EXPONENTIAL LOWER RESOLVENT BOUNDS FAR AWAY FROM TRAPPED SETS

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Abstract. We give examples of semiclassical Schrödinger operators with exponentially large cutoff resolvent norms, even when the supports of the cutoff and potential are very far apart. The examples are radial, which allows us to analyze the resolvent kernel in detail using ordinary differential equation techniques. In particular, we identify a threshold spatial radius where the resolvent behavior changes. We apply these results to wave equations with radial wavespeed, identifying a corresponding threshold radius at which wave decay properties change.

1. Introduction

In the first part of this paper we study semiclassical resolvent estimates, and in the second part we apply the results to wave decay and non-decay.

1.1. **Semiclassical resolvent estimates.**

In this paper we investigate resolvent estimates for the semiclassical Schrödinger operator

$$P = P(h) = -h^2\Delta + V(x),$$

where $V\colon \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$. Initially we suppose $V \in C_c^\infty(\mathbb{R}^n)$, but later we will relax this condition.

Question 1. *For which $E_0 > 0$, and for which bounded and open sets $U \subset \mathbb{R}^n$, can we find an interval I containing E_0 such that the incoming and outgoing cutoff resolvents obey*

$$\sup_{E\in I} \|\mathbf{1}_U(P - E \pm i0)^{-1}\mathbf{1}_U\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C/h, \tag{1.1}$$

for all $h > 0$ sufficiently small?

It is well known that the answer depends upon dynamical properties of the classical flow $\Phi(t) = \exp t(2\xi\partial_x - \partial_x V(x)\partial_\xi)$ in $T^*\mathbb{R}^n$. Of particular importance is the trapped set at energy E_0, which we denote $\mathcal{K}(E_0)$; this is the set of $(x,\xi) \in T^*\mathbb{R}^n$ such that $|\xi|^2 + V(x) = E_0$ and $|\Phi(t)(x,\xi)|$ is bounded as $|t| \to \infty$.

If $\mathcal{K}(E_0)$ is empty, that is to say if E_0 is *nontrapping*, then Robert and Tamura [RoTa1] show that the answer to Question 1 is that U can be arbitrary. Analogous results hold in much more

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general nontrapping situations; see e.g. [Vo, HiZw] for some recent results, and see those papers and also [BoBuRa, Zw] for some pointers to the substantial literature on this topic.

But if $\mathcal{K}(E_0)$ is *not* empty, then Bony, Burq, and Ramond [BoBuRa] show that there is U such that for any interval I containing E_0 we have

$$\sup_{E\in I} \|\mathbf{1}_U(P-E\pm i0)^{-1}\mathbf{1}_U\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \geq \frac{\log(1/h)}{Ch}; \tag{1.2}$$

more specifically it is enough if T^*U contains an integral curve in $\mathcal{K}(E_0)$. Moreover, as we discuss below, the right hand side can sometimes be replaced by $e^{C/h}$.

Nevertheless, regardless of any trapping, for all $I \Subset (0,\infty)$ there exists $r_b > 0$ such that (1.1) holds whenever U is disjoint from $B(0,r_b)$. Thus, if the distance between T^*U and $\mathcal{K}(E_0)$ is large enough, then all losses due to trapping are removed. This was first shown by Cardoso and Vodev [CaVo], refining earlier work of Burq [Bu2], and analogous results hold for much more general operators [CaVo, RoTa2, Vo, Da, DadH, Sh1].

It is not always necessary to cut off so far away: in [DaVa] it is shown that if trapping is sufficiently mild, then we have (1.1) whenever T^*U is disjoint from $\mathcal{K}(E_0)$. (By 'trapping is sufficiently mild' we mean that the resolvent is polynomially bounded in h^{-1}; see [DaVa] and also the survey [Zw, §3.2] and the book [DyZw, Chapter 6] for more details, including sufficient conditions on $\mathcal{K}(E_0)$, and for references to some of the many known results of this kind.) Moreover, in that case (1.1) still holds if we replace $\mathbf{1}_U$ by a microlocal cutoff vanishing only in a small neighborhood of $\mathcal{K}(E_0)$ in $T^*\mathbb{R}^n$. If $\mathcal{K}(E_0)$ is normally hyperbolic, then the vanishing hypothesis can be weakened further: see [HiVa]. Propagation estimates play an important role in such results, and the connection between propagation estimates and polynomial resolvent bounds has been recently studied in [BoFuRaZe].

Our main result is that, when trapping is not mild, the situation can be dramatically different. Namely, losses due to trapping can show up very far away from the support of V:

Theorem 1. *Suppose that $V\in C_c^\infty(B(0,1))$ is radial, $n\geq 2$, and $\min V<0$. Let $R>1$ and let U be a neighborhood of the sphere $\partial B(0,R)$: see Figure 1. There is $C>0$ such that if*

$$0 < E_0 \leq 1/CR^2, \tag{1.3}$$

then

$$\sup_{E\in[E_0-Ch,E_0+Ch]} \|\mathbf{1}_U(P-E\pm i0)^{-1}\mathbf{1}_U\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \geq e^{1/Ch}, \tag{1.4}$$

for all $h>0$ small enough.

The key point is that the distance between U and $\operatorname{supp} V$ (and hence, in particular, between T^*U and the trapped set $\mathcal{K}(E_0)$) can be arbitrarily large. This seems to be a new phenomenon.

Moreover, the conditions on U and V can be weakened, and in this nice radial setting we can say more. See Theorem 3 below for a stronger and more general result, and (4.11) for a more precise version of (1.3).

Lower bounds of the form (1.4) with $\mathcal{K}(E_0)\subset T^*U$ stem from $O(e^{-1/Ch})$ quasimodes, and they are well known to hold in radial situations or under a barrier assumption, namely when $\{x \mid V(x) \leq$

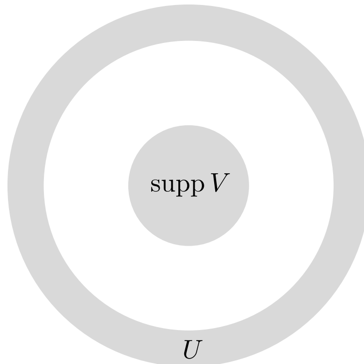

FIGURE 1. The relative positions of U and $\operatorname{supp} V$.

$E_0\}$ has a bounded connected component contained in U. They have been recently investigated in [DaDyZw], where it is shown that, for V satisfying a barrier assumption, $\mathbf{1}_U(P-E\pm i0)^{-1}\mathbf{1}_U$ can be replaced by $\mathbf{1}_{U_L}(P-E\pm i0)^{-1}\mathbf{1}_{U_R}$, with only one of U_L and U_R containing $\operatorname{supp} V$. Lower bounds for the continuation of $\mathbf{1}_U(P-z)^{-1}\mathbf{1}_U$ as z crosses the positive real axis were recently studied in [BoPe, DyWa].

In the setting of Theroem 1, it is clear that no quasimodes can concentrate in U because of the flow invariance of support of semiclassical measures. The quasimodes we use are concentrated in the set where V is negative, and we show below (in Lemma 3) that their exponential decay away from this set is very slow. This is a key difference between our result and previous lower bounds as in [DaDyZw].

The form of the right hand side of (1.4) is essentially optimal: for any $I \Subset (0,\infty)$ there is $C'>0$ such that for any U we have

$$\sup_{E\in I} \|\mathbf{1}_U(P-E\pm i0)^{-1}\mathbf{1}_U\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le Ce^{C'/h}. \tag{1.5}$$

This was first shown by Burq [Bu1, Bu2] in a more general setting, and has been generalized still further in [CaVo, RoTa2, Vo, Da, DadH, Sh1, Ga].

We can define the value $R_* = R_*(E_0;V)$ for more general potentials V by

$$R_* := \inf\{r>0 \mid (1.1) \text{ holds for some } I \text{ containing } E_0 \text{ whenever } U\cap B(0,r)=\varnothing\}. \tag{1.6}$$

As we mentioned previously, in [CaVo, RoTa2, Vo, Da, DadH, Sh1] it is shown that R_* is finite for quite general V and $E_0>0$. In Theorem 1 we show that $R_*\to+\infty$ as $E_0\to 0$ for a family of examples, and in (4.11) we show that $R_*\sim E_0^{-1/2}$. Let us mention here that the finiteness of R_*, in this setting as well as in more general ones, has been applied to a variety of problems in scattering theory: see e.g. [St, Mi1, GuHaSi, Ch]. There are also well-known consequences for Schrödinger and wave evolution: see §1.2 below. These applications motivate the following question.

Question 2. *What can we say about the value of R_* for more general V and E_0?*

For example, it would be interesting to know if the hypothesis that V is radial in Theorem 1 could be weakened. The lower bounds on resonance widths of [DaMa] make it seem unlikely that the radiality hypothesis could be removed altogether.

1.2. **Wave decay and non-decay.** We now give an application of the above results to decay and non-decay estimates for solutions to the wave equation with radial wavespeed, for simplicity restricting attention[1] to the case $n \geq 3$. Thus, let $c \in C^\infty(\mathbb{R}^n; (0,\infty))$ be radial, so that $c(x) = c_0(r)$, and suppose

$$r \geq \rho \quad \Longrightarrow \quad c_0(r) = \kappa, \tag{1.7}$$

for some positive constants ρ and κ. Given initial conditions w_0, $w_1 \in C_c^\infty(\mathbb{R}^n)$, let $w \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ be the solution to

$$(\partial_t^2 - c(x)^2\Delta)w(x,t) = 0, \qquad w(x,0) = w_0, \quad \partial_t w(x,0) = w_1.$$

Then we have conservation of energy in $\mathbb{R}^n$:

$$\begin{aligned} \mathcal{E} = \mathcal{E}[c, w_0, w_1] &= \int_{\mathbb{R}^n} |\nabla w_0(x)|^2 + |c(x)^{-1} w_1(x)|^2 dx \\ &= \int_{\mathbb{R}^n} |\nabla w(x,t)|^2 + |c(x)^{-1}\partial_t w(x,t)|^2 dx, \qquad \forall t \in \mathbb{R}. \end{aligned} \tag{1.8}$$

Our main result in this setting concerns energy on a bounded open set $U \Subset \mathbb{R}^n$:

$$\mathcal{E}_U(t) = \mathcal{E}_U[c, w_0, w_1](t) = \int_U |\nabla w(x,t)|^2 + |c(x)^{-1}\partial_t w(x,t)|^2 dx. \tag{1.9}$$

This energy decays logarithmically in the sense that for all $U \Subset \mathbb{R}^n$ and $k \in \mathbb{N}$ there is $C > 0$ such that

$$\mathcal{E}_U(t) \leq C\langle \log t\rangle^{-k}, \qquad \forall t \in \mathbb{R}. \tag{1.10}$$

In fact, these results are very robust. Proving the conservation of energy (1.8) is simple: one differentiates with respect to t and integrates by parts, and so (1.8) clearly holds for very general symmetric operators. Proving the logarithmic decay (1.10) is more complicated, but it too has been established in great generality. The study of wave decay has a long history and we do not attempt to survey it here. We just mention that the first general logarithmic decay results are due to Burq [Bu1], and refer the reader also to [Bu2, Bo2, Mo, Ga, Sh2] for more recent results on logarithmic decay and for more references.

We now bring in an assumption which ensures (stable) trapping: namely we assume that

$$\min(r/c_0(r))' < 0. \tag{1.11}$$

Such a situation was considered by Ralston [Ra], who showed that then there are sequences of resonances converging exponentially quickly to the real axis, and it is well-known that in particluar this means (1.10) cannot be improved if T^*U contains the trapped set (see [HoSm, §7] for a recent version of such a result in the setting of general relativity). Note also that if instead we had $\min(r/c_0(r))' > 0$ then the problem would be nontrapping (see [Ra, p. 571].)

[1]See §1.4 for a discussion of why we do this.

To state our result we will also need the threshold radius

$$R_c = \kappa \max_{r\in[0,\rho]} r/c_0(r), \tag{1.12}$$

and we assume that

$$R_c > \rho. \tag{1.13}$$

Theorem 2. *Fix c satisfying (1.7) and (1.13).*

(1) If $U \Subset \mathbb{R}^n$ is disjoint from the closed ball $\overline{B(0,R_c)}$, then there is $C>0$ such that

$$\int_{-\infty}^{\infty} \mathcal{E}_U(t)dt \le C\mathcal{E}, \tag{1.14}$$

for all w_0, $w_1 \in C_c^\infty(\mathbb{R}^n)$.

(2) If $U \Subset \mathbb{R}^n$ contains the sphere $\partial B(0,R)$ for some $R\in[\rho,R_c]$, then there is no $C>0$ such that (1.14) holds for all w_0, $w_1 \in C_c^\infty(\mathbb{R}^n)$.

Remarks:

(1) One can check that (1.13) implies (1.11). We make the stronger assumption (1.13) because it simplifies our work, and because the most interesting examples have $R_c \gg \rho$.

(2) It is easy to construct families of examples such that $R_c \to +\infty$ with ρ and κ fixed. One way is to take $\psi \in C_c^\infty([0,1);[0,1])$ such that $\psi \equiv 1$ near 0 and put

$$c_0(r) = 1 - s\psi(r),$$

with $s \in (0,1)$. Then $R_c \to \infty$ as $s \to 1$.

(3) We see that R_c is a threshold at which wave decay behavior changes, just as in Theorem 3 we see that r_2 is a threshold at which resolvent norm behavior changes. Actually, by setting $E_0 = \kappa^{-2}$ and $V_0 = \kappa^{-2} - c_0^{-2}$ we have $R_c = r_2$ (see §4 for more), and so R_c is also a threshold for the behavior of the resolvent $(-c^2\Delta - \lambda^2)^{-1}$: see Lemma 6.

(4) We expect that the second part of Theorem 2 can be strengthened to take into account a possible loss of derivatives as follows: if $U \Subset \mathbb{R}^n$ contains the sphere $\partial B(0,R)$ for some $R\in[\rho,R_c]$, then for any $N\in\mathbb{N}$, there is no $C>0$ such that

$$\int_{-\infty}^{\infty} \mathcal{E}_U(t)dt \le C\left(\|w_0\|^2_{H^N(\mathbb{R}^n)} + \|w_1\|^2_{H^{N-1}(\mathbb{R}^n)}\right),$$

holds for all w_0, $w_1 \in C_c^\infty(\mathbb{R}^n)$. (Thanks to Jason Metcalfe for suggesting this comment.)

We can interpret (1.14) as an 'exterior' wave decay estimate. Many variations of such estimates, including different types of smoothing and Strichartz estimates, have been established. See [BoTz, MaMeTa, MaMeTaTo, BuGuHa, Bo1, Mi2, ChWu, ChMe, RoTa2, MeStTa, BoChMePe] and references therein for results where behavior away from some compact set is better than behavior in sets which overlap trapping. Our result seems to give the first examples of 'bad' behavior extending arbitrarily far from the trapped set.

1.3. **Outline of the rest of the paper.** In §2 we state the main resolvent estimates of the paper, Theorem 3. In §3 we prove pointwise resolvent kernel bounds for a family of semiclassical ordinary differential operators, approximating the solutions by Airy functions using the remainder bounds of Olver [Ol]. In §4 we prove Theorem 3 and use it to prove Theorems 1 and 2.

1.4. **Notation.** In this paper $\Delta \le 0$ is the Euclidean Laplacian on $\mathbb{R}^n$ for some $n \ge 2$, $h > 0$ is a (small) semiclassical parameter, $\mathbf{1}_U$ is the characteristic function of U, $C > 0$ is a constant which may change from line to line, $A \Subset B$ means that the closure of A is a compact subset of B, $B(a, b)$ is the ball with center a and radius b, the sphere $\partial B(a, b)$ is its boundary, $r = |x|$ is the radial coordinate in $\mathbb{R}^n$, $\langle t \rangle = (1 + t^2)^{1/2}$, and $\sum_\pm Q(\pm) = Q(+) + Q(-)$. Ai and Bi are Airy functions, see Appendix A.

The radius ρ and wavespeed κ are defined in (1.7), and the radius R_c is defined in (1.12). The potentials V, V_0, and V_m are defined in (2.1) and the preceding sentences. The angular momentum M_0 and the radii r_2 and r_1 are defined in terms of the potential V and the energy level E_0 in (2.2), (2.4), and (2.6) respectively: see also Figure 2 and Lemma 4 for more on these important quantities. The Schrödinger operator P_m is defined in (3.1), its domain $\mathcal{D}$ is defined in terms of the boundary condition $\mathcal{B}$ immediately afterwards, and its resolvent kernel $K(r, r')$ is then given in (3.2). We also sometimes use the domain $\mathcal{D}_{r_2}$ given in (3.32). The angular momenta m_j are defined in terms of the spherical eigenvalues σ_j in (4.2).

In §4.3 we use the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n)$, defined to be the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $u \mapsto \|\nabla u\|_{L^2(\mathbb{R}^n)}$. When $n = 2$ this is not a space of distributions and various technical difficulties arise (e.g. multiplication by a function in $C_c^\infty(\mathbb{R}^n)$ is not a bounded operator). For simplicity, in §1.2 and §4.3 we stick to the case $n \ge 3$ (so that, in particular, Sobolev embedding implies $\dot{H}^1(\mathbb{R}^n) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^n)$), but see [Sh2] for methods which cover the case $n = 2$. Note that in Theorems 1 and 3 these difficulties do not appear and we allow $n \ge 2$.

2. Main theorem

In this section we state our main semiclassical resolvent estimates. We begin with the assumptions, which are weaker but more complicated than the ones for Theorem 1.

Let $n \ge 2$, and let $V \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be radial, so that $V(x) = V_0(r)$, and suppose[2] $V_0(r) \in r^{-2}C^3([0, \infty))$ is bounded below and $\left|r^{2+k}\partial_r^k V_0(r)\right|$ is bounded for all $r > 0$ and $k \in \{0, 1, 2, 3\}$.

Then $P = -h^2\Delta + V$ is selfadjoint on a domain containing $C_c^\infty(\mathbb{R}^n \setminus \{0\})$, and we fix such a domain. For $E > 0$ we can define and study the incoming and outgoing resolvents $(P - E \pm i0)^{-1}$ using separation of variables as we recall in §4 below.

For $m \in \mathbb{R}$, let

$$V_m(r) = V_0(r) + mr^{-2}. \tag{2.1}$$

[2]To keep things simpler while still capturing the interesting phenomena, one can restrict attention to the case $V_0(r) \in C_c^\infty([0, \infty))$.

This is the effective potential which arises when we write the Laplacian in polar coordinates; we think of m as the angular momentum. For $E_0 > 0$, put

$$M_0 = M_0(E_0) = \sup\{m \geq 0 \mid V_m^{-1}(E_0) \text{ has at least two points}\}, \tag{2.2}$$

and suppose

$$M_0 \text{ is finite.} \tag{2.3}$$

The trapping we use occurs at the angular momentum M_0, and we will also need the following two radii. Put

$$r_2 = r_2(E_0) = \max V_{M_0}^{-1}(E_0), \tag{2.4}$$

and suppose

$$V_{M_0}'(r) < 0 \text{ for all } r \geq r_2. \tag{2.5}$$

Put

$$r_1 = r_1(E_0) = \max\left(V_{M_0}^{-1}(E_0) \setminus \{r_2(E_0)\}\right). \tag{2.6}$$

Note that if V_0 is compactly supported and $\min V_0 < 0$, then the assumptions (2.3) and (2.5) are automatically satisfied for $E_0 > 0$ sufficiently small; we also have more explicit formulas for M_0, r_1, and r_2, which we derive in §4.2 below. These assumptions imply that $V_{M_0}'(r_1) = 0$, so that the trapped set $\mathcal{K}(E_0)$ contains circular orbits in $T^*\partial B(0, r_1)$, and these are the trapped orbits that we will use (in Lemma 3) to prove exponential lower bounds. See Figure 2.

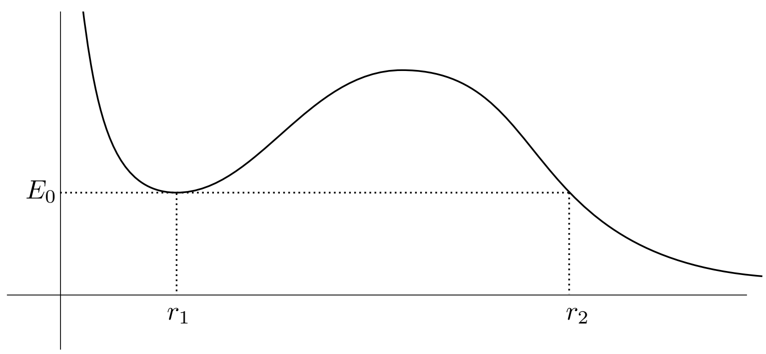

FIGURE 2. A possible graph of V_{M_0}.

Theorem 3. *Let V, E_0, r_1, and r_2 be as above.*

If $U \subset \mathbb{R}^n$ is bounded, open, and disjoint from a neighborhood of $\overline{B(0, r_2)}$, then there is an interval I containing E_0 and a constant C_0 such that

$$\sup_{E \in I} \|\mathbf{1}_U (P - E \pm i0)^{-1} \mathbf{1}_U\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C_0/h, \tag{2.7}$$

for all $h > 0$ small enough.

On the other hand, let U_L and U_R be bounded open sets in $\mathbb{R}^n$ containing spheres $\partial B(0, r_L)$ and $\partial B(0, r_R)$ respectively, such that $\min(r_L, r_R) \in [r_1, r_2]$. Then there are constants C_1 and C_2 such that

$$\sup_{E \in [E_0 - C_1 h, E_0 + C_1 h]} \|\mathbf{1}_{U_L} (P - E \pm i0)^{-1} \mathbf{1}_{U_R}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \geq e^{C_2/h}, \tag{2.8}$$

for all $h > 0$ small enough. If in addition we have $\max V < E_0$, then

$$\|\mathbf{1}_{U_L}(P - E_0 \pm i0)^{-1}\mathbf{1}_{U_R}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \geq e^{C_2/h}, \tag{2.9}$$

for h tending to 0 along a sequence of positive values.

The main point is the value $r_2 = r_2(E_0)$, which corresponds to R_* from (1.6). Comparing (2.7) with (2.8) and (2.9) we see that r_2 is a threshold at which the behavior of the resolvent changes.

We now discuss the values of C_1 and C_2, which come from Lemma 3 below. The former is related to an eigenvalue of an interior problem, and the latter to an Agmon distance, both for the effective potential V_{M_0}. Such eigenvalues are known to approximate real parts of resonances near the real axis, while Agmon distances correspond to imaginary parts of the same resonances: see [HeSj] (especially §11 of that paper) and [FuLaMa] for results in a well-in-an-island setting, and [NaStZw, Corollary, §5] for an abstract statement. Let us emphasize that our lower bounds are in terms of an Agmon distance for V_{M_0} rather than for V, and the former may be much greater than the latter (for example, the latter vanishes if $\max V < E_0$). See also [Se] for one-dimensional resonance asymptotics using methods in some ways similar to ours, and [DaMa] for a more recent higher-dimensional result and more references.

An interesting special case is the one where V has a unique local minimum, located at the origin, and moreover this minumum is nondegenerate and $V_0(0) = E_0 > 0$. Then $M_0(E_0) = 0$, so that $V(x) = V_{M_0}(r)$ is a well-in-an-island type potential, and E_0 is at the bottom of the well. In that case, by [Na, Proposition 4.1], there is $\mu > 0$ such that for any U we have

$$\sup_{E\in[E_0-\mu h, E_0+\mu h]} \|\mathbf{1}_U(P - E)^{-1}\mathbf{1}_U\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C/h. \tag{2.10}$$

This upper bound shows that the form of the interval $[E_0 - C_1h, E_0 + C_1h]$ in (2.8) is optimal in general. See also [BoBuRa, §1] and [DaDyZw, §4] for further discussion.

3. Semiclassical ODE asymptotics

In this section we prove pointwise resolvent estimates, for energies E near E_0, for

$$P_m = P_m(h) = -h^2\partial_r^2 + V_m(r), \tag{3.1}$$

where V_m is given by (2.1) with $m \geq -h^2/4$. Let $\mathcal{D} = \mathcal{D}(m,h) \subset L^2(\mathbb{R}_+)$ be a domain for P_m so that P_m is selfadjoint and $C_c^\infty((0,\infty)) \subset \mathcal{D}$.

We briefly recall some facts about $\mathcal{D}$. By Proposition 2 and Theorems X.7, X.8, and X.10 of [ReSi, Appendix to X.I] we have $\mathcal{D} = \{u \in L^2(\mathbb{R}_+) \mid P_m u \in L^2(\mathbb{R}_+) \text{ and } \mathcal{B}u = 0\}$, where $\mathcal{B}u$ is a boundary condition at 0 with real coefficients, and moreover $\mathcal{B} = 0$ unless $m = O(h^2)$. If $m = O(h^2)$, then we may have $\mathcal{B} \neq 0$; below we will not need further information about $\mathcal{B}$, but see [Re, BuGe] and [Ze, §10.4] for descriptions of the possibilities. Note finally that $\mathcal{D}$ is preserved by complex conjugation because P_m has real coefficients.

The outgoing resolvent kernel at energy $E > 0$ is given by

$$K(r,r') = K(r,r';m;E;h) = -u_0(r)u_1(r')/h^2W, \qquad \text{for } r \leq r', \tag{3.2}$$

and it obeys $K(r,r') = K(r',r)$, where u_0 and u_1 are certain solutions to

$$P_m u_j = E u_j, \tag{3.3}$$

and $W = u_0 u_1' - u_0' u_1$ is their Wronskian. More specifically $u_0 \in L^2((0,1))$ and satisfies $\mathcal{B}u_0 = 0$, and $u_1(r)$ is outgoing, that is to say it is asymptotic to a multiple of $e^{ir\sqrt{E}/h}$ as $r \to \infty$. For convenience we assume without loss of generality that u_0 is real-valued.

In the remainder of §3 we prove three lemmas, each of which bounds $K(r,r')$ for a different range of r, r', m, and E. The first two will be used to prove (2.7), and the third to prove (2.8) and (2.9).

In the first lemma m is small enough that no turning point analysis is needed.

Lemma 1. *Fix $r_{2+} > r_2$, $M > 0$, and $I \Subset (0,\infty)$ such that $V_M(r) < E$ for all $r \ge r_{2+}$ and $E \in I$. Then*

$$|K(r,r')| \le \frac{h^{-1}(1+O(h))}{(E-V_m(r))^{1/4}(E-V_m(r'))^{1/4}}, \tag{3.4}$$

uniformly for all $r \ge r_{2+}$, $r' \ge r_{2+}$, $E \in I$, and $m \in [0,M]$.

In §4 we will specify r_{2+} and M; they will be slightly larger than r_2 and M_0 respectively.

Before giving the proof we give the idea. We will use the fact that u_0 and u_1 are each oscillatory, rather than exponentially growing or decaying, since we are in the classically allowed region $E > V_m$. So the upper bound follows from a lower bound on the Wronskian; this in turn follows from the fact that, roughly speaking, u_1 is like $\exp\left(\frac{i}{h}\int\sqrt{E-V_m}\right)$ since it is outgoing, while u_0 is equal amounts $\exp\left(\frac{i}{h}\int\sqrt{E-V_m}\right)$ and $\exp\left(-\frac{i}{h}\int\sqrt{E-V_m}\right)$ since it is real-valued.

Proof. For any u_0 as above, by [Ol, §6.2.4] there are real constants $A = A(h)$ and $B = B(h)$ such that for $r \ge r_{2+}$ we have

$$u_0(r) = (E-V_m(r))^{-1/4}\left(\sum_{\pm}(A \pm iB)\exp\left(\pm\frac{i}{h}\int_{r_{2+}}^{r}\sqrt{E-V_m(r')}dr'\right)(1+\varepsilon_\pm(r))\right), \tag{3.5}$$

where ε_+ and ε_- satisfy

$$|\varepsilon_\pm(r)| + h|\varepsilon_\pm'(r)| \le Chr^{-1}, \tag{3.6}$$

when $r \ge r_{2+}$.

Again by [Ol, §6.2.4], we can normalize u_1 to be the outgoing solution of (3.3) given by

$$u_1(r) = (E-V_m(r))^{-1/4}\exp\left(\frac{i}{h}\int_{r_{2+}}^{r}\sqrt{E-V_m(r')}dr'\right)(1+\varepsilon_+(r)), \tag{3.7}$$

for $r \ge r_{2+}$.

We compute the Wronskian

$$W = W(u_0,u_1) = \frac{A-iB}{\sqrt{E-V_m(r)}}\cdot\frac{2i}{h}\sqrt{E-V_m(r)}(1+O(hr^{-1})) = 2(B+iA)h^{-1}, \tag{3.8}$$

where we dropped the remainder because W is independent of r. Combining this with (3.2), (3.5), and (3.7) gives (3.4). □

In the second lemma m is large enough that a turning point analysis is needed. For our purposes the following bound which blows up near the turning point is sufficient, even though K is of course continuous there. We state a result for all $r > 0$ and $r' > 0$, even though we only use a smaller range in our application, since the result for the full range is obtained with no extra effort. We remark that a closely related turning point analysis appears in a recent paper of Yafaev on semiclassical asymptotics for eigenfunctions in a potential well [Ya].

Lemma 2. *Fix $M > 0$ such that $V_M'(r) < 0$ for all $r \geq r_2$, and fix $I \Subset (0, \infty)$ containing E_0 such that $V_M(r) > E$ for all $r < r_2$ and $E \in I$. Then*

$$|K(r, r')| \leq \frac{C_A \pi h^{-1}(1 + O(m^{-1/2}h))}{|E - V_m(r)|^{1/4}|E - V_m(r')|^{1/4}}, \tag{3.9}$$

uniformly for all $r > 0$, $r' > 0$, $m \geq M$ and $E \in I$, where C_A is given by (A.6).

Before giving the proof we give the idea. By rescaling, we can use $m^{-1/2}h$ as a new semiclassical parameter, and the turning point R is roughly given by $V_m^{-1}(E) \geq r_2$; the classically forbidden region is $r < R$ and classically allowed region is $r > R$. In the classically allowed region the bound holds for the same reason that it did in Lemma 1. In the classically forbidden region, the solutions are exponentially growing and decaying, rather than oscillatory, because $E < V_m$ there. But u_0 is forced to have only an exponentially decaying component and no exponentially growing one by the condition $u_0 \in L^2((0, 1))$. Since we are estimating an expression of the form $u_0(r)u_1(r')$ with $r \leq r'$, the decay from u_0 beats the growth from u_1. The Wronskian cannot be very small for the same reason as in Lemma 1. Near the turning point these arguments break down, as can be seen from the weakness of (3.9) when r or r' is close to $V_m^{-1}(E)$.

Proof. The proof proceeds in four steps. In the first we introduce some useful notation, including a change of variable $r \mapsto \zeta(r)$ in the manner of [Ol, §11.3]. In the second we express u_0 in terms of Airy functions, and compute asymptotics as $m^{1/2}h^{-1}$ and r become large. In the third we do the same for u_1, and in the fourth we compute the Wronskian and combine the previous results to conclude.

Put

$$m' = m + \tfrac{h^2}{4} \qquad \text{and} \qquad R = V_{m'}^{-1}(E), \tag{3.10}$$

and note that

$$R + |V_{m'}'(R)|^{-1} \leq C m^{1/2}; \tag{3.11}$$

see Figure 3.

Following [Ol, §11.3.1], we rewrite (3.3) as

$$u'' = \Big((m^{1/2}h^{-1})^2 f + g\Big) u, \qquad \text{where} \qquad f = m^{-1}(V_{m'} - E) \quad \text{and} \quad g = -\tfrac{1}{4}r^{-2}. \tag{3.12}$$

As we will see in (3.17) below, this decomposition leads to good asymptotic properties as $r \to 0$.

Define an increasing bijection $(0, \infty) \ni r \mapsto \zeta(r) = \zeta(r; m; h) \in \mathbb{R}$ by

$$\zeta(r) = \pm \left| \frac{3}{2h} \int_R^r \sqrt{E - V_{m'}(r')}dr' \right|^{2/3}, \qquad \text{when } \pm(r - R) \geq 0. \tag{3.13}$$

Note that our ζ differs from the one used in [Ol, §11.3] by a factor of $m^{1/3}h^{-2/3}$.

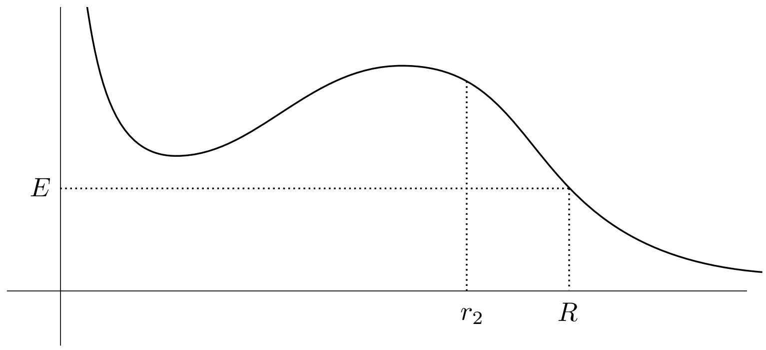

FIGURE 3. A possible graph of $V_{m'}$.

We will need the following bound for ζ: for $r > 0$ sufficiently small we have

$$\frac{2}{3}m^{-1/2}h(-\zeta(r))^{3/2} = m^{-1/2}\int_r^R \sqrt{V_{m'}(r') - E}dr' \geq \ln(1/r)/C. \tag{3.14}$$

By [Ol, §11.3.3], there are constants $A_0 = A_0(h)$ and $B_0 = B_0(h)$ such that

$$u_0(r) = \left(\frac{\zeta(r)}{E - V_{m'}(r)}\right)^{1/4} \left(A_0\left(\operatorname{Ai}(-\zeta(r)) + \varepsilon_A(r)\right) + B_0\left(\operatorname{Bi}(-\zeta(r)) + \varepsilon_B(r)\right)\right), \tag{3.15}$$

where Ai and Bi are given by (A.1), and ε_A and ε_B satisfy

$$|\varepsilon_A(r)| + |\varepsilon_B(r)| \leq Cm^{-1/2}h\langle\zeta(r)\rangle^{-1/4}, \quad \text{when } r \geq R, \tag{3.16}$$

and

$$\frac{|\varepsilon_A(r)|}{\operatorname{Ai}(-\zeta(r))} + \frac{|\varepsilon_B(r)|}{\operatorname{Bi}(-\zeta(r))} \leq Cm^{-1/2}h\langle\zeta(r)\rangle^{-1/4}, \quad \text{when } r \leq R; \tag{3.17}$$

see Appendix B for more details. We use m' rather than m in our definition of ζ because this choice makes (3.17) hold uniformly for all $r \in (0, R]$, rather than merely uniformly on compact subintervals of $(0, R]$: see Appendix B and also [Ol, §11.4.1 and §6.4.3].

Recalling that $u_0 \in L^2((0,1))$, and inserting (3.17), (A.2), and (A.3) into (3.15), we see that $B_0 = 0$. Without loss of generality we normalize u_0 so that $A_0 = 1$ and

$$u_0(r) = \left(\frac{\zeta(r)}{E - V_{m'}(r)}\right)^{1/4} \left(\operatorname{Ai}(-\zeta(r)) + \varepsilon_A(r)\right). \tag{3.18}$$

We now derive the simpler and better asymptotics which hold for large r; these will ease the computation of the Wronskian later. If $r \geq R + 1$, then inserting (3.16) and (A.4) into (3.18) gives

$$u_0(r) = \pi^{-1/2}(E - V_{m'}(r))^{-1/4}\left(\cos\left(\frac{2}{3}\zeta(r)^{3/2} - \frac{\pi}{4}\right) + O(m^{-1/2}h) + O(\zeta(r)^{-3/2})\right). \tag{3.19}$$

On the other hand, as in (3.5), there are real constants $A = A(h)$ and $B = B(h)$ such that for $r \geq R + 1$ we have

$$u_0(r) = (E - V_{m'}(r))^{-1/4}\left(\sum_{\pm}(A \pm iB)\exp\left(\pm\frac{i}{h}\int_R^r \sqrt{E - V_{m'}(r')}dr'\right)(1 + \varepsilon_{\pm}(r))\right), \tag{3.20}$$

where ε_+ and ε_- satisfy

$$|\varepsilon_\pm(r)| + m^{-1/2}h|\varepsilon_\pm'(r)| \le Cm^{-1/2}hr^{-1}, \tag{3.21}$$

when $r \ge R+1$. Setting (3.19) equal to (3.20) and applying (3.21) gives

$$\pi^{-1/2}\cos\left(\frac{1}{h}\varphi(r) - \frac{\pi}{4}\right) + O(m^{-1/2}h) = \sum_\pm (A \pm iB)\exp\left(\pm\frac{i}{h}\varphi(r)\right)(1 + O(m^{-1/2}h)), \tag{3.22}$$

for r large enough (depending on m and h), where we used

$$\varphi(r) = \frac{2h}{3}\zeta(r)^{3/2} = \int_R^r \sqrt{E - V_{m'}(r')}dr' \to \infty \text{ as } r \to \infty.$$

Hence

$$A \pm iB = 2^{-1}\pi^{-1/2}e^{\mp i\pi/4} + O(m^{-1/2}h). \tag{3.23}$$

Now we turn to u_1. As in (3.7), we normalize it by setting

$$u_1(r) = (E - V_{m'}(r))^{-1/4}\exp\left(\frac{i}{h}\int_R^r \sqrt{E - V_{m'}(r')}dr'\right)(1 + \varepsilon_+(r)), \tag{3.24}$$

for $r \ge R+1$. Using [Ol, §11.3.3] again, there are constants $A_1 = A_1(h)$ and $B_1 = B_1(h)$ such that for $r \in (0,\infty)$ we have

$$u_1(r) = \left(\frac{\zeta(r)}{E - V_{m'}(r)}\right)^{1/4}\left(A_1\left(\mathrm{Ai}(-\zeta(r)) + \varepsilon_A(r)\right) + B_1\left(\mathrm{Bi}(-\zeta(r)) + \varepsilon_B(r)\right)\right). \tag{3.25}$$

Proceeding as in the proof of (3.22) gives

$$\begin{aligned} A_1\left(\cos\left(\frac{1}{h}\varphi(r) - \frac{\pi}{4}\right) + O(m^{-1/2}h)\right) - B_1\left(\sin\left(\frac{1}{h}\varphi(r) - \frac{\pi}{4}\right) + O(m^{-1/2}h)\right)& \\ = \sqrt{\pi}\exp\left(\frac{i}{h}\varphi(r)\right)(1 + O(m^{-1/2}hr^{-1}))&. \end{aligned}$$

Hence we have

$$A_1 = \sqrt{\pi}e^{i\pi/4} + O(m^{-1/2}h), \qquad B_1 = \sqrt{\pi}e^{-i\pi/4} + O(m^{-1/2}h), \tag{3.26}$$

which implies

$$|u_1(r)|^2 = \pi\left|\frac{\zeta(r)}{E - V_{m'}(r)}\right|^{1/2}\left(\mathrm{Ai}(-\zeta(r))^2 + \mathrm{Bi}(-\zeta(r))^2\right)(1 + O(m^{-1/2}h)). \tag{3.27}$$

We compute the Wronskian $W = u_0u_1' - u_0'u_1$ as in the proof of (3.8), and apply (3.23):

$$W = 2(B + iA)h^{-1} = \pi^{-1/2}e^{3i\pi/4}h^{-1}(1 + O(m^{-1/2}h)). \tag{3.28}$$

Now (3.9) follows from inserting (3.18), (3.27), and (3.28) into (3.2), applying (3.16), (3.17), and (A.6), and observing that $m' = m + h^2/4$ allows us to replace $V_{m'}$ by V_m in the final statement (actually, keeping $V_{m'}$ gives a slightly better bound). □

In the third lemma the analysis is similar to that in the second, but slightly easier since we consider only a bounded range of m. To obtain good lower bounds we consider only a particular energy level, rather than an interval of energies as in the previous two lemmas. We actually obtain an asymptotic, rather than merely a lower bound, with no extra effort.

Having information about K in (3.30), rather than just $|K|$, is important for our application to wave non-decay in Theorem 2, where we need a lower bound on $|\operatorname{Im} K|$.

Lemma 3. *Fix $r_{1+} < r_{2-} \in (r_1, r_2)$ and fix $r_{2+} > r_2$. For any $m = m(h) = M_0 + O(h)$, there is an energy level $E = E(h) = E_0 + O(h)$, such that*

$$|K(r,r')| = \frac{e^{S(r)/h}h^{-1}(1+O(h^{1/3}))}{2(V_m(r)-E)^{1/4}(E-V_m(r'))^{1/4}}, \text{ when } r \in [r_{1+}, r_{2-}],\ r' \geq r_{2+}, \tag{3.29}$$

and

$$K(r,r') = \frac{-e^{-i\pi/6}e^{S(r)/h}e^{S(r')/h}h^{-1}(1+O(h^{1/3}))}{2(V_m(r)-E)^{1/4}(V_m(r')-E)^{1/4}}, \text{ when } r,\ r' \in [r_{1+}, r_{2-}] \tag{3.30}$$

where

$$S(r) = \int_r^R \sqrt{V_m(r')-E}\,dr', \tag{3.31}$$

with $R = \max V_m^{-1}(E) = r_2 + O(h)$. See Figure 4.

Moreover, it suffices to take E to be an eigenvalue of P_m as an operator on $L^2((0,r_2))$ with domain

$$\mathcal{D}_{r_2} = \{u \in L^2((0,r_2)) \mid P_m u \in L^2((0,r_2)) \text{ and } \mathcal{B}u = u(r_2) = 0\}. \tag{3.32}$$

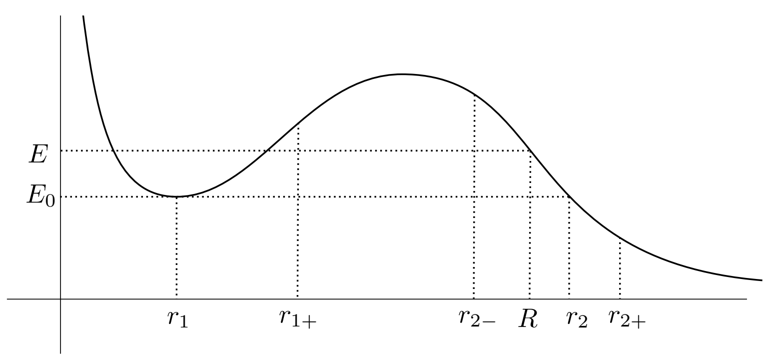

FIGURE 4. A possible graph of V_m in the case $m = M_0$. We think of $r_{2+} - r_{2-}$ and $r_{1+} - r_1$ as being very small, although for clarity this is not so in the picture. In §4 we will take $m = M_0 + O(h)$, and the graph must be suitably changed: in particular, in the proof of (2.9) we will need $E = E_0$ and hence $m < M_0$.

Before giving the proof we give the idea. The operator $P_m - E$ has the same turning point behavior here as in Lemma 2, but this time we must take advantage of the trapping occuring at r_1. We do this by finding an energy level $E = E_0 + O(h)$ which is also an eigenvalue of an interior problem (we use the Dirichlet problem on $(0, r_2)$) and then taking as u_0 the corresponding eigenfunction (this is our quasimode). This way both u_0 and u_1 grow exponentially in the classically forbidden region between r_1 and r_2, giving the desired result.

Proof. We begin by proving that P_m has an eigenvalue $E = E_0 + O(h)$ as an operator on $L^2((0, r_2))$ with domain $\mathcal{D}_{r_2}$. Note first that the spectrum of this operator is discrete since the domain is contained in $H^1((0, r_2))$, and let E be the bottom of the spectrum. Then we have $E \ge E_0 - Ch$ because $E_0 = \min\{V_{M_0}(r) \mid r \in (0, r_2)\}$ by (2.2), the definition of M_0. To see that $E \le E_0 + Ch$, fix $\alpha > 0$ such that

$$V_m(r) - E_0 \le \alpha^2 (r - r_1)^2 + Ch, \tag{3.33}$$

near r_1, and let $w(r) = e^{-\alpha(r-r_1)^2/2h}\chi(r)$ where $\chi \in C_c^\infty((0, r_2); [0, 1])$ is 1 near r_1 and is supported inside the set where (3.33) holds. Then we have

$$0 \le \int (-h^2 w'' + V_m w - Ew)w \le (E_0 - E + Ch) \int w^2, \tag{3.34}$$

which concludes the proof that $|E - E_0| \le Ch$.

Let u_0 be a corresponding real-valued eigenfunction, and extend u_0 to solve (3.3) on all of $\mathbb{R}_+$.

Now we may again write u_0 in terms of Airy functions as in the proof of Lemma 2, with ζ defined by (3.13), but now $m' = m$ and R are as in the statement of the present Lemma. The two main differences for our work here compared to that in the proof of Lemma 2 are that we have good remainder bounds only when $r \ge r_{1+}$, rather than for all $r \in (0, \infty)$, and that m stays bounded.

More precisely, by [Ol, §11.3.3], there are real constants $A_0 = A_0(h)$ and $B_0 = B_0(h)$ such that for $r \ge r_{1+}$ we have (3.15), where ε_A and ε_B satisfy (3.16) for $r \ge R$ and (3.17) for $r \in [r_{1+}, R]$. We will need the following bounds on ζ near the turning point R:

$$h|\zeta(r)|^{3/2} = |V_m'(R)|^{1/2}|R - r|^{3/2}(1 + O(|R - r|)), \quad \text{when } |r - R| \le 1, \tag{3.35}$$

where we used $E - V_m(r) = (R - r)V_m'(R) + O((R - r)^2)$.

Without loss of generality we normalize u_0 so that

$$A_0^2 + B_0^2 = 1, \qquad A_0 \ge 0. \tag{3.36}$$

Since $u_0(r_2) = 0$ we have

$$A_0 \left(\mathrm{Ai}(-\zeta(r_2)) + O(h)\right) + B_0 \left(\mathrm{Bi}(-\zeta(r_2)) + O(h)\right) = 0.$$

Now observe that by $R = r_2 + O(h)$ and (3.35) we have $|\zeta(r_2)| \le Ch^{1/3}$, and since $\mathrm{Bi}(0) = \mathrm{Ai}(0)\sqrt{3} > 0$, we obtain

$$A_0 + B_0(\sqrt{3} + O(h^{1/3})) = 0,$$

and combining with (3.36) gives

$$A_0 = \frac{\sqrt{3}}{2} + O(h^{1/3}), \qquad B_0 = -\frac{1}{2} + O(h^{1/3}). \tag{3.37}$$

When $r \ge r_{2+}$ we have (3.20) with constants A and B, which we can compute as in (3.23) to find

$$A \pm iB = 2^{-1}\pi^{-1/2}e^{\mp i\pi/4}(A_0 \pm iB_0) + O(h). \tag{3.38}$$

We take u_1 to be given by (3.24) for $r \ge r_{2+}$ as before, where now (3.25) holds for $r \ge r_{1+}$ with A_1 and B_1 given by (3.26). We now have (3.27) for $r \ge r_{1+}$. This time the Wronskian

$W = u_0u_1' - u_0'u_1$ obeys

$$W = 2(B+iA)h^{-1} = e^{i11\pi/12}\pi^{-1/2}h^{-1}(1+O(h)), \tag{3.39}$$

where we used (3.38) and (3.37).

Inserting (A.2) and (A.3) into (3.15) and (3.25) and using (3.17) and (3.35) gives

$$\sqrt{\pi}u_j(r) = B_j(V_m(r)-E)^{-1/4}e^{S(r)/h}(1+O(h)), \tag{3.40}$$

for $r \in [r_{1+}, r_{2-}]$, where $j \in \{0,1\}$. Then (3.30) follows from inserting (3.39) and (3.40) into (3.2) and using (3.26) and (3.37). We obtain (3.29) by the same argument with (3.24) in place of (3.40) for u_1. □

4. Proofs of Theorems

4.1. **Proof of Theorem 3.** Let $0 = \sigma_0 < \sigma_1 = \sigma_2 \le \sigma_3 \le \cdots$ be the eigenvalues of the unit sphere of dimension $n-1$, repeated according to multiplicity, and let $Y_0, Y_1, \ldots$ be a corresponding sequence of orthonormal real eigenfunctions.

If v is in the domain of P, then

$$r^{(n-1)/2}Pr^{-(n-1)/2}v(x) = \sum_{j=0}^{\infty} Y_j P_{m_j} v_j(r), \quad \text{where } v(x) = \sum_{j=0}^{\infty} Y_j v_j(r), \tag{4.1}$$

where P_{m_j} is given by (3.1) with

$$m_j = m_j(h) = h^2(\sigma_j + \tfrac{(n-1)(n-3)}{4}), \quad v_j(r) = \int Y_j(\theta)v(r,\theta)d\theta. \tag{4.2}$$

Here $d\theta$ is the usual surface measure on the unit sphere and $v_j \in \mathcal{D}$ for some $\mathcal{D} = \mathcal{D}(m_j, h)$ as in §3.

Similarly, if $v \in L^2(\mathbb{R}^n)$ is compactly supported, then so is $v_j \in L^2(\mathbb{R}_+)$ and

$$r^{(n-1)/2}(P-E\pm i0)^{-1}r^{-(n-1)/2}v(x) = \sum_{j=0}^{\infty} Y_j(P_{m_j} - E \pm i0)^{-1}v_j(r), \tag{4.3}$$

with the outgoing resolvent $(P_{m_j} - E - i0)^{-1}$ having integral kernel K given by (3.2), and the incoming resolvent $(P_{m_j} - E + i0)^{-1}$ having integral kernel given by the complex conjugate of K. (Actually, in (4.3) we could instead take v in a weighted space but we will not need this.)

Hence if χ_L and χ_R are bounded and compactly supported functions on $[0,\infty)$, then

$$\|\chi_L(r)(P-E\pm i0)^{-1}\chi_R(r)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \sup_{j\in\mathbb{N}_0} \|\chi_L(r)(P_{m_j}-E\pm i0)^{-1}\chi_R(r)\|_{L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)}.$$

Proof of (2.8). Suppose without loss of generality that $r_L \le r_R$. Then (2.8) follows from Lemma 3, applied with $m = m_j$ for any $j = j(h)$ chosen such that $m_j = M_0 + O(h)$, and with r_{1+}, r_{2-}, r_{2+} chosen such that $\partial B(0,r^*) \subset U_L$ for some $r^* \in [r_{1+}, r_{2-}]$ and $\partial B(0, r^{**}) \subset U_R$ for some $r^{**} \in [r_{1+}, r_{2-}] \cup [r_{2+}, \infty)$. □

Proof of (2.7). Fix $r_{2+} > r_2$ such that U is disjoint from a neighborhood of $\overline{B(0, r_{2+})}$. Fix $M > M_0$ and $I \Subset (0, \infty)$ containing E_0 such that the hypotheses of Lemmas 1 and 2 are satisfied (it is enough if $M - M_0$ and the length of I are sufficiently small). Then apply the Hilbert–Schmidt bound

$$\|\chi(r)(P_{m_j} - E \pm i0)^{-1}\chi(r)\|^2_{L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)} \le \|\chi\|^4_{L^\infty} \int_{\operatorname{supp}\chi}\int_{\operatorname{supp}\chi} |K(r,r')|^2 dr dr' \le Ch^{-2},$$

which holds uniformly for all $j \in \mathbb{N}_0$. □

Finally, to prove (2.9), by Lemma 3 it suffices to show that E_0 is an eigenvalue of P_{m_j} on $\mathcal{D}_{r_2}$, for some sequence $h_j \to 0$ such that $m_j = m_j(h_j)$ obeys $|m_j - M_0| \le Ch_j$, for j sufficiently large. This follows from a calculation very similar to that in (3.33) and (3.34), but note that now we will necessarily have $m_j \le M_0$; this is reasonable because we are now assuming $\max V < E_0$ and hence $M_0 > 0$ by definition (2.2). We will first define the sequence h_j, then prove that

$$m_j \le M_0, \tag{4.4}$$

and then prove that

$$m_j \ge M_0 - Ch_j. \tag{4.5}$$

We define h_j for j sufficiently large by demanding that h_j^{-2} be the bottom of the spectrum of

$$L_j := (E_0 - V_0(r))^{-1}\left(-\partial_r^2 + (\sigma_j + \tfrac{(n-1)(n-3)}{4})r^{-2}\right),$$

as an operator on $L^2((0, r_2), (E_0 - V_0(r))dr)$ with domain

$$\{u \in L^2((0,r_2)) \mid L_j u \in L^2((0,r_2)) \text{ and } u(r_2) = 0\}.$$

Note that this operator is selfadjoint (that is, there is no need for an analogue of the condition $\mathcal{B}u = 0$ as in the definition of $\mathcal{D}$ in the beginning of §3) as long as $4\sigma_j + (n-1)(n-3) > 3$, that is to say for all but possibly finitely many j: see [Ze, Theorem 10.4.4] and also [ReSi, Theorem X.10]. The spectrum is discrete since the domain is contained $H^1((0, r_2))$. Also, $h_j \to 0$ will follow from (4.4). Hence, to show (2.9) it is enough to prove (4.4) and (4.5).

Proof of (4.4). We have, for $u \in C_c^\infty((0, r_2))$,

$$\int_0^{r_2} (L_j u)(r)\overline{u(r)}(E_0 - V_0(r))dr \ge \int_0^{r_2} (\sigma_j + \tfrac{(n-1)(n-3)}{4})r^{-2}|u(r)|^2 dr$$
$$\ge M_0^{-1}(\sigma_j + \tfrac{(n-1)(n-3)}{4})\int_0^{r_2} |u(r)|^2(E_0 - V_0(r))dr,$$

where we used the fact that by definition

$$M_0^{-1} = \min\{r^{-2}(E_0 - V_0(r))^{-1} \mid r \in (0, r_2)\}.$$

This implies $h_j^{-2} > M_0^{-1}(\sigma_j + \frac{(n-1)(n-3)}{4})$ and hence (4.4). □

Proof of (4.5). Let $w(r) = e^{-\alpha(r-r_1)^2/2h_j}\chi(r)$, with α as in (3.33) and χ as in the line after. Then, as in (3.34),

$$0 \le \int_0^{r_2}\left[(L_j - h_j^{-2})w(r)\right]w(r)(E_0 - V_0(r))dr \le \int_0^{r_2}\left((\sigma_j - M_0 h_j^{-2})r^{-2} + Ch_j^{-1}\right)w(r)^2 dr.$$

This implies $M_0 h_j^{-2} \le \sigma_j + C h_j^{-1}$ and hence (4.5). □

4.2. **Proof of Theorem 1.** To deduce Theorem 1 from 3, we begin with some useful formulas relating M_0, r_1, and r_2. We define

$$\Phi(r) = r^2(E_0 - V_0(r)).$$

Lemma 4. *With the assumptions and notation of Theorem 3, we have*

$$r \ge r_2 \Longrightarrow \Phi'(r) > 0, \tag{4.6}$$

$$\Phi(r_1) = \Phi(r_2) = M_0, \tag{4.7}$$

$$r_1 < r < r_2 \Longrightarrow \Phi(r) < M_0, \tag{4.8}$$

$$r \le r_2 \Longrightarrow \Phi(r) \le M_0. \tag{4.9}$$

See Figure 5.

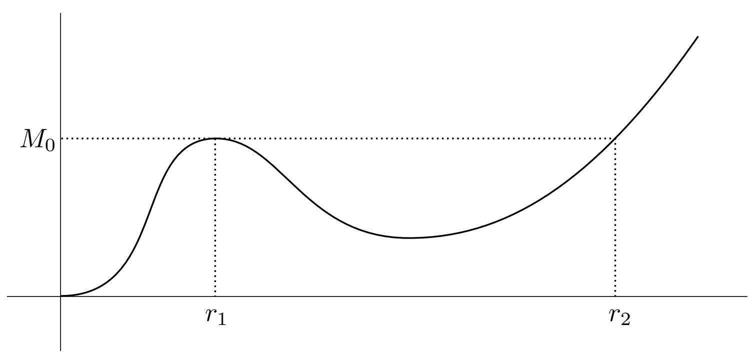

FIGURE 5. A possible graph of Φ.

Proof. To show (4.6) we compute

$$\Phi'(r) = 2r(E_0 - V_0(r)) - r^2 V_0'(r) = 2r(E_0 - V_{M_0}(r)) - r^2 V_{M_0}'(r),$$

and the right hand side is positive when $r \ge r_2$ by (2.5). To prove the other three statements we observe $\Phi(r) \le M_0$ is the same as $V_{M_0(r)} \ge E_0$, with equality always holding simultaneously. □

Now suppose that V_0 is compactly supported in $[0, 1)$ and $\min V_0 < 0$. Then $\Phi(1) = E_0$. If

$$E_0 < \max_{r\in[0,1)} (-r^2 V_0(r)) \le \max_{r\in[0,1)} \Phi(r) = M_0,$$

then $r_1 < 1 < r_2$, and we have moreover

$$r_2 = r_2(E_0) = \sqrt{M_0/E_0}, \tag{4.10}$$

and

$$\lim_{E_0\to 0} E_0 r_2^2 = \lim_{E_0\to 0} M_0 = \min\{m > 0 \mid V_m(r) \ge 0 \text{ for all } r > 0\} = -\min\{r^2 V_0(r)\}, \tag{4.11}$$

as desired.

4.3. **Proof of Theorem 2.** It is convenient to work over the Hilbert space $\mathcal{H} = \dot{H}^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n, dx/c_0(r)^2)$. Let

$$B = -i \begin{pmatrix} 0 & 1 \\ c^2\Delta & 0 \end{pmatrix}.$$

Then B is selfadjoint on $\mathcal{H}$ with domain $\{(u_0, u_1) \in \mathcal{H} \colon \Delta u_0 \in L^2(\mathbb{R}^n), u_1 \in H^1(\mathbb{R}^n)\}$, and we will study the unitary wave propagator $e^{itB} \colon \mathcal{H} \to \mathcal{H}$.

Theorem 2 follows from

Lemma 5. *Let c, ρ, and R_c be as in Theorem 2.*

(1) If $\chi \in C_c^\infty(\mathbb{R}^n)$ has support disjoint from the closed ball $\overline{B(0,R_c)}$, then there is $C > 0$ such that

$$\int_{-\infty}^{\infty} \|\chi e^{itB} u\|_{\mathcal{H}}^2 dt \le C\|u\|_{\mathcal{H}}^2, \tag{4.12}$$

for all $u \in \mathcal{H}$.

(2) If $\chi \in C_c^\infty(\mathbb{R}^n; [0,\infty))$ is positive on the sphere $\partial B(0,R)$ for some $R \in [\rho, R_c]$, then

$$\sup_{\|u\|_{\mathcal{H}}=1} \int_{-\infty}^{\infty} \|\chi e^{itB} u\|_{\mathcal{H}}^2 dt = +\infty. \tag{4.13}$$

Indeed, to prove Theorem 2 from Lemma 5, we observe that if $\operatorname{supp} \chi_0 \subset U$ and $\chi_1 = 1$ near U, then

$$C^{-1}\|\chi_0 e^{itB} u\|_{\mathcal{H}}^2 \le \mathcal{E}_U(t) \le C\|\chi_1 e^{itB} u\|_{\mathcal{H}}^2,$$

with $u = (w, \partial_t w)$, where for the first inequality we used Poincaré's inequality

$$\|v\|_{L^2(U)} \le C\|\nabla v\|_{L^2(U)}, \qquad v \in C_c^\infty(U).$$

To prove Lemma 5, we will need some facts about the resolvent of B, based on the formula

$$(B-\lambda)^{-1} = \begin{pmatrix} \lambda(-c^2\Delta - \lambda^2)^{-1} & -i(-c^2\Delta - \lambda^2)^{-1} \\ i\lambda^2(-c^2\Delta - \lambda^2)^{-1} + i & \lambda(-c^2\Delta - \lambda^2)^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}; \tag{4.14}$$

see [PoVo, p. 265], [Bu3, (2.13)], and [Sh2, (6.6)]. For any $\chi \in C_c^\infty(\mathbb{R}^n)$, the cutoff resolvent $\chi(-c^2\Delta - \lambda^2)^{-1}\chi$ extends continuously from the lower, or upper, half plane to its closure, as an operator from $L^2(\mathbb{R}^n)$ to $H^2(\mathbb{R}^n)$ (see item 1 of [DadH, Lemma 4.1] for a proof using the Sjöstrand–Zworski black box theory [SjZw]). By (4.14) we see that $\chi(B-\lambda)^{-1}\chi$ has corresponding continuous extensions as an operator from $\mathcal{H}$ to $\mathcal{H}$, and we denote these by $\chi(B - \lambda \pm i0)^{-1}\chi$, where $\lambda \in \mathbb{R}$.

Lemma 6. *Let c, ρ, and R_c be as in Theorem 2.*

(1) If $\chi \in C_c^\infty(\mathbb{R}^n)$ has support disjoint from the closed ball $\overline{B(0,R_c)}$, then there is $C > 0$ such that

$$\|\chi(B - \lambda \pm i0)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \le C, \tag{4.15}$$

for all $\lambda \in \mathbb{R}$.

(2) If $\chi \in C_c^\infty(\mathbb{R}^n; [0,\infty))$ is positive on the sphere $\partial B(0,R)$ for some $R \in [\rho, R_c]$, then there are $C > 0$ and a sequence $\lambda_j \to +\infty$ such that

$$\|\chi\left[(B - \lambda_j + i0)^{-1} - (B - \lambda_j - i0)^{-1}\right]\chi\|_{\mathcal{H}\to\mathcal{H}} \ge e^{C\lambda_j}. \tag{4.16}$$

Proof of Lemma 6. We use the identity

$$(-c^2\Delta - (\lambda \pm i0)^2)^{-1} = (-h^2\Delta + V - (\kappa^{-2} \pm i0))^{-1} c^{-2}\lambda^{-2},$$

where $h = \lambda^{-1}$ and $V = \kappa^{-2} - c^{-2}$. To check this, note that if $f \in L^2(\mathbb{R}^n)$ is compactly supported, and

$$(-c^2\Delta - \lambda^2)u = f, \qquad (-h^2\Delta + V - \kappa^{-2})v = c^{-2}\lambda^{-2}f,$$

with u and v both outgoing, then $u = v$ (see [DyZw, Theorem 3.34 and Theorem 4.18]).

With $E_0 = \kappa^{-2}$ and $V_0 = \kappa^{-2} - c_0^{-2}$, we have, in the notation of Theorem 3 and Lemma 4, $\Phi(r) = r^2/c_0^2(r)$. Then

$$M_0 = \max_{r\in[0,\rho]} \Phi(r) = R_c^2/\kappa^2 > \rho^2/\kappa^2 = \Phi(\rho),$$

and hence $r_1 < \rho < r_2 = R_c$.

(1) By (2.7), for all $\chi \in C_c^\infty(\mathbb{R}^n)$ having support disjoint from the closed ball $\overline{B(0, R_c)}$, we have

$$\|\chi(-c^2\Delta - (\lambda \pm i0)^2)^{-1}\chi\|_{L^2\to L^2} \le C\langle\lambda\rangle^{-1}.$$

By a standard argument (see for example the proof of [Bu3, Proposition 2.4]), together with (4.14) this implies (4.15) for all such χ.

(2) To prove (4.16), we argue similarly, but using the following refined version of (2.9):

$$\|\chi\left[(-h_j^2\Delta + V - (\kappa^{-2} + i0))^{-1} - (-h_j^2\Delta + V - (\kappa^{-2} - i0))^{-1}\right]\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \ge e^{C/h_j},$$

where $\chi \in C_c^\infty(\mathbb{R}^n; [0,\infty))$ is positive on the sphere $\partial B(0, R)$ for some $R \in [\rho, R_c]$, and h_j is the same sequence appearing in (2.9). This refined version follows from (3.30) in the same way that (2.9) does, and it implies (4.16) with $\lambda_j = h_j^{-1}$.

□

Proof of Lemma 5. This is a version of Kato smoothing [Ka]; see also [ReSi, §XIII.7] for another general presentation of the theory. We will use an AA^* argument as in [BuGéTz, §2.3] and [Bu3, §2]; see also [DyZw, §7.1]. We define the operator

$$A : \mathcal{H} \ni u \mapsto \chi e^{itB}u \in \mathcal{S}'(\mathbb{R}; \mathcal{H})$$

with adjoint

$$A^* : \mathcal{S}(\mathbb{R}; \mathcal{H}) \ni f \mapsto \int_{\mathbb{R}} \chi e^{-isB} f(s) ds \in \mathcal{H}.$$

Boundedness of A from $\mathcal{H}$ to $L^2(\mathbb{R}; \mathcal{H})$ is equivalent to boundedness of AA^* from $L^2(\mathbb{R}; \mathcal{H})$ to $L^2(\mathbb{R}; \mathcal{H})$. We write

$$AA^*f = \chi \int_{-\infty}^t e^{i(t-s)B}\chi f(s)ds + \chi \int_t^\infty e^{i(t-s)B}\chi f(s)ds =: \chi u_-(t) + \chi u_+(t). \tag{4.17}$$

Now let us suppose temporarily that there is $T > 0$ such that

$$\operatorname{supp} f \subset [-T, T], \tag{4.18}$$

so that in particular

$$\operatorname{supp} u_- \subset [-T, \infty) \text{ and } \operatorname{supp} u_+ \subset (-\infty, T].$$

We use the equations

$$u_\pm'(t) = iBu_\pm(t) \mp \chi f(t)$$

to compute the Fourier–Laplace transforms

$$\hat{u}_\pm(\lambda \pm i\varepsilon) := \int_{\mathbb{R}} e^{-i(\lambda \pm i\varepsilon)t} u_\pm(t)dt = \mp i(B - (\lambda \pm i\varepsilon))^{-1}\chi \hat{f}(\lambda \pm i\varepsilon), \tag{4.19}$$

where $\lambda \in \mathbb{R}$ and $\varepsilon > 0$. By Plancherel's theorem,

$$\begin{aligned} \int_{\mathbb{R}} \|AA^* f\|_{\mathcal{H}}^2 dt &= \frac{1}{2\pi} \int_{\mathbb{R}} \|\chi \hat{u}_-(\lambda - i0) + \chi \hat{u}_+(\lambda + i0)\|_{\mathcal{H}}^2 d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left\| \chi \left[(B - \lambda + i0)^{-1} - (B - \lambda - i0)^{-1} \right] \chi \hat{f}(\lambda) \right\|_{\mathcal{H}}^2 d\lambda, \end{aligned} \tag{4.20}$$

where we used (4.17) and (4.19), and where we allow the integrals to take the value $+\infty$. By density, (4.20) holds also without the assumption (4.18). Now (4.12) follows from (4.15).

To deduce (4.13) from (4.16), we must show that (4.16) implies

$$\left\| \chi \left[(B - \bullet + i0)^{-1} - (B - \bullet - i0)^{-1} \right] \chi \right\|_{L^2(\mathbb{R};\mathcal{H}) \to L^2(\mathbb{R};\mathcal{H})} = +\infty.$$

But this is clear since $\lambda \mapsto \chi \left[(B - \lambda + i0)^{-1} - (B - \lambda - i0)^{-1} \right] \chi$ is continuous by the discussion following (4.14) and unbounded by (4.16).

□

Appendix A. Airy functions

In this appendix we review some needed facts about the Airy functions given by

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{1}{\pi} \int_0^\infty \cos\left(\tfrac{t^3}{3} + xt\right) dt, \\ \mathrm{Bi}(x) &= \frac{1}{\pi} \int_0^\infty \left(e^{-\frac{t^3}{3} + xt} + \sin\left(\tfrac{t^3}{3} + xt\right) \right) dt. \end{aligned} \tag{A.1}$$

These are solutions to $u'' = xu$ satisfying $\mathrm{Bi}(0) = \sqrt{3}\,\mathrm{Ai}(0) > 0$, and as $x \to \infty$ we have

$$2\sqrt{\pi} x^{1/4}\, \mathrm{Ai}(x) = e^{-2x^{3/2}/3}(1 + O(x^{-3/2})), \tag{A.2}$$

$$\sqrt{\pi} x^{1/4}\, \mathrm{Bi}(x) = e^{2x^{3/2}/3}(1 + O(x^{-3/2})), \tag{A.3}$$

$$\sqrt{\pi} x^{1/4}\, \mathrm{Ai}(-x) = \cos\left(\tfrac{2}{3}x^{3/2} - \tfrac{\pi}{4}\right) + O(x^{-3/2}), \tag{A.4}$$

$$-\sqrt{\pi} x^{1/4}\, \mathrm{Bi}(-x) = \sin\left(\tfrac{2}{3}x^{3/2} - \tfrac{\pi}{4}\right) + O(x^{-3/2}). \tag{A.5}$$

These results can be found in [Ol, §11.1], among other places. In particular, there is a constant $C_A > 0$ such that, for all real x and x' satisfying $x' \le x$, we have

$$|x^{1/2} x'^{1/2}\, \mathrm{Ai}(x)^2 (\mathrm{Ai}(x')^2 + \mathrm{Bi}(x')^2)| \le C_A^2. \tag{A.6}$$

Appendix B. Remainder bounds for Airy approximations

In this appendix we prove (3.16) and (3.17). By [Ol, Chapter 11, Theorem 3.1], it is enough to show that $\int_0^\infty |G(r)|dr$ is uniformly bounded for all h and m, where

$$G(r) = |f(r)|^{-1/2}\left(5f(r)^{-2}f'(r)^2 - 4f(r)^{-1}f''(r) - 16g(r) - 5f(r)\zeta_O(r)^{-3}\right). \tag{B.1}$$

with f and g as in (3.12) and

$$\zeta_O(r) = -m^{-1/3}h^{2/3}\zeta(r) = \pm\left|\frac{3}{2}\int_R^r \sqrt{f(r')}dr'\right|^{2/3}, \text{ when } \pm(R-r) \geq 0. \tag{B.2}$$

We will first show that there is $\delta > 0$ such that

$$\int_{R-\delta\sqrt{m}}^{R+\delta\sqrt{m}} |G(r)|dr \leq C, \tag{B.3}$$

and then we will show that

$$\int_{R+\delta\sqrt{m}}^{\infty} |G(r)|dr + \int_0^{R-\delta\sqrt{m}} |G(r)|dr \leq C. \tag{B.4}$$

Proof of (B.3). By Taylor's theorem, for $r \in [R-\delta\sqrt{m}, R+\delta\sqrt{m}]$ we have

$$f(r) = (r-R)f'(R)\left(1 + \frac{f''(R)}{2f'(R)}(r-R) + O(m^{-1}(r-R)^2)\right),$$

where we used $f'(R) \leq -m^{-3/2}/C$ and $|f'''(r)| \leq Cm^{-5/2}$ (these follow from (3.11)). Similar expansions hold for powers and derivatives of f, and inserting the expansion for $\sqrt{f}$ into (B.2) gives

$$\zeta_O(r) = (r-R)f'(R)^{1/3}\left(1 + \frac{f''(R)}{10f'(R)}(r-R) + O(m^{-1}(r-R)^2)\right).$$

Hence

$$f(r)^{-2}f'(r)^2 = (r-R)^{-2}\left(1 + \frac{f''(R)}{f'(R)}(r-R) + O(m^{-1}(r-R)^2)\right),$$

$$f(r)^{-1}f''(r) = (r-R)^{-1}\frac{f''(R)}{f'(R)}\left(1 + O(m^{-1}(r-R)^2)\right),$$

$$f(r)\zeta_O(r)^{-3} = (r-R)^{-2}\left(1 + \frac{f''(R)}{5f'(R)}(r-R) + O(m^{-1}(r-R)^2)\right).$$

Combining these and using $|g(r)| \leq Cm^{-1/2}$ gives

$$|G(r)| \leq Cm^{-1}(r-R)^{-1/2},$$

which implies (B.3). □

Proof of (B.4). To bound the first term in (B.4) we use

$$m|f(r)| + m^{-1}|f(r)^{-1}| + r^2|g(r)| + r^3|f'(r)| + r^4|f''(r)| \leq C,$$

and

$$\left|\tfrac{2}{3}m^{1/2}\zeta_O(r)^{3/2} - r\sqrt{E}\right| = \left|\int_R^r \frac{V_{m'}(r')dr'}{(\sqrt{E - V_{m'}(r')} + \sqrt{E})} + R\sqrt{E}\right| \leq C, \tag{B.5}$$

for $r \geq R + \delta\sqrt{m}$. These imply

$$|G(r)| \leq Cm^{1/2}r^{-2},$$

for $r \geq R + \delta\sqrt{m}$, which implies the bound on the first term in (B.4).

To bound the second term in (B.4), we use

$$r^2 f(r) + \ln^{-2}(R/r)\zeta_O(r)^{-3} \leq C,$$

for $r \leq R - \delta\sqrt{m}$ (for the last term c.f. (3.14)), which implies

$$\int_0^{R-\delta\sqrt{m}} f(r)^{1/2}\zeta_O(r)^{-3}dr \leq C. \tag{B.6}$$

Let $F = \lim_{r\to 0} r^2 f$, so that for $k \in \{0,\, 1,\, 2\}$ we have

$$\partial_r^k f(r) = (-1)^k(k+1)! F r^{-2-k}(1 + O(m^{-1/2}r)),$$

for $r \leq R - \delta\sqrt{m}$. Then, since $F \geq 1/C$, we have

$$f(r)^{-1/2}\left|5f(r)^{-2}f'(r)^2 - 4f(r)^{-1}f''(r) - 16g(r)\right| \leq Cm^{-1/2},$$

and inserting into (B.1) and combining with (B.6) gives the bound on the second term in (B.4). □

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