Quantum Information Processing (2020) 19:61
https://doi.org/10.1007/s11128-019-2563-4

Optimal uniform continuity bound for conditional entropy of classical–quantum states

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Received: 6 September 2019 / Accepted: 23 December 2019

Abstract
In this short note, I show how a recent result of Alhejji and Smith (A tight uniform continuity bound for equivocation, 2019. arXiv:1909.00787v1) regarding an optimal uniform continuity bound for classical conditional entropy leads to an optimal uniform continuity bound for quantum conditional entropy of classical–quantum states. The bound is optimal in the sense that there always exists a pair of classical–quantum states saturating the bound, and so, no further improvements are possible. An immediate application is a uniform continuity bound for the entanglement of formation that improves upon the one previously given by Winter (Commun Math Phys 347(1):291–313, 2016. arXiv:1507.07775). Two intriguing open questions are raised regarding other possible uniform continuity bounds for conditional entropy: one about quantum–classical states and another about fully quantum bipartite states.

Keywords Uniform continuity of entropy · Separable Hilbert space · Entanglement of formation

Recently, the following bound has been established by Alhejji and Smith in [1] for $\varepsilon \in (0, 1 - 1/|\mathcal{Y}|]$:

$$\left|H(Y|X)_p - H(Y|X)_q\right| \leq \varepsilon \log_2(|\mathcal{Y}| - 1) + h_2(\varepsilon), \tag{1}$$

where $h_2(\varepsilon) := -\varepsilon \log_2 \varepsilon - (1-\varepsilon)\log_2(1-\varepsilon)$ is the binary entropy, p_{XY} and q_{XY} are joint probability distributions over the finite-cardinality alphabets $\mathcal{X}$ and $\mathcal{Y}$,

$$H(Y|X)_p := -\sum_{x\in\mathcal{X}} p_X(x) \sum_{y\in\mathcal{Y}} p_{Y|X}(x) \log_2 p_{Y|X}(y|x) \tag{2}$$

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Published online: 02 January 2020

and $H(Y|X)_q$ (defined in a similar way but with q_{XY}) are conditional Shannon entropies, and

$$\varepsilon \geq \frac{1}{2}\|p_{XY} - q_{XY}\|_1 := \frac{1}{2}\sum_{x\in\mathcal{X}, y\in\mathcal{Y}} |p_{XY}(x,y) - q_{XY}(x,y)| \,. \tag{3}$$

The quantity on the right-hand side is known as the total variational distance of the probability distributions p_{XY} and q_{XY}, and it is a measure of their statistical distinguishability. The bound in (1) is called a uniform continuity bound because the right-hand side depends only on ε and the cardinality $|\mathcal{Y}|$. It is optimal in the sense that for every ε and $|\mathcal{Y}|$, there exists a pair of distributions p_{XY} and q_{XY} saturating the upper bound (see Eqs. (27)–(28) of [1]). It generalizes the optimal uniform continuity bound for unconditional Shannon entropy established independently by [2, Eq. (4)] and [3].

Uniform continuity bounds of the form in (1) for both the classical and quantum cases find application in providing estimates for various communication capacities of classical and quantum channels [4–11]. Motivated by this application (as well as fundamental concerns), there has been a large amount of work on this topic over the years [3,12–18].

In this brief note, I show how to employ the bound in (1) to establish the following optimal uniform continuity bound for conditional entropy of finite-dimensional classical–quantum states, improving (optimally) upon one of the cases given in Lemma 2 of [14]:

Proposition 1 *The following inequality holds for $\varepsilon \in (0, 1 - 1/d_B]$:*

$$\left|H(B|X)_\rho - H(B|X)_\sigma\right| \leq \varepsilon \log_2(d_B - 1) + h_2(\varepsilon), \tag{4}$$

where d_B is the dimension of system B, the states ρ_{XB} and σ_{XB} are the following finite-dimensional classical–quantum states:

$$\sum_{x\in\mathcal{X}} r(x)|x\rangle\langle x|_X \otimes \rho_B^x, \quad \sum_{x\in\mathcal{X}} s(x)|x\rangle\langle x|_X \otimes \sigma_B^x, \tag{5}$$

respectively $r(x)$ and $s(x)$ are probability distributions, $\left\{\rho_B^x\right\}_x$ and $\left\{\sigma_B^x\right\}_x$ are sets of states, the conditional entropy is defined in terms of the von Neumann entropy as $H(B|X)_\rho := \sum_x r(x)H(\rho_B^x)$, and

$$\varepsilon \geq \frac{1}{2}\|\rho_{XB} - \sigma_{XB}\|_1 \,. \tag{6}$$

Also, there exists a pair of classical–quantum states saturating the bound for every value of d_B and $\varepsilon \in (0, 1 - 1/d_B]$.

Proof The desired inequality is reduced to the classical case by means of a conditional dephasing channel and data processing. This generalizes an approach recalled in the introduction of [14], which is attributed therein to [19]. Suppose without loss of generality that $H(B|X)_\rho \leq H(B|X)_\sigma$. Let a spectral decomposition of ρ_B^x be as follows:

$$\rho_B^x = \sum_y r(y|x)|\phi^{y,x}\rangle\langle\phi^{y,x}|_B, \tag{7}$$

where $r(y|x)$ is a conditional probability distribution and $\{|\phi^{y,x}\rangle_B\}_y$ is a set of orthonormal states (for fixed x). Define the conditional dephasing channel as

$$\overline{\Delta}^{\text{cd}}_{XB}(\omega_{XB}) = \sum_{x,y} \left(|x\rangle\langle x|_X \otimes |\phi^{y,x}\rangle\langle\phi^{y,x}|_B\right) \omega_{XB} \left(|x\rangle\langle x|_X \otimes |\phi^{y,x}\rangle\langle\phi^{y,x}|_B\right), \tag{8}$$

which we think of intuitively as dephasing or measuring system X and then based on the outcome, dephasing system B in the eigenbasis of ρ_B^x. This is a unital channel, and so, the entropy of any state on systems X and B does not decrease under its action. When this conditional dephasing acts on σ_{XB}, it leads to the following state:

$$\overline{\Delta}^{\text{cd}}_{XB}(\sigma_{XB}) = \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} s(x)s(y|x)|x\rangle\langle x|_X \otimes |\phi^{y,x}\rangle\langle\phi^{y,x}|_B, \tag{9}$$

where $s(y|x)$ is a conditional probability distribution and $\mathcal{Y}$ is an alphabet with the same cardinality as the dimension d_B: $|\mathcal{Y}| = d_B$. Observe that

$$\sigma_X = \text{Tr}_B[\sigma_{XB}] = \text{Tr}_B[\overline{\Delta}^{\text{cd}}_{XB}(\sigma_{XB})]. \tag{10}$$

Furthermore, the state ρ_{XB} is invariant under the action of the conditional dephasing channel:

$$\rho_{XB} = \overline{\Delta}^{\text{cd}}_{XB}(\rho_{XB}). \tag{11}$$

Observe that ρ_{XB} and $\overline{\Delta}^{\text{cd}}_{XB}(\sigma_{XB})$ are commuting states, and thus can be considered as classical–classical states. (To be more precise, the first is classical and the second is classical conditioned on the classical value in the first system.) Define the joint distributions $r_{XY}(x, y) = r(x)r(y|x)$ and $s_{XY}(x, y) = s(x)s(y|x)$. From (10) and the fact that the conditional dephasing channel is unital, it follows that

$$\begin{aligned} H(B|X)_\sigma &= H(BX)_\sigma - H(X)_\sigma && (12)\\ &= H(BX)_\sigma - H(X)_{\overline{\Delta}^{cd}(\sigma)} && (13)\\ &\leq H(BX)_{\overline{\Delta}^{\text{cd}}(\sigma)} - H(X)_{\overline{\Delta}^{\text{cd}}(\sigma)} && (14)\\ &= H(B|X)_{\overline{\Delta}^{\text{cd}}(\sigma)} && (15)\\ &= H(Y|X)_s. && (16) \end{aligned}$$

So we have that

$$H(Y|X)_r = H(B|X)_\rho \leq H(B|X)_\sigma \leq H(Y|X)_s, \tag{17}$$

which means that

$$H(B|X)_\sigma - H(B|X)_\rho \leq H(Y|X)_s - H(Y|X)_r. \tag{18}$$

Meanwhile, we have from data processing for normalized trace distance that

$$\begin{aligned}\frac{1}{2}\left\|\rho_{XB} - \sigma_{XB}\right\|_1 &\geq \frac{1}{2}\left\|\overline{\Delta}^{\text{cd}}_{XB}(\rho_{XB}) - \overline{\Delta}^{\text{cd}}_{XB}(\sigma_{XB})\right\|_1 &(19)\\ &= \frac{1}{2}\left\|\rho_{XB} - \overline{\Delta}^{\text{cd}}_{XB}(\sigma_{XB})\right\|_1 &(20)\\ &= \frac{1}{2}\left\|r_{XY} - s_{XY}\right\|_1. &(21)\end{aligned}$$

In turn, this means that the following bound holds for total variational distance:

$$\frac{1}{2}\left\|r_{XY} - s_{XY}\right\|_1 \leq \varepsilon. \tag{22}$$

Now we have completed the reduction in the classical case and invoke (1) to conclude that

$$\begin{aligned}\left|H(B|X)_\rho - H(B|X)_\sigma\right| &= H(B|X)_\sigma - H(B|X)_\rho &(23)\\ &\leq H(Y|X)_s - H(Y|X)_r &(24)\\ &\leq \varepsilon \log_2(d_B - 1) + h_2(\varepsilon), &(25)\end{aligned}$$

completing the proof of (4). The inequality in (4) is seen to be tight by using the classical example from Eqs. (27)–(28) of [1]. □

By employing the same method of proof given for Corollary 4 in [14] (and observing that $\delta = \sqrt{\varepsilon(2-\varepsilon)}$ and $\delta \in (0, 1 - 1/d]$ imply that $\varepsilon \in (0, 1 - \frac{\sqrt{2d-1}}{d}]$), we arrive at the following uniform continuity bound for the entanglement of formation:

Corollary 2 *Let ρ_{AB} and σ_{AB} be finite-dimensional quantum states such that*

$$\frac{1}{2}\left\|\rho_{AB} - \sigma_{AB}\right\|_1 \leq \varepsilon, \tag{26}$$

where $\varepsilon \in (0, 1 - \frac{\sqrt{2d-1}}{d}]$ and $d = \min\{d_A, d_B\}$. Then

$$\left|E_{\text{F}}(\rho_{AB}) - E_{\text{F}}(\sigma_{AB})\right| \leq \delta \log_2(d-1) + h_2(\delta), \tag{27}$$

where E_{F} is the entanglement of formation and $\delta = \sqrt{\varepsilon(2-\varepsilon)}$. The entanglement of formation of a state ω_{AB} is defined as follows [20]:

$$E_{\text{F}}(\omega_{AB}) := \inf\{H(B|X)_\tau : \tau_{XAB} = \sum_x p(x)|x\rangle\langle x|_X \otimes \phi^x_{AB}, \operatorname{Tr}_X[\tau_{XAB}] = \omega_{AB}\}, \tag{28}$$

where each ϕ^x_{AB} is a pure state and $p(x)$ is a probability distribution.

The statement in Proposition 1 has a straightforward generalization to the case in which the classical conditioning system is countable (thus addressing an open question stated in [1]). To arrive at the corollary, let us define conditional entropy in this case as follows:

$$H(B|X)_\rho := \sum_{x\in\mathcal{X}} p_X(x) H(\rho^x_B), \tag{29}$$

where ρ_{XB} has the same form as in (5), except that $\mathcal{X}$ is now a countable alphabet (correspondingly, X is now a separable Hilbert space). Then we have the following corollary:

Corollary 3 *The following inequality holds for $\varepsilon \in (0, 1 - 1/d_B]$:*

$$\left|H(B|X)_\rho - H(B|X)_\sigma\right| \leq \varepsilon \log_2(d_B - 1) + h_2(\varepsilon), \tag{30}$$

where d_B is the dimension of system B, the states ρ_{XB} and σ_{XB} are the following classical–quantum states:

$$\sum_{x\in\mathcal{X}} r(x)|x\rangle\langle x|_X \otimes \rho^x_B, \quad \sum_{x\in\mathcal{X}} s(x)|x\rangle\langle x|_X \otimes \sigma^x_B, \tag{31}$$

respectively with system B finite-dimensional and the alphabet $\mathcal{X}$ countable, $r(x)$ and $s(x)$ are probability distributions, $\left\{\rho^x_B\right\}_x$ and $\left\{\sigma^x_B\right\}_x$ are sets of states, and

$$\varepsilon \geq \frac{1}{2}\left\|\rho_{XB} - \sigma_{XB}\right\|_1 . \tag{32}$$

Proof Recall that the conditional entropy of a bipartite state ρ_{LM} acting on a separable Hilbert space, with $H(L)_\rho < \infty$, is defined as [21]

$$H(L|M)_\rho := H(L)_\rho - I(L; M)_\rho, \tag{33}$$

where the mutual information is given in terms of the relative entropy $D(\omega\|\tau)$ [22,23] of states ω and τ as

$$I(L; M)_\rho := D(\rho_{LM}\|\rho_L \otimes \rho_M), \tag{34}$$

$$D(\omega\|\tau) := \frac{1}{\ln 2}\sum_{x,y}\left|\langle\phi_x|\psi_y\rangle\right|^2\left[\lambda_x \ln(\lambda_x/\mu_y) + \mu_y - \lambda_x\right], \tag{35}$$

and spectral decompositions of states ω and τ are given by

$$\omega = \sum_x \lambda_x|\phi_x\rangle\langle\phi_x|, \quad \tau = \sum_y \mu_y|\psi_y\rangle\langle\psi_y|. \tag{36}$$

Let us first verify that the formula in (33) reduces to that in (29). Evaluating the formulas in (34) and (35) for the case of interest (the state ρ_{XB} in (31)), while taking spectral decompositions of ρ_{XB} and $\rho_X \otimes \rho_B$ as

$$\rho_{XB} = \sum_{x\in\mathcal{X}} r(x)|x\rangle\langle x|_X \otimes \sum_{y\in\mathcal{Y}} r(y|x)|\phi^{y,x}\rangle\langle\phi^{y,x}|_B, \tag{37}$$

$$\rho_X \otimes \rho_B = \sum_{x'\in\mathcal{X}} r(x')|x'\rangle\langle x'|_X \otimes \sum_{z\in\mathcal{Z}} q(z)|\psi_z\rangle\langle\psi_z|_B, \tag{38}$$

with $\mathcal{X}$ countable, $\mathcal{Y}$ and $\mathcal{Z}$ finite, we find that

$$\begin{aligned} I(X;B)_\rho &= \frac{1}{\ln 2} \sum_{x,y,z,x'} \left|\left(\langle x'|_X \otimes \langle\psi_z|_B\right)\left(|x\rangle_X \otimes |\phi^{y,x}\rangle_B\right)\right|^2 \\ &\quad \times \left[r(x)r(y|x)\ln\left(\frac{r(x)r(y|x)}{[r(x')q(z)]}\right) + r(x')q(z) - r(x)r(y|x)\right] \qquad (39)\\ &= \frac{1}{\ln 2}\sum_{x,y,z} \left|\langle\psi_z|\phi^{y,x}\rangle_B\right|^2 \\ &\quad \times \left[r(x)r(y|x)\ln\left(\frac{r(x)r(y|x)}{[r(x)q(z)]}\right) + r(x)q(z) - r(x)r(y|x)\right] \qquad (40)\\ &= \frac{1}{\ln 2}\sum_x r(x)\sum_{y,z}\left|\langle\psi_z|\phi^{y,x}\rangle_B\right|^2\left[r(y|x)\ln\left(\frac{r(y|x)}{q(z)}\right) + q(z) - r(y|x)\right]. \qquad (41) \end{aligned}$$

For every $x \in \mathcal{X}$, we find that

$$\begin{aligned} &\sum_{y,z}\left|\langle\psi_z|\phi^{y,x}\rangle_B\right|^2\left[r(y|x)\ln\left(\frac{r(y|x)}{q(z)}\right) + q(z) - r(y|x)\right] \qquad (42)\\ &= \sum_{y,z}\left|\langle\psi_z|\phi^{y,x}\rangle_B\right|^2\left[r(y|x)\ln\left(\frac{r(y|x)}{q(z)}\right)\right] \qquad (43)\\ &= \sum_{y,z}\left|\langle\psi_z|\phi^{y,x}\rangle_B\right|^2\left[r(y|x)\ln(r(y|x))\right] + \sum_{y,z}\left|\langle\psi_z|\phi^{y,x}\rangle_B\right|^2\left[r(y|x)\ln\left(\frac{1}{q(z)}\right)\right] \qquad (44)\\ &= \sum_y\left[r(y|x)\ln(r(y|x))\right] + \sum_z\langle\psi_z|\rho_B^x|\psi_z\rangle\ln\left(\frac{1}{q(z)}\right) \qquad (45)\\ &= -(\ln 2)\,H(\rho_B^x) + \sum_z\langle\psi_z|\rho_B^x|\psi_z\rangle\ln\left(\frac{1}{q(z)}\right). \qquad (46) \end{aligned}$$

Then we find that

$$I(X;B)_\rho = \sum_{x\in\mathcal{X}} r(x)\left[-H(\rho_B^x) + \sum_z \langle\psi_z|\rho_B^x|\psi_z\rangle \log_2\left(\frac{1}{q(z)}\right)\right] \tag{47}$$

$$= -\sum_{x\in\mathcal{X}} r(x)H(\rho_B^x) + \sum_z \langle\psi_z|\left[\sum_x r(x)\rho_B^x\right]|\psi_z\rangle \log_2\left(\frac{1}{q(z)}\right) \tag{48}$$

$$= -\sum_{x\in\mathcal{X}} r(x)H(\rho_B^x) + \sum_z \langle\psi_z|\rho_B|\psi_z\rangle \log_2\left(\frac{1}{q(z)}\right) \tag{49}$$

$$= -\sum_{x\in\mathcal{X}} r(x)H(\rho_B^x) + \sum_z q(z)\log_2\left(\frac{1}{q(z)}\right) \tag{50}$$

$$= -\sum_{x\in\mathcal{X}} r(x)H(\rho_B^x) + H(\rho_B). \tag{51}$$

So finally

$$H(B)_\rho - I(X;B)_\rho = \sum_{x\in\mathcal{X}} r(x)H(\rho_B^x), \tag{52}$$

as expected.

Now, it is known from [21] that the following limit holds

$$\lim_{k\to\infty} H(B|X)_{\rho^k} = H(B|X)_\rho, \tag{53}$$

where

$$\rho_{XB}^k := \mathcal{P}_X^k(\rho_{XB}) := \Pi_X^k\rho_{XB}\Pi_X^k + \frac{\Pi_X^k}{\mathrm{Tr}[\Pi_X^k]} \otimes \mathrm{Tr}_X[(I_X - \Pi_X^k)\rho_{XB}], \tag{54}$$

and $\left\{\Pi_X^k\right\}_k$ is a sequence of finite-dimensional projections strongly converging to the identity. Then by taking the projection $\Pi_X^k := \sum_{x=1}^k |x\rangle\langle x|_X$, we find from (32) and data processing for normalized trace distance with respect to the channel defined in (54) that

$$\varepsilon \geq \frac{1}{2}\left\|\rho_{XB}^k - \sigma_{XB}^k\right\|_1, \tag{55}$$

where $\sigma_{XB}^k := \mathcal{P}_X^k(\sigma_{XB})$. Now applying the uniform continuity bound from Proposition 1 to the finite-dimensional states ρ_{XB}^k and σ_{XB}^k, we arrive at the following inequality holding for all $k \in \mathbb{N}$:

$$\left|H(B|X)_{\rho^k} - H(B|X)_{\sigma^k}\right| \leq \varepsilon\log_2(d_B - 1) + h_2(\varepsilon). \tag{56}$$

Finally applying the limit in (53), we arrive at the statement of the corollary. □

Two intriguing questions remain about continuity of conditional entropy. The first is whether the following inequality could hold

$$\left|H(X|B)_\rho - H(X|B)_\sigma\right| \overset{?}{\leq} \varepsilon \log_2(d_X - 1) + h_2(\varepsilon), \tag{57}$$

where ρ_{XB} and σ_{XB} are the same classical–quantum states from (5). (With the systems in the conditional entropy flipped, we could call these states "quantum–classical" now.) The other question is whether the following inequality could hold for fully quantum states ρ_{AB} and σ_{AB} that satisfy $\frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon$ where $\varepsilon \in (0, 1 - 1/d_A^2]$:

$$\left|H(A|B)_\rho - H(A|B)_\sigma\right| \overset{?}{\leq} \varepsilon \log_2(d_A^2 - 1) + h_2(\varepsilon). \tag{58}$$

This inequality is saturated by an example given in Remark 3 of [14]. These questions were raised during the open problem session at the workshop "Algebraic and Statistical ways into Quantum Resource Theories," held in Banff, Canada, during July 2019. It seems that solving them requires techniques beyond what is currently known.

Acknowledgements I acknowledge support from the National Science Foundation under Grant No. 1714215. I am grateful to an anonymous referee for correcting an error and a typo in a previous version of the manuscript.

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