

Measurement errors in the binary instrumental variable model

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SUMMARY

Instrumental variable methods can identify causal effects even when the treatment and outcome are confounded. We study the problem of imperfect measurements of the binary instrumental variable, treatment and outcome. We first consider nondifferential measurement errors, that is, the mismeasured variable does not depend on other variables given its true value. We show that the measurement error of the instrumental variable does not bias the estimate, that the measurement error of the treatment biases the estimate away from zero, and that the measurement error of the outcome biases the estimate toward zero. Moreover, we derive sharp bounds on the causal effects without additional assumptions. These bounds are informative because they exclude zero. We then consider differential measurement errors, and focus on sensitivity analyses in those settings.

Some key words: Complier average causal effect; Misclassification; Noncompliance; Sensitivity analysis; Sharp bound.

1. INTRODUCTION

Instrumental variable methods are powerful tools for causal inference with unmeasured treatment-outcome confounding. Angrist et al. (1996) used potential outcomes to clarify the role of a binary instrumental variable in identifying causal effects. They showed that the classic two-stage least squares estimator is consistent for the complier average causal effect under the monotonicity and exclusion restriction assumptions.

Measurement error, or equivalently misclassification for discrete variables, is common in empirical research. Black et al. (2003) studied the return of a possibly misreported education status. Boatman et al. (2017) studied the effect of a self-reported smoking status. In those settings, the treatments are endogenous and mismeasured. Chalak (2017) considered the measurement error of an instrumental variable. Pierce & VanderWeele (2012) considered a continuous treatment and either a continuous or a binary outcome with measurement errors. The existing literature often relies on modelling assumptions (Schennach, 2007; Pierce & VanderWeele, 2012), auxiliary information (Black et al., 2003; Kuroki & Pearl, 2014; Chalak, 2017; Boatman et al., 2017) or repeated measurements of the unobserved variables (Battistin et al., 2014).

With binary variables, we study all possible scenarios of measurement errors of the instrumental variable, treatment and outcome. With nondifferential measurement errors, we show that the measurement error of the instrumental variable does not result in bias, the measurement error of the treatment moves the estimate away from zero, and the measurement error of the outcome moves the estimate toward zero. This differs

from the result for the total effect (Bross, 1954), where measurement errors of the treatment and outcome both move the estimate toward zero.

For nondifferential measurement errors, we focus on qualitative analysis and nonparametric bounds. For differential measurement errors, we focus on sensitivity analysis. In both cases, we do not impose modelling assumptions or require auxiliary information.

2. NOTATION AND ASSUMPTIONS FOR INSTRUMENTAL VARIABLE ESTIMATION

For unit i , let Z_i denote the treatment assigned, D_i the treatment received and Y_i the outcome. Assume that (Z_i, D_i, Y_i) are all binary, taking values in $\{0, 1\}$. We ignore pretreatment covariates without loss of generality, because all the results hold within strata of covariates. We use potential outcomes to define causal effects. Define the potential values of the treatment received and the outcome as D_{zi} and Y_{zi} if unit i were assigned to treatment arm z ($z = 0, 1$). The observed values are $D_i = Z_i D_{1i} + (1 - Z_i) D_{0i}$ and $Y_i = Z_i Y_{1i} + (1 - Z_i) Y_{0i}$. Angrist et al. (1996) classified the units into four latent strata based on the joint values of (D_{1i}, D_{0i}) . They defined $U_i = a$ if $(D_{1i}, D_{0i}) = (1, 1)$, $U_i = n$ if $(D_{1i}, D_{0i}) = (0, 0)$, $U_i = c$ if $(D_{1i}, D_{0i}) = (1, 0)$ and $U_i = d$ if $(D_{1i}, D_{0i}) = (0, 1)$. The stratum with $U_i = c$ consists of compliers. For notational simplicity we drop the subscript i . We invoke the following assumption for the instrumental variable model.

Assumption 1. Under the instrumental variable model, (a) $Z \perp\!\!\!\perp (Y_1, Y_0, D_1, D_0)$, (b) $D_1 \geq D_0$ and (c) $\text{pr}(Y_1 = 1 \mid U = u) = \text{pr}(Y_0 = 1 \mid U = u)$ for $u = a$ and n .

Assumption 1(a) holds in randomized experiments. Assumption 1(b) means that the treatment assigned has a monotonic effect on the treatment received for all units, which rules out the latent strata $U = d$. Assumption 1(c) implies that the treatment assigned affects the outcome only through the treatment received, which is called exclusion restriction.

Define $\text{RD}_{R|Q} = \text{pr}(R = 1 \mid Q = 1) - \text{pr}(R = 1 \mid Q = 0)$ as the risk difference of Q on R . For example, $\text{RD}_{YD|(1-Z)} = \text{pr}(Y = 1, D = 1 \mid Z = 0) - \text{pr}(Y = 1, D = 1 \mid Z = 1)$. Angrist et al. (1996) showed that the complier average causal effect,

$$\tau_c \equiv E(Y_1 - Y_0 \mid U = c) = \frac{\text{pr}(Y = 1 \mid Z = 1) - \text{pr}(Y = 1 \mid Z = 0)}{\text{pr}(D = 1 \mid Z = 1) - \text{pr}(D = 1 \mid Z = 0)} = \frac{\text{RD}_{Y|Z}}{\text{RD}_{D|Z}},$$

can be identified by the ratio of the risk differences of Z on Y and D if $\text{RD}_{D|Z} \neq 0$.

3. NONDIFFERENTIAL MEASUREMENT ERRORS

Let (Z', D', Y') denote the possibly mismeasured values of (Z, D, Y) . Without the true variables, we use the naive estimator based on the observed variables to estimate τ_c :

$$\tau'_c \equiv \frac{\text{pr}(Y' = 1 \mid Z' = 1) - \text{pr}(Y' = 1 \mid Z' = 0)}{\text{pr}(D' = 1 \mid Z' = 1) - \text{pr}(D' = 1 \mid Z' = 0)} = \frac{\text{RD}_{Y'|Z'}}{\text{RD}_{D'|Z'}}.$$

Assumption 2. All measurement errors are nondifferential: $\text{pr}(D' \mid D, Z', Z, Y, Y') = \text{pr}(D' \mid D)$, $\text{pr}(Y' \mid Y, Z, Z', D, D') = \text{pr}(Y' \mid Y)$ and $\text{pr}(Z' \mid Y, Y', Z, D, D') = \text{pr}(Z' \mid Z)$.

Under Assumption 2, the measurements of the variables do not depend on other variables conditional on the unobserved true variables. We use the sensitivities and specificities to characterize the nondifferential measurement errors:

$$\begin{aligned} \text{SN}_D &= \text{pr}(D' = 1 \mid D = 1), & \text{SP}_D &= \text{pr}(D' = 0 \mid D = 0), & r_D &= \text{SN}_D + \text{SP}_D - 1 \leq 1, \\ \text{SN}_Y &= \text{pr}(Y' = 1 \mid Y = 1), & \text{SP}_Y &= \text{pr}(Y' = 0 \mid Y = 0), & r_Y &= \text{SN}_Y + \text{SP}_Y - 1 \leq 1. \end{aligned}$$

Without measurement errors, $r_D = r_Y = 1$. Assume $r_D > 0$ and $r_Y > 0$, which means that the observed variable is more likely to be 1 if the true variable is 1 rather than 0. We state a simple relationship between τ_c and τ'_c .

THEOREM 1. *Under Assumptions 1 and 2, $\tau_c = \tau'_c \times r_D/r_Y$.*

Theorem 1 shows that measurement errors of Z , D and Y have different consequences. The measurement error of Z does not bias the estimate. The measurement error of D biases the estimate away from zero. The measurement error of Y biases the estimate toward zero. In contrast, measurement errors of the treatment and outcome both bias the estimate toward zero in the total effect estimation (Bross, 1954).

Moreover, the measurement errors of D and Y have mutually independent influences on the estimation of τ_c . Theorem 1 also shows that τ_c and τ'_c have the same sign when $r_D > 0$ and $r_Y > 0$.

4. BOUNDS ON τ_c WITH NONDIFFERENTIAL MEASUREMENT ERRORS

When D or Y is nondifferentially mismeasured, we can identify τ_c if we know r_D and r_Y . Without knowing them, we cannot identify τ_c . Fortunately, the observed data still provide some information about τ_c . We can derive its sharp bounds based on the joint distribution of the observed data. We first introduce a lemma.

LEMMA 1. *Define $SN'_Z = \text{pr}(Z = 1 \mid Z' = 1)$ and $SP'_Z = \text{pr}(Z = 0 \mid Z' = 0)$. Under Assumption 1, given the values of $(SN'_Z, SP'_Z, SN_D, SP_D, SN_Y, SP_Y)$, there is a one-to-one mapping between the set $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = a, n, c\}$ and the set $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$.*

Lemma 1 allows for simultaneous measurement errors of more than one element of (Y, Z, D) . From Lemma 1, given the sensitivities and specificities, we can recover the joint distribution of (Y_z, U, Z) for $z = 0, 1$. Conversely, the conditions $\{0 \leq \text{pr}(Z = z) \leq 1, 0 \leq \text{pr}(U = u) \leq 1, 0 \leq \text{pr}(Y_z = 1 \mid U = u) \leq 1 : z = 0, 1; u = a, n, c\}$ induce sharp bounds on the sensitivities and specificities, which further induce sharp bounds on τ_c . This is a general strategy that we use to derive sharp bounds on τ_c .

First, we discuss the measurement error of Y .

THEOREM 2. *Suppose that $\tau'_c \geq 0$ and only Y is mismeasured with $r_Y > 0$. Under Assumptions 1 and 2, the sharp bounds are $SN_Y \geq M_Y$, $SP_Y \geq 1 - N_Y$ and $\tau'_c \leq \tau_c \leq \tau'_c/(M_Y - N_Y)$, where M_Y and N_Y are the maximum and minimum values of the set*

$$\left\{ \text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 0), \frac{RD_{Y'D|Z}}{RD_{D|Z}}, \frac{RD_{Y'(1-D)|(1-Z)}}{RD_{D|Z}} \right\}.$$

We can obtain the bounds under $\tau'_c < 0$ by replacing Y with $1 - Y$ and Y' with $1 - Y'$ in Theorem 2. Thus, we only consider $\tau'_c \geq 0$ in Theorem 2 and the theorems in later parts of the paper. In Theorem 2, the lower bounds on SN_Y and SP_Y must be smaller than or equal to 1, i.e., $M_Y \leq 1$ and $1 - N_Y \leq 1$. These two inequalities further imply the following corollary on the testable conditions of the instrumental variable model with the measurement error of Y .

COROLLARY 1. *Suppose that only Y is mismeasured with $r_Y > 0$. Under Assumptions 1 and 2,*

$$\begin{aligned} \text{pr}(Y' = y, D = 1 \mid Z = 1) &\geq \text{pr}(Y' = y, D = 1 \mid Z = 0) \quad (y = 0, 1), \\ \text{pr}(Y' = y, D = 0 \mid Z = 0) &\geq \text{pr}(Y' = y, D = 0 \mid Z = 1) \quad (y = 0, 1). \end{aligned}$$

The conditions in Corollary 1 are all testable with observed data (Z, D, Y') , and they are the same under $\tau'_c \geq 0$ and $\tau'_c < 0$. Balke & Pearl (1997) derived the same conditions as in Corollary 1 without

the measurement error of Y . Wang et al. (2017) proposed statistical tests for these conditions. From Corollary 1, the nondifferential measurement error of Y does not weaken the testable conditions of the binary instrumental variable model.

Second, we discuss the measurement error of D .

THEOREM 3. Suppose that $\tau'_c \geq 0$ and only D is mismeasured with $r_D > 0$. Under Assumptions 1 and 2, the sharp bounds are $M_D \leq \text{SN}_D \leq U_D$, $1 - N_D \leq \text{SP}_D \leq 1 - V_D$ and $\tau'_c \times (M_D - N_D) \leq \tau_c \leq \tau'_c \times (U_D - V_D)$, where

$$\begin{aligned} M_D &= \max \left\{ \max_{z=0,1} \text{pr}(D' = 1 \mid Z = z), \max_{y=0,1} \text{pr}(D' = 1 \mid Y = y, Z = 1), \frac{\text{RD}_{(1-Y)D' \mid (1-Z)}}{\text{RD}_{Y \mid Z}} \right\}, \\ N_D &= \min \left\{ \min_{z=0,1} \text{pr}(D' = 1 \mid Z = z), \min_{y=0,1} \text{pr}(D' = 1 \mid Y = y, Z = 0), \frac{\text{RD}_{YD' \mid Z}}{\text{RD}_{Y \mid Z}} \right\}, \\ U_D &= \min \left\{ 1, \frac{\text{RD}_{YD' \mid Z}}{\text{RD}_{Y \mid Z}} \right\}, \quad V_D = \max \left\{ 0, \frac{\text{RD}_{(1-Y)D' \mid (1-Z)}}{\text{RD}_{Y \mid Z}} \right\}. \end{aligned}$$

With a mismeasured D , Ura (2018) derived sharp bounds with and without Assumption 2. The former bounds are equivalent to ours, but the latter bounds are wider. In Theorem 3, the lower bounds on SN_D and SP_D must be smaller than or equal to their upper bounds. This further implies the following corollary on the testable conditions of the binary instrumental variable model with the measurement error of D .

COROLLARY 2. Suppose that $\tau'_c \geq 0$ and only D is mismeasured with $r_D > 0$. Under Assumptions 1 and 2,

$$\begin{aligned} \text{pr}(Y = 1, D' = 1 \mid Z = 1) &\geq \text{pr}(Y = 1, D' = 1 \mid Z = 0), \\ \text{pr}(Y = 0, D' = 0 \mid Z = 0) &\geq \text{pr}(Y = 0, D' = 0 \mid Z = 1), \\ \text{pr}(D' = 1 \mid Y = y, Z = 1) &\leq \text{RD}_{YD' \mid Z} / \text{RD}_{Y \mid Z} \quad (y = 0, 1), \\ \text{pr}(D' = 1 \mid Y = y, Z = 0) &\geq \text{RD}_{(1-Y)D' \mid (1-Z)} / \text{RD}_{Y \mid Z} \quad (y = 0, 1). \end{aligned} \tag{1}$$

We can obtain the conditions under $\tau'_c < 0$ by replacing Y with $1 - Y$. In the Supplementary Material, we show that the conditions in Corollary 2 are weaker than those in Balke & Pearl (1997). Thus, the nondifferential measurement error of D weakens the testable conditions of the binary instrumental variable model.

It is complicated to obtain closed-form bounds with simultaneous measurement errors of more than one element of (Z, D, Y) . In those cases, we can numerically calculate the sharp bounds on τ_c ; the details are given in the Supplementary Material.

5. RESULTS UNDER STRONG MONOTONICITY

Sometimes, units in the control group have no access to the treatment. This is called the one-sided noncompliance problem with the following assumption.

Assumption 3. For all individual i , $D_{0i} = 0$.

Under strong monotonicity, we have only two strata with $U = c$ and $U = n$. Theorem 1 still holds. Moreover, strong monotonicity sharpens the bounds in § 4.

First, we consider the measurement error of Y . We have that

$$\tau'_c = \{\text{pr}(Y' = 1 \mid Z = 1) - \text{pr}(Y' = 1 \mid Z = 0)\} / \text{pr}(D = 1 \mid Z = 1), \quad \tau_c = \tau'_c / r_Y.$$

THEOREM 4. Suppose that $\tau'_c \geq 0$ and only Y is mismeasured with $r_Y > 0$. Under Assumptions 1–3, the sharp bounds are $\text{SP}_Y \geq 1 - N_Y^m$, $\text{SN}_Y \geq M_Y^m$ and $\tau'_c \leq \tau_c \leq \tau'_c / (M_Y^m - N_Y^m)$, where

$$\begin{aligned} N_Y^m &= \min\{\text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 1) - \tau'_c\}, \\ M_Y^m &= \max\{\text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 1)\}. \end{aligned}$$

Second, we consider the measurement error of D . Subtle issues arise. When D is mismeasured, $\text{pr}(D' = 0 \mid D = 0, Z = 0) = 1$ is known, and $\text{pr}(D' = 1 \mid D = 1, Z = 0)$ is not well defined. Thus, Assumption 2 of nondifferential measurement error is implausible, so we need to modify it. Define

$$\text{SN}_D^1 = \text{pr}(D' = 1 \mid D = 1, Z = 1), \quad \text{SP}_D^1 = \text{pr}(D' = 0 \mid D = 0, Z = 1)$$

as the sensitivity and specificity conditional on $Z = 1$. We have

$$\tau'_c = \text{RD}_{Y|Z} / \{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)\}, \quad \tau_c = \tau'_c \times (\text{SN}_D^1 + \text{SP}_D^1 - 1).$$

THEOREM 5. Suppose that $\tau'_c \geq 0$, only D is mismeasured, and

$$\text{pr}(D' = 1 \mid Y = 1, Z = 1) \geq \text{pr}(D' = 1 \mid Y = 0, Z = 1). \quad (2)$$

Under Assumptions 1 and 3, the sharp bounds are

$$\begin{aligned} \text{SP}_D^1 &\geq 1 - \text{pr}(D' = 1 \mid Y = 0, Z = 1), \quad \text{SN}_D^1 \geq \text{pr}(D' = 1 \mid Y = 1, Z = 1), \\ \text{SN}_D^1 &\leq \{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \times \text{pr}(Y = 1 \mid Z = 0)\} / \text{RD}_{Y|Z}, \\ \text{pr}(D' = 1 \mid Y = 1, Z = 1) \times \text{RD}_{Y|Z} / \text{pr}(D' = 1 \mid Z = 1) &\leq \tau_c \leq 1. \end{aligned}$$

Unlike Theorems 2–4, the upper bound on SN_D^1 depends on SP_D^1 in Theorem 5. The condition in (2) is not necessary for obtaining the bounds, but it helps to simplify the expression of the bounds. It holds in our applications in § 7. We give the bounds on τ_c without (2) in the Supplementary Material. The upper bound on τ_c is not informative in Theorem 5, but, fortunately, we are more interested in the lower bound in this case.

It is complicated to obtain closed-form bounds with simultaneous measurement errors of more than one element of (Z, D, Y) . In those cases, we can numerically calculate the sharp bounds; see the Supplementary Material for more detail.

6. SENSITIVITY ANALYSIS FORMULAS WITH DIFFERENTIAL MEASUREMENT ERRORS

Nondifferential measurement error is not plausible in some cases. Section 5 shows that under strong monotonicity, the measurement error of D cannot be nondifferential because it depends on Z in general. In this section we consider differential measurement errors of D and Y without requiring strong monotonicity. We do not consider the differential measurement error of Z , because the measurement of Z often precedes (D, Y) and its measurement error is unlikely to depend on later variables.

We first consider the differential measurement error of Y .

THEOREM 6. Suppose that only Y is mismeasured. Define

$$\text{SN}_Y^1 = \text{pr}(Y' = 1 \mid Y = 1, Z = 1), \quad \text{SN}_Y^0 = \text{pr}(Y' = 1 \mid Y = 1, Z = 0), \quad (3)$$

$$\text{SP}_Y^1 = \text{pr}(Y' = 0 \mid Y = 0, Z = 1), \quad \text{SP}_Y^0 = \text{pr}(Y' = 0 \mid Y = 0, Z = 0). \quad (4)$$

Under Assumption 1,

$$\tau_c = \left\{ \frac{\text{pr}(Y' = 1 \mid Z = 1) - (1 - \text{sp}_Y^1)}{\text{sn}_Y^1 + \text{sp}_Y^1 - 1} - \frac{\text{pr}(Y' = 1 \mid Z = 0) - (1 - \text{sp}_Y^0)}{\text{sn}_Y^0 + \text{sp}_Y^0 - 1} \right\} / \text{RD}_{D \mid Z}.$$

Theorem 6 allows the measurement error of Y to depend on D , but the formula for τ_c only needs the sensitivities and specificities in (3) and (4) conditional on (Z, Y) . It is possible that τ'_c is positive, but τ_c is negative. For example, if $\text{sn}_Y^1 + \text{sp}_Y^1 = \text{sn}_Y^0 + \text{sp}_Y^0 > 1$ and $\text{sp}_Y^0 - \text{sp}_Y^1 > \text{RD}_{Y' \mid Z}$, then τ_c and τ'_c have different signs.

We then consider the differential measurement error of D .

THEOREM 7. Suppose that only D is mismeasured. Define

$$\text{sn}_D^1 = \text{pr}(D' = 1 \mid D = 1, Z = 1), \quad \text{sn}_D^0 = \text{pr}(D' = 1 \mid D = 1, Z = 0), \quad (5)$$

$$\text{sp}_D^1 = \text{pr}(D' = 0 \mid D = 0, Z = 1), \quad \text{sp}_D^0 = \text{pr}(D' = 0 \mid D = 0, Z = 0). \quad (6)$$

Under Assumption 1,

$$\tau_c = \text{RD}_{Y \mid Z} / \left\{ \frac{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{sp}_D^1)}{\text{sn}_D^1 + \text{sp}_D^1 - 1} - \frac{\text{pr}(D' = 1 \mid Z = 0) - (1 - \text{sp}_D^0)}{\text{sn}_D^0 + \text{sp}_D^0 - 1} \right\}.$$

Theorem 7 allows the measurement error of D to depend on Y , but the formula for τ_c only needs the sensitivities and specificities (5) and (6) conditional on Z . Similar to the discussion after Theorem 6, it is possible that τ'_c and τ_c have different signs.

Based on Theorems 6 and 7, if we know or can consistently estimate the sensitivities and specificities in (3)–(6), then we can consistently estimate τ_c ; if we only know the ranges of the sensitivities and specificities, then we can obtain bounds on τ_c .

For simultaneous differential measurement errors of D and Y , the formula for τ_c depends on too many sensitivity and specificity parameters. Thus we omit the discussion.

7. ILLUSTRATIONS

We give three examples and present further details in the Supplementary Material.

Example 1. IMPROVE trial investigators (2014) assessed the effectiveness of emergency endovascular versus open surgical repair strategies for patients with a clinical diagnosis of ruptured aortic aneurism. Patients were randomized to either the emergency endovascular or the open repair strategy. The primary outcome was the survival status after 30 days. Let Z be the treatment assigned, with $Z = 1$ for the endovascular strategy and $Z = 0$ for open repair. Let D be the treatment received. Let Y be the survival status, with $Y = 1$ for dead and $Y = 0$ for alive. If none of the variables are mismeasured, then the estimate of τ_c is 0.131 with 95% confidence interval $(-0.036, 0.298)$ including 0. If only Y is nondifferentially mismeasured then $0.382 \leq \text{sp}_Y \leq 1$, $0.759 \leq \text{sn}_Y \leq 1$ and $0.141 \leq r_Y \leq 1$, and thus $0.131 \leq \tau_c \leq 0.928$ from Theorem 2. If only D is nondifferentially mismeasured, then $0.658 \leq \text{sn}_D \leq 1$, $0.908 \leq \text{sp}_D \leq 1$ and $0.566 \leq r_D \leq 1$, and thus $0.074 \leq \tau_c \leq 0.131$ from Theorem 3.

Example 2. In Hirano et al. (2000), physicians were randomly selected to receive a letter encouraging them to inoculate patients at risk of flu. The treatment was the flu shot, and the outcome an indicator for flu-related hospital visits. However, some patients did not comply with their assignments. Let Z be the indicator of encouragement to receive the flu shot, with $Z = 1$ if the physician receives the encouragement letter and $Z = 0$ otherwise. Let D be the treatment received. Let Y be the outcome, with $Y = 0$ for a flu-related hospitalization during the winter and $Y = 1$ otherwise. If none of the variables are mismeasured, then the estimate of τ_c is 0.116 with 95% confidence interval $(-0.061, 0.293)$ including 0. If only Y is

nondifferentially mismeasured then, from Theorem 2, $SP_Y \geq 1.004 > 1$, and thus the assumptions of the instrumental variable do not hold. If only D is nondifferentially mismeasured then, from Theorem 3, $SN_D \geq 8.676 > 1$, and thus the assumptions of the instrumental variable do not hold either. We reject the testable condition (1) required by both Corollaries 1 and 2 with p -value smaller than 10^{-9} . As a result, the nondifferential measurement error of D or Y cannot explain the violation of the instrumental variable assumptions in this example.

Example 3. Sommer & Zeger (1991) studied the effect of vitamin A supplements on infant mortality in Indonesia. The vitamin supplements were randomly assigned to villages, but some of the individuals in villages assigned to the treatment group did not receive them. Strong monotonicity holds, because the individuals assigned to the control group had no access to the supplements. Let Y denote a binary outcome, with $Y = 1$ if the infant survives to 12 months and $Y = 0$ otherwise. Let Z denote the indicator of assignment to the supplements. Let D denote the actual receipt of the supplements. If none of the variables are mismeasured, then the estimate of τ_c is 0.003 with 95% confidence interval (0.001, 0.005) excluding 0. If only Y is nondifferentially mismeasured then $SP_Y \geq 0.014$ and $SN_Y \geq 0.999$, and thus $0.003 \leq \tau_c \leq 0.252$ from Theorem 4. The 95% confidence interval is (0.001, 1). If only D is nondifferentially mismeasured, then $SP_D \geq 0.739$ and $SN_D \geq 0.802$, and thus $0.003 \leq \tau_c \leq 1$ from Theorem 5. The 95% confidence interval is $(-1 \times 10^{-5}, 1)$. In the Supplementary Material we give the details for constructing confidence intervals for τ_c based on its sharp bounds.

In Examples 1 and 3 the upper bounds on τ_c are too large to be informative, but fortunately the lower bounds are of more interest in these applications.

8. DISCUSSION

8.1. Further comments on the measurement errors of Z

If only Z is mismeasured and the measurement error is nondifferential then $RD_{D|Z'} = r'_Z \times RD_{D|Z}$, where $r'_Z = SN'_Z + SP'_Z - 1$ with SN'_Z and SP'_Z defined in Lemma 1. If r'_Z and $RD_{D|Z}$ are both constants that do not shrink to zero as the sample size n increases, then $RD_{D|Z'}$ does not shrink to zero either. In this case, measurement error of Z does not cause the weak instrumental variable problem (Nelson & Startz, 1990; Staiger & Stock, 1997). Theorem 1 shows that the nondifferential measurement error of Z does not affect the large-sample limit of the naive estimator. We further show in the Supplementary Material that it does not affect the asymptotic variance of the naive estimator either.

Nevertheless, in finite samples the measurement error of Z does result in a smaller estimate for $RD_{D|Z'}$. If we consider the asymptotic regime that $r'_Z = o(n^{-\alpha})$ for some $\alpha > 0$, then it is possible to have the weak instrumental variable problem. In this case we need tools that are tailored to weak instrumental variables (Nelson & Startz, 1990; Staiger & Stock, 1997).

Practitioners sometimes dichotomize a continuous instrumental variable Z into a binary one based on the median or other quantiles. Dichotomized variables based on other quantiles are measurement errors of the dichotomized variable based on the median. However, these measurement errors are differential and thus our results in § 3 and § 4 are not applicable.

8.2. Further comments on the measurement errors of D and Y

If we dichotomize a discrete $D \in \{0, 1, \dots, J\}$ at k , i.e., $D' = 1(D \geq k)$, then we can define two-stage least squares estimators based on D and D' :

$$\tau_{2sls} = \frac{E(Y | Z = 1) - E(Y | Z = 0)}{E(D | Z = 1) - E(D | Z = 0)}, \quad \tau'_{2sls} = \frac{E(Y | Z = 1) - E(Y | Z = 0)}{E(D' | Z = 1) - E(D' | Z = 0)}.$$

Angrist & Imbens (1995) showed that τ_{2sls} is a weighted average of some subgroup causal effects. Analogous to Theorem 1, we show in the Supplementary Material that $\tau_{2sls} = \tau'_{2sls} \times w_k$, where $w_k = \text{pr}(D_1 \geq k >$

$D_0)/\sum_{j=1}^J \text{pr}(D_1 \geq j > D_0) \in [0, 1]$ if Assumptions 1(a) and (b) hold. Therefore, the dichotomization biases the estimate away from zero.

For a continuous outcome, it is common to assume that the measurement error of Y is additive and nondifferential, i.e., $Y' = Y + U$, where U is the error term with mean zero. If the binary Z and D are nondifferentially mismeasured as in Assumption 2, then $\tau_c = \tau'_c \times r_D$. In this case, the measurement error of Y does not bias the estimate for τ_c .

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SUPPLEMENTARY MATERIAL

Supplementary material available at *Biometrika* online includes proofs and further details of the examples.

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Supplementary material for “Measurement errors in the binary instrumental variable model”

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The supplementary material contains six sections §§A–F corresponding to §§3–8.

§A gives the proof for Theorem 1 in §3.

§B gives proofs for Lemma 1, Theorems 2 and 3, and Corollaries 1 and 2 in §4, and details of computing bounds in more complicated cases.

§C gives the proofs of Theorems 4 and 5 in §5, and details of computing bounds in more complicated cases.

§D gives the proofs of Theorems 6 and 7 in §6.

§E gives more details for §7, including the data and a method for constructing confidence intervals for τ_c based on its bounds.

§F gives additional results summarized in §8.

A. PROOF OF THEOREM 1

LEMMA A1. Suppose that (S, S', Q, Q') are binary variables, and

$$\begin{aligned} \Pr(S' = s \mid S = s, Q', Q) &= \Pr(S' = s \mid S = s), \quad \Pr(S = s \mid S' = s) = a_s \quad (s = 0, 1), \\ \Pr(Q' = q \mid Q = q, S', S) &= \Pr(Q' = q \mid Q = q) = b_q \quad (q = 0, 1). \end{aligned}$$

Then

$$\begin{aligned} \Pr(Q = 1 \mid S = 1) &= \frac{a_0 \Pr(Q' = 1 \mid S' = 1) - (1 - a_1) \Pr(Q' = 1 \mid S' = 0)}{(a_1 + a_0 - 1)(b_1 + b_0 - 1)} - \frac{1 - b_0}{b_1 + b_0 - 1}, \\ \Pr(Q = 1 \mid S = 0) &= \frac{a_1 \Pr(Q' = 1 \mid S' = 0) - (1 - a_0) \Pr(Q' = 1 \mid S' = 1)}{(a_1 + a_0 - 1)(b_1 + b_0 - 1)} - \frac{1 - b_0}{b_1 + b_0 - 1}, \\ \text{RD}_{Q'|S'} &= (a_1 + a_0 - 1)(b_1 + b_0 - 1) \text{RD}_{Q|S}. \end{aligned} \tag{A1}$$

The identity (A1) corroborates Bross (1954)’s result that nondifferential measurement error of a binary treatment or outcome biases the estimate of the total effect toward zero if $|a_1 + a_0 - 1| < 1$ or $|b_1 + b_0 - 1| < 1$.

Proof of Lemma A1. From the law of total probability,

$$\begin{aligned} \Pr(Q' = 1 \mid S' = 1) &= (b_1 + b_0 - 1) \Pr(Q = 1 \mid S' = 1) + (1 - b_0), \\ \Pr(Q' = 1 \mid S' = 0) &= (b_1 + b_0 - 1) \Pr(Q = 1 \mid S' = 0) + (1 - b_0), \end{aligned}$$

which imply

$$\text{pr}(Q = 1 \mid S' = 1) = \frac{\text{pr}(Q' = 1 \mid S' = 1) - (1 - b_0)}{b_1 + b_0 - 1}, \quad (\text{A2})$$

$$\text{pr}(Q = 1 \mid S' = 0) = \frac{\text{pr}(Q' = 1 \mid S' = 0) - (1 - b_0)}{b_1 + b_0 - 1}. \quad (\text{A3})$$

Again, from the law of total probability,

$$\begin{aligned} \text{pr}(Q = 1 \mid S' = 1) &= \text{pr}(Q = 1 \mid S = 1)a_1 + \text{pr}(Q = 1 \mid S = 0)(1 - a_1), \\ \text{pr}(Q = 1 \mid S' = 0) &= \text{pr}(Q = 1 \mid S = 1)(1 - a_0) + \text{pr}(Q = 1 \mid S = 0)a_0. \end{aligned}$$

Solving the above two equations, we have

$$\text{pr}(Q = 1 \mid S = 1) = \frac{a_0 \text{pr}(Q = 1 \mid S' = 1) - (1 - a_1) \text{pr}(Q = 1 \mid S' = 0)}{a_1 + a_0 - 1}, \quad (\text{A4})$$

$$\text{pr}(Q = 1 \mid S = 0) = \frac{a_1 \text{pr}(Q = 1 \mid S' = 0) - (1 - a_0) \text{pr}(Q = 1 \mid S' = 1)}{a_1 + a_0 - 1}. \quad (\text{A5})$$

30 Substituting (A2) and (A3) into (A4) and (A5), we obtain

$$\begin{aligned} \text{pr}(Q = 1 \mid S = 1) &= \frac{a_0 \text{pr}(Q' = 1 \mid S' = 1) - (1 - a_1) \text{pr}(Q' = 1 \mid S' = 0)}{(a_1 + a_0 - 1)(b_1 + b_0 - 1)} - \frac{1 - b_0}{b_1 + b_0 - 1}, \\ \text{pr}(Q = 1 \mid S = 0) &= \frac{a_1 \text{pr}(Q' = 1 \mid S' = 0) - (1 - a_0) \text{pr}(Q' = 1 \mid S' = 1)}{(a_1 + a_0 - 1)(b_1 + b_0 - 1)} - \frac{1 - b_0}{b_1 + b_0 - 1}, \\ \text{RD}_{Q'|S'} &= (a_1 + a_0 - 1)(b_1 + b_0 - 1) \text{RD}_{Q|S}. \end{aligned}$$

□

Proof of Theorem 1. From Lemma A1,

$$\tau'_c = \frac{\text{RD}_{Y'|Z'}}{\text{RD}_{D'|Z'}} = \frac{(\text{SN}_Y + \text{SP}_Y - 1)(\text{SN}'_Z + \text{SP}'_Z - 1) \text{RD}_{Y|Z}}{(\text{SN}_D + \text{SP}_D - 1)(\text{SN}'_Z + \text{SP}'_Z - 1) \text{RD}_{D|Z}} = \tau_c \times \frac{r_Y}{r_D}.$$

□

B. BOUNDS ON τ_c UNDER NONDIFFERENTIAL MEASUREMENT ERRORS

B.1. Proofs

Proof of Lemma 1. It is straightforward to write $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$ in terms of the set $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = a, n, c\}$ given $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D, \text{SP}_D, \text{SN}_Y, \text{SP}_Y)$.

We then only need to show that we can express $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = a, n, c\}$ in terms of $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$ given $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D, \text{SP}_D, \text{SN}_Y, \text{SP}_Y)$.

First, we can express $\text{pr}(Z = 1)$ as

$$\text{pr}(Z = 1) = \text{SN}'_Z \text{pr}(Z' = 1) + (1 - \text{SP}'_Z) \text{pr}(Z' = 0). \quad (\text{B1})$$

Second, we express $\{\text{pr}(U = u) : z = 0, 1; u = a, n, c\}$ in terms of $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$ and $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D, \text{SP}_D, \text{SN}_Y, \text{SP}_Y)$. From Assumption 1,

$$\begin{aligned} \text{pr}(U = a) &= \text{pr}(D = 1 \mid Z = 0), \\ \text{pr}(U = n) &= \text{pr}(D = 0 \mid Z = 1), \\ \text{pr}(U = c) &= 1 - \text{pr}(U = a) - \text{pr}(U = n). \end{aligned}$$

Then from Lemma A1, we can further express $\{\text{pr}(U = u) : u = a, n, c\}$ as

$$\text{pr}(U = a) = \frac{\text{SN}'_Z \text{pr}(D' = 1 | Z' = 0) - (1 - \text{SP}'_Z) \text{pr}(D' = 1 | Z' = 1)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_D + \text{SP}_D - 1)} - \frac{1 - \text{SP}_D}{\text{SN}_D + \text{SP}_D - 1}, \quad (\text{B2})$$

$$\text{pr}(U = c) = \frac{\text{pr}(D' = 1 | Z' = 1) - \text{pr}(D' = 1 | Z' = 0)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_D + \text{SP}_D - 1)}, \quad (\text{B3})$$

$$\text{pr}(U = n) = \frac{\text{SN}_D}{\text{SN}_D + \text{SP}_D - 1} - \frac{\text{SP}'_Z \text{pr}(D' = 1 | Z' = 1) - (1 - \text{SN}'_Z) \text{pr}(D' = 1 | Z' = 0)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_D + \text{SP}_D - 1)}. \quad (\text{B4})$$

Third, we express $\{\text{pr}(Y_z = 1 | U = u) : z = 0, 1; u = a, n, c\}$ in terms of $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$ and $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D, \text{SP}_D, \text{SN}_Y, \text{SP}_Y)$. By the law of total probability, we decompose the observed probabilities as

$$\begin{aligned} \text{pr}(Y = 1, D' = 1 | Z = 1) &= \text{pr}(Y_1 = 1 | U = a) \text{pr}(U = a) \text{SN}_D + \text{pr}(Y_1 = 1 | U = c) \text{pr}(U = c) \text{SN}_D \\ &\quad + \text{pr}(Y_1 = 1 | U = n) \text{pr}(U = n) (1 - \text{SP}_D), \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} \text{pr}(Y = 1 | Z = 1) &= \text{pr}(Y_1 = 1 | U = a) \text{pr}(U = a) + \text{pr}(Y_1 = 1 | U = c) \text{pr}(U = c) \\ &\quad + \text{pr}(Y_1 = 1 | U = n) \text{pr}(U = n), \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} \text{pr}(Y = 1, D' = 1 | Z = 0) &= \text{pr}(Y_0 = 1 | U = a) \text{pr}(U = a) \text{SN}_D + \text{pr}(Y_0 = 1 | U = c) \text{pr}(U = c) (1 - \text{SP}_D) \\ &\quad + \text{pr}(Y_0 = 1 | U = n) \text{pr}(U = n) (1 - \text{SP}_D), \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \text{pr}(Y = 1 | Z = 0) &= \text{pr}(Y_0 = 1 | U = a) \text{pr}(U = a) + \text{pr}(Y_0 = 1 | U = c) \text{pr}(U = c) \\ &\quad + \text{pr}(Y_0 = 1 | U = n) \text{pr}(U = n). \end{aligned} \quad (\text{B8})$$

Substituting (B2)–(B4) into (B5)–(B8), we can obtain four equations for $\{\text{pr}(Y_z = 1 | U = u) : z = 0, 1; u = a, n, c\}$. From Assumption 1(c), we can obtain two additional equations $\text{pr}(Y_1 = 1 | U = a) = \text{pr}(Y_0 = 1 | U = a)$ and $\text{pr}(Y_1 = 1 | U = n) = \text{pr}(Y_0 = 1 | U = n)$. Solving them, we have

$$\text{pr}(Y_1 = 1 | U = n) = \text{pr}(Y_0 = 1 | U = n) = \frac{\text{SN}_D \text{pr}(Y = 1 | Z = 1) - \text{pr}(Y = 1, D' = 1 | Z = 1)}{\text{SN}_D - \text{pr}(D' = 1 | Z = 1)}, \quad (\text{B9})$$

$$\text{pr}(Y_1 = 1 | U = a) = \text{pr}(Y_0 = 1 | U = a) = \frac{\text{pr}(Y = 1, D' = 1 | Z = 0) - (1 - \text{SP}_D) \text{pr}(Y = 1 | Z = 0)}{\text{pr}(D' = 1 | Z = 0) - (1 - \text{SP}_D)}, \quad (\text{B10})$$

$$\text{pr}(Y_1 = 1 | U = c) = \frac{\text{RD}_{YD'|Z}}{\text{RD}_{D'|Z}} - (1 - \text{SP}_D) \times \frac{\text{RD}_{Y|Z}}{\text{RD}_{D'|Z}}, \quad (\text{B11})$$

$$\text{pr}(Y_0 = 1 | U = c) = \frac{\text{RD}_{YD'|Z}}{\text{RD}_{D'|Z}} - \text{SN}_D \times \frac{\text{RD}_{Y|Z}}{\text{RD}_{D'|Z}}. \quad (\text{B12})$$

We use Lemma A1 to obtain

$$\begin{aligned} &\text{pr}(Y = 1, D' = 1 | Z = 1) \\ &= \frac{\text{SP}'_Z \text{pr}(Y' = 1, D' = 1 | Z' = 1) - (1 - \text{SN}'_Z) \text{pr}(Y' = 1, D' = 1 | Z' = 0)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_Y + \text{SP}_Y - 1)} \\ &\quad - \frac{\text{SP}'_Z (1 - \text{SP}_Y) \text{pr}(D' = 1 | Z' = 1)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_Y + \text{SP}_Y - 1)} + \frac{(1 - \text{SN}'_Z)(1 - \text{SP}_Y) \text{pr}(D' = 1 | Z' = 0)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_Y + \text{SP}_Y - 1)} \end{aligned} \quad (\text{B13})$$

and

$$\begin{aligned} &\text{pr}(Y = 1, D' = 1 | Z = 0) \\ &= \frac{\text{SN}'_Z \text{pr}(Y' = 1, D' = 1 | Z' = 0) - (1 - \text{SP}'_Z) \text{pr}(Y' = 1, D' = 1 | Z' = 1)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_Y + \text{SP}_Y - 1)} \\ &\quad - \frac{\text{SN}'_Z (1 - \text{SP}_Y) \text{pr}(D' = 1 | Z' = 0)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_Y + \text{SP}_Y - 1)} + \frac{(1 - \text{SP}'_Z)(1 - \text{SP}_Y) \text{pr}(D' = 1 | Z' = 1)}{(\text{SN}'_Z + \text{SP}'_Z - 1)(\text{SN}_Y + \text{SP}_Y - 1)}. \end{aligned} \quad (\text{B14})$$

55 Substituting (B13) and (B14) into (B9)–(B12), we have

$$\begin{aligned} \text{pr}(Y_1 = 1 \mid U = n) &= \text{pr}(Y_0 = 1 \mid U = n) \\ &= \frac{\text{SN}_D}{\text{SN}_Y + \text{SP}_Y - 1} \times \frac{\text{SP}'_Z \text{pr}(Y' = 1 \mid Z' = 1) - (1 - \text{SN}'_Z) \text{pr}(Y' = 1 \mid Z' = 0)}{(\text{SN}'_Z + \text{SP}'_Z - 1) \text{SN}_D - \text{SP}'_Z \text{pr}(D' = 1 \mid Z' = 1) + (1 - \text{SN}'_Z) \text{pr}(D' = 1 \mid Z' = 0)} \\ &\quad - \frac{\text{SP}'_Z \text{pr}(Y' = 1, D' = 1 \mid Z' = 1) - (1 - \text{SN}'_Z) \text{pr}(Y' = 1, D' = 1 \mid Z' = 0)}{(\text{SN}_Y + \text{SP}_Y - 1) \{ (\text{SN}'_Z + \text{SP}'_Z - 1) \text{SN}_D - \text{SP}'_Z \text{pr}(D' = 1 \mid Z' = 1) + (1 - \text{SN}'_Z) \text{pr}(D' = 1 \mid Z' = 0) \}} \\ &\quad - \frac{1 - \text{SP}_Y}{\text{SN}_Y + \text{SP}_Y - 1}, \end{aligned} \quad (\text{B15})$$

$$\begin{aligned} \text{pr}(Y_1 = 1 \mid U = a) &= \text{pr}(Y_0 = 1 \mid U = a) \\ &= \frac{\text{SN}'_Z \text{pr}(Y' = 1, D' = 1 \mid Z' = 0) - (1 - \text{SP}'_Z) \text{pr}(Y' = 1, D' = 1 \mid Z' = 1)}{(\text{SN}_Y + \text{SP}_Y - 1) \{ \text{SN}'_Z \text{pr}(D' = 1 \mid Z' = 0) - (1 - \text{SP}'_Z) \text{pr}(D' = 1 \mid Z' = 1) - (\text{SN}'_Z + \text{SP}'_Z - 1)(1 - \text{SP}_D) \}} \\ &\quad - \frac{1 - \text{SP}_D}{\text{SN}_Y + \text{SP}_Y - 1} \times \frac{\text{SN}'_Z \text{pr}(Y' = 1 \mid Z' = 0) - (1 - \text{SP}'_Z) \text{pr}(Y' = 1 \mid Z' = 1)}{\text{SN}'_Z \text{pr}(D' = 1 \mid Z' = 0) - (1 - \text{SP}'_Z) \text{pr}(D' = 1 \mid Z' = 1) - (\text{SN}'_Z + \text{SP}'_Z - 1)(1 - \text{SP}_D)} \\ &\quad - \frac{1 - \text{SP}_Y}{\text{SN}_Y + \text{SP}_Y - 1}, \end{aligned} \quad (\text{B16})$$

$$\text{pr}(Y_1 = 1 \mid U = c) = \frac{1}{\text{SN}_Y + \text{SP}_Y - 1} \times \left\{ \frac{\text{RD}_{Y'D'|Z'}}{\text{RD}_{D'|Z'}} - (1 - \text{SP}_D) \times \frac{\text{RD}_{Y'|Z'}}{\text{RD}_{D'|Z'}} - (1 - \text{SP}_Y) \right\}, \quad (\text{B17})$$

$$\text{pr}(Y_0 = 1 \mid U = c) = \frac{1}{\text{SN}_Y + \text{SP}_Y - 1} \times \left\{ \frac{\text{RD}_{Y'D'|Z'}}{\text{RD}_{D'|Z'}} - \text{SN}_D \times \frac{\text{RD}_{Y'|Z'}}{\text{RD}_{D'|Z'}} - (1 - \text{SP}_Y) \right\}. \quad (\text{B18})$$

From (B1), (B2)–(B4) and (B15)–(B18), we can express $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = a, n, c\}$ in terms of $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$ and $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D, \text{SP}_D, \text{SN}_Y, \text{SP}_Y)$. $\square$

60 From Lemma 1, if we know the sensitivities and specificities, then we can recover the joint distribution of all the potential outcomes. Furthermore, the conditions

$$\{0 \leq \text{pr}(Z = z) \leq 1, 0 \leq \text{pr}(U = u) \leq 1, 0 \leq \text{pr}(Y_z = 1 \mid U = u) \leq 1 : z = 0, 1; u = a, n, c\} \quad (\text{B19})$$

induce sharp bounds on the sensitivities and specificities, which in turn induce sharp bounds on τ_c .

Proof of Theorem 2. If only Y is mismeasured, $\text{SP}_D = \text{SN}_D = \text{SP}'_Z = \text{SN}'_Z = 1$. In this case, the formulas of $\{\text{pr}(Z = z), \text{pr}(U = u) : z = 0, 1; u = a, n, c\}$ in (B1) and (B2)–(B4) do not depend on $(\text{SN}_Y, \text{SP}_Y)$, and thus do not provide any information about them. We then consider only the inequalities

$$\{0 \leq \text{pr}(Y_z = 1 \mid U = u) \leq 1 : z = 0, 1; u = a, n, c\}. \quad (\text{B20})$$

From (B15)–(B18),

$$\text{pr}(Y_1 = 1 \mid U = n) = \text{pr}(Y_0 = 1 \mid U = n) = \frac{\text{pr}(Y' = 1 \mid D = 0, Z = 1) - (1 - \text{SP}_Y)}{\text{SN}_Y + \text{SP}_Y - 1}, \quad (\text{B21})$$

$$\text{pr}(Y_1 = 1 \mid U = a) = \text{pr}(Y_0 = 1 \mid U = a) = \frac{\text{pr}(Y' = 1 \mid D = 1, Z = 0) - (1 - \text{SP}_Y)}{\text{SN}_Y + \text{SP}_Y - 1}, \quad (\text{B22})$$

$$\text{pr}(Y_1 = 1 \mid U = c) = \frac{1}{\text{SN}_Y + \text{SP}_Y - 1} \times \frac{\text{RD}_{Y'D|Z}}{\text{RD}_{D|Z}} - \frac{1 - \text{SP}_Y}{\text{SP}_Y + \text{SN}_Y - 1}, \quad (\text{B23})$$

$$\text{pr}(Y_0 = 1 \mid U = c) = \frac{1}{\text{SN}_Y + \text{SP}_Y - 1} \times \frac{\text{RD}_{Y'(1-D)|(1-Z)}}{\text{RD}_{D|Z}} - \frac{1 - \text{SP}_Y}{\text{SP}_Y + \text{SN}_Y - 1}. \quad (\text{B24})$$

Solving (B20), we obtain

$$\text{SN}_Y \geq \max \left\{ \text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 0), \frac{\text{RD}_{Y'D|Z}}{\text{RD}_{D|Z}}, \frac{\text{RD}_{Y'(1-D)|(1-Z)}}{\text{RD}_{D|Z}} \right\}, \quad (\text{B25})$$

$$\text{SP}_Y \geq 1 - \min \left\{ \text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 0), \frac{\text{RD}_{Y'D|Z}}{\text{RD}_{D|Z}}, \frac{\text{RD}_{Y'(1-D)|(1-Z)}}{\text{RD}_{D|Z}} \right\}. \quad (\text{B26})$$

After rearrangement, only one of SN_Y and SP_Y appears in each of the inequalities in (B20). As a result, the bounds on SN_Y and SP_Y are both attainable. Thus, we can obtain the sharp bounds on r_Y by summing (B25) and (B26). Then, $\tau'_c \leq \tau_c \leq \tau'_c/(M_Y - N_Y)$. $\square$ 70

Proof of Corollary 1. For SN_Y and SP_Y , the lower bounds must be smaller than or equal to 1. Under $\tau'_c \geq 0$, these require

$$0 \leq \frac{\text{RD}_{Y'D|Z}}{\text{RD}_{D|Z}} \leq 1, \quad 0 \leq \frac{\text{RD}_{Y'(1-D)|(1-Z)}}{\text{RD}_{D|Z}} \leq 1,$$

which are equivalent to the inequalities in Corollary 1. Under $\tau'_c < 0$, we can obtain the same conditions. $\square$

Proof of Theorem 3. If only D is mismeasured, $\text{SP}_Y = \text{SN}_Y = \text{SP}'_Z = \text{SN}'_Z = 1$. In this case, the formula of $\text{pr}(Z = 1)$ does not depend on $(\text{SN}_D, \text{SP}_D)$, and thus does not provide any information about them. We then consider only the inequalities 75

$$\{0 \leq \text{pr}(U = u) \leq 1, 0 \leq \text{pr}(Y_z = 1 | U = u) \leq 1 : z = 0, 1; u = a, n, c\}. \quad (\text{B27})$$

From (B2)–(B4),

$$\begin{aligned} \text{pr}(U = a) &= \frac{\text{pr}(D' = 1 | Z = 0) - (1 - \text{SP}_D)}{\text{SN}_D + \text{SP}_D - 1}, \\ \text{pr}(U = n) &= \frac{\text{SN}_D - \text{pr}(D' = 1 | Z = 1)}{\text{SN}_D + \text{SP}_D - 1}, \\ \text{pr}(U = c) &= \frac{\text{pr}(D' = 1 | Z = 1) - \text{pr}(D' = 1 | Z = 0)}{\text{SN}_D + \text{SP}_D - 1}. \end{aligned}$$

From (B15)–(B18),

$$\begin{aligned} \text{pr}(Y_1 = 1 | U = n) &= \text{pr}(Y_0 = 1 | U = n) = \frac{\text{SN}_D \times \text{pr}(Y = 1 | Z = 1) - \text{pr}(Y = 1, D' = 1 | Z = 1)}{\text{SN}_D - \text{pr}(D' = 1 | Z = 1)}, \\ \text{pr}(Y_1 = 1 | U = a) &= \text{pr}(Y_0 = 1 | U = a) = \frac{\text{pr}(Y = 1, D' = 1 | Z = 0) - (1 - \text{SP}_D) \times \text{pr}(Y = 1 | Z = 0)}{\text{pr}(D' = 1 | Z = 0) - (1 - \text{SP}_D)}, \\ \text{pr}(Y_1 = 1 | U = c) &= \frac{\text{RD}_{YD'|Z}}{\text{RD}_{D'|Z}} - (1 - \text{SP}_D) \times \frac{\text{RD}_{Y|Z}}{\text{RD}_{D'|Z}}, \\ \text{pr}(Y_0 = 1 | U = c) &= \frac{\text{RD}_{YD'|Z}}{\text{RD}_{D'|Z}} - \text{SN}_D \times \frac{\text{RD}_{Y|Z}}{\text{RD}_{D'|Z}}. \end{aligned}$$

Solving (B27), we can obtain the bounds on SN_D and SP_D . When $\tau'_c \geq 0$, we have 80

$$\max_{z,y=0,1} \left\{ \text{pr}(D' = 1 | Z = z), \text{pr}(D' = 1 | Y = y, Z = 1), \frac{\text{RD}_{(1-Y)D'|(1-Z)}}{\text{RD}_{Y|Z}} \right\} \leq \text{SN}_D \leq \min \left\{ 1, \frac{\text{RD}_{YD'|Z}}{\text{RD}_{Y|Z}} \right\}, \quad (\text{B28})$$

$$1 - \min_{z,y=0,1} \left\{ \text{pr}(D' = 1 | Z = z), \text{pr}(D' = 1 | Y = y, Z = 0), \frac{\text{RD}_{YD'|Z}}{\text{RD}_{Y|Z}} \right\} \leq \text{SP}_D \leq 1 - \max \left\{ 0, \frac{\text{RD}_{(1-Y)D'|(1-Z)}}{\text{RD}_{Y|Z}} \right\}. \quad (\text{B29})$$

Because only one of SN_D and SP_D appears in each of the inequalities in (B27) after rearrangement, the bounds on SN_D and SP_D are both attainable. Thus, we can obtain the sharp bounds on r_D by summing (B28) and (B29). Then, we can obtain the sharp bounds on τ_c . $\square$

Before proving Corollary 2, we give a simple lemma.

LEMMA B2. Under Assumption 1(b), if only D is mismeasured with $r_D > 0$, then $\text{RD}_{D'|Z} \geq 0$. 85

Proof of Lemma B2. Assumption 1(b) implies that $\text{RD}_{D|Z} \geq 0$. Using Lemma A1, we have $\text{RD}_{D'|Z} = \text{RD}_{D|Z} \times r_D \geq 0$. $\square$

Proof of Corollary 2. For SN_D and SP_D , the lower bounds must be smaller than or equal to 1 and the upper bounds must be larger than or equal to 0. Moreover, the lower bounds must be smaller than or equal to their upper bounds. These require

$$\frac{\text{RD}_{YD'|Z}}{\text{RD}_{Y|Z}} \geq 0, \quad \frac{\text{RD}_{(1-Y)D'|(1-Z)}}{\text{RD}_{Y|Z}} \leq 1, \quad (\text{B30})$$

$$\text{pr}(D' = 1 | Y = y, Z = 1) \leq \frac{\text{RD}_{YD'|Z}}{\text{RD}_{Y|Z}}, \quad (y = 0, 1), \quad (\text{B31})$$

$$\text{pr}(D' = 1 | Y = y, Z = 0) \geq \frac{\text{RD}_{(1-Y)D'|(1-Z)}}{\text{RD}_{Y|Z}}, \quad (y = 0, 1), \quad (\text{B32})$$

$$\frac{\text{RD}_{(1-Y)D'|(1-Z)}}{\text{RD}_{Y|Z}} \leq \frac{\text{RD}_{YD'|Z}}{\text{RD}_{Y|Z}}, \quad (\text{B33})$$

$$\text{pr}(D' = 1 | Z = z) \geq \frac{\text{RD}_{(1-Y)D'|(1-Z)}}{\text{RD}_{Y|Z}}, \quad (z = 0, 1), \quad (\text{B34})$$

$$\text{pr}(D' = 1 | Z = z) \leq \frac{\text{RD}_{YD'|Z}}{\text{RD}_{Y|Z}}, \quad (z = 0, 1). \quad (\text{B35})$$

When $\tau'_c \geq 0$, Lemma B2 ensures (B33). Moreover, (B32) implies (B34) with $z = 0$, and (B31) implies (B35) with $z = 1$. Lemma B2 further ensures (B34) with $z = 1$ and (B35) with $z = 0$. Therefore, (B33) to (B35) are redundant. The remaining conditions (B30)–(B32) are equivalent to the inequalities in Corollary 2. $\square$

Next, we show that the conditions in Corollary 2 are weaker than the following conditions in Balke & Pearl (1997):

$$\text{pr}(Y = y, D' = 1 | Z = 1) \geq \text{pr}(Y = y, D' = 1 | Z = 0), \quad (y = 0, 1), \quad (\text{B36})$$

$$\text{pr}(Y = y, D' = 0 | Z = 0) \geq \text{pr}(Y = y, D' = 0 | Z = 1), \quad (y = 0, 1). \quad (\text{B37})$$

These are the testable conditions for the binary instrumental variable model without measurement errors.

Proof. First, (B30) is equivalent to (B36) with $y = 1$ and (B37) with $y = 0$.

Second, we show (B36) and (B37) imply (B31). From (B36) and (B37) with $y = 1$, we have

$$\frac{\text{pr}(Y = 1, D' = 1 | Z = 1)}{\sum_{d'=0,1} \text{pr}(Y = 1, D' = d' | Z = 1)} \geq \frac{\text{pr}(Y = 1, D' = 1 | Z = 0)}{\sum_{d'=0,1} \text{pr}(Y = 1, D' = d' | Z = 0)}.$$

Therefore, $\text{pr}(D' = 1 | Y = 1, Z = 1) \geq \text{pr}(D' = 1 | Y = 1, Z = 0)$, which is equivalent to (B31) with $y = 1$. From $\text{RD}_{Y|Z} \geq 0$, we have

$$\begin{aligned} & \text{pr}(Y = 0 | Z = 0) \text{pr}(D' = 1 | Y = 0, Z = 1) \\ &= \text{pr}(Y = 0 | Z = 0) \{1 - \text{pr}(D' = 0 | Y = 0, Z = 1)\} \\ &= \text{pr}(Y = 0 | Z = 0) - \text{pr}(Y = 0 | Z = 0) \text{pr}(D' = 0 | Y = 0, Z = 1) \\ &\leq \text{pr}(Y = 0 | Z = 0) - \text{pr}(Y = 0 | Z = 1) \text{pr}(D' = 0 | Y = 0, Z = 1) \\ &= \text{pr}(Y = 0, D' = 1 | Z = 0). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{pr}(D' = 1 | Z = 1) &\geq \text{pr}(D' = 1 | Z = 0) \\ &= \text{pr}(Y = 1, D' = 1 | Z = 0) + \text{pr}(Y = 0, D' = 1 | Z = 0) \\ &\geq \text{pr}(Y = 1, D' = 1 | Z = 0) + \text{pr}(Y = 0 | Z = 0) \text{pr}(D' = 1 | Y = 0, Z = 1), \end{aligned}$$

which is equivalent to (B31) with $y = 0$.

Third, we show (B36) and (B37) imply (B32). From (B36) with $y = 0$ and $\text{RD}_{Y|Z} \geq 0$, we have

$$\begin{aligned} \text{pr}(D' = 1 \mid Z = 0) &= \text{pr}(Y = 1, D' = 1 \mid Z = 0) + \text{pr}(Y = 0, D' = 1 \mid Z = 0) \\ &\leq \text{pr}(Y = 1 \mid Z = 0)\text{pr}(D' = 1 \mid Y = 1, Z = 0) + \text{pr}(Y = 0, D' = 1 \mid Z = 1) \\ &\leq \text{pr}(Y = 1 \mid Z = 1)\text{pr}(D' = 1 \mid Y = 1, Z = 0) + \text{pr}(Y = 0, D' = 1 \mid Z = 1), \end{aligned}$$

which is equivalent to (B32) with $y = 1$. From (B36) with $y = 0$ and $\text{RD}_{Y|Z} \geq 0$, we have

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$$\frac{\text{pr}(Y = 0, D' = 1 \mid Z = 1)}{\text{pr}(Y = 0 \mid Z = 1)} \geq \frac{\text{pr}(Y = 0, D' = 1 \mid Z = 0)}{\text{pr}(Y = 0 \mid Z = 0)}.$$

Therefore, $\text{pr}(D' = 1 \mid Y = 0, Z = 1) \geq \text{pr}(D' = 1 \mid Y = 0, Z = 0)$, which is equivalent to (B32) with $y = 0$. $\square$

B.2. Bounds on τ_c under simultaneous measurement errors

It is complicated to obtain closed-form bounds with simultaneous measurement errors of more than one element of (Z, D, Y) . We provide a general strategy for calculating the sharp bounds numerically.

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From Lemma 1, we can express $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$ in terms of $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = a, n, c\}$ and $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D, \text{SP}_D, \text{SN}_Y, \text{SP}_Y)$. Therefore, we obtain $8 - 1 = 7$ equality constraints for $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = a, n, c\}$ and $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D, \text{SP}_D, \text{SN}_Y, \text{SP}_Y)$. Using linear or non-linear programming, we can numerically calculate the bounds by minimizing and maximizing τ_c under the equality constraints and the inequality constraints (B19).

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C. RESULTS UNDER STRONG MONOTONICITY

C.1. Proofs

Proof of Theorem 4. If only Y is mismeasured,

$$\tau'_c = \frac{\text{pr}(Y' = 1 \mid Z = 1) - \text{pr}(Y' = 1 \mid Z = 0)}{\text{pr}(D = 1 \mid Z = 1)}, \quad \tau_c = \tau'_c / r_Y.$$

In this case, the formulas of $\{\text{pr}(Z = z), \text{pr}(U = u) : z = 0, 1; u = n, c\}$ do not depend on $(\text{SN}_Y, \text{SP}_Y)$, and thus do not provide any information about them. Therefore, we consider only the inequalities in (B20) based on the following probabilities:

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$$\text{pr}(Y_1 = 1 \mid U = n) = \text{pr}(Y_0 = 1 \mid U = n) = \frac{\text{pr}(Y' = 1 \mid D = 0, Z = 1) - (1 - \text{SP}_Y)}{\text{SN}_Y + \text{SP}_Y - 1}, \quad (\text{C1})$$

$$\text{pr}(Y_1 = 1 \mid U = c) = \frac{\text{pr}(Y' = 1 \mid D = 1, Z = 1) - (1 - \text{SP}_Y)}{\text{SN}_Y + \text{SP}_Y - 1}, \quad (\text{C2})$$

$$\text{pr}(Y_0 = 1 \mid U = c) = \frac{\text{pr}(Y' = 1 \mid D = 1, Z = 1) - \tau'_c}{\text{SN}_Y + \text{SP}_Y - 1} - \frac{1 - \text{SP}_Y}{\text{SP}_Y + \text{SN}_Y - 1}. \quad (\text{C3})$$

Using (C1)–(C3) to solve (B20), we obtain

$$\text{SP}_Y \geq 1 - \min\{\text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 1) - \tau'_c\}, \quad (\text{C4})$$

$$\text{SN}_Y \geq \max\{\text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 1)\}. \quad (\text{C5})$$

After rearrangement, only one of SN_Y and SP_Y appears in each of the inequalities in (B20). As a result, the bounds (C4) and (C5) are both attainable. Thus, we obtain the sharp bounds on r_Y by summing (C4) and (C5), i.e., $r_Y \geq (M_Y^m - N_Y^m)$, where

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$$\begin{aligned} M_Y^m - N_Y^m &= \max\{\text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 1)\} \\ &\quad - \min\{\text{pr}(Y' = 1 \mid D = 0, Z = 1), \text{pr}(Y' = 1 \mid D = 1, Z = 1) - \tau'_c\}. \end{aligned}$$

Then, $\tau'_c \leq \tau_c \leq \tau'_c / (M_Y^m - N_Y^m)$. □

We give a more general version of Theorem 5 without the condition in (2).

THEOREM C1. *Suppose that $\tau'_c \geq 0$ and only D is mismeasured. Define*

$$S_D = \left\{ \frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - \text{RD}_{Y|Z} \times \max_{y=0,1} \{\text{pr}(D' = 1 \mid Y = y, Z = 1)\}}{\text{pr}(Y = 1 \mid Z = 0)}, \text{pr}(D' = 1 \mid Y = y, Z = 1) \right\}.$$

Under Assumptions 1 and 3, the sharp upper bound on τ_c is

$$\max \left\{ \text{pr}(D' = 1 \mid Y = 1, Z = 1), \frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - \min S_D \times \text{pr}(Y = 1 \mid Z = 1)}{\text{pr}(D' = 1 \mid Z = 1) - \min S_D} \right\},$$

and the sharp lower bound on τ_c is

$$\min \left\{ \frac{\max_{y=0,1} \text{pr}(D' = 1 \mid Y = y, Z = 1) \times \text{RD}_{Y|Z}}{\text{pr}(D' = 1 \mid Z = 1)}, \frac{\{\max_{y=0,1} \text{pr}(D' = 1 \mid Y = y, Z = 1) - \min S_D\} \times \text{RD}_{Y|Z}}{\text{pr}(D' = 1 \mid Z = 1) - \min S_D} \right\}.$$

Proof of Theorem C1. First, $\text{pr}(Z = z)$ does not depend on $(\text{SN}_D^1, \text{SP}_D^1)$, and thus the condition $0 \leq \text{pr}(Z = 1) \leq 1$ do not provide any information about them. We need only to express $\{\text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = n, c\}$ in terms of $\{\text{pr}(Z = z, D' = d', Y = y) : z, d', y = 0, 1\}$ and $(\text{SN}_D^1, \text{SP}_D^1)$. The proportions of principal strata are

$$\text{pr}(U = c) = \frac{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)}{\text{SN}_D^1 + \text{SP}_D^1 - 1}, \quad \text{pr}(U = n) = \frac{\text{SN}_D^1 - \text{pr}(D' = 1 \mid Z = 1)}{\text{SN}_D^1 + \text{SP}_D^1 - 1}.$$

We decompose the observed probabilities into

$$\begin{aligned} \text{pr}(Y = 1, D' = 1 \mid Z = 1) &= \text{pr}(Y_1 = 1 \mid U = c) \text{pr}(U = c) \text{SN}_D^1 + \text{pr}(Y_1 = 1 \mid U = n) \text{pr}(U = n) (1 - \text{SP}_D^1), \\ \text{pr}(Y = 1 \mid Z = 1) &= \text{pr}(Y_1 = 1 \mid U = c) \text{pr}(U = c) + \text{pr}(Y_1 = 1 \mid U = n) \text{pr}(U = n). \end{aligned}$$

Solving the above two equations, we have

$$\begin{aligned} \text{pr}(Y_1 = 1 \mid U = c) &= \frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \times \text{pr}(Y = 1 \mid Z = 1)}{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)}, \\ \text{pr}(Y_0 = 1 \mid U = n) &= \text{pr}(Y_1 = 1 \mid U = n) = \frac{\text{SN}_D^1 \times \text{pr}(Y = 1 \mid Z = 1) - \text{pr}(Y = 1, D' = 1 \mid Z = 1)}{\text{SN}_D^1 - \text{pr}(D' = 1 \mid Z = 1)}. \end{aligned}$$

From the following decomposition of the outcome distribution in the control group

$$\text{pr}(Y = 1 \mid Z = 0) = \text{pr}(Y_0 = 1 \mid U = c) \text{pr}(U = c) + \text{pr}(Y_0 = 1 \mid U = n) \text{pr}(U = n),$$

we obtain

$$\text{pr}(Y_0 = 1 \mid U = c) = \frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \text{pr}(Y = 1 \mid Z = 1)}{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)} - \frac{(\text{SN}_D^1 + \text{SP}_D^1 - 1) \text{RD}_{Y|Z}}{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)}.$$

Second, we derive the bounds on SN_D^1 and SP_D^1 by solving the inequalities in

$$\{0 \leq \text{pr}(U = u) \leq 1, 0 \leq \text{pr}(Y_z = 1 \mid U = u) \leq 1 : z = 0, 1; u = c, n\}.$$

Solving $\{0 \leq \text{pr}(U = u) \leq 1 : u = c, n\}$, we have

$$\text{SN}_D^1 \geq \text{pr}(D' = 1 \mid Z = 1), \quad \text{SP}_D^1 \geq 1 - \text{pr}(D' = 1 \mid Z = 1). \quad (\text{C6})$$

Solving $0 \leq \text{pr}(Y_1 = 1 \mid U = c) \leq 1$ and $0 \leq \text{pr}(Y_1 = 1 \mid U = n) \leq 1$, we have

$$\text{SN}_D^1 \geq \text{pr}(D' = 1 \mid Y = y, Z = 1), \quad (y = 0, 1), \quad (\text{C7})$$

$$\text{SP}_D^1 \geq 1 - \text{pr}(D' = 1 \mid Y = y, Z = 1), \quad (y = 0, 1). \quad (\text{C8})$$

The inequalities in (C7) and (C8) are stronger than those in (C6). Therefore, we can omit (C6). Solving $0 \leq \text{pr}(Y_0 = 1 \mid U = c) \leq 1$, we have 140

$$\begin{aligned} \text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \text{pr}(Y = 1 \mid Z = 0) - \text{SN}_D^1 \text{RD}_{Y|Z} &\geq 0, \\ \text{pr}(Y = 0, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \text{pr}(Y = 0 \mid Z = 0) + \text{SN}_D^1 \text{RD}_{Y|Z} &\geq 0, \end{aligned}$$

which imply

$$\text{SN}_D^1 \leq \frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \text{pr}(Y = 1 \mid Z = 0)}{\text{RD}_{Y|Z}}, \quad (\text{C9})$$

$$\text{SN}_D^1 \geq -\frac{\text{pr}(Y = 0, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \text{pr}(Y = 0 \mid Z = 0)}{\text{RD}_{Y|Z}}. \quad (\text{C10})$$

Next, we show that (C7) and (C8) imply (C10). From (C8),

$$\begin{aligned} &\text{pr}(D' = 1 \mid Y = 0, Z = 1) \times \text{RD}_{Y|Z} \\ &= \text{pr}(D' = 1 \mid Y = 0, Z = 1) \{ \text{pr}(Y = 0 \mid Z = 0) - \text{pr}(Y = 0 \mid Z = 1) \} \\ &= \text{pr}(D' = 1 \mid Y = 0, Z = 1) \text{pr}(Y = 0 \mid Z = 0) - \text{pr}(Y = 0, D' = 1 \mid Z = 1) \\ &\geq (1 - \text{SP}_D^1) \text{pr}(Y = 0 \mid Z = 0) - \text{pr}(Y = 0, D' = 1 \mid Z = 1). \end{aligned}$$

Therefore,

$$\text{pr}(D' = 1 \mid Y = 0, Z = 1) \geq -\frac{\text{pr}(Y = 0, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \text{pr}(Y = 0 \mid Z = 0)}{\text{RD}_{Y|Z}},$$

which means that (C7) implies (C10). As a result, we can omit (C10). 145

Combining (C7) with (C9),

$$\max_{y=0,1} \text{pr}(D' = 1 \mid Y = y, Z = 1) \leq \text{SN}_D^1 \leq \frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \times \text{pr}(Y = 1 \mid Z = 0)}{\text{RD}_{Y|Z}}. \quad (\text{C11})$$

In (C11), the upper bound must be larger than or equal to 0, the lower bound must be smaller than or equal to 1, and the upper bound must be larger than or equal to the lower bound. These require

$$\frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \times \text{pr}(Y = 1 \mid Z = 0)}{\text{RD}_{Y|Z}} \geq \max_{y=0,1} \text{pr}(D' = 1 \mid Y = y, Z = 1), \quad (\text{C12})$$

$$\frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \times \text{pr}(Y = 1 \mid Z = 0)}{\text{RD}_{Y|Z}} \geq 0. \quad (\text{C13})$$

Under $\text{RD}_{Y|Z} \geq 0$, (C13) holds. Thus, we can omit (C13). Combining (C12) with (C8), we have $\text{SP}_D^1 \geq 1 - \min S_D$. 150

Finally, we derive the bounds on

$$\tau_c = \frac{(\text{SN}_D^1 + \text{SP}_D^1 - 1) \text{RD}_{Y|Z}}{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)}, \quad (\text{C14})$$

where only D is mismeasured. If $\tau_c' \geq 0$, then from (C14), τ_c is increasing in SN_D^1 . Replacing SN_D^1 with its bound limits in (C14), we obtain

$$\begin{aligned} &\frac{\{ \max_{y=0,1} \text{pr}(D' = 1 \mid Y = y, Z = 1) + \text{SP}_D^1 - 1 \} \times \text{RD}_{Y|Z}}{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)} \\ &\leq \tau_c \leq \frac{\text{pr}(Y = 1, D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1) \times \text{pr}(Y = 1 \mid Z = 1)}{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)}. \end{aligned}$$

Because the above bound limits are monotone in SP_D^1 , we can obtain the sharp upper and lower bounds on τ_c by replacing SP_D^1 with its bound limits. If $\text{pr}(D' = 1 \mid Y = 1, Z = 1) \geq \text{pr}(D' = 1 \mid Y = 0, Z = 1)$, then the bounds simplify to those in Theorem 5. $\square$

C.2. Bounds on τ_c under simultaneous measurement errors

Under strong monotonicity, it is complicated to obtain closed-form bounds with simultaneous measurement errors of more than one element of (Z, D, Y) . We propose the general strategy for calculating the sharp bounds numerically.

We can express $\{\text{pr}(Z' = z', D' = d', Y' = y') : z', d', y' = 0, 1\}$ in terms of $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = n, c\}$ and $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D^1, \text{SP}_D^1, \text{SN}_Y, \text{SP}_Y)$. Therefore, we obtain 8 – 1 equality constraints for $\{\text{pr}(Z = z), \text{pr}(U = u), \text{pr}(Y_z = 1 \mid U = u) : z = 0, 1; u = n, c\}$ and $(\text{SN}'_Z, \text{SP}'_Z, \text{SN}_D^1, \text{SP}_D^1, \text{SN}_Y, \text{SP}_Y)$. Using linear or non-linear programming, we can numerically calculate the bounds by minimizing and maximizing τ_c under the equality constraints and the inequality constraints $\{0 \leq \text{pr}(Z = z) \leq 1, 0 \leq \text{pr}(U = u) \leq 1, 0 \leq \text{pr}(Y_z = 1 \mid U = u) \leq 1 : z = 0, 1; u = n, c\}$.

D. RESULTS WITH DIFFERENTIAL MEASUREMENT ERRORS

Proof of Theorem 6. From (A2) and (A3),

$$\begin{aligned}\text{pr}(Y = 1 \mid Z = 1) &= \frac{\text{pr}(Y' = 1 \mid Z = 1) - (1 - \text{SP}_Y^1)}{\text{SN}_Y^1 + \text{SP}_Y^1 - 1}, \\ \text{pr}(Y = 1 \mid Z = 0) &= \frac{\text{pr}(Y' = 1 \mid Z = 0) - (1 - \text{SP}_Y^0)}{\text{SN}_Y^0 + \text{SP}_Y^0 - 1},\end{aligned}$$

which imply the formula of τ_c in Theorem 6. $\square$

Proof of Theorem 7. From (A2) and (A3),

$$\begin{aligned}\text{pr}(D = 1 \mid Z = 1) &= \frac{\text{pr}(D' = 1 \mid Z = 1) - (1 - \text{SP}_D^1)}{\text{SN}_D^1 + \text{SP}_D^1 - 1}, \\ \text{pr}(D = 1 \mid Z = 0) &= \frac{\text{pr}(D' = 1 \mid Z = 0) - (1 - \text{SP}_D^0)}{\text{SN}_D^0 + \text{SP}_D^0 - 1},\end{aligned}$$

which imply the formula of τ_c in Theorem 7. $\square$

E. MORE DETAILS FOR §7

E.1. Data

Table 1 shows the data in §7 in the main text.

E.2. A method for constructing confidence intervals for τ_c

The bounds on τ_c have the form

$$\begin{aligned}\tau'_c \times l_1 &\leq \tau_c \leq \tau'_c \times u_1, & \text{if } \tau'_c \geq 0, \\ \tau'_c \times l_2 &\leq \tau_c \leq \tau'_c \times u_2, & \text{if } \tau'_c < 0,\end{aligned}$$

where l_1, l_2, u_1 and u_2 are maximums or minimums of the functions of the observed distribution. This form of bounds is different from most partially identified parameter in the literature (Imbens & Manski, 2004; Chernozhukov et al., 2013; Jiang & Ding, 2018). Motivated by Berger & Boos (1994)'s method for hypothesis testing, we propose the following strategy for constructing confidence interval.

Table 1: Data

(a) Example 1: Investigators et al. (2014)'s study				
	$Z = 1$		$Z = 0$	
	$D = 1$	$D = 0$	$D = 1$	$D = 0$
$Y = 1$	107	68	24	131
$Y = 0$	42	42	8	79

(b) Example 2: Hirano et al. (2000)'s study				
	$Z = 1$		$Z = 0$	
	$D = 1$	$D = 0$	$D = 1$	$D = 0$
$Y = 1$	31	85	30	99
$Y = 0$	424	944	237	1041

(c) Example 3: Sommer & Zeger (1991)'s study				
	$Z = 1$		$Z = 0$	
	$D = 1$	$D = 0$	$D = 1$	$D = 0$
$Y = 1$	9663	2385	0	11514
$Y = 0$	12	34	0	74

In the first step, construct CI' , a $1 - \gamma$ confidence interval for τ'_c . In the second step, construct $CI(\tilde{\tau}'_c)$, a $1 - (\alpha - \gamma)$ confidence interval for τ_c when the parameter τ'_c is fixed at the value $\tilde{\tau}'_c$, for all $\tilde{\tau}'_c \in CI'$. In the third step, construct the final confidence interval by taking the union of these $CI(\tilde{\tau}'_c)$'s: $CI = \cup_{\tilde{\tau}'_c \in CI'} CI(\tilde{\tau}'_c)$. 185

This strategy is easy to implement. In the first step, we can construct CI' based on standard techniques. In the second step, we can construct $CI(\tilde{\tau}'_c)$ using the method of Imbens & Manski (2004) or Jiang & Ding (2018) for partially identified parameters with interval bounds.

We then prove that this confidence interval has a coverage rate at least as large as $1 - \alpha$. 190

Proof. The conclusion follows from

$$\begin{aligned}
\text{pr}(\tau_c \notin CI) &= \text{pr}\{\tau_c \notin \cup_{\tilde{\tau}'_c \in CI'} CI(\tilde{\tau}'_c), \tau'_c \in CI'\} + \text{pr}\{\tau_c \notin \cup_{\tilde{\tau}'_c \in CI'} CI(\tilde{\tau}'_c), \tau'_c \notin CI'\} \\
&\leq \text{pr}\{\tau_c \notin \cup_{\tilde{\tau}'_c \in CI'} CI(\tilde{\tau}'_c), \tau'_c \in CI'\} + \text{pr}(\tau'_c \notin CI') \\
&\leq \text{pr}\{\tau_c \notin CI(\tau'_c)\} + \gamma \\
&= \alpha - \gamma + \gamma \\
&= \alpha.
\end{aligned}$$

The proof above is based on finite-sample exact confidence intervals. It carries over to large-sample confidence intervals. □

F. OTHER RESULTS

F.1. More discussion on the non-differential measurement error of Z

Let $\widehat{RD}_{Y|Z}$ and $\widehat{RD}_{D|Z}$ be the estimators of $RD_{Y|Z}$ and $RD_{D|Z}$, respectively. Without measurement error, applying the central limit theorem, we have 195

$$n^{1/2} \begin{pmatrix} \widehat{RD}_{Y|Z} \\ \widehat{RD}_{D|Z} \end{pmatrix} \longrightarrow \mathcal{N}_2 \left\{ \begin{pmatrix} RD_{Y|Z} \\ RD_{D|Z} \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right\}$$

in distribution, where $\sigma_1^2, \sigma_2^2, \rho\sigma_1\sigma_2$ are the asymptotic variances and covariance of $n^{1/2}\widehat{\text{RD}}_{Y|Z}$ and $n^{1/2}\widehat{\text{RD}}_{D|Z}$, respectively. Using the delta method, we obtain that the asymptotic variance of the naive estimator $\widehat{\text{RD}}_{Y|Z}/\widehat{\text{RD}}_{D|Z}$ is $(\sigma_1^2 - \tau_c\rho\sigma_1\sigma_2 + \tau_c^2\sigma_2^2)/\text{RD}_{D|Z}^2$.

When Z is mismeasured, let $r'_Z = \text{SN}'_Z + \text{SP}'_Z - 1$. From Theorem 1, $\text{RD}_{Y|Z'} = r'_Z \times \text{RD}_{Y|Z}$ and $\text{RD}_{D|Z'} = r'_Z \times \text{RD}_{D|Z}$. Therefore,

$$n^{1/2} \left(\frac{\widehat{\text{RD}}_{Y|Z'}}{\widehat{\text{RD}}_{D|Z'}} \right) \longrightarrow \mathcal{N}_2 \left\{ \begin{pmatrix} r'_Z \times \text{RD}_{Y|Z} \\ r'_Z \times \text{RD}_{D|Z} \end{pmatrix}, (r'_Z)^2 \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right\},$$

in distribution. Using the delta method, we obtain the asymptotic variance of the naive estimator $\widehat{\text{RD}}_{Y|Z'}/\widehat{\text{RD}}_{D|Z'}$ is $(\sigma_1^2 - \tau_c\rho\sigma_1\sigma_2 + \tau_c^2\sigma_2^2)/\text{RD}_{D|Z}^2$. Therefore, the non-differential measurement error of Z does not affect the asymptotic variance of the naive estimator.

F.2. Dichotomization of a discrete treatment

We show that $\tau_{2\text{sls}} = \tau'_{2\text{sls}} \times w_k$, where $w_k = \text{pr}(D_1 \geq k > D_0) / \sum_{j=1}^J \text{pr}(D_1 \geq j > D_0)$ if Assumptions 1(a) and (b) hold. This follows from

$$\begin{aligned} \tau_{2\text{sls}} &= \tau'_{2\text{sls}} \times \frac{E(D' | Z = 1) - E(D' | Z = 0)}{E(D | Z = 1) - E(D | Z = 0)} \\ &= \tau'_{2\text{sls}} \times \frac{\text{pr}(D_1 \geq k) - \text{pr}(D_0 \geq k)}{\sum_{j=1}^J \{\text{pr}(D_1 \geq j) - \text{pr}(D_0 \geq j)\}} \\ &= \tau'_{2\text{sls}} \times \frac{\text{pr}(D_1 \geq k > D_0)}{\sum_{j=1}^J \text{pr}(D_1 \geq j > D_0)} \\ &= \tau'_{2\text{sls}} \times w_k. \end{aligned}$$

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