Gromov-Hausdorff Limits of Kähler Manifolds with Ricci Curvature Bounded Below II

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Abstract

We study noncollapsed Gromov-Hausdorff limits of Kähler manifolds with Ricci curvature bounded below. Our main result is that each tangent cone is homeomorphic to a normal affine variety. This extends a result of Donaldson-Sun, who considered noncollapsed limits of polarized Kähler manifolds with two-sided Ricci curvature bounds. © 2019 Wiley Periodicals, Inc.

1 Introduction

Consider a sequence (M_i, ω_i, p_i) of pointed complete Kähler manifolds of dimension n, with $\mathrm{Ric}(\omega_i) > -\omega_i$ and $\mathrm{Vol}(B(p_i, 1)) > \nu > 0$. Suppose that the sequence converges to a metric space (Z, d, p) in the pointed Gromov-Hausdorff sense. By the work of Cheeger-Colding [3], and more recently Cheeger-Jiang-Naber [6] and others, we have a detailed understanding of the structure of Z, even if the M_i are merely Riemannian. A starting point for this structure theory is Cheeger-Colding's result [2] that the limit space Z admits tangent cones at each point that are metric cones. In this paper we are interested in studying the additional structure of the tangent cones of Z in the Kähler case.

There are few general results that exploit the Kähler condition: by Cheeger-Colding-Tian [5], the tangent cones are of the form $C(Y) \times \mathbf{C}^k$, where $C(Y)$ does not split off a factor of $\mathbf{R}$, while the first author [17] showed that each tangent cone admits a one-parameter group of isometries. Under two-sided Ricci curvature bounds it follows from Anderson [1] that the regular set in Z is a Kähler manifold. In our previous work [18] we showed that under Ricci lower bounds, for sufficiently small $\epsilon > 0$, the ϵ-regular set $\mathcal{R}_\epsilon \subset Z$ has the structure of a complex manifold, although the metric may not be smooth. Here $x \in \mathcal{R}_\epsilon$ if $V_{2n} - \lim_{r\to 0} r^{-2n}\,\mathrm{vol}(B(x, r)) < \epsilon$, where V_{2n} is the volume of the Euclidean unit ball. An important problem is whether the complex structure can be extended across the singular set of Z to give it the structure of an analytic space. When the (M_i, ω_i) are polarized, i.e., the ω_i are curvature forms of line bundles, then

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Donaldson-Sun [11] (under two-sided Ricci bounds) and the authors [18] (under just lower Ricci bounds) showed that this is the case. Without polarizations the question is still open; however, we have the following;

THEOREM 1.1. *Every tangent cone of Z is homeomorphic to a normal affine algebraic variety such that under a suitable embedding into $\mathbf{C}^N$, the homothetic action on the tangent cone extends to a linear torus action.*

Theorem 1.1 was shown previously by Donaldson-Sun [12] under the assumptions that the ω_i are curvature forms of line bundles $L_i \to M_i$, and $|\mathrm{Ric}(\omega_i)| < 1$. An important application of their result is that in their setting the holomorphic spectrum of the tangent cones is rigid, which in turn they used to show the uniqueness of tangent cones. While we are not able to show uniqueness, our result does imply the rigidity of the holomorphic spectrum under two-sided Ricci curvature bounds, even when the (M_i, ω_i) are not polarized.

More precisely, recall that for a Kähler cone $C(Y)$, possibly with singularities, the holomorphic spectrum is defined by

$$\mathcal{S} = \{\deg(f) : f \text{ is homogeneous and holomorphic on } C(Y)\} \subset \mathbf{R}.$$

We then have:

COROLLARY 1.2. *Suppose that we have two-sided bounds $|\mathrm{Ric}(\omega_i)| < 1$ along the sequence above. Then for any $q \in Z$ the holomorphic spectrum of every tangent cone at q is the same. In addition, the volume ratio $V_{2n}^{-1}\,\mathrm{Vol}(B(o,1))$ is an algebraic number for every tangent cone (Z_q, o).*

As in [12], the rigidity of the holomorphic spectrum follows from the fact that the space of tangent cones at each point is connected, and the holomorphic spectrum consists of algebraic numbers. Note that these results hold in particular for tangent cones at infinity of Calabi-Yau manifolds with Euclidean volume growth.

The method of proof follows the overall strategy of Donaldson-Sun [11, 12] for constructing holomorphic functions on limit spaces. A crucial difference is that in our case the holomorphic functions on a tangent cone Z_q are not obtained as limits of holomorphic functions on smooth manifolds. Instead we prove a version of Hörmander's L^2-estimate on the tangent cone; see Proposition 3.1. In the setting of two-sided Ricci curvature bounds there are substantial simplifications using that the singular set has codimension 4, and the L^2-estimate can be proven directly on the tangent cone. We give this argument separately in Section 2, since it may be of independent interest. The case of lower Ricci bounds is treated in Section 3 by proving approximate versions of the basic estimate on smooth spaces converging to the tangent cone. Using this, in Section 4 we follow the strategy of Donaldson-Sun [11, 12] to prove Theorem 1.1.

2 Two-Sided Ricci Bounds

In this section, we show that when our sequence (M_i, ω_i) has two-sided Ricci curvature bounds $|\mathrm{Ric}(\omega_i)| < 1$, then we can directly prove the Hörmander L^2 estimate on the tangent cone. While the result is subsumed by the more general setting treated in Section 3, the proof may be of independent interest.

Let X be a tangent cone of the noncollapsed Gromov-Hausdorff limit of a sequence of complete Kähler manifolds with two-sided Ricci curvature bounds. Let $S \subset X$ denote the singular set (see Cheeger-Colding [3] for details on the structure of X). The regular set $\mathcal{R} = X \setminus S$ admits a Ricci flat Kähler metric $\omega = \frac{1}{2}\sqrt{-1}\partial\overline{\partial}r^2$, where r is the distance from the vertex of the cone X. Suppose that we have a function φ on $\mathcal{R}$ satisfying $c\omega \le \sqrt{-1}\partial\overline{\partial}\varphi \le C\omega$ on $\mathcal{R}$ for a nonnegative function c and constant $C > 0$. More generally, C could be a locally bounded function. For instance, we could use $\varphi = r^2$ or $\varphi = \log(1 + r^2)$.

PROPOSITION 2.1. *Suppose that α is a smooth $(0,1)$-form compactly supported in $\mathcal{R}$ such that $\overline{\partial}\alpha = 0$. Then there exists a function f on $\mathcal{R}$ satisfying $\overline{\partial} f = \alpha$, and*

$$\int_{\mathcal{R}} |f|^2 e^{-\varphi}\,\omega^n \le \int_{\mathcal{R}} c^{-1}|\alpha|^2 e^{-\varphi}\,\omega^n.$$

PROOF. We follow the argument from Demailly [9, theorem 5.1] to prove Hörmander's L^2-estimate [14], the difference being that in our case we do not know that $\mathcal{R}$ admits a complete Kähler metric, and so we need a more careful argument for approximating L^2 forms with smooth forms of compact support.

As in Demailly's proof, our goal is to prove the inequality

$$\left|\int_{\mathcal{R}} \langle \alpha, v\rangle e^{-\varphi}\,\omega^n\right|^2 \le \left(\int_{\mathcal{R}} c^{-1}|\alpha|^2 e^{-\varphi}\,\omega^n\right)\|\overline{\partial}^*_{\varphi} v\|^2 \tag{2.1}$$

for all smooth $(0,1)$-forms v with compact support in $\mathcal{R}$. To define $\overline{\partial}^*_{\varphi} v$ we are viewing v as a $(0,1)$-form valued in the trivial bundle with metric $e^{-\varphi}$. The existence of the required function f then follows from the Riesz representation theorem.

Given such a smooth $(0,1)$-form v compactly supported away from S, we can decompose $v = v_1 + v_2$ under the L^2-orthogonal decomposition

$$L^2 = \ker\overline{\partial} \oplus (\ker\overline{\partial})^{\perp}.$$

Note that $\overline{\partial}^*_{\varphi} v_2 = 0$, and also $\overline{\partial}^*_{\varphi} v = 0$ near S, and therefore $\overline{\partial}^*_{\varphi} v_1 = 0$ near S. Since also by definition $\overline{\partial} v_1 = 0$, it follows that v_1 is a harmonic $(0,1)$-form near S. If u is any harmonic $(0,1)$-form valued in a line bundle L with curvature $F_{j\bar{k}}$, in an orthonormal frame we have

$$\begin{aligned}0 = \Delta_{\bar\partial} u &= (\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)u \\ &= \sum_j -\nabla_j\nabla_{\bar j}u_{\bar k} + \nabla_j\nabla_{\bar k}u_{\bar j} - \nabla_{\bar k}\nabla_j u_{\bar j} \\ &= \sum_j -\nabla_j\nabla_{\bar j}u_{\bar k} + \sum_{j,p} R^{\bar p}_{\bar j j\bar k}u_{\bar p} + \sum_j F_{j\bar k}u_{\bar j},\end{aligned}$$

and so

$$\sum_j \nabla_j\nabla_{\bar j}u_{\bar k} = \sum_{j,p} R^{\bar p}_{\bar j j\bar k}u_{\bar p} + \sum_j F_{j\bar k}u_{\bar j}.$$

We also have

$$\begin{aligned}\sum_j \nabla_{\bar j}\nabla_j u_{\bar k} &= \sum_j \nabla_j\nabla_{\bar j}u_{\bar k} - \sum_{j,p} R^{\bar p}_{\bar k j\bar j}u_{\bar p} - \sum_j F_{j\bar j}u_{\bar k} \\ &= \sum_j F_{j\bar k}u_{\bar j} - F_{j\bar j}u_{\bar k}.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta|u|^2 &= \sum_j \nabla_j\nabla_{\bar j}|u|^2 \\ &= \sum_{j,k}(\nabla_j\nabla_{\bar j}u_{\bar k})\overline{u_{\bar k}} + |\nabla^{1,0}u|^2 + |\nabla^{0,1}u|^2 + \sum_{j,k} u_{\bar k}\overline{\nabla_{\bar j}\nabla_j u_{\bar k}} \\ &= |\nabla u|^2 + \sum_{k,p} R_{p\bar k}u_{\bar p}\overline{u_{\bar k}} + \sum_{j,k} F_{j\bar k}u_{\bar j}\overline{u_{\bar k}} + \sum_{j,k}\overline{F_{j\bar k}}u_{\bar k}\overline{u_{\bar j}} \\ &\quad - \sum_{j,k}\overline{F_{j\bar j}}u_{\bar k}\overline{u_{\bar k}} \\ &= |\nabla u|^2 + \sum_{p,k} R_{p\bar k}u_{\bar p}\overline{u_{\bar k}} + \sum_{j,k} 2F_{j\bar k}u_{\bar j}\overline{u_{\bar k}} - \sum_j F_{j\bar j}|u|^2.\end{aligned} \tag{2.2}$$

In our setting the Ricci curvature vanishes, and $F_{j\bar k} = \partial_j\partial_{\bar k}\varphi$. In particular, in a neighborhood of S we have

$$\Delta|v_1|^2 \geq |\nabla v_1|^2 - Cn|v_1|^2.$$

It follows that $\Delta|v_1| \geq -C|v_1|$ near S for a locally bounded function C. Since $|v_1| \in L^2(\mathcal{R})$, we can apply Lemma 2.2 below to obtain that $|v_1| \in L^\infty_{\mathrm{loc}}$ on a neighborhood of S in X (i.e., $|v_1|$ cannot blow up as we approach the singular set).

Let us now define cutoff functions η_R and χ_ϵ as follows. The function η_R equals 1 in $B_R(0)$, vanishes outside of $B_{R+1}(0)$, and satisfies $|\nabla\eta_R| < 2$. The function χ_ϵ equals 1 outside of the 2ϵ-neighborhood of S and vanishes in the ϵ-neighborhood of S. In addition, since S has codimension 4 (see [5]), we can arrange that on compact sets K we have $\|\nabla\chi_\epsilon\|_{L^2(K)} \to 0$ as $\epsilon \to 0$. Since $\chi_\epsilon\eta_R v_1$ has compact support in $\mathcal{R}$, by the Bochner-Kodaira inequality,

$$\int_{\mathcal{R}} c|\chi_\epsilon\eta_R v_1|^2 e^{-\varphi}\,\omega^n \leq \|\overline{\partial}(\chi_\epsilon\eta_R v_1)\|^2 + \|\overline{\partial}^*_\varphi(\chi_\epsilon\eta_R v_1)\|^2,$$

and it follows using the Cauchy-Schwarz inequality that

$$\begin{aligned} &\left|\int_{\mathcal{R}} \langle\alpha, \chi_\epsilon\eta_R v_1\rangle e^{-\varphi}\,\omega^n\right|^2 \\ &\quad\leq \int_{\mathcal{R}} c^{-1}|\alpha|^2 e^{-\varphi}\,\omega^n \int_{\mathcal{R}} c|\chi_\epsilon\eta_R v_1|^2 e^{-\varphi}\,\omega^n \\ &\quad\leq \left(\int_{\mathcal{R}} c^{-1}|\alpha|^2 e^{-\varphi}\,\omega^n\right)(\|\overline{\partial}(\chi_\epsilon\eta_R v_1)\|^2 + \|\overline{\partial}^*_\varphi(\chi_\epsilon\eta_R v_1)\|^2). \end{aligned} \tag{2.3}$$

Now recall that v_1 is locally bounded near S, and $\|\nabla\chi_\epsilon\|_{L^2}$ can be made arbitrarily small. It follows that for fixed R if we let $\epsilon \to 0$, we have $\chi_\epsilon\eta_R v_1 \to \eta_R v_1$, $\overline{\partial}(\chi_\epsilon\eta_R v_1) \to \overline{\partial}(\eta_R v_1)$, and $\overline{\partial}^*_\varphi(\chi_\epsilon\eta_R v_1) \to \overline{\partial}^*_\varphi(\eta_R v_1)$ in L^2. In addition, using that $|\nabla\eta_R| < 2$ and that the supports of η_R exhaust $\mathcal{R}$ as $R \to \infty$, we have that $\eta_R v_1 \to v_1$, $\overline{\partial}(\eta_R v_1) \to \overline{\partial} v_1$, and $\overline{\partial}^*(\eta_R v_1) \to \overline{\partial}^* v_1$ in L^2 as $R \to \infty$. It follows from (2.3) that

$$\left|\int_{\mathcal{R}} \langle\alpha, v_1\rangle e^{-\varphi}\,\omega^n\right|^2 \leq \left(\int_{\mathcal{R}} c^{-1}|\alpha|^2 e^{-\varphi}\,\omega^n\right)(\|\overline{\partial} v_1\|^2 + \|\overline{\partial}^*_\varphi v_1\|^2).$$

The required inequality (2.1) now follows since the assumption $\overline{\partial}\alpha = 0$ implies that α is orthogonal to v_2, and at the same time $\overline{\partial} v_1 = 0$ and $\overline{\partial}^*_\varphi v_1 = \overline{\partial}^*_\varphi v$. □

We used the following lemma in the proof above. Note that this estimate fails when the singular set has codimension 2.

LEMMA 2.2. *Let B be a unit ball in X, and suppose that $u \in L^2(B)$ is such that on $B \setminus S$ the function u is smooth, nonnegative, and $\Delta u \geq -Au$ for a constant $A > 0$. Then we have*

$$\sup_{\frac{1}{2}B\setminus S} u \leq C\|u\|_{L^2(B)}$$

for a constant C depending on A, the dimension, and asymptotic volume ratio of X.

PROOF. The function $\widetilde{u}(s, x) = e^{\sqrt{A}s}u(x)$ on the product $\mathbf{R} \times B$ satisfies $\Delta_{\mathbf{R}\times X}\widetilde{u} = Ae^{\sqrt{A}s}u + e^{\sqrt{A}s}\Delta_X u \geq 0$, and the L^2-norm of $\widetilde{u}$ on $[-1, 1] \times B$ can be bounded in terms of the L^2-norm of u on B and the constant A. Using this we can reduce to the case $A = 0$.

Given $y \in \frac{1}{2}B \setminus S$, let $H_t(x, y)$ be the heat kernel on X satisfying $\partial_t H_t = \Delta_x H_t$ (see Ding [10] for details on the heat kernel on tangent cones). In addition, let η be a cutoff function such that $\eta = 1$ on $\frac{2}{3}B$ and $\eta = 0$ outside of $\frac{3}{4}B$. We have

$$
\begin{aligned}
&\partial_t \int_B u(x)\eta(x)H_t(x,y)dx \\
&\quad = \int_B u\eta\Delta H_t \\
&\quad = \int_B u\Delta(\eta H_t) - \int_B [uH_t\Delta\eta + 2u\nabla\eta\cdot\nabla H_t] \\
&\quad \geq \int_B u\Delta(\eta H_t) - C\|u\|_{L^2}.
\end{aligned} \tag{2.4}
$$

Here we used that $\Delta\eta \in L^2$, and ∇H_t is bounded independently of t on the support of $\nabla\eta$.

We claim that the first term is nonnegative for any $t > 0$. For this let χ_ϵ be cutoff functions such that $1 - \chi_\epsilon$ is supported in the ϵ-neighborhood of S, and in addition $\nabla\chi_\epsilon, \Delta\chi_\epsilon \in L^2$. Such a choice is possible because of remark 1.15 on page 6 of [15]. For a fixed $t > 0$ let us write $\psi = \eta H_t$. Then

$$\int_B \chi_\epsilon u\Delta\psi = \int_B u\Delta(\chi_\epsilon\psi) - \int_B [u\psi\Delta\chi_\epsilon + 2u\nabla\psi\cdot\nabla\chi_\epsilon].$$

Since now $\chi_\epsilon\psi \geq 0$ is compactly supported in $B\setminus S$, the first term is nonnegative, while at the same time the last two terms tend to 0 as $\epsilon \to 0$. It follows that

$$\int_B u\Delta(\eta H_t) \geq 0,$$

and so (2.4) implies

$$\int_B u\eta H_1 \geq \lim_{t\to 0}\int_B u\eta H_t - C\|u\|_{L^2} = u(y) - C\|u\|_{L^2}.$$

Since H_1 is bounded above, we obtain the required estimate for $u(y)$. □

3 The L^2 Estimate on Tangent Cones

Suppose now that (X, o) is a tangent cone at a point of a noncollapsed limit space of n-dimensional Kähler manifolds with lower bounds on the Ricci curvature. In particular, for all $R > 0$, the ball $B(o,R) \subset X$ is the Gromov-Hausdorff limit of a sequence of balls $B(p_i, R)$ in Kähler manifolds with $\mathrm{Ric} > -i^{-1}$, and $\mathrm{vol}(B(p_i,R)) > \nu R^{2n}$ for the noncollapsing constant $\nu > 0$, with $p_i \to o$. For sufficiently small ϵ we know from [18] that the ϵ-regular set $\mathcal{R}_\epsilon$ is a complex manifold, and from Cheeger-Jiang-Naber [6] that $X\setminus\mathcal{R}_\epsilon$ is $(2n-2)$-rectifiable with locally finite $(2n-2)$-dimensional Minkowski content. From now on we choose an $\epsilon = \epsilon(n)$ sufficiently small, but fixed.

For $t \geq 0$, let $w(t)$ satisfy

$$w' > 0, \quad w'' \leq 0, \quad \text{and} \quad w' + tw'' > 0. \tag{3.1}$$

For instance, we can take $w(t) = C\log(t+1)$ for any constant $C > 0$. In the L^2-estimate we will use a weight function of the form $\varphi = w(r^2)$, where r is the distance function from the vertex of the cone X. We denote by μ the natural measure on X. It will also be useful to denote by μ_0 a smooth volume measure on $\mathcal{R}_\epsilon$ with respect to the holomorphic charts (up to bounded factors). It follows that on compact sets we have a lower bound $d\mu > C^{-1}d\mu_0$.

PROPOSITION 3.1. *Suppose that $\alpha = \overline{\partial}h$, where h is a smooth function compactly supported in $\mathcal{R}_\epsilon$. Then there exists a function f on $\mathcal{R}_\epsilon$ satisfying $\overline{\partial}f = \alpha$, and*

$$\int_{\mathcal{R}_\epsilon} |f|^2 e^{-\varphi}\, d\mu \leq \int_{\mathcal{R}_\epsilon} \frac{|\alpha|^2}{w'(r^2) + r^2 w''(r^2)} e^{-\varphi}\, d\mu.$$

Remark 3.2. It will be clear from the proof that we can add weight functions with logarithmic poles on $\mathcal{R}_\epsilon$. This will be useful to separate tangents in Proposition 4.1.

Remark 3.3. We must restrict ourselves to exact forms α for technical reasons, but this will not matter in our application. We will define the norm $|\alpha|$ precisely below by taking a limit of the corresponding norms on smooth spaces.

As in the proof of Proposition 2.1, we follow the approach in Demailly [9] to Hörmander's L^2-estimate, but we need to take care with working on the space $\mathcal{R}_\epsilon$ since the metric we are using is not smooth. We construct f satisfying the equation $\overline{\partial}f = \alpha$ in a weak sense, i.e., satisfying

$$\int_{\mathcal{R}_\epsilon} f\,\overline{\partial}\eta = \int_{\mathcal{R}_\epsilon} h\,\overline{\partial}\eta \tag{3.2}$$

for all smooth $(n, n-1)$-forms η with compact support in $\mathcal{R}_\epsilon$. In order to construct f as an element of $L^2(e^{-\varphi}\, d\mu)$, we need to define the function $*\overline{\partial}\eta \in L^2(e^{-\varphi}\, d\mu)$. For this, as well as for the definition of the norm $|\alpha|$, we observe that in any holomorphic chart on $\mathcal{R}_\epsilon$ we have a well-defined tensor $g^{j\bar{k}}$ with bounded measurable components, corresponding to the inverse of the metric. More precisely, we have the following:

LEMMA 3.4. *Suppose that $q_i \in B(p_i, R)$ satisfy $q_i \to q \in \mathcal{R}_\epsilon$, and z_{ij} are holomorphic charts on small balls $B(q_i, \rho)$ converging to a holomorphic chart z_j on $B(q, \rho)$. We use the charts to identify functions on $B(q_i, \rho)$ with functions on $B(q, \rho)$. Then the inverses $g_i^{j\bar{k}}$ of the metric tensors on $B(q_i, \rho)$ converge in $L^p(d\mu)$ for all p to a tensor $g^{j\bar{k}}$ on $B(q, \rho)$ with bounded components.*

PROOF. We show that we have

$$\int_{B(q,\rho)} \left|g_a^{j\bar{k}} - g_b^{j\bar{k}}\right| d\mu \to 0,$$

as $a, b \to \infty$. Note first that we have a uniform upper bound $|g_a^{j\bar{k}}| < C$. Using that the regular points in $\mathcal{R}_\epsilon$ have full measure, together with a covering argument, it is enough to show that for all regular $x \in B(q, \rho)$ we have

$$\lim_{\kappa\to 0}\lim_{a,b\to\infty} \fint_{B(x,\kappa)} \left|g_a^{j\bar{k}} - g_b^{j\bar{k}}\right| d\mu = 0. \tag{3.3}$$

Consider a ball $B(x, \kappa)$ and $x_i \in B(p_i, R)$ such that $x_i \to x$. We can find holomorphic coordinates w_{ip} on $B(x_i, \kappa)$ converging to w_p on $B(x, \kappa)$, which give Gromov-Hausdorff approximations to the corresponding Euclidean balls. Let us rescale distances by κ^{-1}. The Cheeger-Colding estimate [3] shows that the corresponding metric components $g_i^{p\bar{q}}$ satisfy

$$\fint_{\kappa^{-1}B(x_i,\kappa)} \left|g_i^{p\bar{q}} - \delta^{p\bar{q}}\right| d\mu_i < \Psi(\kappa, i^{-1}|x). \tag{3.4}$$

Here, and below, $\Psi(\delta_1, \ldots, \delta_k | a_1, \ldots, a_l)$ denotes a function converging to 0 as $\delta_i \to 0$ while the a_j are fixed.

In addition, we have $g_{i,p\bar{q}} > (1 - \Psi(\kappa, i^{-1}|x))\delta_{p\bar{q}}$. It follows from this and Colding's volume convergence [8] that in (3.4) we can replace the measure $d\mu_i$ with $d\mu$. The gradient estimate for the coordinates z_j implies that the Jacobian matrix dz_j/dw_p is uniformly bounded. It follows that

$$\fint_{\kappa^{-1}B(x_i,\kappa)} \left|g_i^{j\bar{k}} - (dz_j/dw_p)\overline{(dz_k/dw_q)}\delta^{p\bar{q}}\right| d\mu < \Psi(\kappa, i^{-1}|x),$$

which implies (3.3). The sequence $g_a^{j\bar{k}}$ therefore converges to a limit $g^{j\bar{k}}$ in $L^1(d\mu)$, and since these components are uniformly bounded, the convergence is in L^p for all p as well.

If we have another set of charts z'_{ij} converging to z'_j, then the corresponding components $g'^{j\bar{k}}$ are related to $g^{j\bar{k}}$ in the usual way. This follows from the fact that the transition functions $z_{ij} \circ z'^{-1}_{ij}$ converge to $z_j \circ z'^{-1}_j$ as $i \to \infty$. □

In terms of the metric components $g^{j\bar{k}}$ we can now define $|\alpha|^2 = g^{j\bar{k}}\alpha_{\bar{k}}\overline{\alpha_{\bar{j}}}$ in the usual way. Note that if we have local coordinates z_{ij} converging to z_j on $B(q, \rho)$ as above such that φ_i, ρ_i converge uniformly to φ, r^2, then we have

$$\lim_{i\to\infty} \int_{B(q_i,\rho)} \frac{|\alpha|^2_{g_i} e^{-\varphi_i}}{w'(\rho_i) + \rho_i w''(\rho_i)}\, d\mu_i = \int_{B(q,\rho)} \frac{|\alpha|^2 e^{-\varphi}}{w'(r^2) + r^2 w''(r^2)}\, d\mu.$$

This follows from the convergence of the metric components $g_i^{j\bar{k}}$ to $g^{j\bar{k}}$, together with the convergence [8] of the measures μ_i to μ.

Similarly, we can define $*\bar{\partial}\eta$ in local coordinates by letting

$$\overline{*\bar{\partial}\eta} = e^{\varphi}(\bar{\partial}\eta)_{1\bar{1}\cdots n\bar{n}} \det(g^{j\bar{k}}).$$

In terms of coordinates z_{ij} converging to z_j as above, we then have that $*_{\varphi_i}\overline{\partial}\eta$ converges in L^p to $*\overline{\partial}\eta$, and in addition we have

$$\int_{\mathcal{R}_\epsilon} f\,\overline{*\overline{\partial}\eta}e^{-\varphi}\,d\mu = \int_{\mathcal{R}_\epsilon} f\,\overline{\partial}\eta$$

for any $f \in L^2(e^{-\varphi}d\mu)$. Note that such an f is also locally in L^2 with respect to $d\mu_0$ since φ is locally bounded, and so the right hand side is well defined.

Proof of Proposition 3.1. Given a smooth $(n, n-1)$-form η with compact support in $\mathcal{R}_\epsilon$, our goal is to prove the inequality

$$\begin{aligned}\left(\int_{\mathcal{R}_\epsilon} h\,\overline{*\overline{\partial}\eta}e^{-\varphi}\,d\mu\right)^2 &= \left(\int_{\mathcal{R}_\epsilon} h\,\overline{\partial}\eta\right)^2\\ &\leq \int_{\mathcal{R}_\epsilon}\frac{|\alpha|^2 e^{-\varphi}}{w'(r^2)+r^2w''(r^2)}\,d\mu\int_{\mathcal{R}_\epsilon}|*\overline{\partial}\eta|^2e^{-\varphi}\,d\mu.\end{aligned}\tag{3.5}$$

The existence of the required function f then follows in the standard way using the Hahn-Banach and Riesz representation theorems. Indeed, if we denote by $E \subset L^2(e^{-\varphi}d\mu)$ the closure of the subspace of functions of the form $\overline{*\overline{\partial}\eta}$ with compact support, then (3.5) implies that h defines a linear functional $\lambda : E \to \mathbf{C}$ with norm

$$\|\lambda\| \leq \left(\int_{\mathcal{R}_\epsilon}\frac{|\alpha|^2e^{-\varphi}}{w'(r^2)+r^2w''(r^2)}\,d\mu\right)^{1/2}.$$

We then deduce the existence of $f \in E \subset L^2(e^{-\varphi}d\mu)$ with the same bound on its norm such that (3.2) holds for all smooth η with compact support. It then follows that $\overline{\partial}f = \overline{\partial}h$.

We prove the inequality (3.5) using approximations by smooth spaces. Let $\mathcal{U} \subset \mathcal{R}_\epsilon$ be an open relatively compact subset containing the closure of the support of η. Let us fix a large radius $R > 0$ such that $\mathcal{U} \subset B(o, R/2)$. Recall that $B(o, R)$ is a noncollapsed Gromov-Hausdorff limit of a sequence of balls $B(p_i, R)$ in Kähler manifolds with $\mathrm{Ric} > -i^{-1}$. For every $q \in B(o, R) \cap \mathcal{U}$, by definition we have a radius δ such that $B(q, \delta)$ is $\Psi(\epsilon)$-Gromov-Hausdorff close to the Euclidean ball.

Suppose that $q_i \in B(p_i, R)$ are such that $q_i \to q$. By theorem 2.1 in [18], we have δ' such that on each $B(q_i, \delta')$ there is a holomorphic chart giving a Gromov-Hausdorff approximation to the Euclidean ball, and a Kähler potential close to $\frac{1}{2}d(q_i, \cdot)^2$. These holomorphic charts converge, as $i \to \infty$, to a holomorphic chart on $B(q, \delta')$ defining the holomorphic structure on $\mathcal{U}$.

We first construct suitable weight functions φ_i on $B(p_i, R)$. Since $B(p_i, R) \to B(o, R)$ in the Gromov-Hausdorff sense, by Cheeger-Colding [3] we have functions b_i on $B(p_i, R)$ such that $\Delta b_i^2/2 = n$, $|\nabla b_i| < 1 + \Psi(i^{-1})$ on $B(p_i, R-1)$,

$$\int_{B(p_i,R)}|\sqrt{-1}\partial\overline{\partial}b_i^2/2 - g_i|^2 + \big||\nabla b_i| - 1\big|^2 < \Psi(1/i),\tag{3.6}$$

and $|b_i - r_i| < \Psi(1/i)$, in terms of the distance r_i from p_i. In other words, $b_i^2/2$ is plurisubharmonic in an L^2 sense, but this is not good enough to get an inequality of the form (3.5). We will therefore modify $b_i^2/2$ on the set $\mathcal{U}$ to make it plurisubharmonic in a pointwise sense there.

Let us cover $\mathcal{U} \subset B(o, R/2)$ using charts U_j such that the smaller charts $\frac{1}{10}U_j$ still cover. Under the Gromov-Hausdorff convergence these define charts U_{ij} centered at points $q_{ij} \in B(p_i, R)$. On each U_{ij} there is a Kähler potential ψ_{ij}, close to the function $\frac{1}{2}d(q_{ij}, \cdot)^2$, the closeness determined by how close to the Euclidean ball our chart is in the Gromov-Hausdorff sense (this is controlled by our choice of $\epsilon(n)$ above).

At the same time, on each U_{ij} we also have Kähler potentials φ_{ij}, which are close to $\frac{1}{2}r_i^2$ (where r_i is the distance from p_i in the ball $B(p_i, R)$) in the sense that

$$\left|\varphi_{ij} - r_i^2/2\right| < \Psi(1/i),$$

where $r_i = d(p_i, \cdot)$ as above. This is because if we let $\psi_{\infty j}$ be the limit of the potentials ψ_{ij} as $i \to \infty$ along a subsequence, then $\partial\bar{\partial}(\psi_{\infty j} - r^2/2) = 0$ on U_j (see [18, claim 3.1] for a similar result). We can then define $\varphi_{ij} = \psi_{ij} + (r^2/2 - \psi_{\infty j})$ using the local coordinates to identify U_{ij} with U_j.

It is clear that on the overlap $U_{ij} \cap U_{ik}$, $|\varphi_{ij} - \varphi_{ik}| < \Psi(1/i)$. We can ensure that on a smaller compact set of the intersection, $|\nabla(\varphi_{ij} - \varphi_{ik})|$ is small. Using a partition of unity on $\bigcup_j \frac{1}{2}U_{ij}$, we can glue the φ_{ij} to obtain a function $\widetilde{\rho}_i$ that satisfies

$$\begin{aligned}\left|\widetilde{\rho}_i - \frac{r_i^2}{2}\right| + |\partial\bar{\partial}\widetilde{\rho}_i - g_i| < \Psi(1/i),\\ |\nabla\widetilde{\rho}_i| \le r_i + \Psi(1/i), |\nabla(\widetilde{\rho}_i - b_i^2/2)| < \Psi(1/i),\end{aligned} \tag{3.7}$$

on $\bigcup_j \frac{1}{3}U_{ij}$. Note that since the partition of unity is defined using the charts, the functions giving the partition of unity have uniformly bounded derivatives.

Similarly, we can define a cutoff function μ_i, supported on $\bigcup_j \frac{1}{3}U_{ij}$, so that $\mu_i = 1$ on $\bigcup_j \frac{1}{6}U_{ij}$. Furthermore,

$$|\nabla\mu_i| + |\partial\bar{\partial}\mu_i| \le C, \tag{3.8}$$

where C is independent of i. Now we define

$$\rho_i = \mu_i\widetilde{\rho}_i + (1 - \mu_i)b_i + \epsilon_i, \tag{3.9}$$

where ϵ_i is a positive sequence converging to 0 so that $\rho_i \ge 0$. Set

$$\varphi_i = w(\rho_i). \tag{3.10}$$

From (3.7), (3.8), (3.9), (3.10), and standard computation, we find that on $\bigcup_j \frac{1}{6}U_{ij}$

$$\sqrt{-1}\partial\bar{\partial}\varphi_i \ge (w'(\rho_i) + \rho_i w''(\rho_i) - \Psi(1/i))g_i, \tag{3.11}$$

and we also have

$$\int_{B(p_i,R-1)} |\sqrt{-1}\partial\overline{\partial}\varphi_i - (w'(\rho_i)g_i + \sqrt{-1}w''(\rho_i)\partial\rho_i \wedge \overline{\partial}\rho_i)|^2 \leq \Psi(1/i). \tag{3.12}$$

Let us consider now the $(n, n-1)$-form η. Let γ_j be a partition of unity subordinate to the cover U_j of $\mathcal{U}$. We can view each $\gamma_j \eta$ as an $(n, n-1)$ form on $U_{ij} \subset B(p_i, R)$, and define the form η_i as their sum. Define $v_i = *_{\varphi_i} \eta_i$, where we are viewing η_i as an $(n, n-1)$-form with values in the trivial bundle L^* with metric e^{φ_i} (i.e., the dual of the trivial bundle L with metric $e^{-\varphi_i}$), and v_i as a $(0,1)$-form with values in L. On $B(p_i, R)$, under the L^2-product with weight $e^{-\varphi_i}$ we decompose $v_i = v_i^{(1)} + v_i^{(2)}$, where $\overline{\partial} v_i^{(1)} = 0$, and $v_i^{(2)} \perp \ker \overline{\partial}$. It follows that $\overline{\partial}^*_{\varphi_i} v_i^{(2)} = 0$, and so also $\overline{\partial}^*_{\varphi_i} v_i^{(1)} = 0$ outside of the support of v_i. By Lemma 3.5 below we have that

$$|v_i^{(1)}|^2 e^{-\varphi_i} < C \|v_i\|^2_{L^2(e^{-\varphi_i})} \tag{3.13}$$

on the set $B(p_i, R-1) \setminus \bigcup \frac{1}{6} U_{ij}$, for a constant C independent of i.

Let χ_R be a cutoff function, supported in $B(p_i, R-1)$, and equal to 1 on $B(p_i, R/2)$ such that $|\nabla \chi_R| < C'/R$ for a constant C' independent of R. We can regard the $(0,1)$-form $\chi_R v_i^{(1)}$ as an $(n,1)$-form-valued in the anticanonical line bundle. From the Bochner-Kodaira formula (see (4.7) of [9]), we have

$$\begin{aligned} &\int_{B(p_i,R)} \big(\sqrt{-1}\partial_j \partial_{\overline{k}} \varphi_i + \mathrm{Ric}^{g_i}_{j\overline{k}}\big) \big(\chi_R v_i^{(1)}\big)_{\overline{j}} \overline{\big(\chi_R v_i^{(1)}\big)_{\overline{k}}} e^{-\varphi_i} \\ &\qquad \leq \big\|\overline{\partial}\big(\chi_R v_i^{(1)}\big)\big\|^2_{L^2(e^{-\varphi_i})} + \big\|\overline{\partial}^*_{\varphi_i}\big(\chi_R v_i^{(1)}\big)\big\|^2_{L^2(e^{-\varphi_i})}. \end{aligned} \tag{3.14}$$

To estimate the left-hand side, note that writing $\mathcal{V}_i = \bigcup \frac{1}{6} U_{ij}$, (3.11) implies that we have

$$\begin{aligned} &\int_{\mathcal{V}_i} \big(\sqrt{-1}\partial_j \partial_{\overline{k}} \varphi_i + \mathrm{Ric}^{g_i}_{j\overline{k}}\big) \big(\chi_R v_i^{(1)}\big)_{\overline{j}} \overline{\big(\chi_R v_i^{(1)}\big)_{\overline{k}}} e^{-\varphi_i} \\ &\qquad \geq \int_{\mathcal{V}_i} (w'(\rho_i) + \rho_i w''(\rho_i) - \Psi(1/i)) \big|\chi_R v_i^{(1)}\big|^2 e^{-\varphi_i}, \end{aligned}$$

while also using (3.1), (3.6), (3.12), and (3.13) we have

$$\begin{aligned} &\int_{B(p_i,R)\setminus\mathcal{V}_i} \big(\sqrt{-1}\partial_j \partial_{\overline{k}} \varphi_i + \mathrm{Ric}^{g_i}_{j\overline{k}}\big) \big(\chi_R v_i^{(1)}\big)_{\overline{j}} \overline{\big(\chi_R v_i^{(1)}\big)_{\overline{k}}} e^{-\varphi_i} \\ &\quad \geq -\Psi(1/i) \int_{B(p_i,R)\setminus\mathcal{V}_i} |\chi_R v_i^{(1)}|^2 e^{-\varphi_i} \\ &\qquad + \int_{B(p_i,R)\setminus\mathcal{V}_i} (w'(\rho_i) + \rho_i w''(\rho_i)) |\chi_R v_i^{(1)}|^2 e^{-\varphi_i} - \end{aligned}$$

$$
\begin{aligned}
&- \int_{B(p_i,R)\setminus\mathcal{V}_i} |\sqrt{-1}\partial\overline{\partial}\varphi_i - (w'(\rho_i)g_i + \sqrt{-1}w''(\rho_i)\partial\rho_i\wedge\overline{\partial}\rho_i)| \\
&\qquad\qquad \cdot \big|\chi_R v_i^{(1)}\big|^2 e^{-\varphi_i} \\
&- \int_{B(p_i,R)\setminus\mathcal{V}_i} w''(\rho_i)(\rho_i(g_i)_{j\overline{k}} - \sqrt{-1}\partial_j\rho_i\wedge\partial_{\overline{k}}\rho_i) \\
&\qquad\qquad \cdot \big(\chi_R v_i^{(1)}\big)_{\overline{j}} \overline{\big(\chi_R v_i^{(1)}\big)_{\overline{k}}} e^{-\varphi_i} \\
\geq & \int_{B(p_i,R)\setminus\mathcal{V}_i} (w'(\rho_i) + \rho_i w''(\rho_i))\big|\chi_R v_i^{(1)}\big|^2 e^{-\varphi_i} - \Psi(i^{-1}|R).
\end{aligned}
$$

Notice that we used the assumption $w'' \leq 0$ above. As for the right-hand side of (3.14), we use that $\overline{\partial} v_i^{(1)} = 0$ and $\overline{\partial}^*_{\varphi_i} v_i^{(1)} = \overline{\partial}^*_{\varphi_i} v_i$, as well as the bound $|\nabla\chi_R| < C'/R$, to get

$$
\begin{aligned}
&\big\|\overline{\partial}\big(\chi_R v_i^{(1)}\big)\big\|^2_{L^2(e^{-\varphi_i})} + \big\|\overline{\partial}^*_{\varphi_i}\big(\chi_R v_i^{(1)}\big)\big\|^2_{L^2(e^{-\varphi_i})} \\
&\quad \leq \big\|\overline{\partial}^*_{\varphi_i} v_i\big\|^2_{L^2(e^{-\varphi_i})} + \frac{C'}{R}\|v_i\|^2_{L^2(e^{-\varphi_i})}.
\end{aligned}
$$

Note that by construction the v_i are bounded independently of i, and so their L^2-norms are uniformly bounded. It follows that for sufficiently large i we have

$$
\int_{B(p_i,R)} (w'(\rho_i) + \rho_i w''(\rho_i))\big|\chi_R v_i^{(1)}\big|^2 e^{-\varphi_i} \leq \big\|\overline{\partial}^*_{\varphi_i} v_i\big\|^2_{L^2(e^{-\varphi_i})} + \frac{C'}{R}.
$$

We now use the assumption that $\alpha = \overline{\partial} h$ is exact. Using the cutoff functions γ_j from before, we define smooth functions h_i on $B(p_i, R)$ analogously to the way we defined η_i. We then let $\alpha_i = \overline{\partial} h_i$. By taking R large, we can assume that the supports of h_i, α_i are in $B(p_i, R/2)$.

Since $\overline{\partial}^*_{\varphi_i} v_i^{(2)} = 0$, we have

$$
\begin{aligned}
&\left(\int_{B(p_i,R)} h_i \overline{\overline{\partial}^*_{\varphi_i} v_i} e^{-\varphi_i}\right)^2 \\
&\quad = \left(\int_{B(p_i,R/2)} \langle\overline{\partial} h_i, \chi_R v_i^{(1)}\rangle e^{-\varphi_i}\right)^2 \\
&\quad \leq \int_{B(p_i,R/2)} \frac{|\alpha_i|^2 e^{-\varphi_i}}{w'(\rho_i) + \rho_i w''(\rho_i)} \\
&\qquad \cdot \int_{B(p_i R/2)} (w'(\rho_i) + \rho_i w''(\rho_i))|\chi_R v_i^{(1)}|^2 e^{-\varphi_i} \\
&\quad \leq \int_{B(p_i,R/2)} \frac{|\alpha_i|^2 e^{-\varphi_i}}{w'(\rho_i) + \rho_i w''(\rho_i)} \left(\|\overline{\partial}^*_{\varphi_i} v_i\|^2_{L^2(e^{-\varphi_i})} + \frac{C'}{R}\right),
\end{aligned}
\tag{3.15}
$$

for i large enough. Note that in the first equality we used that the support of h_i is contained in $B(p_i, \frac{R}{2})$ where $\chi_R = 1$.

Next we want to take the limit as $i \to \infty$. For this note first that

$$\int_{B(p_i,R)} h_i \overline{\overline{\partial}^*_{\varphi_i} v_i} e^{-\varphi_i} = \int_{B(p_i,R)} h_i \overline{*_{\varphi_i} \overline{\partial} \eta_i} e^{-\varphi_i} \, d\mu_i .$$

Let us consider one of our charts U_j, with coordinates z_{ik} on U_{ij} converging to z_k on U_j. Note that in terms of these coordinates η_i is not identified exactly with η because of the way we defined η_i in terms of the cutoff functions γ_j. However, if U'_j is another chart, with corresponding coordinates z'_{ik} converging to z'_k, then the transition functions $z'_{ik} \circ z_{ik}^{-1}$ converge smoothly to $z'_k \circ z_k^{-1}$. It follows that if we use our coordinates to identify U_{ij} with U_j, then η_i converges smoothly to η as $i \to \infty$. Similarly h_i converges smoothly to h and in addition φ_i converges uniformly to φ. It then follows from Lemma 3.4 that

$$\lim_{i \to \infty} \int_{B(p_i,R)} h_i \overline{*_{\varphi_i} \overline{\partial} \eta_i} e^{-\varphi_i} \, d\mu_i = \int_{B(p,R)} h \overline{* \overline{\partial} \eta} e^{-\varphi} \, d\mu .$$

Similarly we can take the limit as $i \to \infty$ of the right-hand side of (3.15), noting that $\overline{\partial}^*_{\varphi_i} v_i = *_{\varphi_i} \overline{\partial} \eta_i$ and using Lemma 3.4. We therefore have

$$\begin{aligned}
&\left(\int_{B(p,R)} h \overline{* \overline{\partial} \eta} e^{-\varphi} \, d\mu \right)^2 \\
&\quad \leq \int_{B(p,R/2)} \frac{|\alpha|^2 e^{-\varphi}}{w'(r^2) + r^2 w''(r^2)} \, d\mu \left(\| * \overline{\partial} \eta \|^2_{L^2(e^{-\varphi})} + \frac{C'}{R} \right),
\end{aligned}$$

Finally, we obtain the required inequality (3.5) by letting $R \to \infty$. □

We used the following estimate in the proof above.

LEMMA 3.5. *Let $B(p, 2)$ be a relatively compact ball in a Kähler manifold with* $\mathrm{Ric} > -1$ *and* $\mathrm{vol}(B(p, 2)) > \nu > 0$*. Let us denote by L the trivial bundle with metric $e^{-\psi}$, and suppose that $|\nabla \psi|$ is bounded. Suppose that u is a harmonic* $(0, 1)$*-form valued in L. Then*

$$\sup_{B(p,1)} |u| \leq C \left(\int_{B(p,2)} |u|^2 \right)^{1/2},$$

where C depends on ν, the dimension, and $\sup_{B(p,2)} |\nabla \psi|$.

PROOF. The proof is by Moser iteration. Note first that by (2.2) the norm of u satisfies the differential inequality

$$
\begin{aligned}
\Delta|u|^2 &= |\nabla u|^2 + R^{j\bar{k}}u_{\bar{k}}\overline{u_{\bar{j}}} + 2g^{j\bar{k}}g^{p\bar{q}}(\nabla_j\nabla_{\bar{q}}\varphi)u_{\bar{k}}\overline{u_{\bar{p}}} \\
&\quad - g^{j\bar{k}}g^{p\bar{q}}(\nabla_j\nabla_{\bar{k}}\varphi)u_{\bar{q}}\overline{u_{\bar{p}}} \\
&\geq |\nabla u|^2 - |u|^2 + 2g^{j\bar{k}}g^{p\bar{q}}(\nabla_j\nabla_{\bar{q}}\varphi)u_{\bar{k}}\overline{u_{\bar{p}}} \\
&\quad - g^{j\bar{k}}g^{p\bar{q}}(\nabla_j\nabla_{\bar{k}}\varphi)u_{\bar{q}}\overline{u_{\bar{p}}}.
\end{aligned}
\tag{3.16}
$$

Let us write $B_r = B(p,r)$ for any $r \leq 2$. Let χ be a cutoff function supported in B_2, with $\chi = 1$ on $B_{3/2}$ and $|\nabla\chi| + |\Delta\chi| < C$. Then we have (each integral being on B_2)

$$
\begin{aligned}
&\int -\chi^2\Delta|u|^2 \\
&\quad \leq -\int \chi^2|\nabla u|^2 + \int \chi^2|u|^2 - 2\int \chi^2 g^{j\bar{k}}g^{p\bar{q}}(\nabla_j\nabla_{\bar{q}}\varphi)u_{\bar{k}}\overline{u_{\bar{p}}} \\
&\qquad + \int \chi^2 g^{j\bar{k}}g^{p\bar{q}}(\nabla_j\nabla_{\bar{k}}\varphi)u_{\bar{q}}\overline{u_{\bar{p}}} \\
&\quad = -\int \chi^2|\nabla u|^2 + \int \chi^2|u|^2 + 2\int 2\chi g^{j\bar{k}}g^{p\bar{q}}\nabla_j\chi\nabla_{\bar{q}}\varphi\, u_{\bar{k}}\overline{u_{\bar{p}}} \\
&\qquad + 2\int \chi^2 g^{j\bar{k}}g^{p\bar{q}}\nabla_{\bar{q}}\varphi\big(\overline{u_{\bar{p}}}\nabla_j u_{\bar{k}} + u_{\bar{k}}\overline{\nabla_{\bar{j}}u_{\bar{p}}}\big) \\
&\qquad - \int 2\chi g^{j\bar{k}}g^{p\bar{q}}\nabla_j\chi\nabla_{\bar{k}}\varphi\, u_{\bar{q}}\overline{u_{\bar{p}}} \\
&\qquad - \int \chi^2 g^{j\bar{k}}g^{p\bar{q}}\nabla_{\bar{k}}\varphi\big(\overline{u_{\bar{p}}}\nabla_j u_{\bar{q}} + u_{\bar{q}}\overline{\nabla_{\bar{j}}u_{\bar{p}}}\big) \\
&\quad \leq -\frac{1}{2}\int \chi^2|\nabla u|^2 + C\int |u|^2,
\end{aligned}
\tag{3.17}
$$

for a constant C depending on the dimension, ν, and $\sup_{B(p,2)}|\nabla\varphi|$. At the same time

$$
\int \chi^2\Delta|u|^2 = \int \Delta(\chi^2)|u|^2 \leq C\int |u|^2,
$$

and combining this with our previous inequality, we get

$$
\int \chi^2|\nabla u|^2 < C\int |u|^2. \tag{3.18}
$$

Under our assumptions we have a bound for the Sobolev constant, and so

$$\begin{aligned}\left(\int |\chi|u||^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq \int |\nabla(\chi|u|)|^2\\ &= \int |\nabla\chi|^2|u|^2 + \int \chi^2\big|\nabla|u|\big|^2 + 2\chi|u|\nabla\chi.\nabla|u|\\ &\leq C\int |u|^2,\end{aligned}$$

using the Cauchy-Schwarz inequality and also (3.18). It follows that

$$\|u\|_{L^{\frac{2n}{n-2}}(B_{3/2})} \leq C\,\|u\|_{L^2(B_2)}. \tag{3.19}$$

To estimate higher L^p norms, let $p \geq \frac{2n}{n-2}$, and let χ be a smooth function compactly supported in B_2. We can compute, using the Sobolev inequality,

$$\begin{aligned}&\left(\int \Big|\chi|u|^{\frac{p}{2}}\Big|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\ &\quad\leq C\int \big|\nabla(\chi|u|^{\frac{p}{2}})\big|^2\\ &\quad= C\int |\nabla\chi|^2|u|^p + \chi^2\big|\nabla|u|^{\frac{p}{2}}\big|^2 + 2\chi|u|^{\frac{p}{2}}\nabla\chi.\nabla|u|^{\frac{p}{2}}.\end{aligned}\tag{3.20}$$

We have

$$\begin{aligned}&\int \chi^2\Big|\nabla|u|^{\frac{p}{2}}\Big|^2\\ &\quad= \int \frac{p^2}{8(p-2)}\chi^2\nabla|u|^{p-2}.\nabla|u|^2\\ &\quad= -\int \frac{p^2}{4(p-2)}\chi|u|^{p-2}\nabla\chi.\nabla|u|^2 - \int \frac{p^2}{8(p-2)}\chi^2|u|^{p-2}\Delta|u|^2,\end{aligned}$$

and

$$\begin{aligned}&\int 2\chi|u|^{\frac{p}{2}}\nabla\chi.\nabla|u|^{\frac{p}{2}}\\ &\quad= \int \frac{p}{4}|u|^{p-2}\nabla\chi^2.\nabla|u|^2\\ &\quad= -\int \frac{p(p-2)}{8}\chi^2|u|^{p-4}\nabla|u|^2.\nabla|u|^2 - \int \frac{p}{4}\chi^2|u|^{p-2}\Delta|u|^2\\ &\quad\leq -\int \frac{p}{4}\chi^2|u|^{p-2}\Delta|u|^2.\end{aligned}$$

Combining these with (3.20) we get

$$\left(\int \left|\chi|u|^{\frac{p}{2}}\right|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C\int |\nabla\chi|^2|u|^p - \frac{p^2}{2(p-2)}\chi|u|^{p-1}\nabla\chi.\nabla|u| - \left(\frac{p^2}{8(p-2)}+\frac{p}{4}\right)\chi^2|u|^{p-2}\Delta|u|^2. \tag{3.21}$$

Using (3.16) we have, similarly to (3.17),

$$\begin{aligned}&-\int \chi^2|u|^{p-2}\Delta|u|^2\\ &\leq -\frac{1}{2}\int \chi^2|u|^{p-2}|\nabla u|^2 + Cp^2\int(\chi^2+|\nabla\chi|^2)|u|^p,\end{aligned} \tag{3.22}$$

where C depends on $\sup|\nabla\varphi|$. Since $p \geq \frac{2n}{n-2}$, we have some constant $c_n > 0$ such that

$$c_n p < \frac{p^2}{2(p-2)} < c_n^{-1}p, \quad c_n p < \left(\frac{p^2}{8(p-2)}+\frac{p}{4}\right) < c_n^{-1}p.$$

It then follows, combining (3.21) and (3.22), and using the Cauchy-Schwarz inequality, that

$$\left(\int \left|\chi|u|^{\frac{p}{2}}\right|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq Cp^3\int(\chi^2+|\nabla\chi|^2)|u|^p. \tag{3.23}$$

We now choose a sequence of cutoff functions χ_i such that χ_i is supported in $B_{1+2^{-i}}$, and $\chi_i = 1$ on $B_{1+2^{-i-1}}$. We can arrange that $|\nabla\chi_i| < C2^i$. We also define $\gamma = \frac{n}{n-2}$. Equation (3.23), setting $\chi = \chi_i$ and $p = 2\gamma^i$, implies that

$$\left(\int \chi_i^{2\gamma}|u|^{2\gamma^{i+1}}\right)^{\frac{1}{2\gamma^{i+1}}} \leq (8C\gamma^{3i}2^{2i})^{\frac{1}{2\gamma^i}}\left(\int_{B_{1+2^{-i}}}|u|^{2\gamma^i}\right)^{\frac{1}{2\gamma^i}}.$$

From this we have

$$\|u\|_{L^{2\gamma^{i+1}}(B_{1+2^{-i-1}})} \leq C^{\frac{3i}{\gamma^i}}\|u\|_{L^{2\gamma^i}(B_{1+2^{-i}})}.$$

Iterating this, we find that

$$\sup_{B_1}|u| \leq C\|u\|_{L^{2\gamma}(B_{3/2})} \leq C'\|u\|_{L^2(B_1)},$$

using (3.19). □

We will use Proposition 3.1 to construct holomorphic functions on the tangent cone X. We will need the following result, which implies that the functions that we construct are actually harmonic across the singular set, and in particular satisfy local L^∞ and Lipschitz estimates. Note that in the setting of Section 2 with two-sided Ricci bounds, this step is much more straightforward since the singular set has codimension 4.

PROPOSITION 3.6. *Let $f : X \to \mathbf{C}$ be a function such that $\overline{\partial} f = 0$ on $\mathcal{R}_\epsilon$, and f has polynomial growth in the sense that for some $k, D > 0$ we have*

$$\int_{B(o,R)} |f|^2 < DR^k \tag{3.24}$$

for all $R > 0$. Then f is harmonic on X. In particular, we have local estimates

$$\sup_{y\in B(x,1/2)} |f(y)| + \sup_{y,z\in B(x,1/2)} \frac{|f(y)-f(z)|}{d(y,z)} \leq C\left(\int_{B(x,1)} |f|^2\right)^{1/2}$$

for a constant C depending only on the dimension and the noncollapsing constant ν. In addition, f has polynomial growth in the pointwise sense that $|f(x)| \leq C(1+d(o,x))^d$ for some $C, d > 0$ depending on k, D.

Remark 3.7. In the statement we require the global growth condition (3.24) since we use the heat flow to smooth out f. The argument could be localized by working with Green's functions on balls instead, but we will not need this.

PROOF. Let $H(x,y,t)$ be the heat kernel on X. We let f_t be the evolution of f under the heat flow:

$$f_t(x) = \int f(y)H(x,y,t)d\mu(y).$$

Note that the polynomial growth assumption on f together with the exponential decay of $H(x,y,t)$ ensures that this is well-defined. Our goal is to show that $f_t = f$ for all $t > 0$, which implies that f is harmonic. We will use results about the convergence of heat kernels under Gromov-Hausdorff convergence due to Ding [10], and argue by approximating f_t with corresponding flows on smooth spaces.

We fix $T \in (0,1)$ and $q \in \mathcal{R}_\epsilon \cap B(o,1)$. We will show that $f_T(q) = f_1(q)$. Let $R > 0$ be large, and set $\delta = e^{-R}$. The balls $B(p_i, R)$ converge to $B(o,R) \subset X$ in the Gromov-Hausdorff sense. By the results of Ding, the heat kernels $H_i(x,y,t)$ on $B(p_i,R)$ (with Dirichlet boundary conditions) satisfy

$$|H_i(x,y,t) - H(x,y,t)| < \Psi(i^{-1}, R^{-1})$$

for $t = T, 1$ and $x, y \in B(o,R)$, where we use Gromov-Hausdorff approximations to identify points in $B(p_i,R)$ and $B(o,R)$.

Using that $X \setminus \mathcal{R}_\epsilon$ has locally finite codimension 2 Minkowski measure, we can find a cutoff function η with compact support in $\mathcal{R}_\epsilon$ such that $\eta(q) = 1$ and $\int_{B(o,R)} |\nabla\eta|^2 < \delta$. In addition, we can ensure that $\int_{B(o,R)}(1-\eta)^2 < \delta$. We have corresponding cutoff functions η_i on $B(p_i,R)$ converging to η and satisfying the same estimates.

Using the local holomorphic charts, we can find functions f_i on $B(p_i,R)$, converging uniformly to f on the support of η, such that $|\overline{\partial} f_i| < \Psi(i^{-1})$ on $\mathrm{supp}(\eta_i)$.

In particular, this implies that

$$\int_{B(p_i,R)} |\overline{\partial}(\eta_i f_i)| < \Psi(i^{-1}, R^{-1}).$$

Let χ_i denote cutoff functions supported in $B(p_i, R)$ and equal to 1 on $B(p_i, R-1)$ such that $|\nabla\chi_i| < C$ for a uniform constant C. Consider the heat flow for the functions $\chi_i\eta_i f_i$. We let

$$f_{i,R,t}(q) = \int_{B(p_i,R)} \chi_i(y)\eta_i(y) f_i(y) H_i(q, y, t) d\mu_i(y).$$

We have

$$\begin{aligned}
\frac{d}{dt} f_{i,R,t}(q) &= \int_{B(p_i,R)} \chi_i\eta_i f_i g_i^{a\bar{b}} \overline{\partial}_b \partial_a H_i(q, y, t) d\mu_i(y) \\
&= -\int_{B(p_i,R)} (\overline{\partial}_b\chi_i)\eta_i f_i g_i^{a\bar{b}} \partial_a H_i(q, y, t) d\mu_i(y) \\
&\quad - \int_{B(p_i,R)} \chi_i \overline{\partial}_b(\eta_i f_i) g_i^{a\bar{b}} \partial_a H_i(q, y, t) d\mu_i(y).
\end{aligned}$$

On the support of $\nabla\chi_i$ we have the estimate $|\nabla_y H_i(q, y, t)| < C_T e^{-cR^2}$ for $t \in [T, 1]$ for some $c > 0$ and $C_T > 0$ depending on T, while on $B(p_i, R)$ we have $|\nabla_y H(q, y, t)| < C_T$ for $t \in [T, 1]$. It follows that

$$\left|\frac{d}{dt} f_{i,R,t}(q)\right| < C_T \Psi(i^{-1} R^{-1}),$$

for $t \in [T, 1]$ and so

$$|f_{i,R,1}(q) - f_{i,R,T}(q)| < C_T \Psi(i^{-1}, R^{-1}). \tag{3.25}$$

The convergence of $\eta_i f_i$ to ηf, the convergence of the heat kernels as $i \to \infty$, and their exponential decay imply that

$$|f_{i,R,t}(q) - f_t(q)| < \Psi(i^{-1}|R) \tag{3.26}$$

for $t = T, 1$. Choosing R sufficiently large (depending on T) and i large (depending on T, R), from (3.25) and (3.26) we see that $|f_1(q) - f_T(q)|$ can be made arbitrarily small, and therefore $f_1(q) = f_T(q)$. □

4 Tangent Cones Are Affine Varieties

As before, we suppose that X is a tangent cone at a point of a noncollapsed limit of Kähler manifolds with Ricci curvature bounded below. Our main goal in this section is to prove that X is homeomorphic to an affine variety. Given the L^2-existence result Proposition 3.1 or Proposition 2.1 when we have two-sided Ricci bounds, we can more or less follow the argument in Donaldson-Sun [11, 12] with suitable modifications as in our previous work [18], where only a lower bound for the Ricci curvature is assumed.

Recall that we have a subset $\mathcal{R}_\epsilon \subset X$ which has the structure of a complex manifold, and whose complement has locally finite codimension-2 Minkowski content. Our first goal is to show that X is locally homeomorphic to a complex variety by constructing holomorphic functions that embed a neighborhood of the vertex $o \in X$ into $\mathbf{C}^N$. The basic ingredient for this is the following.

PROPOSITION 4.1. *We can use holomorphic functions of polynomial growth on X to separate points and also to separate tangents at points in $\mathcal{R}_\epsilon$. More precisely*:

(1) *Let $x_1, x_2 \in X$. There exists a holomorphic function f of polynomial growth on X such that $f(x_1) \neq f(x_2)$.*
(2) *Let $x \in \mathcal{R}_\epsilon$. There are holomorphic functions $f_1, \dots, f_n$ of polynomial growth on X that define an embedding of a neighborhood of x into $\mathbf{C}^n$.*

PROOF. The proof of this closely follows arguments in Donaldson-Sun [11] as well as the authors' work [18], given the L^2-existence result Proposition 3.1 together with Proposition 3.6. We can construct holomorphic functions of polynomial growth by using a logarithmic weight $C\log(1+r^2)$; however, in this case the curvature of the metric $e^{-\varphi}$ is not proportional to the metric on the cone. We can, however, use a slightly different weight function $\varphi = w(r^2)$, where $w(t) = t+1$ for t close to 0, and $w(t) = C\log(1+t)$ for t large, still satisfying the inequalities (3.1). Then in a neighborhood of the vertex the curvature of $e^{-\varphi}$ will be the metric on X. Then we can proceed as in [11] or [18] to finish the proof of (1) at least for x_1, x_2 close to the vertex. By scaling this can be extended to all of X. Part (2) can be proven in a standard way using weight functions with logarithmic poles (see Remark 3.2). $\square$

As in Chen-Donaldson-Sun [7, sec. 2.5] or [18, prop. 3.2], we can show that if at a point $x \in X$ there is a tangent cone that splits off $\mathbf{R}^{2n-2}$, then polynomial growth holomorphic functions can be used to embed a neighborhood of x into $\mathbf{C}^n$.

LEMMA 4.2. *A small neighborhood of $o \in X$ is homeomorphic to a normal analytic space.*

PROOF. The argument is very similar to [12]. By using the separation of points, we can find N polynomial growth holomorphic functions $f_1, \dots, f_N$ which all vanish at o so that

$$\sum_{j=1}^{N} |f_j|^2 \geq c > 0 \quad \text{on } \partial B(o,1).$$

One can verify, by using cutoff functions along the singular set, that the image of $(f_1, \dots, f_N)$ near the origin of $\mathbb{C}^N$ defines a d-closed positive locally rectifiable current of type $(N-n, N-n)$. According to the main theorem in [16] (or equivalently, theorem 1.3(a') of [13]), the image of $(f_1, \dots, f_N)$ defines a complex analytic variety of dimension n near the origin in $\mathbb{C}^N$. By adding more holomorphic functions, we can ensure as in [12, props. 2.3, 2.4] that this map is a homeomorphism locally, and the image is a normal variety. $\square$

Remark 4.3. Note that unlike in [11, 12] or [18], the holomorphic functions that we construct on X do not necessarily arise as limits of sequences of holomorphic functions on the approximating manifolds.

In order to show that X is homeomorphic to a normal affine variety, we can use the method in Donaldson-Sun [12], which in turn is based on an idea of Van Coevering [20] in the setting of Kähler cones with smooth links. The main point is to decompose polynomial growth holomorphic functions into sums of homogeneous holomorphic functions under the homothetic vector field $r\partial_r$. Recall that by [17, theorem 2] there is a one parameter group σ_t of isometries acting on the tangent cone X, preserving the distance function from the vertex $o \in X$. Note that since it acts by isometries, the action of σ_t preserves $\mathcal{R}_\epsilon$. We need the following.

PROPOSITION 4.4. *The group σ_t acts by biholomorphisms on $\mathcal{R}_\epsilon$. In addition, the vector field v on $\mathcal{R}_\epsilon$ generating this action satisfies $v = Jr\partial_r$.*

PROOF. Let us briefly recall the construction of the isometric action from [17]. We consider the geodesic annuli $A_i = B(p_i, 10) \setminus B(p_i, 1/3)$. From Cheeger-Colding [3] there are smooth functions ρ_i on A_i such that

$$\begin{aligned} &\int_{A_i} |\nabla^2\rho_i - g_i|^2 + |\nabla\rho_i - \nabla r_i^2/2|^2 < \Psi(i^{-1}), \\ &\left|\rho_i - r_i^2/2\right| < \Psi(i^{-1}), \quad |\nabla\rho_i| \le C, \end{aligned} \tag{4.1}$$

where r_i is the distance function from p_i. Define vector fields $v_i = J\nabla\rho_i$, and let $\sigma_{i,t}$ be the diffeomorphisms generated by v_i. It is shown in [17] that for small t we can extract a limit σ_t as $i \to \infty$, which gives rise to a one-parameter group of isometries on $B(o, 6) \setminus B(o, 5)$ in X. Since the action commutes with the homothetic transformations of X, it can be extended to all of X.

We will show that for small t the limit σ_t is a biholomorphism. Let $q \in \mathcal{R}_\epsilon$ such that $q \in B(o, 6) \setminus B(o, 5)$. Let $q_i \in B(p_i, 10)$ such that $q_i \to q$. We have a small $\delta > 0$ and holomorphic coordinates z_{ij} on $B(q_i, \delta)$ converging to holomorphic coordinates z_j on $B(q, \delta)$.

Abusing notation, we will also denote by v_i the $(1, 0)$-part of the real vector field defined above. The estimate (4.1) implies that

$$\int_{A_i} |\bar\partial v_i|^2 < \Psi(i^{-1}). \tag{4.2}$$

Let us write $v_i = v_i^p \partial_{z_{ip}}$ on $B(q_i, \delta)$ in terms of the holomorphic coordinates. Since in these coordinates we have a lower bound $g_{i,a\bar b} > C^{-1}\delta_{a\bar b}$, it follows from (4.2) that the components v_i^p satisfy

$$\int_{B(q_i,\delta)} \left|\bar\partial v_i^p\right|^2 < \Psi(i^{-1}).$$

On the ball $B(q_i, \delta)$ we can apply the L^2-estimate to obtain holomorphic functions $\widetilde{v}_i^p$ with

$$\int_{B(q_i,\delta/2)} \left|v_i^p - \widetilde{v}_i^p\right|^2 < \Psi(i^{-1}). \tag{4.3}$$

At the same time, using (4.1), we have uniform bounds $|v_i^p|, |\widetilde{v}_i^p| < C$. Let us denote by $\widetilde{\sigma}_{i,t}$ the flow of (the real part of) $\widetilde{v}_i = \widetilde{v}_i^p \partial_{z_{ip}}$. As long as $t < T$ for sufficiently small $T > 0$, this is well-defined on $B(q_i, \delta/3)$, say. Up to choosing a subsequence we can assume that the $\widetilde{v}_i$ converge to a holomorphic vector field $\widetilde{v}$ on $B(q, \delta/3)$, generating a one-parameter group of biholomorphisms $\widetilde{\sigma}_t$.

We claim that the diffeomorphisms $\sigma_{i,t}$ are approximated by $\widetilde{\sigma}_{i,t}$ for $t < T$, and therefore the limiting flow σ_t coincides with the flow $\widetilde{\sigma}_t$ by biholomorphisms. More precisely, define the function

$$u(x,t) = |\sigma_{i,t}(x) - \widetilde{\sigma}_{i,t}(x)|^2$$

for $x \in B(q_i, \delta/3)$ and $t < T$. We will show that we have $u(x,t) < \Psi(i^{-1})$ for x outside of a set of measure at most $\Psi(i^{-1})$. Note that here we are viewing $B(q_i, \delta)$ as a subset of $\mathbf{C}^n$ using our coordinates. Define

$$F(t) = \int_{B(q,\delta/3)} u(x,t).$$

Note that

$$\begin{aligned}\frac{d}{dt}u(x,t) &= 2\big(\sigma_{i,t}(x) - \widetilde{\sigma}_{i,t}(x)\big)\cdot\big(v_i(\sigma_{i,t}(x)) - \widetilde{v}_i(\widetilde{\sigma}_{i,t}(x))\big)\\ &= 2\big(\sigma_{i,t}(x) - \widetilde{\sigma}_{i,t}(x)\big)\\ &\quad \cdot\big(v_i(\sigma_{i,t}(x)) - \widetilde{v}_i(\sigma_{i,t}(x)) + \widetilde{v}_i(\sigma_{i,t}(x)) - \widetilde{v}_i(\widetilde{\sigma}_{i,t}(x))\big).\end{aligned}$$

Using that we have a gradient bound for the holomorphic vector field $\widetilde{v}_i$, this implies

$$\begin{aligned}\frac{d}{dt}u(x,t) &\le 2\sqrt{u(x,t)}|v_i - \widetilde{v}_i|(\sigma_{i,t}(x)) + Cu(x,t)\\ &\le Cu(x,t) + |v_i - \widetilde{v}_i|^2_{\mathrm{Euc}}(\sigma_{i,t}(x)),\end{aligned}$$

where we emphasize that the difference $|v_i - \widetilde{v}_i|$ is measured using the Euclidean metric given by our coordinates. We can now use the fact that $\sigma_{i,t}$ distorts volumes by at most $\Psi(i^{-1})$ (see the proof of theorem 2 in [17]), together with the estimate (4.3) and the uniform bounds for $v_i^p, \widetilde{v}_i^p$ to see that

$$\frac{d}{dt}F(t) \le CF(t) + \Psi(i^{-1}).$$

Since $F(0) = 0$, we get $F(t) < \Psi(i^{-1})$ for $t < T$, and from this it follows in turn that $u(x,t) < \Psi(i^{-1})$ outside of a set of measure at most $\Psi(i^{-1})$, as required.

Note that if in terms of the notation above we let $w_i = \nabla\rho_i$, and we let $\varphi_{i,t}$ denote the one-parameter group of diffeomorphisms generated by w_i, then $\varphi_{i,t}$

converges to the homothetic expansion map on X, generated by $r\partial_r$ on $\mathcal{R}_\epsilon$. Since $Jw_i = v_i$, it is not hard to see that in the limit we have $Jr\partial_r = v$. $\square$

The action of σ_t defines a subgroup isomorphic to $\mathbf{R}$ in the isometry group of X preserving the vertex o. The latter group is a Lie group by [4]. Taking the closure of this $\mathbf{R}$-subgroup defines a torus T acting isometrically on X, which also acts by biholomorphisms on $\mathcal{R}_\epsilon$. Fixing $d > 0$, let $\mathcal{H}_d$ denote the vector space of polynomial growth holomorphic functions f on X of degree at most d, i.e., satisfying $|f(x)| < C(1 + d(o,x))^d$ for a constant $C > 0$. Since such f are harmonic, $\mathcal{H}_d$ is finite dimensional, and decomposes into weight spaces under the T-action. It follows that any $f \in \mathcal{H}_d$ can be decomposed into a sum of eigenfunctions $f = f_{\alpha_1} + \cdots + f_{\alpha_m}$ for $\alpha_k \in \mathrm{Lie}(T)^*$, where $e^{\sqrt{-1}t} \cdot f_{\alpha_k} = e^{\sqrt{-1}\langle \alpha_k, t\rangle} f_{\alpha_k}$ for $t \in \mathrm{Lie}(T)$. The one-parameter group of isometries σ_t is generated by a vector $\xi \in \mathrm{Lie}(T)$. The relation $v = Jr\partial_r$ then implies that $r\partial_r f_{\alpha_k} = \langle \xi, \alpha_k\rangle f_{\alpha_k}$, i.e., each f_{α_k} is homogeneous. We can therefore embed a neighborhood of $o \in X$ into $\mathbf{C}^N$ using homogeneous holomorphic functions, and this embedding naturally extends to an embedding of all of X. As in [12, lemma 2.19] it follows that the image of X is an affine variety and by construction the torus T acts linearly. This completes the proof of Theorem 1.1.

The proof of Corollary 1.2 follows exactly as proposition 2.21 and lemma 4.2 of Donaldson-Sun [12], based on the work of Martelli-Sparks-Yau [19].

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