |     | 0   | $a \geq 0$             |            | $a < 0$                                            |                             |                                               |
|                 | 0   | 0   | $a > 0$                |            | $a < 0$                                            |                             |                                               |

Table 1

Number of fixed points of the map (2.9) depending upon  $\varepsilon_2$  and parameters  $(a, b, c)$ . Since this number is an even function of  $b$  and  $c$ , we can assume that both are nonnegative. The functions  $b_{\pm}(a, c)$  are given by (3.7), and  $a_{\pm}(c)$  by (3.4) and (3.6). The additional rows delineate special cases when either  $b$  or  $c$  or both are zero.

Note that when  $\varepsilon_2 = 1$  there are four real fixed points whenever  $a < 0$ , and two in the range  $0 < a < a_-$ . If  $\varepsilon_2 = -1$ , then the first pair is real when  $a \leq 0$ , and the second pair is real only if  $a \geq -a_-$ . Thus there are four real fixed points when  $-a_- < a < 0$ .

For  $\varepsilon_2 = 0$  and  $c \neq 0$ , the polynomial (3.3) is cubic, so there is always at least one fixed point. The critical points of  $P$  are at  $v = 0$  and  $v = -2a/3c$ , and critical values  $P(0) = \frac{1}{4}b^2 \geq 0$  and  $P(-2a/3c) = \frac{4}{27}\frac{a^3}{c^2} + \frac{1}{4}b^2$ . Thus there are three fixed points when

$$(3.10) \quad a < -a_0 \equiv -3 \left| \frac{bc}{4} \right|^{2/3}.$$

If  $c = 0$ , then the polynomial (3.3) is at most quadratic. It has two solutions if  $a < 0$ . Special cases are again shown in Table 1. Finally, when  $a = b = c = 0$ , there is a line of fixed points at  $\xi_2 = 0$ . This case (not shown in Table 1) is the only one for which there are infinitely many fixed points.

**3.2. Stability.** The stability properties of fixed points of the map (2.9) are most easily computed using the second difference form (2.13). Linearization about a fixed point gives the  $2 \times 2$  eigenvalue problem

$$(3.11) \quad (\lambda C^T + \lambda^{-1}C)q = Wq,$$

where  $W = C + C^T + D^2U(\xi)$  is the symmetric matrix

$$(3.12) \quad W = \begin{pmatrix} 2\alpha & \mu \\ \mu & 2\delta \end{pmatrix} + \begin{pmatrix} c + 6\varepsilon_2\xi_1 & 2\xi_2 \\ 2\xi_2 & 2\xi_1 \end{pmatrix}.$$

Given the coordinate eigenvector  $q$ , the momentum components are  $p = \lambda^{-1}Cq$ . A similar analysis can be used, more generally, for a period- $n$  orbit, see [36, eqs. (21)-(22)].

Thus for a nontrivial solution of (3.11), the  $2 \times 2$  matrix

$$N(\lambda) = \lambda C^T + \lambda^{-1}C - W$$

must be singular. Since the map is symplectic its eigenvalues must satisfy the reflexive property: if  $\lambda$  is an eigenvalue, then so is  $\lambda^{-1}$ . This follows for (3.11) because  $W^T = W$  implies that  $N^T(\lambda) = N(\lambda^{-1})$ . As a consequence the characteristic polynomial can be written as a quadratic polynomial in the partial trace  $\rho = \lambda + \lambda^{-1}$ :

$$(3.13) \quad \det(N(\lambda)) = \varepsilon_1(\rho^2 - A\rho + B - 2),$$

where we recall that  $\varepsilon_1 = \det C$ . The parameters  $A, B$  are Broucke's stability parameters [37, 38]. More generally, these parameters are determined by the linearized map  $DM$  at a fixed point, by  $A = \text{tr}(DM)$  and  $B = \frac{1}{2}[(\text{tr}(DM))^2 - \text{tr}(DM^2)]$ ; equivalently, in terms of the eigenvalues  $\rho_{1,2}$  of the reduced characteristic polynomial one has  $A = \rho_1 + \rho_2$  and  $B = \rho_1\rho_2 + 2$ , or explicitly

$$(3.14) \quad \rho_{1,2} = \frac{1}{2} \left( A \pm \sqrt{A^2 + 8 - 4B} \right) \equiv \lambda_{1,2} + \lambda_{1,2}^{-1},$$

where  $\lambda_{1,2}, \lambda_{1,2}^{-1}$  are the two reciprocal pairs of eigenvalues of the characteristic polynomial of the linearized map.

The  $(A, B)$ -plane is divided into seven stability regions as shown in Fig. 4. These are bounded by the saddle-center ( $SC$ ), and period-doubling ( $PD$ ) lines:

$$(3.15) \quad \begin{aligned} SC &= B - 2A + 2 = 0, \\ PD &= B + 2A + 2 = 0, \end{aligned}$$

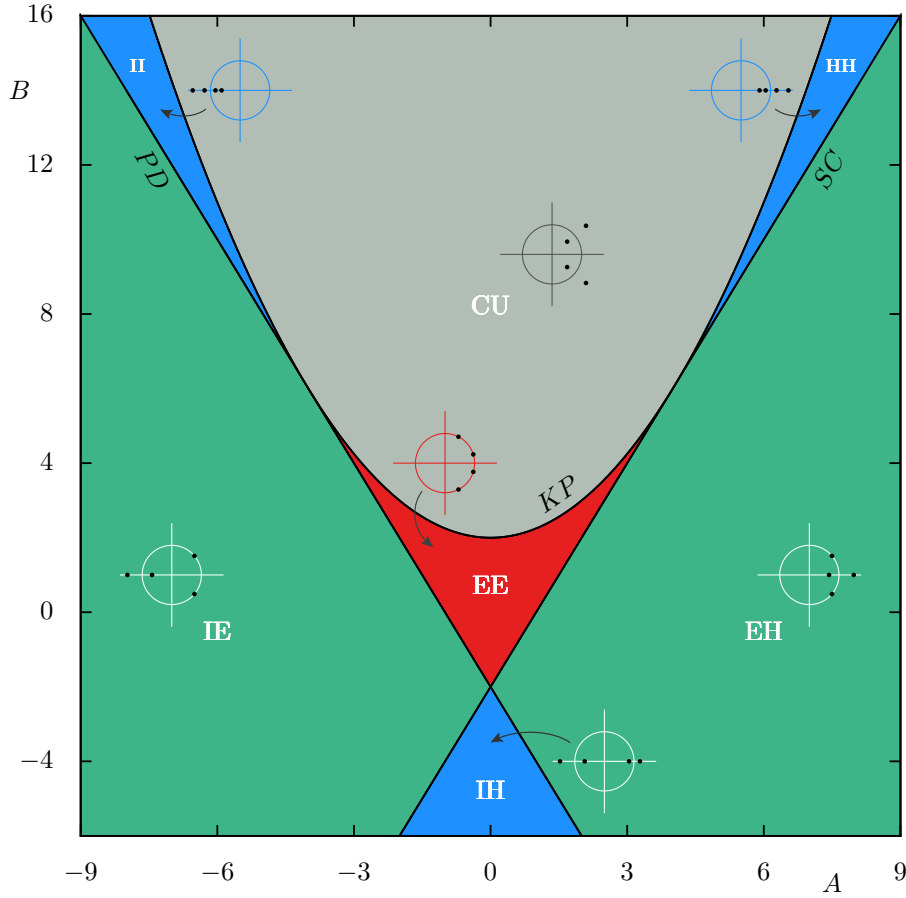
on which there is a pair of eigenvalues at  $+1$  or  $-1$ , respectively, and the Krein parabola ( $KP$ )

$$(3.16) \quad KP = B - A^2/4 - 2 = 0$$

on which there are double eigenvalues on the unit circle (for  $B < 6$  and  $|A| < 4$ ) or real axis (for  $B \geq 6$  and  $|A| \geq 4$ ). The point  $(A, B) = (4, 6)$  corresponds to four unit eigenvalues. The seven stability regions with different types of linearized dynamics around the fixed point are labeled by combinations of E (elliptic), H (hyperbolic), and I (inverse hyperbolic), each involving a pair of eigenvalues  $(\lambda, 1/\lambda)$ , and in the CU (complex unstable) region, where  $KP > 0$ , there is a complex quartet of eigenvalues.

For a general symmetric  $W$ , the stability parameters are

$$\begin{aligned} A &= \varepsilon_1(w_{22}\alpha + w_{11}\delta - w_{12}\mu), \\ B &= \varepsilon_1(\det(W) - (\beta - \gamma)^2) + 2. \end{aligned}$$



**Figure 4.** Stability regions for a reflexive, quartic, characteristic polynomial. There are seven regions,  $EE$ ,  $EH$ ,  $IE$ ,  $IH$ ,  $II$ ,  $HH$ , and  $CU$ , with the configuration of the four eigenvalues as shown in the representative complex plane insets.

For the matrix (3.12) these become

$$(3.17) \quad \begin{aligned} A &= 4 + \varepsilon_1 [\delta c - (\beta - \gamma)^2 + (6\delta\varepsilon_2 + 2\alpha)\xi_1 - 2\mu\xi_2], \\ B &= 6 + \varepsilon_1 [2(\delta c - (\beta - \gamma)^2) + 12\varepsilon_2(\xi_1 + \delta)\xi_1 + 2(2\alpha + c)\xi_1 - 4(\xi_2 + \mu)\xi_2], \end{aligned}$$

and the saddle-center and period-doubling parameters are

$$(3.18) \quad \begin{aligned} SC &= \varepsilon_1 \det(D^2U) = \varepsilon_1(12\varepsilon_2\xi_1^2 + 2c\xi_1 - 4\xi_2^2), \\ PD &= 16 + 4\varepsilon_1 (\delta c - (\beta - \gamma)^2 + 3\varepsilon_2\xi_1^2 + (6\delta\varepsilon_2 + 2\alpha + \tfrac{1}{2}c)\xi_1 - \xi_2^2 - 2\mu\xi_2). \end{aligned}$$

In particular, note that the sign of  $SC$  depends on the fixed points only through the sign of the Hessian of the potential (2.10); for example, if  $\varepsilon_1 = 1$ , then  $SC > 0$  at extrema and

$SC < 0$  at saddle points of  $U$ . Finally, the Krein parameter is

$$(3.19) \quad \begin{aligned} KP = & -\frac{1}{4}(\delta c - (\beta - \gamma)^2)^2 \\ & + (c\delta - (\beta - \gamma)^2)(\mu\xi_2 - (3\delta\varepsilon_2 + \alpha)\xi_1) + 2\varepsilon_1 c\xi_1 \\ & + (12\varepsilon_1\varepsilon_2 - (3\delta\varepsilon_2 + \alpha)^2)\xi_1^2 + 2\mu(3\delta\varepsilon_2 + \alpha)\xi_1\xi_2 - (\mu^2 + 4\varepsilon_1)\xi_2^2. \end{aligned}$$

At the quadfurcation, where  $a = b = c = \xi_1 = \xi_2 = 0$ , (3.17) gives

$$(3.20) \quad \begin{aligned} A^Q &= 4 - \varepsilon_1(\beta - \gamma)^2, \\ B^Q &= 6 - 2\varepsilon_1(\beta - \gamma)^2. \end{aligned}$$

This implies that the quadfurcation point lies on the saddle-center line; indeed from (3.18),  $SC = 0$  at this point. When  $\varepsilon_1 = 1$  ( $\varepsilon_1 = -1$ ), then  $(A^Q, B^Q)$  lies below and to the left of (above and to the right of) the point  $(4, 6)$ . The quadfurcation occurs at  $(4, 6)$  only when  $\beta = \gamma$ , i.e., when the matrix  $C$  is symmetric, see the discussion in §3.5 below. The quadfurcation occurs below the period-doubling line, i.e., for  $PD < 0$ , only if  $\varepsilon_1 = 1$  and  $|\beta - \gamma| > 2$ .

More generally, since fixed points are critical points of  $U$ , (3.1), they can be created or destroyed only when  $\det(D^2U) = 0$ , which is equivalent to  $SC = 0$  by (3.18). This can also be seen upon computing the *resultant* of  $SC$  and  $P$  (3.3)—recall that the resultant gives the set of parameters on which two polynomials simultaneously vanish. This resultant is proportional to  $b^4(b^2 - b_-^2)(b^2 - b_+^2)$ . Of course, this is what we saw in Fig. 3—pairs of fixed points are created or destroyed upon crossing the surfaces  $b = b_{\pm}$  (3.7).

**3.3. Quadfurcation along a Line in Parameter Space.** Near the quadfurcation, if we assume that  $a, b, c = \mathcal{O}(\Delta)$  for  $\Delta \ll 1$ , then  $\xi_i = \mathcal{O}(\sqrt{\Delta})$ , the cubic term involving  $c$  in (3.3) is negligible to lowest order, and the fixed points are given by (3.8) to  $\mathcal{O}(\sqrt{\Delta})$ . Substitution into the stability criteria then gives

$$(3.21a) \quad SC = \mp 4\varepsilon_1 \sqrt{a^2 - 3\varepsilon_2 b^2} + \mathcal{O}(\Delta^{3/2}),$$

$$(3.21b) \quad PD = 16 - 4\varepsilon_1(\beta - \gamma)^2 + 8\varepsilon_1[(3\delta\varepsilon_2 + \alpha)\xi_1 - \mu\xi_2]\sqrt{\Delta} + \mathcal{O}(\Delta),$$

$$(3.21c) \quad KP = (\beta - \gamma)^2 \left[ -\frac{1}{4}(\beta - \gamma)^2 + (3\delta\varepsilon_2 + \alpha)\xi_1 - \mu\xi_2 \right] + \mathcal{O}(\Delta).$$

Note that the  $\mp$  signs in (3.21a) correspond to the inner  $\pm$  in (3.8), the sign inside the square root. Using these results, we can get an overview of all possible stability scenarios of the fixed points created in a quadfurcation, see Table 2 and Fig. 5. As the quadfurcation point shifts along the  $SC$  line, different stabilities occur, but since  $SC \sim \Delta$  and generically  $PD - PD^Q \sim \sqrt{\Delta}$ , the branches emerge tangentially to the  $SC$  line (the  $\mathcal{O}(\sqrt{\Delta})$  term could vanish, but this is exceptional). Moreover, the sign of  $A - A^Q$  depends on the choice of the outer  $\pm$  sign in (3.8), so this pair of fixed points form a parabolic curve that is tangent to the  $SC$  line at the quadfurcation point.

Finally, note that when  $\beta \neq \gamma$ , then  $KP < 0$  at the quadfurcation, and hence a direct transition to the complex unstable (CU) region is only possible in the symmetric case for which  $(A^Q, B^Q) = (4, 6)$ ; this is discussed in §3.5.

| $(A^Q, B^Q)$<br>on $SC$ | Condition       |                    | Fixed Points and Stability                      |                                              |
|-------------------------|-----------------|--------------------|-------------------------------------------------|----------------------------------------------|
|                         | $\varepsilon_1$ | $ \beta - \gamma $ | $\varepsilon_2 = 1, a < -\sqrt{3} b $           | $\varepsilon_2 = -1, a \text{ or } b \neq 0$ |
| $> (4, 6)$              | $-1$            | $\neq 0$           | $2 \text{ EH} + 2 \text{ HH}$                   | $2 \text{ HH}$                               |
| $= (4, 6)$              | $\pm 1$         | $0$                | see §3.5                                        | see §3.5                                     |
| $< (4, 6)$              | $1$             | $< 2$              | $2 \text{ EE} + 2 \text{ EH}$                   | $2 \text{ EH}$                               |
| $= (0, -2)$             | $1$             | $2$                | $\text{IE} + \text{EE} + \text{IH} + \text{EH}$ | $\text{IH} + \text{EH}$                      |
| $< (0, -2)$             | $1$             | $> 2$              | $2 \text{ IE} + 2 \text{ IH}$                   | $2 \text{ IH}$                               |

Table 2

Overview of the location of the quadfurcation along the  $SC$  line depending on the the value of  $\varepsilon_1$  and the asymmetry of  $C$ . Stabilities of the fixed points are shown in the last two columns for a path of the form  $(a, b, c) = \Delta(a^*, b^*, c^*)$ , which has a quadfurcation at  $\Delta = 0$ . For  $\varepsilon_2 = 1$ , four fixed points are created as  $a$  becomes negative if  $a < -\sqrt{3}|b|$ . Their stabilities are shown in column four. When  $\varepsilon_2 = -1$ , two fixed points exist whenever  $a$  or  $b \neq 0$  and collide at  $\Delta = 0$ ; their stabilities are shown in the last column.

For  $\varepsilon_2 = 1$  the basic structure is the one shown in the fourth column of Table 2 and Fig. 5(a, c): four new branches emerge from a point on the saddle-center line. In each case, two of the fixed points are above ( $SC > 0$ ), and two are below ( $SC < 0$ ), this line. The implication is that when  $\varepsilon_1 = 1$ , and  $|\beta - \gamma| > 2$  the quadfurcation occurs below the  $PD$  line and corresponds to the transition

$$\emptyset \rightarrow 2 \text{ IE} + 2 \text{ IH},$$

i.e., two of the created fixed points are of type IE and two of type IH (this is not shown in Fig. 5, but compare with Fig. 4). Perhaps the most interesting quadfurcation creates stable fixed points. This occurs for  $\varepsilon_1 = 1$  and  $0 < |\beta - \gamma| < 2$ , where we have the transition

$$(3.22) \quad \emptyset \rightarrow 2 \text{ EE} + 2 \text{ EH}.$$

This is the case shown in Fig. 5(a): as  $\Delta$  decreases from zero, the two created EE points move along the green and black curves in the figure and the two EH points move along the red and blue curves.

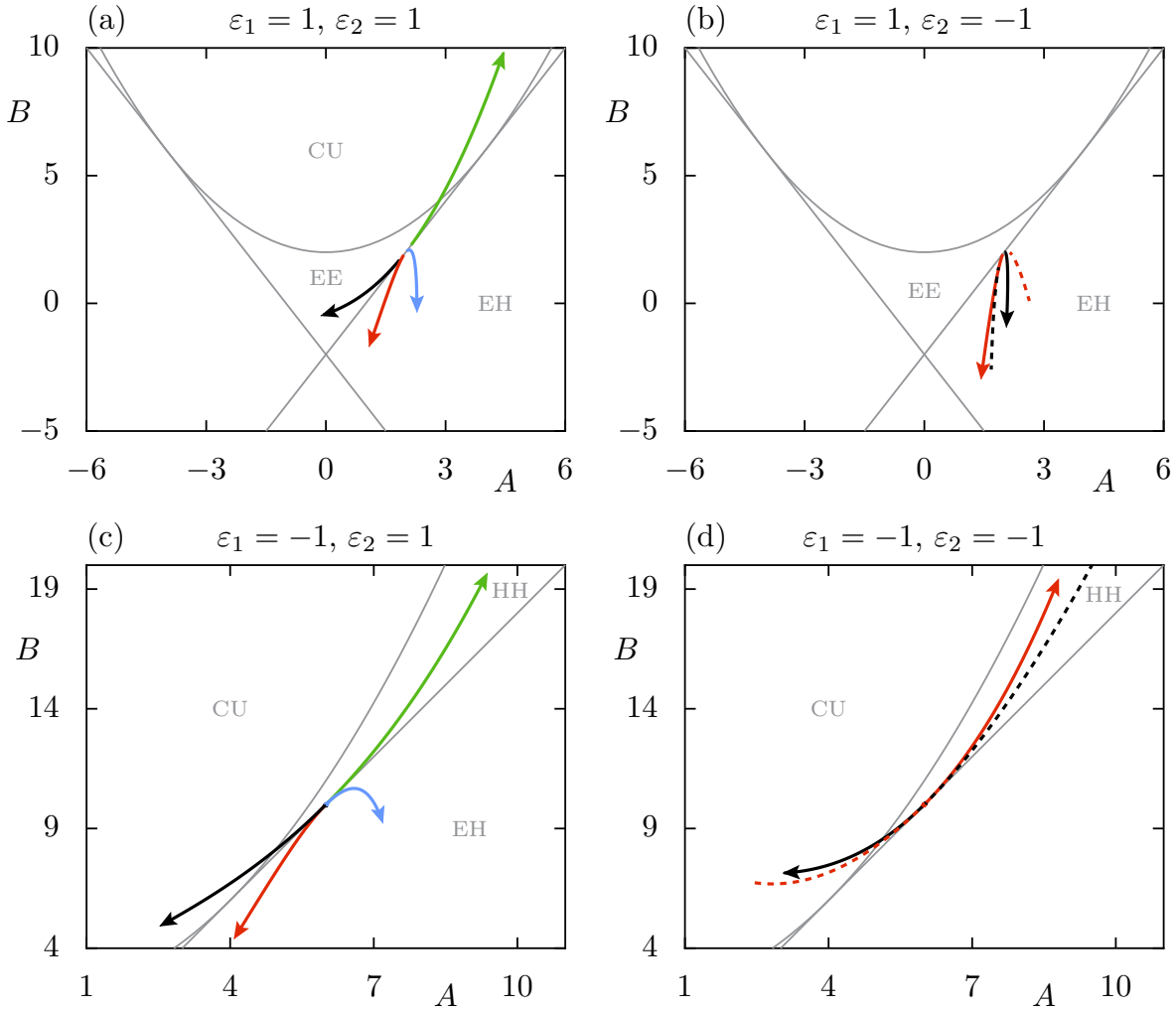
Note that one of the EE points in this figure eventually undergoes a Krein bifurcation [39], moving into the CU region. The implication is that the Krein signature of this point must have been indefinite when it was created in the quadfurcation, since this signature is constant under parameter variations so long as the stability remains in the interior of the EE-region in Fig. 4 [38, Sec. III].

Finally, when  $\varepsilon_1 = -1$  the quadfurcation point is above  $(A, B) = (4, 6)$  whenever  $\beta \neq \gamma$ , so the transition is

$$(3.23) \quad \emptyset \rightarrow 2 \text{ EH} + 2 \text{ HH}.$$

This case is shown in Fig. 5(c). Again, a Krein bifurcation,  $\text{HH} \rightarrow \text{CU}$ , eventually occurs.

For  $\varepsilon_2 = -1$  the basic structure is shown in the last column of Table 2 and in Fig. 5(b, d). There are two fixed points before and after the quadfurcation with positions given by the inner + sign in (3.8). Using this and (3.2), there is no sign choice that smoothly connects the  $(\xi_1^*, \xi_2^*)$  branches for  $\Delta < 0$  to  $\Delta > 0$ : the fixed points lose their identity when they collide. The sign

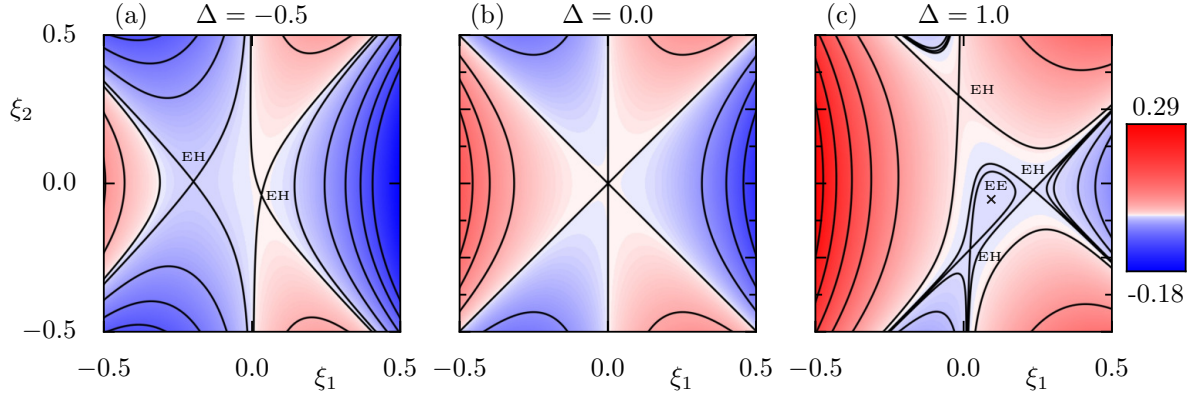


**Figure 5.** Stability of fixed points for  $(a, b, c) = \Delta(1.5, 0.5, 1)$  with  $\Delta \in [-1, 1]$  and different choices of  $C$ ,  $\varepsilon_2$ , and  $\varepsilon_1$ : (a, b) Case  $(\alpha, \mu, \delta, \varepsilon_1) = (1, 0.1, 0.5, 1)$ . Since  $|\beta - \gamma| \approx 1.42$ , the quadrupling occurs at  $(A^Q, B^Q) = (1.99, 1.98)$ . (a)  $\varepsilon_2 = +1$ : creation of four fixed points in the transition (3.22) for  $\Delta < 0$ . (b)  $\varepsilon_2 = -1$ : the transition (3.24), before and after the bifurcation one has two fixed points. (c, d) Case  $(\alpha, \mu, \delta) = (1, 2.0, -0.5)$ ,  $\varepsilon_1 = -1$ , giving  $(\beta, \gamma) \simeq (1.71, 0.29)$  and the quadrupling occurs at  $(A^Q, B^Q) = (6, 10)$ . (c)  $\varepsilon_2 = 1$ : creation of four fixed points (3.23) for  $\Delta < 0$ . (d)  $\varepsilon_2 = -1$ : the transition (3.25) two fixed points before and after the bifurcation. The arrows indicate the direction towards more negative  $\Delta$  and in (b, d) the branches for  $\Delta > 0$  are shown as dashed lines.

choice implies that  $\text{sgn}(SC) = -\text{sgn}(\varepsilon_1)$ . When  $\varepsilon_1 = 1$ , and hence  $(A^Q, B^Q) < (4, 6)$ , the fixed points both before and after the quadfurcation are below the  $SC$  line, so the transition is

$$(3.24) \quad 2 \text{ EH} \rightarrow 2 \text{ EH}$$

if  $|\beta - \gamma| < 2$ . As shown in Fig. 5(b), the fixed points move in towards the  $SC$  line (black and red dashed curves) as  $\Delta \rightarrow 0^+$  colliding at  $\Delta = 0$ , and splitting apart again for  $\Delta < 0$  (full



**Figure 6.** Contour plots of the potential  $U$ , Eq. (2.10), for the path (3.26) with  $(\alpha, \mu, \delta) = (1, 0.1, 0.5)$ ,  $\varepsilon_1 = 1$ , and  $\varepsilon_2 = -1$ . The three panels show  $\Delta = -0.5, 0$ , and  $1$ . Two  $EH$  fixed points merge in the quadfurcation at  $\Delta = 0$  and for  $\Delta > 0$  there are four fixed points, one being  $EE$  and three  $EH$ .

curves). Similarly, when  $|\beta - \gamma| > 2$  the transition corresponds to

$$2 \text{ IH} \rightarrow 2 \text{ IH}.$$

Finally, when  $\varepsilon_1 = -1$  and  $(A^Q, B^Q) > (4, 6)$ , the transition is

$$(3.25) \quad 2 \text{ HH} \rightarrow 2 \text{ HH},$$

as shown in Fig. 5(d).

It is interesting that all of this structure is quite different from what would be expected from a pair of decoupled, area-preserving maps undergoing saddle-center bifurcations, where there can be at most one  $EE$  point. This case corresponds to the special point  $(A^Q, B^Q) = (4, 6)$ , which will be treated in §3.5 and applied to the case of decoupled maps in §6.

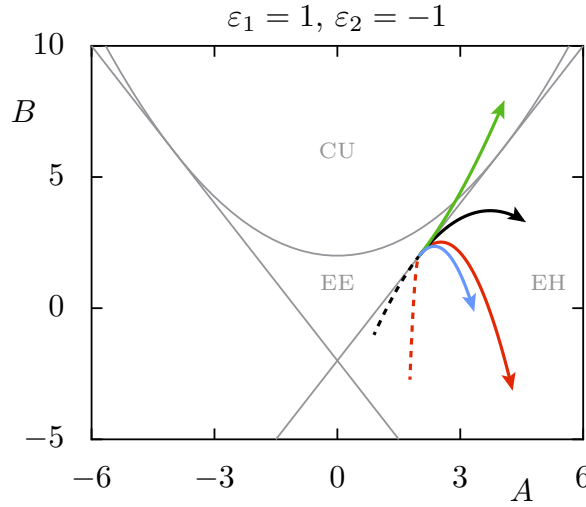
**3.4. Two to Four Fixed Point Transitions.** When a parameter path crosses one of the surfaces  $b_{\pm}(a, c)$  then  $SC = 0$  in (3.18), and the resulting saddle-center bifurcation typically creates or annihilates a pair of new fixed points, one with  $E$  eigenvalues and one with  $H$  eigenvalues. When  $\varepsilon_2 = -1$ , there are two fixed points outside the wedge between  $b_+$  and  $b_-$  shown in Fig. 3(b), and so if the parameter path enters the wedge then two new fixed points are created. If the path enters the wedge at the quadfurcation point  $a = b = c = 0$ , then the two existing fixed points merge, and the bifurcation—now a quadfurcation—occurs at the origin. Depending upon  $\beta - \gamma$  and the sign  $\varepsilon_1$ , the quadfurcation can occur at any point along the  $SC$  line, and so a number of different stability cases can arise.

An example is shown in Fig. 6 for the parabolic path

$$(3.26) \quad (a, b, c) = (-0.07 \cdot \Delta|\Delta|, 0.01 \cdot \Delta^2, \Delta)$$

as  $\Delta$  varies, with the remaining parameters as shown in the caption. Figure 6 shows the contours of the potential for  $\Delta = -0.5, 0$ , and  $1$  and Fig. 7 shows the corresponding stability





**Figure 7.** Stability diagram for the path (3.26) with  $\Delta \in [-2, 5]$  and all other parameters as in Fig. 6. The two EH fixed points for negative  $\Delta$  (dashed curves) merge at the quadfurcation point  $\Delta = 0$  and lead to four fixed points for  $\Delta > 0$ , one with EE and three with EH stability. For larger  $\Delta$  the EE point becomes type CU.

diagram. There are two fixed points when  $\Delta < 0$  both of type EH; these merge at  $\Delta = 0$ . The quadfurcation corresponds to a transition

$$2 \text{ EH} \rightarrow 3 \text{ EH} + \text{EE}.$$

Effectively the original EH pair is reformed and the contour lines near the new EE–EH pair in Fig. 6(c) resemble those for a local saddle-center bifurcation.

**3.5. Krein Collisions, Symmetric  $C$  and Reversibility.** As we noted in (3.20), the quadfurcation occurs for a multiplicity-four unit eigenvalue only when the matrix  $C$  of (2.3) is symmetric. This is the only case in which the quadfurcation can immediately create fixed points of type CU, recall Fig. 4.

As discussed in §2.3 the inverse of the Moser map is conjugate to the original map upon the replacement  $C \rightarrow C^T$ . Therefore, when  $C$  is symmetric, the map (2.2) is reversible, i.e., it is conjugate to its inverse [40]:  $S \circ f = f^{-1} \circ S$  for a homeomorphism  $S$ . For example, the Hénon map (2.1) is reversible with  $S(x, y) = (y, x)$ . In general, the inverse of (2.9) is

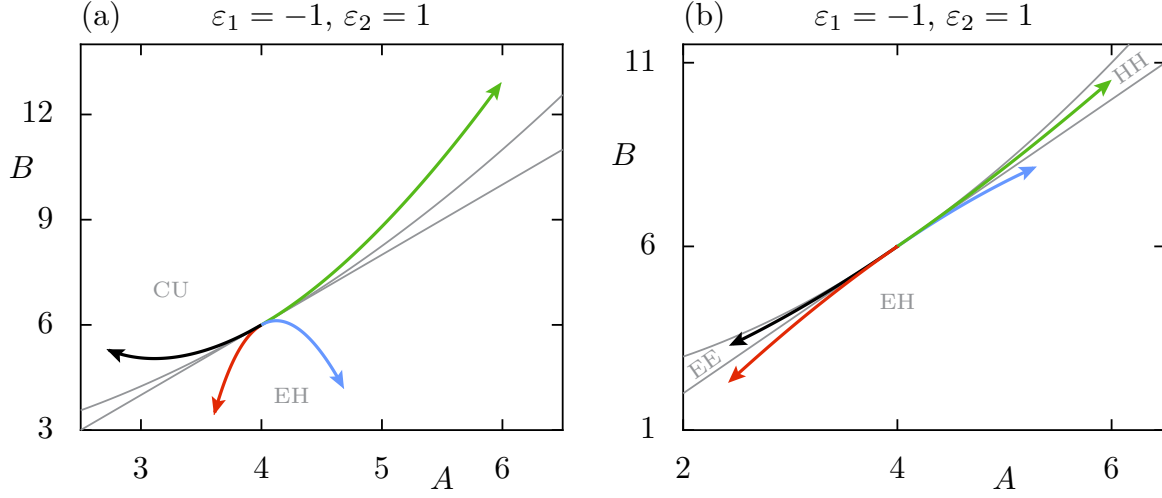
$$M^{-1}(\xi, \eta) = (C^{-1}\eta, \eta + C^T(-\xi + C^{-1}\eta) + \nabla U(C^{-1}\eta)).$$

This map is conjugate to  $M$  when  $C = C^T$  using the reversor

$$S(\xi, \eta) = (C^{-1}\eta, C\xi).$$

Thus, as Moser showed [31], if  $C$  is symmetric the map (2.2), or equivalently (2.9), is reversible; we do not know if the converse of this statement is true. This reversor is an involution with the fixed set  $\text{Fix}(S) = \{(\xi, C\xi) : \xi \in \mathbb{R}^2\}$ , a 2D plane. All of the fixed points are thus symmetric.

The quadfurcation is especially interesting in the reversible case since, by (3.17) and (3.21), only then does it occur for the stability parameters  $(A^Q, B^Q) = (4, 6)$ . Indeed as



**Figure 8.** Stability of fixed points for symmetric  $C$  with  $\varepsilon_1 = -1$  and  $\varepsilon_2 = +1$ . The parameters vary along the line  $(a, b, c) = \Delta(\frac{3}{2}, \frac{1}{2}, 1)$  with  $\Delta \in [-1, 0]$ . (a) The transition (3.27) for  $(\alpha, \mu, \delta) = (-\frac{3}{2}, 1, \frac{1}{2})$ , giving  $\beta = \gamma = \frac{1}{2}$ . (b) The transition (3.28) for  $(\alpha, \mu, \delta) = (\frac{3}{2}, \sqrt{\frac{23}{2}}, \frac{5}{4})$ , giving  $\beta = \gamma = \sqrt{\frac{23}{8}}$ .

in §3.3, assuming that  $a, b, c = \mathcal{O}(\Delta)$  and  $\xi_i = \mathcal{O}(\sqrt{\Delta})$  near the quadfurcation, the Krein criterion (3.21c) is zero to  $\mathcal{O}(\Delta^{1/2})$ , and the first nonzero terms are

$$KP = 4\alpha\delta a - 2(\alpha + 3\varepsilon_2\delta)\gamma b + (12\varepsilon_1\varepsilon_2 - (\alpha - 3\varepsilon_2\delta)^2)\xi_1^{*2} + \mathcal{O}(\Delta^{3/2}, (\beta - \gamma)^2)$$

where  $\xi_1^*$  is given by (3.8). The sign of this parameter can change, depending upon the details. However, note that since  $KP$  depends only on  $\xi_1^{*2}$ , it does not depend upon the outer sign in (3.8). Thus when  $c = \mathcal{O}(\Delta)$ , the fixed points come in pairs with the same sign of  $KP$ .

When  $\varepsilon_2 = 1$ , the four created fixed points come in pairs with opposite signs of  $SC$  from (3.21a). Thus there will be a pair of fixed points of type EH. Since the curves generically emerge tangent to the the  $SC$  line, the second pair will both have type CU or one will be EE and the other HH. For example, a quadfurcation of the form

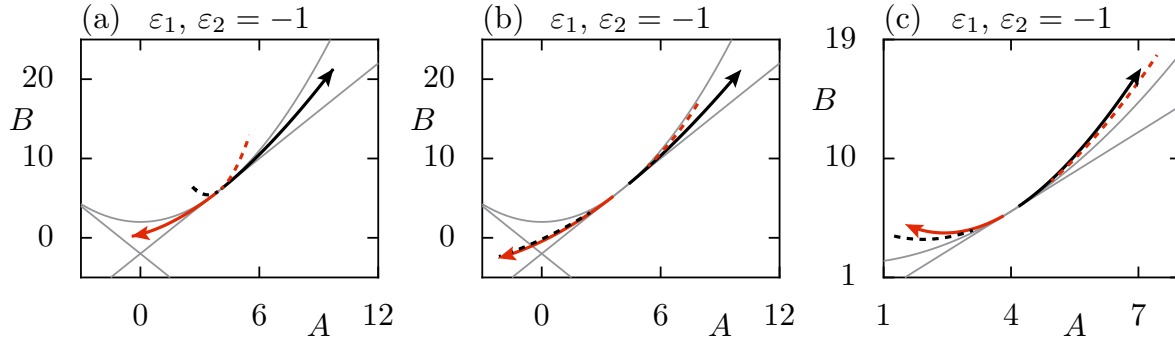
$$(3.27) \quad \emptyset \rightarrow 2 \text{ CU} + 2 \text{ EH}$$

is shown in Fig. 8(a). For this case  $KP = \frac{1}{4}(3 \pm 7\sqrt{6})\Delta$  to lowest order, where the sign is the inner  $\pm$  sign in (3.8), implying that the two fixed points with the  $+$  sign have  $KP > 0$  when  $\Delta < 0$  near the quadfurcation and are thus of type CU. In contrast, for the example shown in Fig. 8(b) the quadfurcation is

$$(3.28) \quad \emptyset \rightarrow \text{EE} + \text{HH} + 2 \text{ EH}$$

as here  $KP = \frac{\Delta}{64}(993 - 84\sqrt{46} \pm 91\sqrt{6}) < 0$  when the fixed points exist,  $\Delta < 0$ . This last case is what would happen in a pair of uncoupled 2D maps, and will be seen below in §6.

As in §3.3, when  $\varepsilon_2 = -1$ , the quadfurcation at  $\Delta = 0$  corresponds to a collision and re-emergence of a pair of fixed points with  $\text{sgn}(SC) = -\text{sgn}(\varepsilon_1)$ . When  $\varepsilon_1 = 1$ , all of the fixed



**Figure 9.** Stability of fixed points for symmetric  $C$  and  $\varepsilon_1 = \varepsilon_2 = -1$ . The parameters  $(a, b, c) = \Delta(\frac{3}{2}, \frac{1}{2}, 1)$  for (a) and (b), and  $\Delta(2, \frac{1}{2}, 1)$  for (c), with  $\Delta \in [-\frac{1}{2}, \frac{1}{2}]$ . The arrows indicate the direction towards more negative  $\Delta$ , and branches for  $\Delta > 0$  are shown as dashed lines. (a) The transition (3.29) for  $(\alpha, \mu, \delta) = (\frac{1}{2}, \sqrt{6}, 1)$  giving  $\beta = \gamma = \sqrt{\frac{3}{2}}$ . (b) The transition (3.30) for  $(\alpha, \mu, \delta) = (\frac{99}{16}, -\frac{5}{2}, \frac{1}{11})$  giving  $\beta = \gamma = -\frac{5}{4}$ . (c) The transition (3.31) for  $(\alpha, \mu, \delta) = (-\frac{3}{2}, 1, \frac{1}{2})$ , giving  $\beta = \gamma = \frac{1}{2}$ .

points will have stability type EH, and so the transition will be

$$2 \text{ EH} \rightarrow 2 \text{ EH}$$

and thus follows the pattern shown in Fig. 5(b).

However, when  $\varepsilon_1 = -1$ , then  $\text{sgn}(SC) = +1$  implying that EE, HH and CU are all possible. However, since the two fixed points have the + inner sign in (3.8) they will have the same sign of  $KP$ , so we either have a CU pair or an EE+HH pair. The possible transitions are

$$(3.29) \quad 2 \text{ CU} \rightarrow \text{EE} + \text{HH},$$

for which the stability diagram is shown in Fig. 9(a)

$$(3.30) \quad \text{EE} + \text{HH} \rightarrow \text{EE} + \text{HH},$$

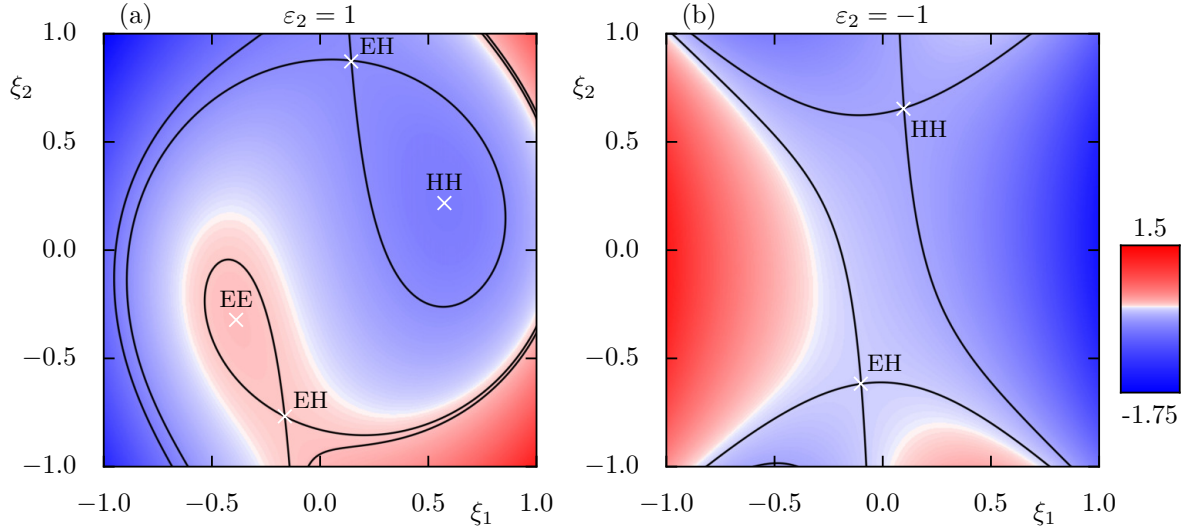
with stability diagram shown in in Fig. 9(b) and

$$(3.31) \quad 2 \text{ CU} \rightarrow 2 \text{ CU},$$

with stability diagram shown in Fig. 9(c).

Another technique for analyzing stability in the neighborhood of the quadfurcation is to use the ODE limit (2.16) with the Hamiltonian (2.18). Recall that this limit assumes that we assume the scaling  $(a, b, c) \rightarrow (h^4 a, h^4 b, h^2 c)$ , for  $h \ll 1$ . This scaling differs from the  $\Delta$ -scaling by allowing larger relative values for  $c$ . The implication of this is that the term  $\frac{1}{2} c q_1^2$  in  $U$ , which was negligible when  $c = \mathcal{O}(\Delta)$ , is formally important when we take  $c = \mathcal{O}(h^2)$ . When  $C$  is symmetric,  $C^a = 0$ , and the Coriolis and centripetal-like terms vanish, simply giving

$$(3.32) \quad H(q, p) = \frac{1}{2} p^T C^{-1} p - U(q).$$



**Figure 10.** Contour plot of the potential  $U$ , Eq. (2.10). (a) For  $(a, b, c) = (-\frac{3}{4}, -\frac{1}{4}, -\frac{1}{2})$  and  $\varepsilon_2 = 1$  there are four critical points, which are equilibria of (3.32). When  $C > 0$ , the maximum, at  $\xi = (-0.38825, -0.32196)$ , corresponds to a doubly elliptic equilibrium and the minimum, at  $\xi = (0.57441, 0.21761)$ , to a doubly hyperbolic equilibrium. As specific example  $(\alpha, \mu, \delta) = (\frac{5}{2}, \sqrt{6}, 1)$  is used for the matrix  $C$  determining the stabilities. (b) Potential for  $(a, b, c) = (-\frac{3}{8}, -\frac{1}{8}, -\frac{1}{4})$  and  $\varepsilon_2 = -1$  where there are two critical points; the matrix  $C$ , determining the stabilities, is the same as in (a).

When  $C = C^s$  is positive or negative definite, then the stability is governed entirely by the classification of the critical point of  $U$ . In particular if  $C > 0$  then since the potential in (2.18) is  $-U$ , a minimum of  $U$  is an HH point, and a maximum is an EE point. Saddles, correspond to EH points. When  $C < 0$ , the minima are HH and the maxima are EE. This is consistent more generally with (3.18), which shows that  $SC = \varepsilon_1 \det(D^2U)$ . Two example contour plots for  $U$  are shown in Fig. 10.

**4. Elliptic Bubbles.** To visualize the dynamics near the fixed points of the 4D map we use a 3D *phase space slice* [41]. In its simplest form one considers a thickened 3D hyperplane in the 4D phase space defined by fixing one of the coordinates, e.g.,  $\eta_2 = \eta_2^*$ , to define the slice of thickness  $\epsilon$  by

$$\{(\xi_1, \xi_2, \eta_1, \eta_2) \mid |\eta_2 - \eta_2^*| \leq \epsilon\},$$

Whenever the points of an orbit lie within the slice, the remaining coordinates  $(\xi_1, \xi_2, \eta_1)$  are displayed in a 3D plot. The parameter  $\epsilon$  determines the resolution of the resulting plot; decreasing  $\epsilon$  requires the computation of longer trajectories as the slice condition is fulfilled less often, but the resulting intersections will be more precise. For example, if a two-torus intersects the hyperplane, it will typically do so in one or more loops. As  $\epsilon$  grows these loops thicken into annuli in the slice. For further examples and detailed discussion see [41, 42, 43, 44, 45, 46].

For our purposes a slightly more general, rotated slice, defined so as to contain all of the fixed points, will be more convenient. Because the momenta of the fixed points are determined by the coordinates through  $\eta = C\xi$ , all fixed points of the 4D map (2.9) are contained in a 2D plane. Following the ideas of [41, App. 3], we define new coordinates  $(\xi, \chi)$ , with  $\chi = \eta - C\xi$ ,

so that the fixed points lie in the two-plane  $\chi = 0$ . Thus we define the 3D slice

$$\Gamma_\epsilon = \{(\xi_1, \xi_2, \chi_1, \chi_2) \mid |\chi_2| \leq \epsilon\}$$

so that we get  $(\xi_1, \xi_2, \chi_1)$  as 3D coordinates. Equivalently, this corresponds to using non-orthogonal basis vectors given as columns of the block matrix

$$B = (v_1|v_2|v_3|v_4) = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}.$$

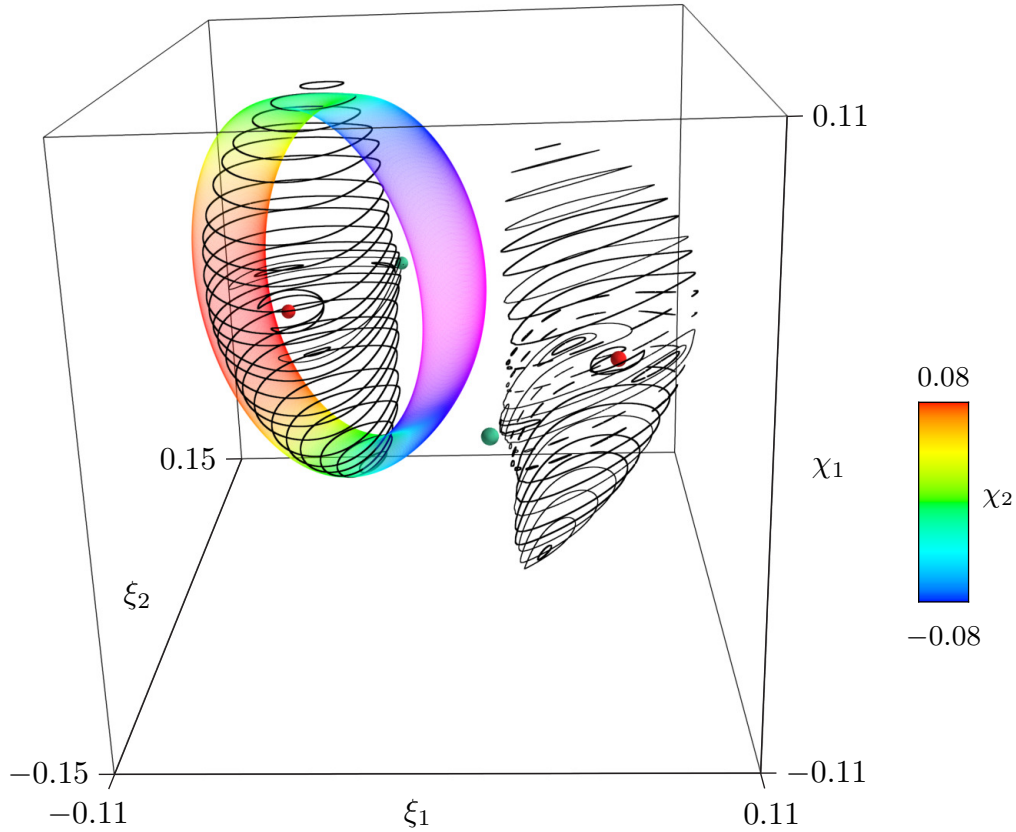
As these are linearly independent, they can be used to express any point as linear combination with coefficients  $(\xi_1, \xi_2, \chi_1, \chi_2)$ . These coefficients can be computed from the scalar products with the dual basis vectors  $\{v^i\}$ , which are the columns of

$$B^{-T} = (v^1|v^2|v^3|v^4) = \begin{pmatrix} I & -C^T \\ 0 & I \end{pmatrix}.$$

Figure 11 shows an example of a slice for the map (2.9) with the parameters of Fig. 5(a) when  $\Delta = -0.01$ . For these parameters the quadfurcation has created two EE and two EH fixed points that, by construction of the 3D slice, lie in the 2D plane  $\chi_1 = 0$ . As expected from KAM theory, the EE fixed points should be surrounded by a Cantor family of two-tori on which the dynamics is conjugate to incommensurate rotation. By analogy with Moser's theorem for 2D maps [47], the density of these tori should approach one as they limit on the EE points providing that the linearized frequencies are not in a low-order resonance. Indeed, the formal normal form expansion around a nonresonant EE point is an integrable twist map to all orders [21], and higher-dimensional results along the lines of Moser's twist theorem have been proven for elliptic equilibria of Hamiltonian flows [48, 49]. In the 3D slice each of these tori becomes two (or more) thin annular rings, which, since we have set  $\epsilon = 10^{-6}$ , appear as 1D loops in the figure [41].

The numerical approximations of regular tori in Fig. 11 were obtained by manually choosing initial conditions in several 2D planes in the 3D slice so that they appeared to represent a kind of “outer” boundary of the regular region. Some initial conditions escape quickly and are discarded. We numerically verified that these sets are 2D tori using frequency analysis [50, 51, 41]. Here one computes, for a finite trajectory segment, two fundamental frequencies from a windowed Fourier transform; on a true torus, these should be independent of the segment chosen. All but two of the tori in Fig. 11 have frequencies that vary by less than  $7 \times 10^{-7}$  when computed over successive segments of 4096 iterates, and for most the variation is of order  $10^{-10}$ . The two exceptional “tori” in the figure have a frequency variation of order  $2 \times 10^{-5}$  and are probably very weakly chaotic orbits that are trapped near a resonance for long times by Nekhoroshev stability.

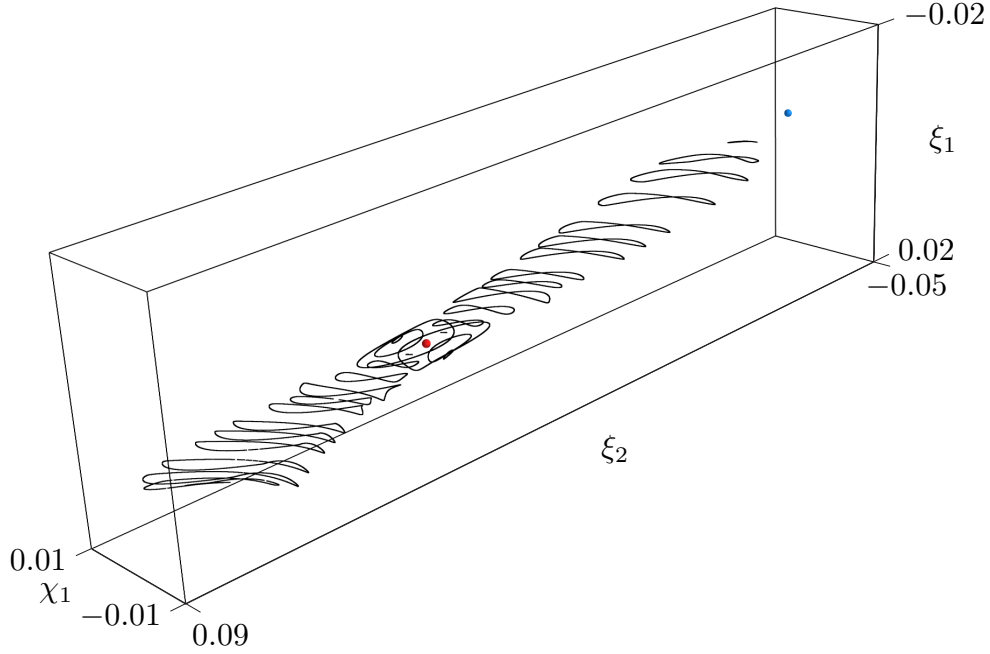
This family of two-tori appears to approximately be limited by the locations of the EH fixed points. Indeed, since the center-stable and center-unstable manifolds of the EH points are 3D, they should form boundaries for the elliptic dynamics. Of course, we expect there will be chaotic orbits near these manifolds, and so the regular 2D tori will not extend into the chaotic zone. Also shown in the plot is the full orbit of one of these 2D tori, now *projected*



**Figure 11.** Three-dimensional phase space slice of the 4D map corresponding to Fig. 5(a) with parameters  $(\alpha, \mu, \delta) = (1, 0.1, 0.5)$ ,  $\varepsilon_1 = \varepsilon_2 = 1$ , and  $(a, b, c) = (-0.015, -0.005, -0.01)$ . The small spheres show two EE (red) and two EH (green) fixed points. Also shown are several selected regular tori (black lines) surrounding the EE fixed points. Each torus is represented by  $10^4$  points in the slice with  $\epsilon = 10^{-6}$ . These tori are 2D in the 4D phase space and therefore (usually) lead to a pair of loops in the 3D phase space slice. A projection of one 2D torus is shown as  $10^6$  semi-transparent points with  $\chi_2$  encoded in color (see color bar). For a rotating view see <http://www.comp-phys.tu-dresden.de/supp/>.

onto the slice; the projected coordinate  $\chi_2$  is encoded in color as indicated in the color bar at the right [52].

Figure 12 shows an example of a slice for the transition (3.30), when the matrix  $C$  is symmetric; the parameters correspond to Fig. 9(b) with  $\Delta = -0.001$ , close to the quadfurcation. For this case there are only two fixed points, one of type EE and the other of type HH. Here again we see a family of 2D tori surrounding the EE point. This family has a larger extent in the  $\xi_2$  direction than in  $\xi_1$ , and the loops shrink in size as they become closer to the HH point. Note that now the stable and unstable manifolds of the HH point are two-dimensional, and so do not form barriers in 4D.



**Figure 12.** 3D phase space slice of the 4D map for parameters of Fig. 9(b) with  $\Delta = -0.001$ . The small spheres show an EE (red) and an HH (blue) fixed point. Also shown are slices of several selected regular tori as in Fig. 11. For a rotating view see <http://www.comp-phys.tu-dresden.de/supp/>.

**5. Bounded Orbits.** Moser showed, under a nondegeneracy condition on the quadratic terms, that the domain of the quadratic map containing bounded orbits is itself bounded [31]. To obtain an explicit bound we consider the second-difference form (2.13), rewriting it as

$$(5.1) \quad C^T \xi_{t+1} + C \xi_{t-1} = A + D\xi_t + Q(\xi_t),$$

where  $A = (a, b)^T$  is a constant vector and the linear and quadratic terms are

$$D\xi \equiv \begin{pmatrix} 2\alpha + c & \mu \\ \mu & 2\delta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

$$Q(\xi) \equiv \begin{pmatrix} 3\varepsilon_2 \xi_1^2 + \xi_2^2 \\ 2\xi_1 \xi_2 \end{pmatrix}.$$

Using this form we can prove the following.

**Theorem 1.** *When  $\varepsilon_2 = \pm 1$ , all orbits that are bounded for both positive and negative time of the map (5.1) are contained in the disk  $\|\xi\| \leq \kappa$ , where*

$$(5.2) \quad \kappa \equiv \frac{1}{2\tau} \left( \kappa_2 + 2\kappa_3 + \sqrt{(\kappa_2 + 2\kappa_3)^2 + 4\tau \sqrt{a^2 + b^2}} \right),$$

with  $\tau = 1$  for  $\varepsilon_2 = 1$  and  $\tau = \sqrt{\frac{2}{3}}$  for  $\varepsilon_2 = -1$ , and we define

$$(5.3) \quad \begin{aligned} \kappa_2 &\equiv \|D\|_F = \sqrt{(2\alpha + c)^2 + 2\mu^2 + 4\delta^2}, \\ \kappa_3 &\equiv \|C\|_F = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}. \end{aligned}$$

*Proof.* When  $\varepsilon_2 = 1$  the norm of the quadratic terms has the lower bound

$$\|Q(\xi)\|^2 = 9\xi_1^4 + 10\xi_1^2\xi_2^2 + \xi_2^4 = (9\xi_1^2 + \xi_2^2)\|\xi\|^2 \geq \|\xi\|^4 = \rho^4,$$

where we denote  $\|\xi\| = \rho$ .

Using the triangle inequality on (5.1) gives

$$(5.4) \quad \|C^T \xi_{t+1}\| + \|C \xi_{t-1}\| \geq \|Q(\xi_t)\| - \|D \xi_t\| - \|A\|.$$

Define  $\kappa_{2,3} > 0$  so that

$$\|D \xi_t\| \leq \kappa_2 \rho_t; \quad \|C^T \xi_t\|, \|C \xi_t\| \leq \kappa_3 \rho_t.$$

For example we can use the Frobenius norms of these matrices to give (5.3) (one could also use the operator norm, giving results in terms of the singular values). Putting the bounds into (5.4) gives

$$(5.5) \quad \kappa_3(\rho_{t+1} + \rho_{t-1}) \geq \rho_t^2 - \kappa_2 \rho_t - \|A\|.$$

Let  $\kappa > 0$  be chosen such that whenever  $\rho > \kappa$ , then  $\rho^2 - \kappa_2 \rho - \|A\| > 2\kappa_3 \rho$ . Solving for the largest root of this quadratic, as an equality, gives (5.2). Using this in (5.5) implies that whenever  $\rho_t > \kappa$ ,

$$(5.6) \quad \rho_{t+1} + \rho_{t-1} > 2\rho_t.$$

Thus, if  $\rho_t > \kappa$ , there are two possible cases:

- Suppose that  $\rho_t \geq \rho_{t-1}$ . Then by (5.6), we have  $\rho_{t+1} > 2\rho_t - \rho_{t-1} \geq \rho_t$ . This implies that the sequence  $\{\rho_t\}$  is strictly increasing with  $t$ . If this monotone sequence is bounded, it must approach a limit  $\rho_t \rightarrow \rho^*$ , which must be a solution of (5.5) as an equality. But this implies  $\rho^*$  is no more than the largest root  $\kappa$ , contradicting our assumption. Thus  $\{\rho_t\}$  is unbounded as  $t \rightarrow \infty$ .
- Alternatively, suppose that  $\rho_t < \rho_{t-1}$ . Relabeling the  $t$  index in (5.6) then implies that  $\rho_{t-2} > 2\rho_{t-1} - \rho_t > \rho_{t-1}$ . Thus the sequence  $\{\rho_t\}$  strictly increases as  $t$  decreases. Again, since  $\rho_t > \kappa$  this implies that  $\rho_t$  is unbounded, now as  $t \rightarrow -\infty$ .

Together, these imply the theorem when  $\varepsilon_2 = 1$ .

If  $\varepsilon_2 = -1$ , then we can see that  $\|Q(\xi)\|^2 \geq \frac{2}{3}\rho^4$ , indeed

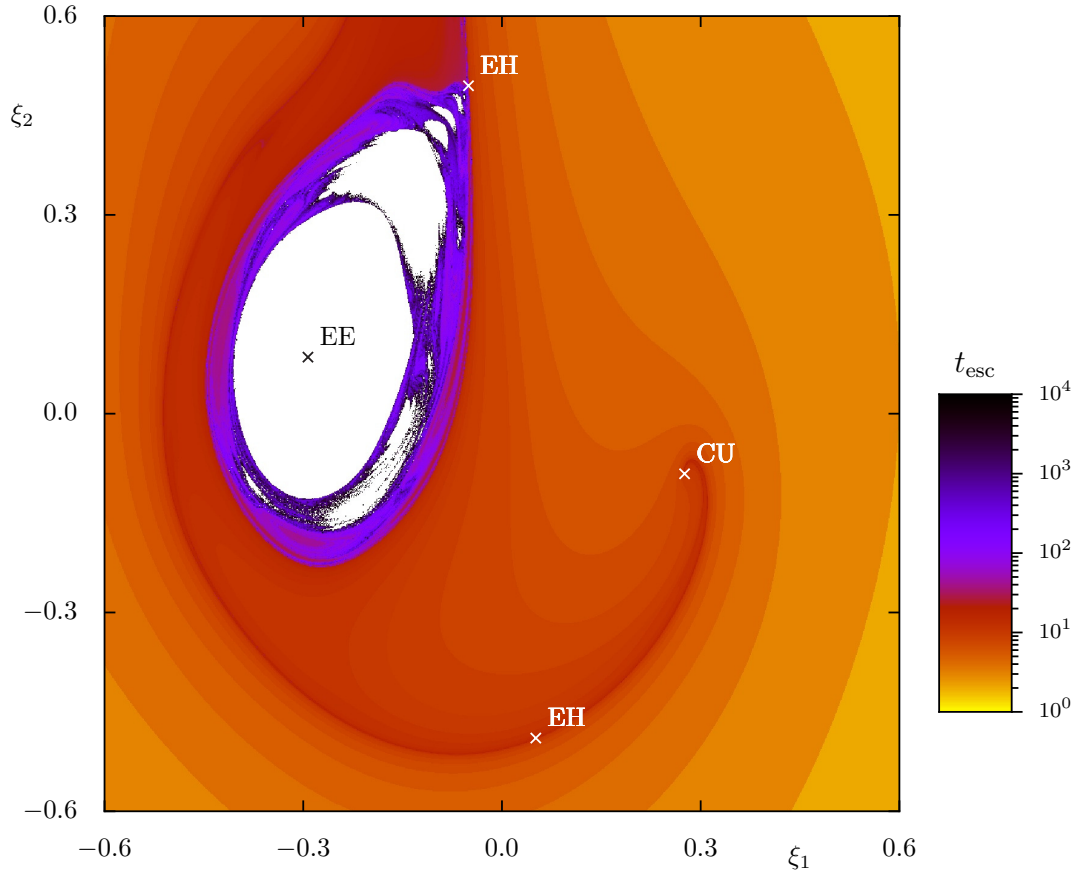
$$\|Q(\xi)\|^2 - \frac{2}{3}(\xi_1^2 + \xi_2^2)^2 = \frac{25}{3}\xi_1^4 - \frac{10}{3}\xi_1^2\xi_2^2 + \frac{1}{3}\xi_2^4 = \frac{1}{3}(5\xi_1^2 - \xi_2^2)^2 \geq 0.$$

Thus the analysis above works, if we replace  $\kappa$  by the larger solution to

$$\tau\rho^2 - (\kappa_2 + 2\kappa_3)\rho - \|A\| = 0$$

with  $\tau = \sqrt{\frac{2}{3}}$ , giving (5.2) again. ■





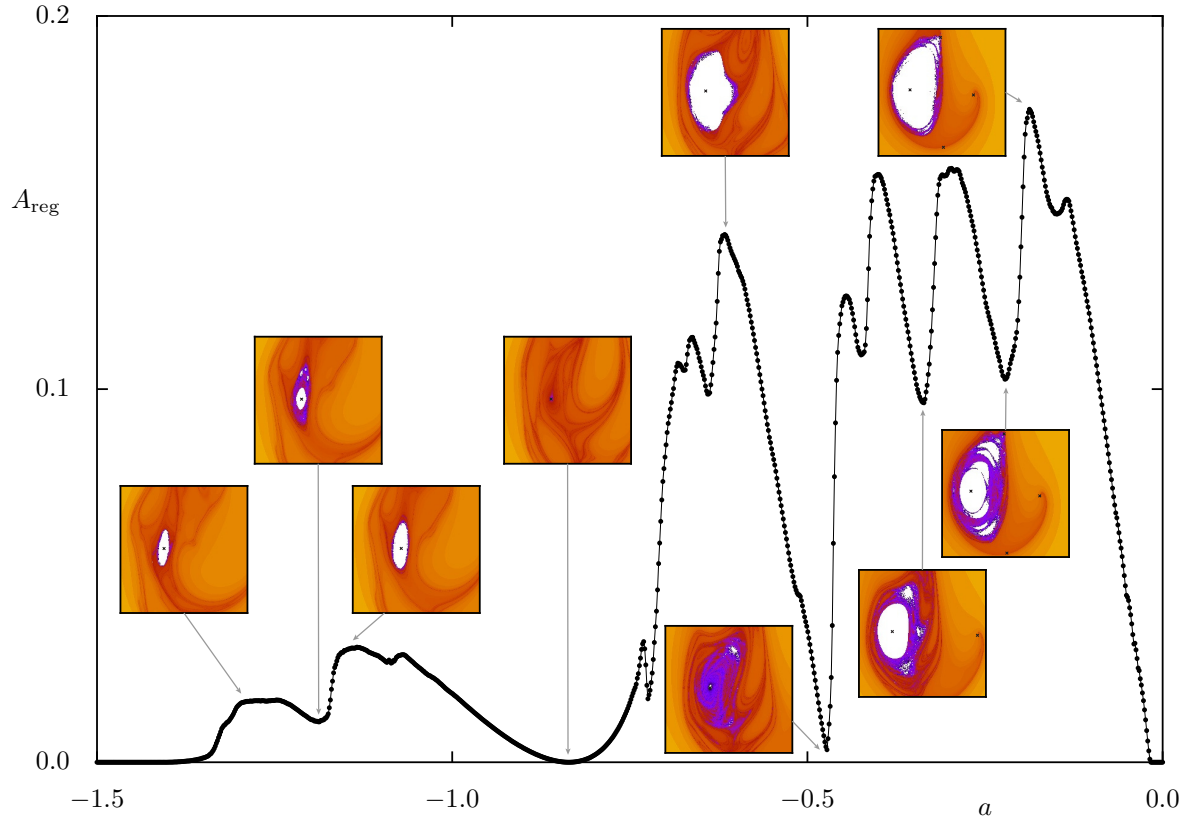
**Figure 13.** Plot of the escape time  $t_{\text{esc}}$  encoded in color for initial conditions defined via  $(\xi_1, \xi_2)$  with  $\eta = C\xi$ . Initial conditions whose orbits have not escaped within  $10^4$  iterations are colored white. There is a large region of non-escaping orbits surrounding the EE fixed point. Further from the EE point one observes a complicated fine-scale structure of escaping and non-escaping orbits. Parameters are:  $(\alpha, \mu, \delta) = (1, 0.1, 0.5)$ ,  $\varepsilon_1 = \varepsilon_2 = 1$ , and  $(a, b, c) = (-0.25, 0.05, 0.05)$ .

Note that if  $\varepsilon_2 = 0$ , then

$$\|Q(\xi)\|^2 = 4\xi_1^2\xi_2^2 + \xi_2^4 = \xi_2^2(4\xi_1^2 + \xi_2^2),$$

which does not obey a bound of the form needed in the proof of Th. 1. Thus the theorem does not apply to this case. Indeed, we showed in §3.1 that if  $\varepsilon_2 = 0$ , then when  $a = b = c = 0$  there is a line of fixed points, recall the discussion in §3.1.

A good way to visualize the distinction between bounded and unbounded orbits for a given parameter set is an escape time plot, see Fig. 13. In this plot, a grid of initial points of the form  $(\xi, \eta) = (\xi, C\xi)$  are iterated until  $\|\xi\| > \kappa$  and the required time to escape is encoded in color. Points that have not escaped within  $10^4$  iterations are displayed in white. Some of these points lie on a family of regular 2D tori in the neighborhood of the EE fixed point; these will never escape. Points near the boundary of the white region may eventually escape, and indeed, even arbitrarily close to an EE fixed point there are initial conditions that are

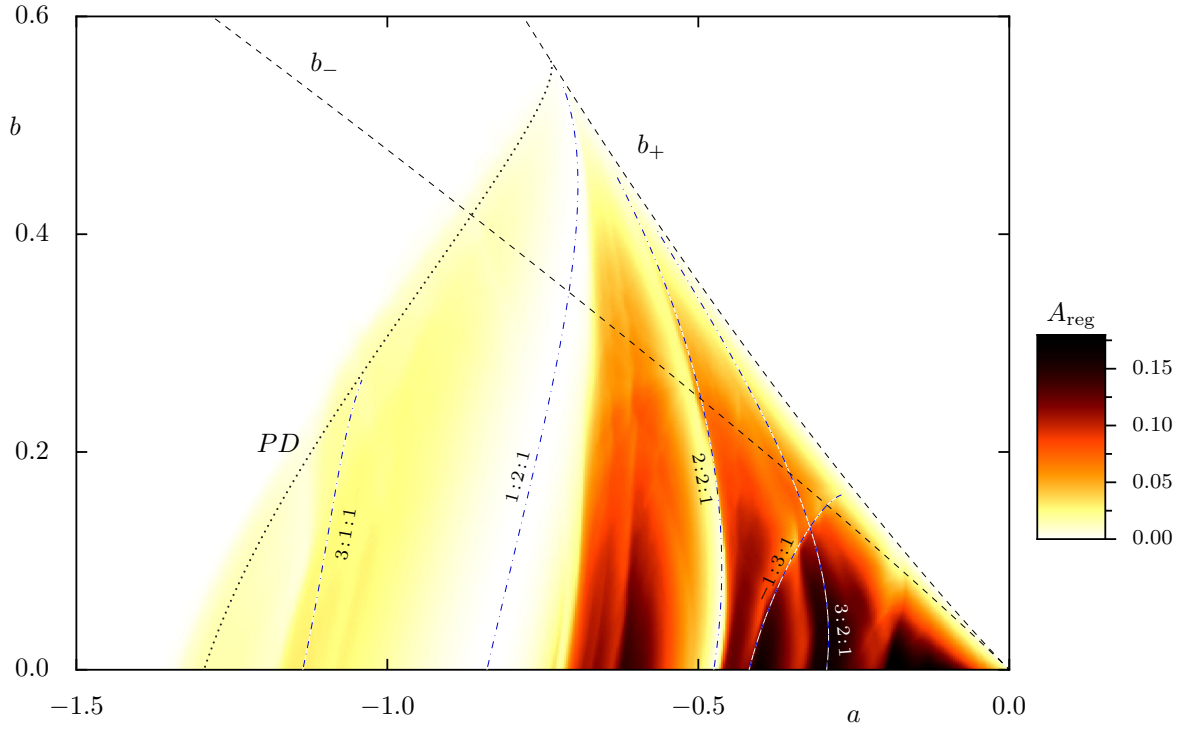


**Figure 14.** Area of bounded initial conditions  $(\xi, C\xi)$ , determined from a grid in the  $\xi$ -plane for  $|\xi_i| < \kappa/2$ , under variation of  $a$  with  $b = c = 0.01$ , and  $\varepsilon_2 = 1$ ; the matrix  $C$  is as in Fig. 13. The insets show escape time plots for a neighborhood of the EE fixed point. Note that for all  $a$ -values in the figure there are four fixed points (shown as small black crosses in the insets), which, as  $a$  decreases, move out of the shown square. A movie of escape time plots as a function of  $a$  can be found in the supplementary material at <http://www.comp-phys.tu-dresden.de/supp/>.

expected to escape for extremely large times by means of Arnold’s exponentially-slow diffusion mechanism [53, 54, 55, 56].

We can quantify the size of the region of “bounded orbits” under parameter variation by computing the area of the white region in a 2D-plane of initial conditions like that in Fig. 13. For this we choose initial conditions in the two-plane  $(\xi, C\xi)$ , with  $\xi$  varied on a grid of  $3000 \times 3000$  points within the box  $-\frac{1}{2}\kappa < \xi_i < \frac{1}{2}\kappa$ , and iterate at most 5000 steps. Orbits that remain within the disk  $\|\xi\| \leq \kappa$  are counted, and the resulting area is denoted  $A_{\text{reg}}$ . This area varies as fixed points undergo various bifurcations. A 1D cut through parameter space, Fig. 14, shows how  $A_{\text{reg}}$  varies with the parameter  $a$  (here  $b = c = 0.01$ ). Insets in the figure show escape time plots, like that in Fig. 13, for some selected parameters. There is a strong correlation of the area with the structure of the region of stable orbits around the EE point, as we discuss further below.

In Fig. 15 we show  $A_{\text{reg}}$  as a function of the two parameters  $a$  and  $b$ , setting  $c = b$ . Note that there are apparently no bounded orbits when  $b > b_+$  (3.7), where there are no fixed

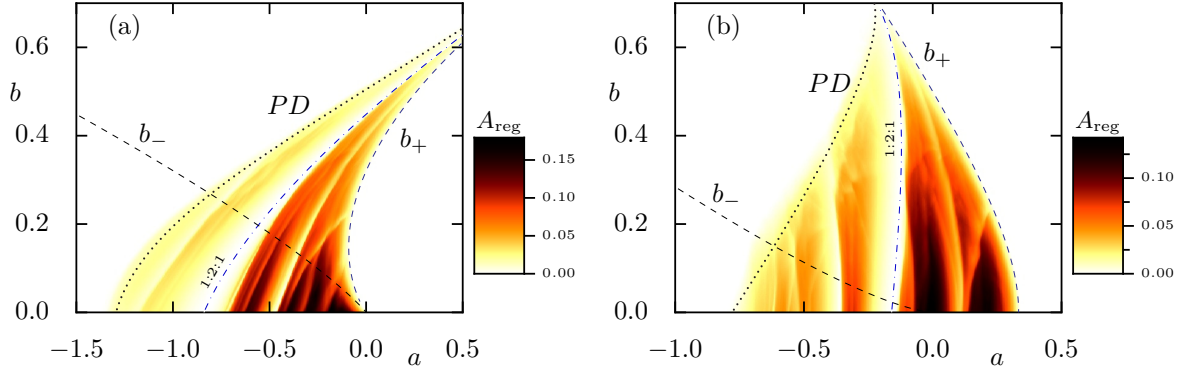


**Figure 15.** Area  $A_{\text{reg}}$  of bounded initial conditions as a function of  $(a, b)$  with  $c = b$  and  $\varepsilon_2 = 1$ . The matrix  $C$  is defined by  $(\alpha, \mu, \delta) = (1, 0.1, 0.5)$ , and  $\varepsilon_1 = 1$ , as in Fig. 5(a, b). The curves  $b_{\pm}$  represent the crossing of the surfaces (3.7) with the plane  $c = b$ . Thus there are four fixed points when  $b$  is below  $b_-$  and two when  $b_- < b < b_+$ . Upon crossing the (dotted)  $PD$  curve from right to left, the  $EE$  fixed point becomes  $IE$  and  $A_{\text{reg}}$  rapidly drops to zero. The blue dashed-dotted curves show several resonances (5.7), labeled  $n_1 : n_2 : m$ , of the  $EE$  fixed point.

points, nor when  $b \gtrsim 0.6$  or  $a < -1.5$ . The largest bounded area occurs in the region near the origin in the  $(a, b)$  plane; recall that the origin corresponds to the quadfurcation since  $c = b$ . The dotted curve corresponds to parameters for which  $PD = 0$ , the period-doubling bifurcation (3.21b). To the left of the  $PD$  curve, the  $EE$  fixed point becomes  $IE$  and  $A_{\text{reg}}$  decreases quickly to 0. To the right of the  $PD$  curve and for  $b < b_-$ , the four fixed points have the stabilities  $EE$ ,  $EH$ ,  $EH$ , and  $CU$ . On the curve  $b = b_-$  the  $EH$  and  $CU$  fixed points coalesce; however, since there are no bounded, regular orbits in a neighborhood of these fixed points, this transition does not influence  $A_{\text{reg}}$ .

At several places in Fig. 14 and along several curves in Fig. 15 one observes a substantial decrease in the bounded area. Several of these decreases can be related to those parameters for which the linearization about the elliptic-elliptic fixed point fulfills a low-order resonance. Of course, such behavior is well-known for the case of 2D area-preserving maps. The variation is related to the fractal structure of the stability region which changes significantly, in particular when passing through the low-order resonances, see e.g. [57, 58, 59].

At a resonance in the 4D case, the four eigenvalues have the form  $\lambda_{1,2} = e^{2\pi i \nu_{1,2}}$ , and the conjugate/inverse values  $\bar{\lambda}_i = 1/\lambda_i$ . Each frequency,  $\nu_i$ , describes the rate of rotation around



**Figure 16.** Area  $A_{\text{reg}}$  of initial conditions on the 2D plane  $\eta = C\xi$  inside the box  $|\xi_i| \leq \kappa/2$  (on a  $3000 \times 3000$  point grid) that remain within the disk  $\|\xi_t\| \leq \kappa$  for  $t \leq 5000$ . Variation of  $(a, b)$  with (a)  $c = 5b$  and (b)  $c = 2$ . The remaining parameters are the same as in Fig. 15. The curves  $b_{\pm}$  represent the crossing of the surfaces (3.7) with the plane  $c = 5b$  or 2, respectively. Thus there are four fixed points when  $b$  is below  $b_-$  and two when  $b$  is between  $b_-$  and  $b_+$ . To the left of the (dotted) PD line the EE fixed point has become EI.

the fixed point in the 2D invariant planes spanned by the eigenvectors of the corresponding conjugate pair of eigenvalues. These can be written in terms of the partial traces,  $\rho_{1,2}$ , recall (3.14), as

$$\nu_{1,2} = \frac{1}{2\pi} \arccos\left(\frac{1}{2}\rho_{1,2}\right).$$

The frequencies  $(\nu_1, \nu_2)$  of an EE fixed point fulfill a resonance condition when

$$(5.7) \quad n_1\nu_1 + n_2\nu_2 = m, \quad (n_1, n_2) \in \mathbb{Z}^2 \setminus \{0, 0\}, \quad m \in \mathbb{Z}.$$

Without loss of generality, we can set  $\gcd(n_1, n_2, m) = 1$ , and  $m \geq 0$ . We refer to such a relation as an  $n_1 : n_2 : m$  resonance. Curves of several such resonances are shown in Fig. 15

While resonances are dense in frequency space, those with small order,  $|n_1| + |n_2|$ , are of particular relevance. For example the large white region in Fig. 15 starting at  $b = 0$  and  $a \approx -0.75$  corresponds to the  $1 : 2 : 1$  resonance. This is also manifested in the broad minimum with  $A_{\text{reg}} \approx 0$  in Fig. 14. The other prominent minimum near  $a \approx -0.5$  is caused by the  $2 : 2 : 1$  resonance. In these two cases  $A_{\text{reg}}$  is reduced in a parameter neighborhood of the resonance curve. For other indicated resonances, i.e.,  $3 : 2 : 1$ ,  $-1 : 3 : 1$  and  $3 : 1 : 1$ , the density is only reduced on one side of the bifurcation. In the examples this happens for smaller  $a$ , and sometimes—as for the  $3 : 2 : 1$  resonance—it occurs quite some distance away.

Higher-order resonances,  $|n_1| + |n_2| \geq 5$ , should have less influence on  $A_{\text{reg}}$ . The results of [21] lead to the expectation that the EE point will remain stable, and that for a single resonance the bifurcation creates a pair of invariant 1D tori, one normally hyperbolic and one normally elliptic (at least in the normal form). Further away from the bifurcation of the EE fixed point the geometry is described by bifurcations of families of 1D tori, see [43] and references therein. When the frequency passes through a double resonance, so that  $\nu_i = p_i/q_i$  are rational, then one expects four periodic orbits to be created [21, 60]. According to [21] their stability is either  $\text{EE} + 2 \text{EH} + \text{HH}$  or  $2 \text{EH} + 2 \text{CU}$ , see [42] for an illustration of the geometry in the first case.

Two further examples of  $A_{\text{reg}}$  are shown in Fig. 16 with the same parameters as in Fig. 15, except for Fig. 16(a)  $c = 5b$  and for Fig. 16(b) the parameter  $c = 2$  is fixed. In the latter case the parameter plane no longer intersects the quadfurcation point ( $a = b = c = 0$ ), so that the curves  $b_{\pm}$  do not intersect at the origin in the figure. In Fig. 16(a) the line  $b = 0$  corresponds to that in Fig. 15 so that the same resonances are still relevant. These now extend to the region  $b > 0$ , bending strongly to the right. The same overall resonance structure is also visible in Fig. 16(b).

**6. Coupled Hénon Maps.** Since Moser's map, (2.2) or equivalently (2.9), is the generic, four-dimensional quadratic symplectic map, there must be parameters for which it corresponds to a pair of uncoupled quadratic maps. In this section, we show that this is possible for  $\varepsilon_2 = 1$  and special choices of the matrix  $C$ , depending on  $\varepsilon_1$ . The sign  $\varepsilon_1$  corresponds to positive and negative Krein signatures. These Hénon maps are uncoupled when  $c = 0$ . Whether there are other possibilities for which the Moser map has uncoupled dynamics with respect to some invariant canonical planes on which the dynamics is conjugate to Hénon maps is presently not clear and left for future study.

Dynamics of a pair of coupled Hénon maps has been studied previously for example in [19, 20, 21, 27, 22, 23, 24, 28, 29, 30], particularly in regard to models of storage rings for particle accelerators.

**6.1. Decoupled Limits: Hénon Maps.** To find parameter values for which the Moser map is decoupled, we search for a coordinate transformation that reveals the invariant planes. This transformation should be affine in order to maintain the quadratic form. So that the resulting map is symplectic with the standard Poisson matrix (1.1) and to maintain the momentum-coordinate split for the Hénon form, we start with the linear transformation:

$$(\xi, \eta) = S(q, p) = (Aq, \rho A^{-T}p),$$

where  $A$  is an invertible matrix and  $\rho > 0$ . This transformation is symplectic-with-multiplier,  $DS^T J DS = \rho J$ . In the new coordinates the map (2.9) becomes

$$\begin{aligned} q' &= q + \hat{C}^{-T}(-p + \hat{C}q + \nabla_q \hat{U}(q)), \\ p' &= \hat{C}q, \end{aligned}$$

where

$$\hat{C} = \frac{1}{\rho} A^T C A, \quad \hat{U}(q) = \frac{1}{\rho} U(Aq),$$

so that  $\nabla_q \hat{U}(q) = \frac{1}{\rho} A^T \nabla_{\xi} U(Aq)$ . In order that the map be decoupled in the new coordinates, any cross terms in the new potential  $\hat{U}$  should be zero. This can be accomplished for (2.10) only if  $\varepsilon_2 = 1$ ,  $c = 0$ . We can normalize the amplitude of the quadratic terms in the new map by setting  $\rho = 1/\sqrt{12}$  choosing

$$A = \rho \begin{pmatrix} 1 & 1 \\ -\sqrt{3} & \sqrt{3} \end{pmatrix}$$

to give

$$\hat{U}(q) = (a - \sqrt{3}b)q_1 + (a + \sqrt{3}b)q_2 + \frac{1}{3}(q_1^3 + q_2^3).$$

The transformation is thus fixed by this choice. So that the resulting map be decoupled,  $\hat{C}$  must be diagonal, e.g.,

$$(6.1) \quad \hat{C} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & 1 \end{pmatrix},$$

where  $\det C = \det \hat{C} = \varepsilon_1 = \pm 1$ . In order for this to be the case, the original  $C$  must take one of two forms:

$$(6.2a) \quad \varepsilon_1 = 1 : \quad C = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$(6.2b) \quad \varepsilon_1 = -1 : \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that one could also replace  $\hat{C}$  by  $-\hat{C}$  in (6.1), but by the symmetries discussed in §2.3, this gives nothing new.

With this we get the transformed map

$$\begin{aligned} q'_1 &= 2q_1 + \varepsilon_1(-p_1 + a - \sqrt{3}b + q_1^2) \\ p'_1 &= \varepsilon_1 q_1 \\ q'_2 &= 2q_2 - p_2 + a + \sqrt{3}b + q_2^2 \\ p'_2 &= q_2. \end{aligned}$$

This is not quite in the Hénon form (2.1), but a final affine transformation  $q \rightarrow (\hat{q}_1 - \varepsilon_1, \hat{q}_2 - 1)$  and  $p \rightarrow \hat{p} - (1, 1)$  brings the map to the form

$$(6.3) \quad \begin{aligned} \hat{q}'_1 &= \varepsilon_1(-\hat{p}_1 + 1 + a - \sqrt{3}b + \hat{q}_1^2) \\ \hat{p}'_1 &= \varepsilon_1 \hat{q}_1 \\ \hat{q}'_2 &= -\hat{p}_2 + 1 + a + \sqrt{3}b + \hat{q}_2^2 \\ \hat{p}'_2 &= \hat{q}_2. \end{aligned}$$

Note that after this transformation we obtain, when  $\varepsilon_1 = 1$ , a pair of uncoupled maps of the Hénon-form (2.1). However, when  $\varepsilon_1 = -1$ , the first canonical pair has the Hénon form only upon a mirroring transformation, e.g.,  $(\hat{q}_1, \hat{p}_1) \rightarrow (-\hat{q}_1, \hat{p}_1)$ ; the point is that in the canonical coordinates, the  $(\hat{q}_1, \hat{p}_1)$  components “rotate” under the map in the opposite sense from  $(\hat{q}_2, \hat{p}_2)$ .

The component maps have saddle-center bifurcations along the lines  $a = \pm\sqrt{3}b$ , creating pairs of fixed points for each component when  $a < \pm\sqrt{3}b$ , respectively. However, in order that the 4D map have a fixed point, both components must have fixed points, implying that  $a < -\sqrt{3}|b|$ . This bifurcation occurs on the same line that appears in Fig. 3(a) when  $c = 0$ .

When the maps are decoupled, and  $a < -\sqrt{3}|b|$  the four newly created fixed points have types EE, EH, EH and HH. In particular the EE point is located at

$$\begin{aligned} \varepsilon_1 q_1^* &= p_1^* = 1 - \sqrt{\sqrt{3}b - a}, \\ q_2^* &= p_2^* = 1 - \sqrt{-\sqrt{3}b - a}. \end{aligned}$$

In the original variables, the fixed points of these maps are given by (3.8) since this corresponds to  $c = 0$ . The doubly elliptic fixed point remains stable in the rectangle

$$-4 + \sqrt{3}|b| < a < -\sqrt{3}|b|$$

since the individual maps have period-doubling bifurcations at  $a = -4 \pm \sqrt{3}b$ , respectively.

Recall that for a symplectic map (1.1), with a doubly elliptic fixed point  $z^*$ , the quadratic form  $q(v) = v^T J Df(z^*) v$  is an invariant of the linearized dynamics. This implies stability when the form  $q$  is definite [61, 38]: an EE fixed point cannot cross the  $KP = 0$  line in Fig. 4 into the CU region. Equivalently, the Krein bifurcation cannot occur if the symmetric matrix

$$\mathcal{Q} = \frac{1}{2}[J Df(z^*) - Df^T(z^*) J]$$

is definite. At an elliptic-elliptic point for (6.3),  $\mathcal{Q}$  becomes

$$\mathcal{Q} = \left( \begin{array}{cc|cc} \hat{C} & & -\varepsilon_1 q_1^* & 0 \\ & & 0 & -q_2^* \\ \hline -\varepsilon_1 q_1^* & 0 & & \\ 0 & -q_2^* & & \hat{C} \end{array} \right),$$

where  $\hat{C}$  is given in (6.1). When (6.3) has an EE point then  $q_i^{*2} < 1$ . This implies that  $\mathcal{Q}$  is positive definite when  $\varepsilon_1 = 1$ , but has a pair of negative eigenvalues when  $\varepsilon_1 = -1$ . Thus, only in the latter case, can coupling lead to a Krein bifurcation.

Indeed, we can see this if we re-introduce coupling by allowing  $c \neq 0$ . The same transformations that lead to (6.3) can be applied if we still take  $\varepsilon_2 = 1$  and  $C$  to be one of the matrices (6.2). The result is the pair of coupled Hénon maps

$$\begin{aligned} \hat{q}'_1 &= \varepsilon_1 \left( -\hat{p}_1 + a_{h1} + \hat{q}_1^2 + \frac{c}{2\sqrt{3}}(\hat{q}_1 + \hat{q}_2) \right), \\ \hat{p}'_1 &= \varepsilon_1 q_1, \\ \hat{q}'_2 &= -\hat{p}_2 + a_{h2} + \hat{q}_2^2 + \frac{c}{2\sqrt{3}}(\hat{q}_1 + \hat{q}_2), \\ \hat{p}'_2 &= \hat{q}_2, \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} a_{h1} &= 1 + a - \sqrt{3}b - \frac{\varepsilon_1 + 1}{2\sqrt{3}}c, \\ a_{h2} &= 1 + a + \sqrt{3}b - \frac{\varepsilon_1 + 1}{2\sqrt{3}}c. \end{aligned} \tag{6.5}$$

The stability of the fixed points for this map can be conveniently obtained from the results for the Moser map (2.9). For  $\varepsilon_1 = \varepsilon_2 = 1$  and the diagonal matrix in (6.2a), the stability parameters (3.17) become

$$\begin{aligned} A &= 4(1 + \sqrt{3}\xi_1) + \frac{c}{\sqrt{3}}, \\ B &= 6 + \frac{2c}{\sqrt{3}}(1 + \sqrt{3}\xi_1) + 4\sqrt{3}\xi_1(2 + \sqrt{3}\xi_1) - 4\xi_2^2. \end{aligned}$$



Since the quadfurcation occurs for  $a = b = c = 0$ ,  $\xi_1 = \xi_2 = 0$ , it always occurs at the point  $(A^Q, B^Q) = (4, 6)$  for this  $C$ , as we showed in §3.5 is more generally true for the symmetric case. For this case the Krein parameter (3.19) is always nonpositive:

$$(6.6) \quad KP = -4\xi_2^2 - c^2/12,$$

even when the maps are coupled again by nonzero  $c$ . This reflects the fact that for the two 2D uncoupled maps the elliptic motions in  $(\hat{p}_1, \hat{q}_1)$  and  $(\hat{p}_2, \hat{q}_2)$  have the the same orientation. This is not changed by the coupling when  $\varepsilon_1 = +1$ , and a CU instability is not possible. This case corresponds to the transition (3.28) with a stability diagram like that shown in Fig. 8(b) and neither the EE nor the HH fixed point may turn CU.

For the case  $\varepsilon_2 = 1$ , and  $\varepsilon_1 = -1$ , for the matrix in (6.2b), we similarly obtain

$$\begin{aligned} A &= 4(1 + \xi_2), \\ B &= 6 - 12\xi_1^2 + 8\xi_2 + 4\xi_2^2 - 2c\xi_1. \end{aligned}$$

Now the Krein parameter becomes

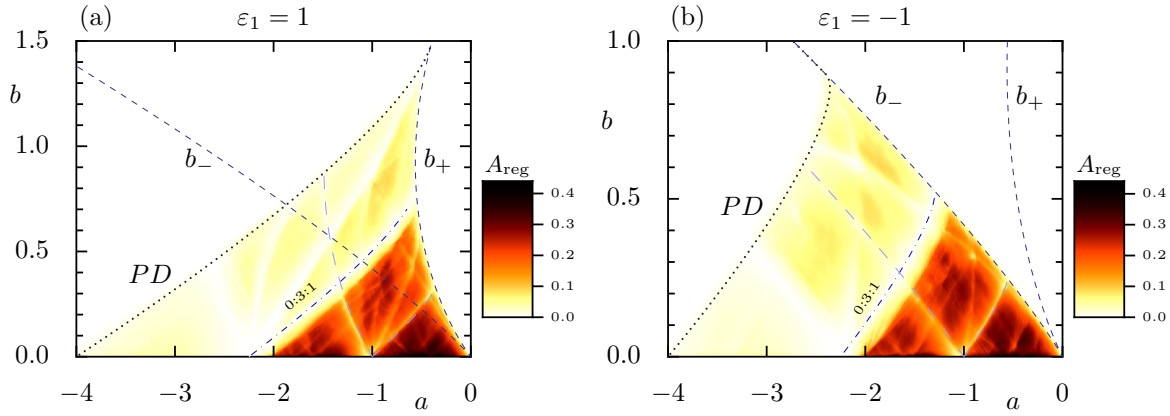
$$(6.7) \quad KP = -2\xi_1(6\xi_1 + c),$$

which potentially may have either sign. So when the rotation directions for the two canonical planes are opposed, the coupling terms make a Krein bifurcation possible for nonzero  $c$ . Thus the initial quadfurcation may correspond to the transition  $\emptyset \rightarrow \text{EE} + \text{HH} + 2 \text{EH}$ , as in Fig. 8(b), and both the EE and HH might later become CU. In addition a direct transition to  $2 \text{CU} + 2 \text{EH}$  is possible, which is analogous to the  $A$ - $B$  diagram shown in Fig. 8(a) for the fully coupled case.

**6.2. Numerical Illustration.** Let us now illustrate the dynamics near the uncoupled case. Figure 17 shows the area,  $A_{\text{reg}}$ , of bounded orbits for the two cases (6.2) of the matrix  $C$ . In these figures, the maps are uncoupled along the line  $b = 0$  since we choose  $c = 2b$ . As in Figs. 15 and 16 one observes clear drops of  $A_{\text{reg}}$  along curves in the  $(a, b)$  plane. For the uncoupled case, when  $b = 0$ , the two plots in Fig. 17 agree because the individual 2D maps in (6.4) have the same parameters  $a_{h1} = a_{h2}$ . For both panels, the  $b_-$  line corresponds to a saddle-center bifurcation, but in Fig. 17(a) two fixed points with stabilities EH and HH disappear upon reaching  $b_-$  from below; thus the decrease in the number of fixed points has no significant influence on the size of  $A_{\text{reg}}$ . In contrast, for Fig. 17(b) the fixed points with stabilities EE and EH disappear upon reaching  $b_-$  from below, so that only the EH and HH fixed points are left in the region between  $b_-$  and  $b_+$  and  $A_{\text{reg}}$  decreases abruptly. As  $\varepsilon_1 = 1$  for Fig. 17(a) none of fixed points can become CU. For  $\varepsilon_1 = -1$ , as in Fig. 17(b), the HH fixed point becomes CU when  $a$  is sufficiently negative, but again this does not significantly influence  $A_{\text{reg}}$ .

In both panels of Fig. 17 the  $0 : 3 : 1$  resonance curve of the EE fixed point leads to a strong reduction in bounded area as  $a$  decreases through the resonance. There is also a strong reduction for both cases in  $A_{\text{reg}}$  along two lines that meet in  $(a, b) = (-1, 0)$ . At this parameter value the uncoupled Hénon maps have a fourth-order resonance, see e.g. [59] for a detailed study of the 2D case. The right line correspond to the  $0 : 4 : 1$  resonance and the





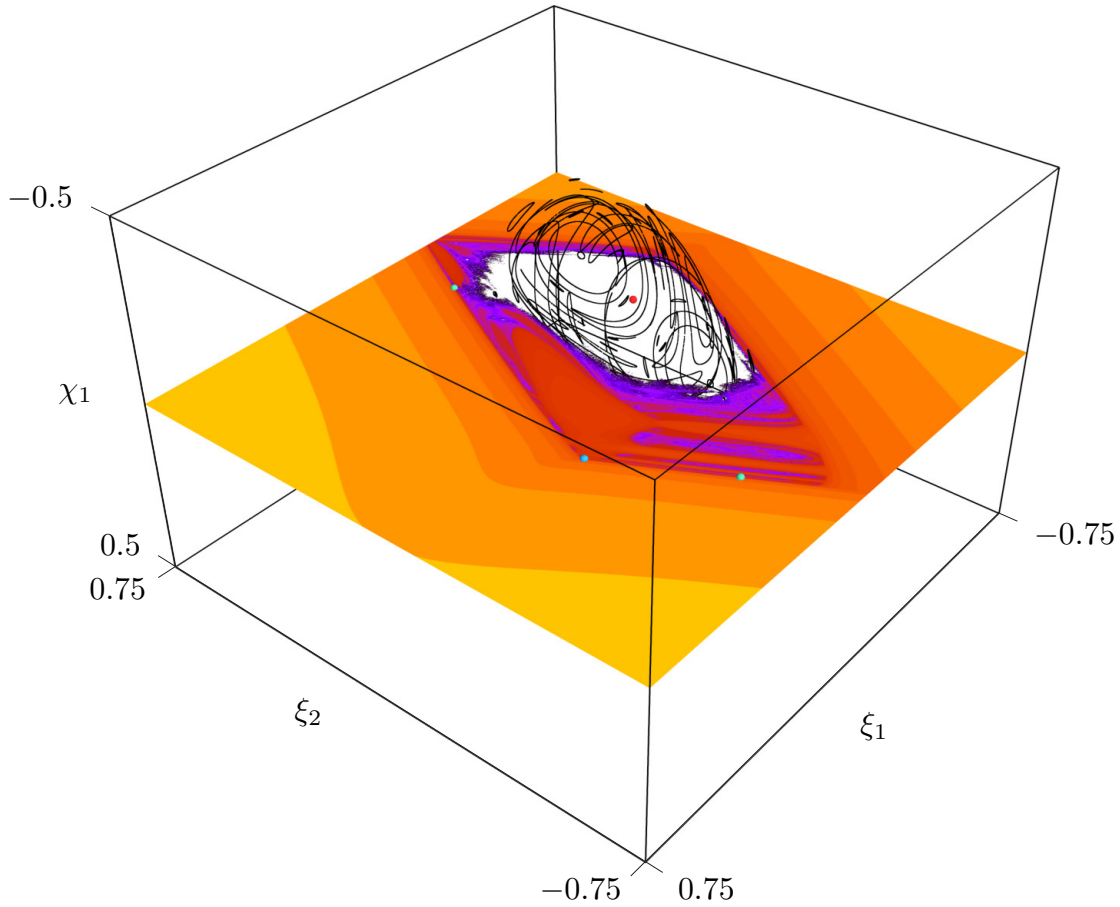
**Figure 17.** Area  $A_{\text{reg}}$  of bounded orbits on the plane  $\eta = C\xi$  for the matrices (6.2) with  $\varepsilon_2 = 1$ ,  $c = 2b$  and for (a)  $\varepsilon_1 = 1$ , and (b)  $\varepsilon_1 = -1$ . The maps reduce to the uncoupled case only along the axis  $b = 0$ , since then  $c = 0$  as well. The curves  $b_{\pm}$  represent the crossing of the surfaces (3.7) with the plane  $c = 2b$ . Thus there are four fixed points when  $b$  is below  $b_-$  and two when  $b$  is between  $b_-$  and  $b_+$ . For both examples, to the left of the (dotted) PD line the EE fixed point becomes EI. The dashed-dotted line shows the  $0:3:1$  resonance near where there is a strong decrease in  $A_{\text{reg}}$ . The long-dashed (light blue) lines show the  $4:0:1$  resonance (left branch) and the  $0:4:1$  resonance (right branch) that emanate from  $(a, b) = (-1, 0)$  corresponding to the quadrupling bifurcation in the uncoupled case.

left to the  $4:0:1$  resonance. For both resonances the reduction happens to the left of the lines, i.e. for smaller values of  $a$ . This corresponds to the direction in which the quadrupling bifurcation creates new elliptic and hyperbolic fixed points and 1D tori.

Figure 18 shows a 3D phase space slice plot and an escape time plot in the  $(\xi_1, \xi_2)$ -plane for  $(a, b, c) = (-0.3, 0.1, 0.2)$  and  $\varepsilon_1 = 1$ . The elliptic-elliptic fixed point is surrounded by a region of predominantly regular motion as seen by the white region of non-escaping orbits (within  $10^4$  iterations). Corresponding regular 2D tori are shown as black curves in the slice. Some of these are secondary tori around periodic orbits and appear as sequences of disjoint loops in the 3D phase space slice. The HH fixed point and the two EH fixed points approximately limit the region in which the regular orbits and orbits with longer escape times are contained.

**7. Summary and Outlook.** In this paper we have studied some of the dynamics of Moser’s 4D quadratic, symplectic maps, which have a normal form (2.9) with six parameters  $a, b, c, \alpha, \delta, \mu$  and two discrete parameters  $\varepsilon_1, \varepsilon_2$ . We showed that there is a codimension-three submanifold in parameter space for which Moser’s map has a single fixed point with a pair of unit eigenvalues. Bifurcations that occur on this submanifold correspond to creation and destruction of up to four fixed points, the maximum possible number for the map (except for one singular case). Along paths in parameter space that pass through this singularity it is possible that four fixed points are created from none—a quadfurcation. For other paths, two fixed points may be created or collide and emerge as two or four, recall Fig. 3 and Table 1. Intuitively, the simplest case corresponds to a pair of uncoupled 2D Hénon maps, where a quadfurcation corresponds to choosing parameters so that the maps have simultaneous, co-located saddle-center bifurcations.

When a fixed point has four distinct eigenvalues on the unit circle (has type “EE”), then it



**Figure 18.** 3D phase space slice and corresponding escape time plot in the  $\eta = C\xi$  plane for the diagonal matrix  $C$  of (6.2a) with  $\varepsilon_1 = 1$  and  $(a, b, c) = (-0.3, 0.1, 0.2)$ . Shown are several selected regular tori (black lines) near the “outer edge” of the “regular region” surrounding the EE fixed point. Each torus is represented by  $10^4$  points in the slice with  $\epsilon = 10^{-6}$ . The four fixed points are shown as small spheres: EE (red), 2 EH (green), and HH (blue). The coloring of the escape times is the same as in Fig. 13. For a rotating view see <http://www.comp-phys.tu-dresden.de/supp/>.

is linearly stable, and according to KAM theory, is generically surrounded by a Cantor family of invariant two-tori. We have seen that it is possible for one or two EE-fixed points to emerge from the quadfurcation. The remaining two fixed points have at least one hyperbolic pair of eigenvalues. For the case of uncoupled Hénon maps, the four fixed points correspond to the cross-products of the saddles and centers of the two area-preserving maps, giving rise to a single EE fixed point, two EH points, and one HH point. This scenario persists when coupling is added, and, as we will show in a future paper, can describe the creation of accelerator modes in a 4D standard map near zero coupling. However, this scenario is rather special from the point view of Moser’s map, where one more typically has the creation of two EE and two EH fixed points, unless the matrix  $C$  is symmetric.

For symmetric  $C$ , where the map is reversible, the quadfurcation has the special feature that the fixed points are born with four unit eigenvalues. This allows, for example, the direct creation of complex unstable, CU, fixed points. It is also of interest that when  $C$  is symmetric, the limiting form of Moser's map near the quadfurcation is a natural Hamiltonian system (3.32). It is still an open question whether the map is reversible *only* when  $C$  is symmetric.

We showed in Th. 1 that, when  $\varepsilon_2 \neq 0$ , there is a ball that contains all bounded orbits. We observe that orbits that remain bounded are typically associated with the EE fixed points. The computations suggest that the center-stable manifolds of the EH points are likely candidates for partial barriers that delineate the boundary of the region of orbits that have long escape times. In a future paper, we hope to compute these manifolds to understand better the geometry of the barriers.

There are several other interesting questions left for future studies. For the 2D case, where the Hénon map provides the universal form for any quadratic area-preserving map, the algebraic decay of the survival probability for the escape from a neighborhood of the regular region is well established and understood in terms of partial barriers and approximately described by Markov models. While for higher-dimensional maps such power-law stickiness is numerically well established, the mechanism for this is still not understood. Moser's map is the prototypical example for the study of the stickiness of a regular region in 4D. Of course, in this context, Arnold diffusion will also be important.

Finally, it would be of interest to study similar bifurcations for polynomial maps of higher degree; for example, the cubic case can be written in the Moser form as a composition of affine maps with a symplectic shear [33]. And of course, one can wonder what other exotic local bifurcations may happen in even higher-dimensional symplectic maps.

**Acknowledgements.** We would like to thank Robert Easton, Roland Ketzmerick, Rafael de la Llave, and Martin Richter for useful discussions. JDM was supported as a Dresden Senior Fellow of the Technische Universität Dresden, and the authors acknowledge support from the Mathematical Sciences Research Institute in Berkeley California during the Fall 2018 semester under NSF Grant DMS-1440140. The visualizations of the 3D phase space slices were created using MAYAVI [62].

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