HOLOMORPHIC QUADRATIC DIFFERENTIALS ON GRAPHS AND THE CHROMATIC POLYNOMIAL

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Abstract. We study "holomorphic quadratic differentials" on graphs. We relate them to the reactive power in an LC circuit, and also to the chromatic polynomial of a graph. Specifically, we show that the chromatic polynomial χ of a graph $\mathcal{G}$, at negative integer values, can be evaluated as the degree of a certain rational mapping, arising from the defining equations for a holomorphic quadratic differential. This allows us to give an explicit integral expression for $\chi(-k)$.

1. Introduction

Let $\mathcal{G} = (V, E)$ be a connected graph together with a specified set $V_b \subset V$ of *boundary vertices*. We let $V_{int} = V - V_b$ be the *interior vertices*. We assume that every edge has at least one interior vertex.

Definition 1. *A* holomorphic quadratic differential (HQD) *on a graph $\mathcal{G} = (V, E)$ is the data consisting of: a function $q : E \to \mathbb{R}$ defined on unoriented edges, i.e. $q_{uv} = q_{vu}$, and a mapping (called* realization*) $z : V \to \mathbb{C}$ to the complex plane, satisfying for every interior vertex v*

$$\sum_u q_{uv} = 0 \tag{1}$$

$$\sum_u \frac{q_{uv}}{z_u - z_v} = 0. \tag{2}$$

We sometimes drop condition (1). While condition (1) is important for geometric applications, since it implies a certain Möbius invariance (see Section 2.1 below), it is less relevant in some applications, notably for the applications to circuits. Since both cases with and without (1) have interesting applications, we will consider both possibilities. (This issue is further discussed in Section 2.1 below.)

As examples of geometric applications, in the case of a surface graph, HQDs (satisfying both (1) and (2)) appear in the study of discrete minimal surfaces (see [9] and Section 2.2). In the case of complete graphs, they arise from energy minimization for points on the sphere (see Section 2.4).

There are several applications of HQDs to circuits. For example, the equation (2) was studied in [1] in the context of the Dirichlet problem on graphs with fixed edge energies. Another application is to LC circuits (circuits with inductors and capacitors): we show in Section 4 that q has a concrete physical interpretation as the *reactive power* in an LC circuit associated to the graph.

Key words and phrases. Laplacian, LCR circuit, chromatic polynomial, quadratic differential.

Even though the definition of HQD involves both the function q and realization z, we refer to q as "the holomorphic quadratic differential", since in many applications the realization z is given *a priori*. In this paper however we are mainly interested in the reverse problem, regarding existence and enumeration of realizations z for a given q.

Problem 2. *Given a function $q : E \to \mathbb{R}$ satisfying (1), and fixed boundary values $z : V_b \to \mathbb{C}$, can we find a realization $z : V \to \mathbb{C}$ satisfying (2) and how many such realizations are there?*

This problem is related to the entropic discriminant studied by Sanyal, Sturmfels and Vinzant [10]. In the case where q is positive everywhere (and thus failing (1)), their result showed that the number of realizations z is equal to the Möbius invariant of an associated matroid.

In Section 5 we show, under a genericity assumption on q and for certain special boundary conditions, that the number of solutions to Problem 2 is precisely $|\chi_{\mathcal{G}}(-k)|$ where $\chi_{\mathcal{G}}$ is the chromatic polynomial of $\mathcal{G}$. This gives a novel way to compute $\chi_{\mathcal{G}}(-k)$ as the number of solutions over $\mathbb{C}$ to a system of rational equations. From work in [1] it also shows that $|\chi_{\mathcal{G}}(-k)|$ enumerates a certain set of acyclic orientations of a related graph. This enumeration is closely related to the one given by Stanley in [11], who also enumerated $\chi_{\mathcal{G}}(-k)$ with a set of orientation/coloring pairs. A bijection is given below.

We give an explicit integral formula for $\chi_{\mathcal{G}}(-k)$ in Theorem 8 below. Computing $\chi_{\mathcal{G}}(k)$ is generally #P-complete; for $k \geq 3$, even approximating $\chi_{\mathcal{G}}(k)$ to within a constant factor (in polynomial time) is known to be NP-hard [6]. In principle one can approximate $\chi_{\mathcal{G}}$ at negative integer values using our integral formula (12) which has positive integrand. However since the integrand is oscillatory it is not clear how quickly one can approximate this integral for general graphs.

2. Background

The term "holomorphic quadratic differential" in the continuous setting is a tool used in Teichmüller theory; it is an object on a Riemann surface which in a local complex coordinate z has the form $\phi(z)dz^2$. Such differentials define the cotangent space to the Teichmüller space (or to the moduli space) of a Riemann surface. As such one can think of a holomorphic quadratic differential as an *infinitesimal change in conformal structure.*

In the discrete setting, HQDs on surface graphs were introduced in [9]. For a triangulated disk in the plane, it was shown there that there is a one-to-one correspondence between holomorphic quadratic differentials, infinitesimal deformations of circle packings and discrete minimal surfaces.

2.1. **Möbius invariance.** Given a realization (z_v) of $\mathcal{G}$ with HQD (q_{uv}), and a Möbius transformation $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the tuple $(\phi(z_v))_{v \in V}$ is also a realization with the same HQD. To see this, note that it is true when $\phi(z) = az + b$ is affine, so it suffices to prove when $\phi(z) = 1/z$ (since every Möbius transformation is a composition of affine transformations and inversions).

However when $\phi(z) = 1/z$ we can write

$$\begin{aligned}\sum_u \frac{q_{uv}}{\phi(z_u) - \phi(z_v)} &= -z_v \sum_u \frac{q_{uv} z_u}{z_u - z_v} \\ &= -z_v \sum_u \frac{q_{uv}(z_u - z_v + z_v)}{z_u - z_v} \\ &= -z_v \left(\sum_u q_{uv} + z_v \sum_u \frac{q_{uv}}{z_u - z_v} \right) \\ &= 0.\end{aligned}$$

As a special case, suppose $\mathcal{G}$ has a boundary vertex v_0 attached to *every* interior vertex, with value $z(v_0) = z_0$ in some realization. Compose this realization with a Möbius transformation taking z_0 to ∞. Then we obtain a realization of the smaller graph $\mathcal{G} \setminus \{v_0\}$ satisfying (2), but without the condition (1). Conversely let q be an *arbitrary* function on the edges of $\mathcal{G} \setminus \{v_0\}$ and find a realization of $\mathcal{G} \setminus \{v_0\}$ satisfying (2). Then on $\mathcal{G}$ defining q_{uv_0} for each u so that (1) is satisfied, we find a realization of $\mathcal{G}$ with $z_0 = \infty$.

2.2. **Surface graphs.** In the case where $\mathcal{G}$ is a cell decomposition of a surface, Problem 2 is equivalent to finding a polyhedral surface in space (with combinatorics dual to $\mathcal{G}$) with prescribed curvature on edges

$$q_{uv} = \ell_{uv} \tan \frac{\alpha_{uv}}{2}$$

where ℓ and α denote edge lengths and dihedral angles, respectively. Indeed, the dihedral angles of such a surface are determined by the face normals, whose stereographic projection yields a realization z in the plane satisfying (2). The condition $\sum_v q_{uv} = 0$ indicates that the "discrete mean curvature" on each face vanishes and thus the polyhedral surface is a *discrete minimal surface* [8].

2.3. **HQDs with a given realization.** Note that not every realization $(z_v)_{v \in V}$ of a graph possesses a holomorphic quadratic differential q. For a fixed realization (z_v), finding a holomorphic quadratic differential (q_{uv}) involves solving a system of linear equations with E unknowns and $3V_{int}$ constraints, and $E - 3V_{int}$ can be positive or negative. Even when it is negative, however, there may be nontrivial solutions. For example, realizations of the $\mathbb{Z}^2$-lattice equipped with $q = 1$ on horizontal and $q = -1$ on vertical edges include *orthogonal circle patterns*, see [3].

2.4. **Complete graphs.** Although we focus on generic HQDs and realizations z in the complex plane in Section 4 and 5, it should be pointed out that realizations in $\mathbb{R}^d$ for $d > 2$ are also interesting.

Problem 2 (extension): Given a function $q : E \to \mathbb{R}$ satisfying (1) and fixed boundary values $f : V_b \to \mathbb{R}^d \cup \{\infty\}$, can we find a realization $f : V \to \mathbb{R}^d \cup \{\infty\}$ satisfying

$$\sum_u \frac{q_{uv}}{|f_u - f_v|^2}(f_u - f_v) = 0 \quad \forall v \in V_{int} \tag{3}$$

and how many such realizations are there?

Such HQDs can arise from energy-minimizing point configurations on the sphere. Imagine that a set of particles are distributed on the sphere and interact with each other via a pairwise potential. What configurations minimize the total energy? Such problems appear in many optimization problems in the physical and biological sciences. We use the *logarithmic energy*.
Question: Find a configuration of N points $f_i \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ that minimizes the logarithmic energy

$$E_{\log}(f_1, f_2, \dots, f_N) = -\sum_{u \neq v} \log |f_u - f_v|. \tag{4}$$

Even when $d = 3$, the exact determination of the set of optimal configurations is known only for a handful of cases, since the number of critical configurations grows exponentially with the number of particles N. However, critical configurations on the sphere correspond exactly to realizations that carry a specific HQD:

Proposition 3. *Let $\mathcal{G} = (V, E)$ be the graph obtained by connecting all the vertices of the complete graph K_N to two boundary vertices v_0, v_∞. A configuration $f_1, f_2, \dots, f_N \in \mathbb{R}^3$ is a critical configuration of the logarithmic energy on a sphere centered at the origin if and only if the configuration together with boundary vertices $f_0 := (0,0,0)$, $f_\infty := \infty$ has a holomorphic quadratic differential of the form*

$$q = \begin{cases} 1 & \text{on } E(K_N) \\ -(N-1)/2 & \text{on } E(\mathcal{G}) - E(K_N) \end{cases}$$

Proof. Suppose $f_1, f_2 \dots f_N$ are points on a sphere that form a critical configuration for the logarithmic energy. Then by the method of Lagrange multipliers, we have for each $v = 1, 2, \dots N$

$$\lambda_v \frac{f_v}{|f_v|^2} = \sum_{u \neq v} \frac{f_u - f_v}{|f_u - f_v|^2}$$

for some $\lambda_v \in \mathbb{R}$ and the summation is over all possible values u in $\{1, 2, \dots, N\}$. Taking the dot product with f_v on both sides, we have $\lambda_v = -(N-1)/2$. Thus the configuration of points together with f_0 and f_∞ carries the HQD as described.

Conversely, suppose $f_1, f_2 \dots f_N \in \mathbb{R}^3$ together with boundary vertices f_0, f_∞ carry the HQD as described. We need to show that $f_1, f_2 \dots f_N \in \mathbb{R}^3$ lies on a sphere centered at the origin $f_0 = (0,0,0)$. To see this, for each v we have

$$0 = -\frac{N-1}{2} \frac{f_0 - f_v}{|f_0 - f_v|^2} + \sum_{u \neq v} \frac{f_u - f_v}{|f_u - f_v|^2}.$$

Taking the dot product with f_v, we deduce for each v

$$0 = \sum_{u \neq v} \frac{|f_u|^2 - |f_v|^2}{|f_u - f_v|^2}.$$

This implies that $|f_1| = |f_2| = \cdots = |f_N|$: otherwise there would be a vertex f_v having the maximal distance from the origin, and the sum on the right hand would be strictly negative, a contradiction. □

See [5] for more details on point configurations under the logarithmic energy.

3. Preliminaries

3.1. **Space of q-values on a graph.**

Lemma 4. *When $\mathcal{G}$ has a boundary and is connected, the space of solutions to (1) is of dimension $|E| - |V_{int}|$.*

Proof. In fact we prove slightly more, that the conclusion holds even if $\mathcal{G}$ is disconnected, but each vertex is connected to some boundary vertex. The condition (1) says that q is in the kernel of the incidence matrix $M : \mathbb{R}^E \to \mathbb{R}^{V_{int}}$. So it suffices to show that M has full rank $|V_{int}|$, that is, that M is surjective. Given $\vec{v} \in \mathbb{R}^{V_{int}}$ let us find $q \in \mathbb{R}^E$ with $Mq = \vec{v}$. It suffices to take q with support on a *wired spanning tree* T of $\mathcal{G}$, that is a spanning forest every component of which contains a unique boundary vertex (equivalently, a spanning tree of the graph in which all boundary vertices of $\mathcal{G}$ have been identified). Now proceed by induction on the number of internal vertices: if T has an internal leaf i, then q on its edge is determined by v_i; remove this leaf and vertex and finish by induction. □

3.2. **Graph reductions.** We have assumed that $\mathcal{G}$ is a simple graph; if $\mathcal{G}$ has self-loops then equation (2) is not defined.

We can allow multiple edges. However if $\mathcal{G}$ has multiple edges then a realization of $\mathcal{G}$ gives a realization of the graph $\mathcal{G}'$ in which multiple edges have been replaced by single edges, with the q-value on the new edge being the sum of the q-values on the multiple edges.

Suppose $\mathcal{G}$ has an interior vertex v of degree 2, with neighbors u, w. Consider a realization (q, z). Then at v we have (when $q_{uv} \neq 0$)

$$\frac{q_{uv}}{z_v - z_u} - \frac{q_{uv}}{z_v - z_w} = 0,$$

or $z_u = z_w$. In particular a realization of $\mathcal{G}$ is also a realization of $\mathcal{G}'$, the graph in which edge uv and wv have been contracted (and the values q_{uv}, q_{wv} have been discarded). It is therefore convenient to assume that interior vertices of $\mathcal{G}$ have degree at least 3.

3.3. **Example with a family of realizations.** When $\mathcal{G}$ is bipartite, and all boundary vertices are on the same side of the bipartition (say, all boundary vertices are white), then Problem 2 for certain nongeneric q can have a one or more parameter family of solutions.

Consider for example the graph of Figure 1, with boundary vertices v_1, v_2, v_3 (marked $1, 2, 3$ in the diagram). Suppose that q satisfies (1) at all vertices including the boundary vertices. Then we claim that for fixed boundary values z_1, z_2, z_3, equations (2) have a

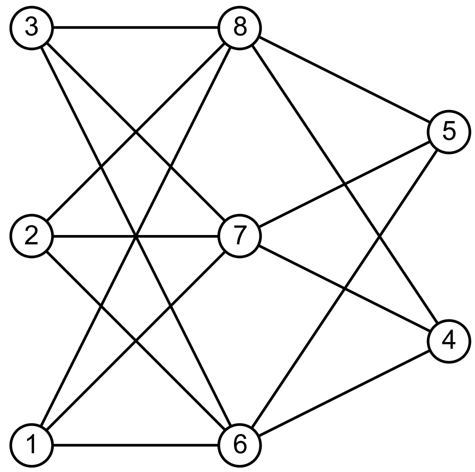

Figure 1. A graph with a continuous family of realizations.

solution in which $z_6 = z_7 = z_8 = b$ for any value of $b \neq z_1, z_2, z_3$. The equations at v_4 and v_5 drop out when $z_6 = z_7 = z_8$, and the equations at v_6, v_7, v_8 are

$$\frac{q_{64}}{b-z_4} + \frac{q_{65}}{b-z_5} + \frac{q_{61}}{b-z_1} + \frac{q_{62}}{b-z_2} + \frac{q_{63}}{b-z_3} = 0 \tag{5}$$

$$\frac{q_{74}}{b-z_4} + \frac{q_{75}}{b-z_5} + \frac{q_{71}}{b-z_1} + \frac{q_{72}}{b-z_2} + \frac{q_{73}}{b-z_3} = 0 \tag{6}$$

$$\frac{q_{84}}{b-z_4} + \frac{q_{85}}{b-z_5} + \frac{q_{81}}{b-z_1} + \frac{q_{82}}{b-z_2} + \frac{q_{83}}{b-z_3} = 0 \tag{7}$$

and the last is a consequence of the first two (since $q_{6j}+q_{7j}+q_{8j} = 0$ for $j = 1, 2, 3, 4, 5$). Given b we can solve (5) and (6) for z_4, z_5 on condition that $q_{64}q_{75} - q_{65}q_{74} \neq 0$, since they are linear in $\frac{1}{b-z_4}, \frac{1}{b-z_5}$.

4. LC circuits

We show in this section that a holomorphic quadratic differential (satisfying (2) but not necessarily (1)) can be regarded as the *reactive power* in an AC circuit consisting of inductors and capacitors, where the z_v's represent the complex voltages.

For information on RLC circuits see [7]. An *RLC circuit* is an electric circuit consisting of inductors (represented by an L value, or inductance value), capacitors (represented by a C value, or capacitance value) and resistors (represented by an R value, or resistance value). The L, C, R values are positive real numbers. Mathematically an RLC circuit is a graph with boundary, on each edge of which we have some subset of an inductor, a capacitor, or a resistor, connected in series. For such a ciruit we consider a certain *driving voltage* applied at the boundary vertices. This voltage is a real function depending on time, and in the present case will be a sinusoidal wave with a common frequency ω but various phases at different boundary vertices (the AC in "AC circuit" means alternating current, by which we mean there is a sinusoidal driving voltage of this type). This driving voltage will induce voltages at each interior vertex, and currents on edges, according to the usual Kirchhoff circuit laws which are explained below.

An LC circuit is an RLC circuit without any resistors. However to explain the concept of real power and reactive power, we assume for the moment that there are resistors in the circuit as well.

We consider a circuit with the combinatorics of the graph $\mathcal{G} = (V, E)$. The voltages $\{U(t)\}$ at the boundary vertices are imposed and satisfy:

$$U_v(t) = \mathrm{Re}(u_v e^{i\omega t})$$

for some $u : V_b \to \mathbb{C}$ and fixed real ω. Let us discuss how to compute the voltages at the interior vertices; these will have the same form for some function $u : V \to \mathbb{C}$.

Under a sinusoidal waveform, associated to a capacitor with capacitance $C > 0$, an inductor with inductance $L > 0$ and a resistor $R > 0$ is an *impedance*

$$\begin{aligned} Z_{capacitor} =& \frac{1}{i\omega C} \\ Z_{inductor} =& i\omega L \\ Z_{resistor} =& R. \end{aligned}$$

The impedance on an edge is by definition the sum of the impedances of the individual elements. We thus have $Z : E \to \mathbb{C}$ defined on unoriented edges. Note that if no resistor is present, Z is pure imaginary.

We define at each vertex the *complex voltage* $\mathbf{U} := ue^{i\omega t}$ and $U = \mathrm{Re}(\mathbf{U})$. Ohm's law relates complex voltage $\mathbf{U}$, complex current $\mathbf{I}$ on edges, and impedance via

$$\mathbf{U}_v - \mathbf{U}_w = \mathbf{I}(e_{vw}) Z_{vw}$$

and Kirchhoff's current law say that at each interior vertex v,

$$\sum_{w \sim v} \mathbf{I}(e_{vw}) = 0,$$

that is, there is no current lost at v. Combining these two laws, we see that $\mathbf{U}$ must satisfy $\Delta \mathbf{U} = 0$ at interior vertices for the associated Laplacian operator $\Delta : \mathbb{C}^V \to \mathbb{C}^V$ defined by

$$(\Delta \mathbf{U})_v = \sum_{w \sim v} \frac{1}{Z_{vw}} (\mathbf{U}_v - \mathbf{U}_w). \tag{8}$$

To solve for $\mathbf{U}$ for given boundary values, we need to be able to invert Δ', where Δ' is Δ restricted to the internal vertices (for the natural basis, Δ' is the submatrix of Δ indexed by internal vertices). Typically Δ' is nonsingular: a frequency ω such that $\det \Delta' = 0$ is called a *resonant frequency*. The set of resonant frequencies is finite and nonempty if there are no resistances, see Theorem 5 below.

For nonresonant frequencies, the function $\mathbf{U}$ exists and is harmonic at interior vertices, that is, $\Delta U(v) = 0$ there. The *real* current through the oriented edge e_{uv} is

$$I(e_{vw}) = \mathrm{Re}(\mathbf{I}(e_{vw})) = \mathrm{Re}(e^{i\omega t}(u_w - u_v)/Z_{vw}).$$

We have

$$U_w - U_v = \mathrm{Re}(e^{i\omega t}(u_w - u_v)) = |u_w - u_v| \cos(\omega t + \phi) \tag{9}$$

$$I(e_{vw}) = \mathrm{Re}(e^{i\omega t}(u_w - u_v)/Z_{vw}) = \frac{|u_w - u_v|}{|Z_{vw}|} \cos(\omega t + \phi + \tilde{\phi}) \tag{10}$$

where $Z_{vw} = |Z_{vw}|e^{-i\tilde{\phi}}$. We now calculate the *instantaneous power*, by multiplying (9) and (10):

$$P_{vw}(t) := (U_w - U_v)I(e_{vw}) = \frac{|u_w - u_v|^2}{2|Z_{vw}|}\Big(\cos(\tilde{\phi})(1+\cos(2\omega t+2\phi)) - \sin\tilde{\phi}\sin(2\omega t+2\phi)\Big).$$

In terms of *complex power*

$$S_{vw} := (\mathbf{U}_w - \mathbf{U}_v)\overline{\mathbf{I}(e_{vw})} = \frac{|u_w - u_v|^2}{|Z_{vw}|}(\cos\tilde{\phi} - i\sin\tilde{\phi}),$$

the instantaneous power can be written as

$$P_{vw}(t) = \frac{1}{2}(\mathrm{Re}\, S_{vw}(1 + \cos(2\omega t + 2\phi)) + \mathrm{Im}\, S_{vw}\ \sin(2\omega t + 2\phi)).$$

The first part of this equation

$$\frac{1}{2}\,\mathrm{Re}\, S_{vw}(1 + \cos(2\omega t + 2\phi))$$

is non-negative and regarded as power dissipated in resistors. In fact if the edge consists of resistors only, it is the same as the instantaneous power since $\mathrm{Im}\, S = 0$. In general, the average power over a period is $\frac{1}{2}\,\mathrm{Re}\, S_{vw}$, which is called *real power*.

On the other hand, the second part of the instantaneous power

$$\frac{1}{2}\,\mathrm{Im}\, S_{vw}\ \sin(2\omega t + 2\phi)$$

does not contribute any energy over a period. If the edge does not contain any resistors, then this term is exactly the instantaneous power as $\mathrm{Re}\, S_{vw} = 0$. The quantity $\mathrm{Im}\, S_{vw}$ is called the *reactive power*.

After this explanation of the physical meaning, we can now go back to our original setting where there are no resistors in the circuit and hence no energy loss. In this case the complex power is purely imaginary

$$S_{vw} = (\mathbf{U}_w - \mathbf{U}_v)\overline{\mathbf{I}(e_{vw})} = |u_w - u_v|^2/\overline{Z}_{vw} =: iq_{vw}$$

where $q : E \to \mathbb{R}$ and $q_{vw} = q_{wv}$. Kirchhoff's current law implies that for every interior vertex v

$$0 = \sum_w \mathbf{I}(e_{vw}) = \sum_w -\frac{iq_{vw}}{\overline{u_w} - \overline{u_v}} e^{i\omega t}$$

which is equivalent to

$$0 = \sum_w \frac{q_{vw}}{u_w - u_v}.$$

On the other hand, the impedance is

$$Z_{vw} = -\frac{|u_w - u_v|^2}{iq_{vw}}$$

and hence the Laplacian operator (8) can be written in terms of the complex voltage u and q

$$(\Delta f)_v = \sum_{w\sim v} \frac{1}{Z_{vw}}(f_v - f_w) = -i\sum_{w\sim v} \frac{q_{vw}}{|u_w - u_v|^2}(f_v - f_w). \tag{11}$$

Given an HQD satisfying (2) (but not necessarily (1)) equation (11) (with u_* there equal to z_*) is a naturally associated Laplacian-type operator. In this case equation (2) says that z is a harmonic function, that is $\Delta(z)(v) = 0$ for internal vertices v. By analogy with the usual Laplacian we define $c_e = \frac{-iq_{uv}}{|z_u - z_v|^2}$ to be the *conductance*, $I_{uv} = \frac{-iq_{uv}}{\bar{z}_u - \bar{z}_v}$ to be the *current*, and q_{uv} to be the *energy* associated to this function. Equation (2) says there is no current lost at an internal vertex.

In standard potential theory the conductances are positive real, so the above should be considered as only an analogy with the real case.

Theorem 5. *In an LC circuit in which not all wired spanning trees have the same number of capacitors, the set of resonant frequencies is finite and nonempty.*

Proof. This argument was provided to us by Robin Pemantle. Recall that a multivariate real polynomial $p(z_1, \ldots, z_n)$ is *stable* if $p(z_1, \ldots, z_n) \neq 0$ whenever $\mathrm{Im}(z_i) > 0$ for each i. See [12] for background on stable polynomials. Stability is preserved under certain operations: setting some variables to be real constants; replacing a variable z_i by cz_i for $c > 0$; setting some variables equal to each other.

On a finite graph $\mathcal{G}$ with variables z_e on edges, let $Z = Z(\{z_e\})$ be the weighted sum of wired spanning trees; it is a homogeneous stable polynomial in the z_e, see [4, Prop. 2.4]: in fact this is one of the canonical examples of multivariate stable polynomials. Moreover the Matrix-Tree Theorem [2] shows that $Z = \det \Delta'$ where Δ' is the Laplacian matrix of $\mathcal{G}$, restricted to the internal vertices, with conductances z_e on edges.

Now suppose $\mathcal{G}$ is an LC circuit. For those edges e with inductors, replace each z_e by $1/L_e$ where L_e is the inductance, and for those edges e with capacitors, replace z_e with z_eC_e where C_e is the capacitance. These operations preserve stability since $C_e, L_e > 0$. Now set all remaining factors of z_e to a single value z. This also preserves stability. The new one-variable polynomial of z, $\tilde{Z}(z)$, has only real roots, by stability (and the fact that it has real coefficients). Moreover these roots must be negative since all coefficients are positive. By our hypothesis that not all spanning trees have the same number of inductors, $\tilde{Z}$ is not a monomial in z, so has at least one negative root. Now replace z with $-\omega^2$, and divide by $(i\omega)^n$ where n is the degree. This modified (Laurent) polynomial $\tilde{Z}(-\omega^2)/(i\omega)^n$ is the determinant of the Laplacian Δ' associated to the HQD (see (8)), and thus its zeros are precisely the resonant frequencies. $\square$

Question. Under what general circumstances will it be the case that (1) holds (at internal vertices) as well? Is there a "physical" meaning for such a circuit?

5. Counting realizations

5.1. **Compatible orientations.** For a graph $\mathcal{G}$ with boundary V_b and a function $z : V_b \to \mathbb{R}$, a *compatible orientation* is an acyclic orientation of $\mathcal{G}$, with no internal sources or sinks, and with no oriented path from a boundary vertex to another boundary vertex of higher or equal u-value.

In [1] it was shown that the number of compatible orientations for a graph with distinct boundary values does not depend on the boundary values themselves (or even their relative order). Moreover there is a bijection between the set of compatible orientations and the number of solutions to (2) under the assumption $q > 0$: here is a statement of the main result of that paper.

Theorem 6 ([1])**.** *Let $\mathcal{G}$ be a graph with boundary, with fixed distinct real boundary values z_{v_i}, and $q : E \to \mathbb{R}_{>0}$. Then any solution $\{z_v\}_{v\in V_{int}}$ to the system (2) is real, and there is a bijection between solutions and compatible orientations, where the orientation of an edge uv is $sign(z_v - z_u)$.*

When the boundary values are distinct *complex* numbers, rather than real numbers, the number of solutions is still equal to the number of compatible orientations, by Zariski density, but there is no obvious bijection in this case.

5.2. **Chromatic polynomial.** Recall that the *chromatic polynomial* $\chi_{\mathcal{G}}(x)$ of a graph $\mathcal{G}$ is a polynomial with the property that for nonnegative integer x, $\chi_{\mathcal{G}}(x)$ is the number of proper colorings of $\mathcal{G}$ with x colors (a proper coloring is one with no two adjacent vertices having the same color).

Theorem 7. *Let $\mathcal{G}_{k+2}$ be obtained from a graph $\mathcal{G}$ without boundary by adding $k+2$ additional vertices $v_0, v_1, \dots, v_{k+1}$, each attached to all vertices of $\mathcal{G}$, and playing the role of the boundary vertices of $\mathcal{G}_{k+2}$. Fix distinct values $z_i \in \mathbb{C}$ at v_i. Then the number $N_{\mathcal{G}_{k+2}}$ of realizations z for fixed generic q satisfying (1) and (2) is $N_{\mathcal{G}_{k+2}} = (-1)^{|V_{int}|}\chi(-k)$ where χ is the chromatic polynomial of $\mathcal{G}$.*

In particular when the boundary has size $k+2 = 2$ there are no realizations.

Proof. As discussed at the end of Section 2.1, we send the boundary value z_{k+1} to ∞ using a Möbius transformation, and assume that q is positive on the remaining graph $\mathcal{G}_{k+1}$, which now has boundary of size $k+1$. Since the number of solutions to (2) does not depend on the exact boundary values $z_0, \dots, z_k$, as long as they vary in a Zariski dense subset, we may assume that the remaining boundary values $z_0, \dots, z_k$ are real and satisfy $z_0 < z_1 < \cdots < z_k$.

By Theorem 6, the number of realizations is equal to the number of compatible orientations of $\mathcal{G}_{k+1}$.

Stanley showed in [11] that for a graph $\mathcal{G}$, $|\chi(-k)|$ counts the number of pairs (σ, C) where $C : V \to \{1, 2, \dots k\}$ is a (not necessarily proper) coloring of $\mathcal{G}$ with k colors and σ is an acyclic orientation of $\mathcal{G}$ which is *compatible with C* in the sense that an edge uv is directed from u to v if $C(u) > C(v)$ (for the prescribed total order on the colors). Edges uv for which $C(u) = C(v)$ can have either orientation, subject to acyclicity.

Here is a bijection between such pairs (σ, C) and compatible orientations of $\mathcal{G}_{k+1}$; we thank the referee for this argument. Let τ be a compatible orientation of $\mathcal{G}_{k+1}$. Note that v_0 is a sink and the restriction of τ to $\mathcal{G}$ is acyclic. Let V_1 be those vertices of $\mathcal{G}$ which can be reached from v_1 going positively along τ. Let V_2 be those vertices of $\mathcal{G} \setminus V_1$ which can be reached from v_2 going positively along τ, and so on for $V_3, \dots, V_k$. For $1 \le i \le k$, color vertices in V_i with the i-th color. This yields a coloring $C : V(\mathcal{G}) \to \{1, 2, \dots k\}$ which is compatible with σ in the above sense. This gives a pair (σ, C) where σ is just τ restricted to $\mathcal{G}$. To get the inverse map from such pairs to compatible orientations, orient the remaining edges so that edges at v_i are oriented away from v_i if and only if their other endpoint is in V_i. □

5.3. **Integral formulation.** We construct an integral whose value is $|\chi_{\mathcal{G}}(-k)|$ as follows. Construct the graph $\mathcal{G}_{k+1}$ as above, and assign it distinct boundary values $x_0, \dots, x_k \in \mathbb{R}$. For any choice of conductances $(c_1, \dots, c_m)$ on the edges of $\mathcal{G}_{k+1}$, let h be the harmonic extension, that is, the solution to the Dirichlet problem with boundary values $x_0, \dots, x_k$. Let $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ be the map from tuples of edge conductances $(c_1, \dots, c_m)$ of $\mathcal{G}_{k+1}$ to tuples of energies $(q_1, \dots, q_m)$ for the harmonic extension: here $q_{uv} = c_{uv}(h(u) - h(v))^2$. In [1] it is shown that the Jacobian of Ψ is $\pm \prod_e \frac{q_e}{c_e}$, and the degree of Ψ as a map on projective space is the number of compatible orientations. We can compute the degree of Ψ by integrating its Jacobian.

Because Ψ is homogeneous and $\Phi^{-1}((0,\infty)^E) \subset (0,\infty)^E$ we can simply integrate $\tilde{\Psi}$ over the simplex Δ_m of (positive) conductances whose sum is 1, where $\tilde{\Psi} = \pi \circ \Psi$ is Ψ projected back to the unit simplex. Since the Jacobian of π is Z^{-m} where $Z = \sum_e q_e$, the Jacobian of $\tilde{\Psi}$ is $\prod_{e=uv} \frac{(h(u)-h(v))^2}{Z}$. We have proved:

Theorem 8. *The chromatic polynomial at $-k$ satisfies*

$$|\chi_{\mathcal{G}}(-k)| = \frac{1}{|\Delta_m|} \int_{\Delta_m} \prod_{e=uv} \frac{(h(u) - h(v))^2}{Z} \, d\mathrm{vol} \tag{12}$$

where the integral is over the m-simplex of conductances on $\mathcal{G}_k$ normalized to sum to 1, h is the harmonic extension with the given conductances and boundary values $x_0, \dots, x_k$, $Z = \sum_e q_e$ is the total Dirichlet energy of h and dvol *is the Lebesgue measure on Δ_m.*

As an example, let $\mathcal{G}$ consist of a single vertex v. Then $\mathcal{G}_3$ is a star with three boundary vertices v_0, v_1, v_2. Assign them boundary values $0, 1, 2$ respectively and let their corresponding edge conductances be c_0, c_1, c_2 which sum to 1. The harmonic extension h then has $h(v) = c_1 + 2c_2$. The above integral is

$$\begin{aligned}
|\chi_{\mathcal{G}}(-2)| &= 2 \int_0^1 \int_0^{1-c_1} \frac{h(v)^2 (h(v)-1)^2 (h(v)-2)^2}{(c_0 h(v)^2 + c_1 (h(v)-1)^2 + c_2 (h(v)-2)^2)^3} dc_2 \, dc_1 \\
&= 2 \int_0^1 \int_0^{1-c_1} \frac{(2c_2 + c_1)^2 (2c_2 + c_1 - 1)^2 (2c_2 + c_1 - 2)^2}{(c_1 - c_1^2 + 4c_2 - 4c_1 c_2 - 4c_2^2)^3} dc_2 \, dc_1 \\
&= 2.
\end{aligned}$$

The value of the integral does not depend on the precise boundary values $0, 1, 2$ we used; can we further simplify this integral by taking particular boundary values, for example when the boundary values are highly skewed?

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