Journal of Functional Analysis 278 (2020) 108306

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Convex hulls of unitary orbits of normal elements in C^*-algebras with tracial rank zero

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ARTICLE INFO

Article history:
Received 18 July 2018
Accepted 16 August 2019
Available online 23 August 2019
Communicated by S. Vaes

Keywords:
Unitary orbit
Convex hull
C^*-algebra with tracial rank zero

ABSTRACT

Let A be a unital separable simple C^*-algebra with tracial rank zero and let $x, y \in A$ be two normal elements. We show that x is in the closure of the convex hull of the unitary orbit of y if and only if there exists a sequence of unital completely positive linear maps φ_n from A to A such that the sequence $\varphi_n(y)$ converges to x in norm and also approximately preserves the trace values. A purely measure theoretical description for normal elements in the closure of the convex hull of unitary orbit of y is also given. In the case that A has a unique tracial state some classical results about the closure of the convex hull of the unitary orbits in von Neumann algebras are proved to hold in the C^*-algebraic setting.

1. Introduction

In the algebra of n by n matrices over the complex field, two normal elements are unitarily equivalent inly if they have the same eigenvalues counting multiplicities. One

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https://doi.org/10.1016/j.jfa.2019.108306
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can also describe the convex hull of unitary orbit of a given self-adjoint matrix using eigenvalue distribution. Indeed, by the classical Horn's theorem ([11], see also [2] and [14]), if x and y are two self-adjoint matrices, then x is in the convex hull of unitary orbit of y if the eigenvalues are majorized by those of x.

In infinite dimensional spaces, one studies the closure of the convex hull of unitary orbit of an operator. These results were extended to von Neumann algebras (see, for example, [10] and [13]). Moreover, in [9] (and [3]), the closure of the convex hull of unitary orbits of normal elements in II_1 factors were also studied. Since these descriptions are closely related to the measure theory, via spectral theory, one may expect that the original description in finite matrix algebras carries out to von Neumman algebras or at least II_1 factors.

The situation is rather different in C^*-algebras since spectral theory no longer holds. However, more recently, in [31], [28] and [29], the closure of the convex hull of self-adjoint elements in unital simple C^*-algebras with tracial rank zero (or real rank zero and stable rank one, as well as other regularities) has been studied. The current research was inspired by these papers with [12].

One of the conveniences of studying self-adjoint elements is that the C^*-subalgebra generated by a self-adjoint element has a certain weak semi-projective property. Moreover, the assumption that C^*-algebras have real rank zero means that self-adjoint elements can be approximated by those with finite spectrum. These advantages disappear when x and y are only assumed to be normal.

In the current paper, we study the normal elements in the closure of the convex hull of unitary orbit of normal elements in unital simple C^*-algebras with tracial rank zero. The weak semi-projectivity property can be partially recovered by the theorem of [7]. However, normal elements in general simple C^*-algebras with tracial rank zero may not be approximated by normal elements with finite spectrum. Nevertheless, a theorem in [19] shows that the normal elements in general simple C^*-algebras with tracial rank zero can actually be approximated by those with finite spectrum if a K_1-related index vanish. Moreover, unitary orbits of normal elements in general simple C^*-algebras with tracial rank zero were characterized in [23]. Using these results, in this paper, we characterize the normal elements in the closure of the convex hull of unitary orbits of normal elements in a general simple C^*-algebra with tracial rank zero (see 4.8 below). This is in the same spirit as results in II_1-factors as in [9] and [3] even though the simple C^*-algebra A may have rich tracial simplex. On the other hand, say, if we assume that A also has a unique tracial state, then a purely measure theoretical description of normal elements in the closure of convex hull of normal elements can be presented (see 5.12 below). We also extend the result slightly beyond the case that A has tracial rank zero (see 5.10).

Suppose that x is a normal element in the closure of the convex hull of the unitary orbit of a normal element y, and y is in the closure of the convex hull of the unitary orbit of x. Then, in a II_1-factor M, x and y are approximately unitarily equivalent (see Theorem 5.1 of [3]). In a general unital simple C^*-algebra A with tracial rank zero, this no longer holds simply because the presence of non-trivial K_1 as well as infinitesimal elements in

$K_0(A)$. However, when these K-theoretical obstacles disappear, we show that these two notions still coincide in simple C^*-algebra of tracial rank zero. In particular, we show that, in a unital simple AF-algebra A with a unique trace if both $\mathrm{sp}(x)$ and $\mathrm{sp}(y)$ are connected, and x is in the closure of convex hull of the unitary orbit of a normal element y, and y is in the closure of convex hull of the unitary orbit of x, then x and y are approximately unitarily equivalent.

Acknowledgments Much of this research work was done when both authors were in the Research Center of Operator Algebras at East China Normal University which is partially supported by Shanghai Key Laboratory of PMMP, The Science and Technology Commission of Shanghai Municipality (STCSM), grant #13dz2260400 and by a NNSF grant (11531003). During the research, the second named author was also supported by a NSF grant (DMS 1665183).

2. Notations

Let A be a unital C^*-algebra. We will use the following convention:

(1) $U(A)$ is the unitary group of A.

(2) $\mathcal{N}(A)$ is the set of all normal elements of A, $A_{s.a}$ is the set of all self-adjoint elements of A and A_+ is the set of positive elements of A.

(3) $\mathcal{N}_0(A) = \{x \in \mathcal{N}(A) : \forall \lambda \notin \mathrm{sp}(x), [\lambda - x] = 0 \text{ in } K_1(A)\}$.

(4) For any $a \in A$, $\mathcal{U}(a) = \{u^*au : u \in U(A)\}$ is the unitary orbit of a.

(5) For any $a \in A$, $\mathrm{conv}(\mathcal{U}(a))$ is the convex hull of the unitary orbit $\mathcal{U}(a)$.

(6) If p is a projection of A, $a \in pAp$, $\mathrm{conv}(\mathcal{U}_p(a))$ is the convex hull of the unitary orbit in pAp.

(7) $T(A)$ is the set of all tracial states. If $\tau \in T(A)$, then $\tau \otimes \mathrm{tr}$ is a tracial state of $M_n(A)$, where tr is the tracial state of $M_n(\mathbb{C})$. We shall continue to use τ for $\tau \otimes \mathrm{tr}$.

Denote by $QT(A)$ the set of all normalized 2-quasi-traces (II.1.1 of [5]). By II.4.3 of [5], if A is a simple C^*-algebra which has only one quasi-trace, then it is a 2-quasi-trace. If A is a unital C^*-algebra, then $QT(A)$ is a simplex (see II. 4.4 of [5]). By a result of Haagrup [8], every 2-quasi-trace on an exact C^*-algebra is a trace. We will also use the fact that every quasi-trace of a unital simple separable C^*-algebra of tracial rank zero is a trace (see, for example, Corollary 6.3 of [26]).

(8) Let $a, b \in A$ and let $\epsilon > 0$. Let us write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. Suppose that $S \subset A$ is a subset. Let us write $a \in_\epsilon S$ if $\inf\{\|a - s\| : s \in S\} < \epsilon$. We may write $a \in_{\epsilon'} S$ including the case $\epsilon' = 0$ which we mean that $a \in S$.

(9) Denote by $GL(A)$ the set of invertible elements. Recall that A has stable rank one, if $GL(A)$ is dense in A.

(10) Let $p, q \in A$ be two projections. We write $[p] = [q]$ if there exists a $v \in A$ such that $v^*v = p$ and $vv^* = q$. We write $[p] \le [q]$, if $[p] = [q']$ for some projection $q' \le q$.

We write $[p] \le_u [q]$, if there exists a unitary $u \in A$ such that $u^*pu \le q$, and $[p] =_u [q]$, if $u^*pu = q$. For any integer $K > 0$, we write $K[p] =_u [q]$ if there are mutually orthogonal projections $q_1, q_2, ..., q_K \in A$ such that $q_i \le q, [q_i] =_u [p], i = 1, 2..., K$, and $\sum_{i=1}^K q_i = q$, and write $K[p] \le_u [q]$ if there exists a projection $q' \le q$ such that $K[p] =_u [q']$.

If A has stable rank one, then $[p] \le [q]$ is the same as $[p] \le_u [q]$ and $[p] = [q]$ is the same as $[p] =_u [q]$. Note, almost all the cases in this paper, A has stable rank one.

(11) Let $x, y \in A_+$ be positive elements. We write $x \lesssim y$, if there exists $r_n \in A$ such that $\lim_{n\to\infty} \|r_n^* y r_n - x\| = 0$. In case x and y are projections, then there exists a partial isometry $v \in A$ such that $v^*v = x$ and $vv^* \le y$. If A has stable rank one and $x \lesssim y$, then there exists $z \in A$ such that $z^*z = x$ and $zz^* \in \overline{yAy}$.

Let $K \ge 1$ be an integer. We write $K\langle x\rangle \le \langle y\rangle$, if there are K mutually orthogonal positive elements $x_1, x_2, ..., x_K \in M_m(A)$ for some $m \ge 1$,

$$x_1 + x_2 + \cdots + x_K \lesssim \mathrm{diag}(y, \overbrace{0, ..., 0}^{m-1}),$$

and $x_i \lesssim x$ and $x \lesssim x_i$, $i = 1, 2, ..., K$.

If $p \in A$ is a projection and $p \lesssim x$, then there exists partial isometry $v \in A$ such that $v^*v = p$ and $vv^* \in \overline{xAx}$.

(12) A linear map $\varphi : A \to A$ is said to trace preserving if $\tau \circ \varphi = \tau$ for all $\tau \in T(A)$.

(13) Let $\mathcal{F} \subset A$ be a finite subset and $\epsilon > 0$. Suppose that B is another C^*-algebra. A positive linear map $L : A \to B$ is said to be $\mathcal{F}$-ϵ-multiplicative if $\|L(xy) - L(x)L(y)\| < \epsilon$ for all $x, y \in \mathcal{F}$.

3. Preliminaries

The following lemma is well-known.

Lemma 3.1. *Suppose that A is a unital C^*-algebra.*

(1) *Let $\epsilon_1, \epsilon_2 > 0$, and let $a, b, c \in A$ such that $a \in_{\epsilon_1} \mathrm{conv}(\mathcal{U}(b))$ and $b \in_{\epsilon_2} \mathrm{conv}(\mathcal{U}(c))$. Then*

$$a \in_{\epsilon_1+\epsilon_2} \mathrm{conv}(\mathcal{U}(c)).$$

(2) *Let $\{p_i, i = 1, 2, ..., n\}$ be projections in A with $\sum_{i=1}^n p_i = 1_A$ and let $a_i, b_i \in p_iAp_i$, $i = 1, 2, ..., n$. Suppose that for some $\epsilon_i > 0$, $a_i \in_{\epsilon_i} \mathrm{conv}(\mathcal{U}_{p_i}(b_i)), i = 1, 2, ..., n$. Then*

$$\sum_{i=1}^n a_i \in_\epsilon \mathrm{conv}(\mathcal{U}(\sum_{i=1}^n b_i)),$$

where $\epsilon = \max\{\epsilon_i : i = 1, 2, ..., n\}$.

Proof. (1) There are $\{u_i, v_j : i = 1, ..., m; j = 1, ..., n\} \subset U(A)$ and $\{t_i, s_j : i = 1, ..., m; j = 1, ..., n\} \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$ and $\sum_{j=1}^n s_j = 1$ satisfying

$$a \approx_{\epsilon_1} \sum_{i=1}^m t_i u_i^* b u_i, \quad \text{and} \quad b \approx_{\epsilon_2} \sum_{j=1}^n s_j v_j^* c v_j.$$

Let

$$a' = \sum_{i=1}^m \sum_{j=1}^n t_i s_j u_i^* v_j^* c v_j u_i,$$

then $a' \in \mathrm{conv}(\mathcal{U}(c))$ and

$$a' \approx_{\epsilon_2} \sum_{i=1}^m t_i u_i^* b u_i \approx_{\epsilon_1} a.$$

(2) It is enough to prove the case $n = 2$. Note that $p_1 p_2 = 0$. Suppose that

$$a_1 \approx_{\epsilon_1} \sum_{i=1}^n s_i u_i^* b_1 u_i, \quad \text{and} \quad a_2 \approx_{\epsilon_2} \sum_{j=1}^m t_j v_j^* b_2 v_j, \tag{e 3.1}$$

where $s_i, t_j \in [0,1]$ with $\sum_{i=1}^n s_i = 1$ and $\sum_{j=1}^m t_j = 1$, and $u_i \in U(p_1 A p_1), v_j \in U(p_2 A p_2)$. For any i, j, let $w_{ij} = u_i + v_j \in U(A)$, and let $h_{ij} = s_i t_j \in [0,1]$. Then one can check $\sum_{i,j} h_{ij} = 1$, and

$$\begin{aligned}\sum_{i,j} h_{ij} w_{ij}^* (b_1 + b_2) w_{ij} &= \sum_{i=1}^m (\sum_{j=1}^n s_i t_j (u_i^* b_1 u_i)) + \sum_{i=1}^m \sum_{j=1}^n s_i t_j (v_j^* b_2 v_j) \\ &= \sum_{i=1}^m s_i (u_i^* b_1 u_i) + \sum_{j=1}^n \sum_{i=1}^m s_i t_j (v_j^* b_2 v_j) = \sum_{i=1}^m s_i u_i^* b_1 u_i + \sum_{j=1}^n t_j v_j^* b_2 v_j.\end{aligned}$$

Combining this with (e 3.1), one obtains that $a_1 + a_2 \approx_\epsilon \mathrm{conv}(\mathcal{U}(b_1 + b_2))$. □

Definition 3.2. Recall that $D \in M_n(\mathbb{C})$ is called a doubly stochastic matrix if $D = (d_{ij})$ with $d_{ij} \in [0,1]$ with $\sum_{i=1}^n d_{ij} = \sum_{j=1}^n d_{ij} = 1$ for all i, j. Denote by $\mathcal{D}_n$ the set of all doubly stochastic matrices in $M_n(\mathbb{C})$.

Definition 3.3. For any $x = (\lambda_1, \cdots, \lambda_n), y = (\mu_1, \cdots, \mu_n) \in \mathbb{C}^n$, we write $x \prec y$ if there is $D = (d_{i,j}) \in \mathcal{D}_n$ such that

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}. \tag{e 3.2}$$

We may also write

$$(\lambda_1, \cdots, \lambda_n)^T = D(\mu_1, \cdots, \mu_n)^T$$

instead of (e 3.2).

The following is a variation of Horn's theorem (see also [2]).

Lemma 3.4. *Suppose $x = \sum_{i=1}^n \lambda_i p_i, y = \sum_{i=1}^n \mu_i q_i$ are normal elements in $M_n(\mathbb{C})$, where $\{\lambda_i, \mu_i : i = 1, 2..., n\} \subset \mathbb{C}$, and $\{p_i, q_i : 1 = 1, 2, ..., n\}$ are rank one projections such that $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1_{M_n(\mathbb{C})}$. Then the following conditions are equivalent:*

(1) $(\lambda_1, \cdots, \lambda_n) \prec (\mu_1, \cdots, \mu_n)$.
(2) $x \in \mathrm{conv}(\mathcal{U}(y))$.
(3) *There is a unital trace preserving completely positive linear mapping Φ on $M_n(\mathbb{C})$ such that $\Phi(y) = x$.*
(4) *There is a unital positive linear mapping $\Psi : C^*(y) \to C^*(x)$ such that $\Psi(y) = x$ and $\tau(\Psi(f)) = \tau(f)$ for any $f \in C^*(y)$, where τ is the tracial state of $M_n(\mathbb{C})$, and where $C^*(y), C^*(x)$ are the C^*-subalgebras in $M_n(\mathbb{C})$ generated by $\{1_{M_n(\mathbb{C})}, y\}$ and $\{1_{M_n(\mathbb{C})}, x\}$ respectively.*

Proof. (1) $\Rightarrow$ (2): First, we consider that in the case $x = \mathrm{diag}(\lambda_1, \cdots, \lambda_n), y = \mathrm{diag}(\mu_1, \cdots, \mu_n)$. If $(\lambda_1, \cdots, \lambda_n) \prec (\mu_1, \cdots, \mu_n)$, then there is $D = (d_{ij}) \in \mathcal{D}_n$ such that $(\lambda_1, \cdots, \lambda_n)^T = D(\mu_1, \cdots, \mu_n)^T$. By Birkhoff's Theorem [4],

$$D = \sum_{\sigma \in \Sigma_n} t_\sigma v_\sigma,$$

where Σ_n is the permutation group of $\{1, ..., n\}$, $t_\sigma \in [0, 1]$ with $\sum_{\sigma \in \Sigma_n} t_\sigma = 1$, and v_σ is the permutation on $\mathbb{C}^n$. One may check that

$$\mathrm{diag}(\lambda_1, \cdots, \lambda_n) = \sum_{\sigma \in \Sigma_n} t_\sigma u_\sigma^* (\mathrm{diag}(\mu_1, \cdots, \mu_n)) u_\sigma, \tag{e 3.3}$$

where u_σ is the unitary of $M_n(\mathbb{C})$ induced by v_σ. That is, viewing element u_σ as a linear operator on $\mathbb{C}^n$, for any $(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, $u_\sigma((\lambda_1, \cdots, \lambda_n)) = (\lambda_{\sigma_1}, ..., \lambda_{\sigma_n})$. In general case, let e_j be the element in $\mathbb{C}^n$ with 1 in j-th coordinate and zero elsewhere, $j = 1, 2, ..., n$. There are $u, v \in U(M_n(\mathbb{C}))$ such that $u^* p_i u = e_i$ and $v^* q_j v = e_j$, $i, j = 1, 2, ..., n$. Then, $u^* x u = \mathrm{diag}(\lambda_1, \cdots, \lambda_n), v^* y v = \mathrm{diag}(\mu_1, \cdots, \mu_n)$. Then by (e 3.3),

$$x = \sum_{\sigma \in \Sigma_n} t_\sigma (v u_\sigma^* u^*)^* y (v u_\sigma u^*),$$

or $x \in \mathrm{conv}(\mathcal{U}(y))$.

(2) $\Rightarrow$ (3): If $x = \sum_{i=1}^{N} t_i u_i^* y u_i$, where $t_i \in [0,1]$ with $\sum_{i=1}^{N} t_i = 1$ and $u_i \in U(M_n(\mathbb{C})), i = 1,2,...,N$. Define $\Phi(z) = \sum_{i=1}^{N} t_i u_i^*(z) u_i$ for any $z \in M_n(\mathbb{C})$. Then Φ is a trace preserving completely positive contractive linear map with $\Phi(y) = x$.

(3) $\Rightarrow$ (1): Let u, w be unitaries in $M_n(\mathbb{C})$ such that $u^* y u = \mathrm{diag}(\mu_1, \cdots, \mu_n)$ and $w^* x w = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. By replacing Φ by $\mathrm{Ad}\, w \circ \Phi \circ \mathrm{Ad}\, u^*$, we assume that $\Phi(\mathrm{diag}(\mu_1, \cdots, \mu_n)) = (\lambda_1, \cdots, \lambda_n)$. Let $\tau \in T(M_n(\mathbb{C}))$ be the unique tracial state. Let e_i be as above. Then for any $i, j = 1,2,...,n$, let define

$$f_{ij} = e_i \Phi(e_j) e_i \text{ and } d_{ij} = \mathrm{Tr}(f_{ij}) = n\tau(f_{ij}), \;\; i,j = 1,2,...,n.$$

Then $f_{ij} \in (e_i M_n(\mathbb{C}) e_i)_+$ with $\|f_{ij}\| \le 1$, and $0 \le d_{ij} \le 1$. One checks

$$\sum_{i=1}^{n} d_{ij} = \sum_{i=1}^{n} n\tau(e_i \Phi(e_j)) = n\tau(\Phi(e_j)) = n\tau(e_j) = 1;$$
$$\sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n} n\tau(e_i \Phi(e_j)) = n\tau(e_i \Phi(1_A)) = n\tau(e_i) = 1.$$

Moreover, for any i,

$$\sum_{j=1}^{n} \mu_j d_{ij} = \sum_{j=1}^{n} \mu_j n\tau(e_i \Phi(e_j)) = n\tau(e_i \Phi(\sum_{j=1}^{n} \mu_j e_j))$$
$$= n\tau(e_i(\sum_{j=1}^{m} \lambda_j e_j)) = n\tau(\lambda_i e_i) = \lambda_i n\tau(e_i) = \lambda_i.$$

In other words, with $D = (d_{ij}) \in \mathcal{D}_n$,

$$(\lambda_1, \cdots, \lambda_n)^T = D(\mu_1, \cdots, \mu_n)^T.$$

(4) $\Rightarrow$ (3): Write $y = \sum_{i=1}^{k} \alpha_i Q_i$, where $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ is distinct eigenvalues and $Q_1, Q_2, ..., Q_k$ are mutually orthogonal non-zero projections such that $\sum_{i=1}^{k} Q_i = 1$. Let $\Psi_1 : M_n(\mathbb{C}) \to C^*(y)$, be the conditional expectation,

$$\Psi_1(a) = \sum_{i=1}^{k} \frac{\tau(Q_i a Q_i)}{\tau(Q_i)} Q_i \;\text{ for all }\; a \in M_n(\mathbb{C}).$$

Then Ψ_1 is a unital positive linear map preserving the trace such that $\Psi_1(y) = y$. Then $\Phi \circ \Psi_1(y) = x$. Moreover $\Psi = \Phi \circ \Psi_1$ is a unital trace preserving positive linear map.

(3) $\Rightarrow$ (4): Let Ψ_2: $M_n(\mathbb{C}) \to C^*(x)$, be the condition expectation, and let $\Phi = \Psi_2 \circ \Psi|_{C^*(y)}$, then $\Phi(y) = x$, and Φ is a unital trace preserving positive linear map from $C^*(y)$ to $C^*(x)$. $\square$

Corollary 3.5. *Suppose A is a unital C^*-algebra with $T(A) \neq \emptyset$. $\{\lambda_i, \mu_i : i = 1, 2, ..., n\}$ are complex numbers.*

(1) *If $\{p_i : i = 1, ..., n\}$ and $\{q_i : i = 1, ..., n\}$ are projections in A with $[p_i] =_u [q_j]$ for any i, j, $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1_A$. If*

$$x = \sum_{i=1}^n \lambda_i p_i, \quad y = \sum_{i=1}^n \mu_i q_i,$$

then $x \in \mathrm{conv}(\mathcal{U}(y))$ if and only if $(\lambda_1, \cdots, \lambda_n) \prec (\mu_1, \cdots, \mu_n)$.

(2) *If $\{p_i : i = 1, 2, ..., n\}$ are mutually orthogonal projections in A with $[p_i] =_u [p_1]$ for all i and $p = \sum_{i=1}^n p_i$. Let $B = \{\sum_{i=1}^n \eta_i p_i : \eta_i \in \mathbb{C}, i = 1, ..., n\}$. If there is $D \in \mathcal{D}_n$ such that $(\lambda_1, \cdots, \lambda_n)^T = D(\mu_1, \cdots, \mu_n)$, then there is a unital trace preserving completely positive linear map φ_D on pAp which maps B into B such that*

$$\varphi_D(\sum_{i=1}^n \lambda_i p_i) = \sum_{i=1}^n \mu_i p_i.$$

Proof. (1) There exists $u \in U(A)$ such that $u^* p_i u = q_i, i = 1, ..., n$. Without loss of generality, we may assume $p_i = q_i, i = 1, 2, ..., n$. Furthermore, since $[p_i] = [p_1]$ for all i, and $\sum_{i=1}^n p_i = 1_A$, A is isomorphic to $M_n(B_1)$, where $B_1 = p_1 A p_1$.

Therefore it suffices to show

$$\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathrm{conv}(\mathcal{U}(\mathrm{diag}(\mu_1, \mu_2, ..., \mu_n))) \tag{e 3.4}$$

if and only if $(\lambda_1, \cdots, \lambda_n) \prec (\mu_1, \cdots, \mu_n)$. By 3.4, we only need to show that (e 3.4) implies $(\lambda_1, \cdots, \lambda_n) \prec (\mu_1, \cdots, \mu_n)$. Let $\{e_{ij}\}_{1 \le i,j \le n}$ be a system of matrix unit. Suppose that

$$\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i=1}^n \lambda_i e_{ii}$$
$$= \sum_{s=1}^m t_s w_s^* (\sum_{i=1}^n \mu_i e_{ii}) w_s = \sum_{s=1}^m t_s w_s^* (\mathrm{diag}(\mu_1, \mu_2, ..., \mu_n)) w_s,$$

where $w_s \in U(M_n(B_1)), s = 1, ..., m$, and $t_s \in (0, 1)$ with $\sum_{s=1}^m t_s = 1$. Suppose $w_s = (w_{ij}^{(s)})$, where $w_{ij}^{(s)} \in B$. For $s = 1, 2, ..., m$, we have

$$\lambda_i e_{ii} = \sum_{s=1}^m t_s e_{ii} (c_{kj}^{(s)})_{n \times n} e_{ii},$$

where $c_{kj}^{(s)} = \sum_{l=1}^n \mu_l (w_{lk}^{(s)})^* w_{lj}^{(s)}$, $k, j = 1, 2, ..., n$, $s = 1, 2, ..., m$. In other words,

$$\lambda_i e_{ii} = \sum_{s=1}^{m} t_s c_{ii}^{(s)} \otimes e_{ii} = \sum_{s=1}^{m} t_s (\sum_{l=1}^{n} \mu_l (w_{li}^{(s)})^* w_{li}^{(s)}) \otimes e_{ii}$$

$$= (\sum_{l=1}^{n} \mu_l \sum_{s=1}^{m} t_s (w_{li}^{(s)})^* w_{li}^{(s)}) \otimes e_{ii}. \tag{e 3.5}$$

Let $t \in T(A)$ and let $\tau \in T(B_1)$ such that $\tau = n \cdot t|_{B_1}$. Set $f_{li} = \sum_{s=1}^{m} t_s (w_{li}^{(s)})^* w_{li}^{(s)}$ and $d_{li} = \tau(f_{li})$. Then $f_{li} \in B_+, d_{li} \in [0,1]$, $l, i = 1, ..., n$. We have, since w_s is a unitary,

$$\sum_{l=1}^{n} d_{li} = \tau(\sum_{s=1}^{m} t_s \sum_{l=1}^{n} (w_{li}^{(s)})^* w_{li}^{(s)}) = \tau(\sum_{s=1}^{m} t_s 1_{B_1}) = 1,$$

and

$$\sum_{i=1}^{n} d_{li} = \sum_{s=1}^{m} t_s \tau\Big(\sum_{i=1}^{n} (w_{li}^{(s)^*}) w_{li}^{(s)}\Big)$$
$$= \sum_{s=1}^{m} t_s \tau\Big(\sum_{i=1}^{n} w_{li}^{(s)} (w_{li}^{(s)})^*\Big) = \sum_{s=1}^{m} t_s \tau(1_B) = 1.$$

Thus (d_{ij}) is a stochastic matrix. Then, by (e 3.5), (e 3.2) holds. We obtain $(\lambda_1, \cdots, \lambda_n) \prec (\mu_1, \cdots, \mu_n)$.

(2) Since pAp is isomorphic to $M_n(p_1 A p_1)$, $U(M_n(\mathbb{C}))$ can be viewed as an subset $U(pAp)$. For any $D \in \mathcal{D}_n$, by the proof of (1) ⇒ (4) in 3.4, $\varphi_D = \sum_\sigma t_\sigma Ad(u_\sigma)$, defined on $M_n(\mathbb{C})$, can be extended to pAp. □

Lemma 3.6. *Suppose A is a unital C^*-algebra and suppose that $e, p \in A$ are two mutually orthogonal projections.*

(1) *If $K > 1$ is an integer, $e_1, e_2, ..., e_l$ are mutually orthogonal projections in eAe, and $p_1, p_2, ..., p_l$ are mutually orthogonal projections in pAp with $K[e_i] =_u [p_i], i = 1, 2, ..., l$, such that $\sum_{i=1}^{l} e_i = e$ and $\sum_{i=1}^{l} p_i = p$, and, if*

$$x = \sum_{i=1}^{l} \lambda_i p_i \in pAp, \quad x' = \sum_{i=1}^{l} \lambda_i e_i \in eAe,$$

then

$$x + x' \in_{\epsilon_1} \mathrm{conv}(\mathcal{U}_{p+e}(x)),$$

where $\epsilon_1 = \frac{\|x\|}{K}$.

(2) *If, in addition, $q \in A$ is another projection with $q + e + p = 1_A$ and $K[e] \leq_u [q]$. Then, for any $y' \in eAe$ and any $y \in \mathcal{N}(pAp)$, one has that*

$$y \in_{\epsilon_2} \operatorname{conv}(\mathcal{U}(y'+y)),$$

where $\epsilon_2 = \frac{\|y'\|}{K+1}$.

(3) *In (1), with* $K[e_i] =_u [p_i]$ *replaced by* $K[e_i] \leq_u [p_i], i = 1, 2, ..., l,$ *and* $e + p = 1_A$, *then for any* $y' \in eAe$,

$$x + x' \in_{\epsilon_3} \operatorname{conv}(\mathcal{U}(x+y')),$$

where $\epsilon_3 = \frac{\|y'\|+3\|x\|}{K}$.

Proof. (1) We write $x = \sum_{i=1}^{l} x_i, x' = \sum_{i=1}^{l} x'_i$, where $x_i = \lambda_i p_i, x'_i = \lambda_i e_i, i = 1, 2, ..., l$.

By the definition of (10) in section 2, there exist projections $p_{1j}, j = 1, ..., K$ such that $p_1 = \sum_{j=1}^{K} p_{1j}$ and $[p_{1j}] = [e_1]$. Then there are $v_j \in (p_1 + e_1)A(p_1 + e_1)$ such that $v_j^* v_j = p_{1j}, v_j v_j^* = e_1, j = 1, 2, ..., K$. Let $u_j = v_j + v_j^* + \sum_{i \neq j} p_{1i}$, then $u_j \in U((e_1 + p_1)A(e_1 + p_1))$ with

$$u_j^* p_{1j} u_j = e_1 \text{ and } u_j^* p_{1j'} u_j = p_{1j'},\ j' \neq j;\ j = 1, 2, ..., K.$$

Let $t_j = 1/K, j = 1, 2, ..., K$. Then

$$\lambda_1(e_1 + \frac{(K-1)}{K} p_1) = \sum_{j=1}^{K} t_j u_j^* (\lambda_1 p_1) u_j \in \operatorname{conv}(\mathcal{U}_{e_1+p_1}(x_1)).$$

Since $x_1 + x'_1 = \lambda_1(p_1 + e_1)$, we have

$$x_1 + x'_1 \approx_{|\lambda_1|/K} \lambda_1(e_1 + \frac{(K-1)}{K} p_1) \in \operatorname{conv}(\mathcal{U}_{p_1+e_1}(x_1)).$$

Similarly, $x_i + x'_i \approx_{|\lambda_i|/K} \operatorname{conv}(\mathcal{U}_{p_i+e_i}(x_i)), i = 2, ..., l$. Note $\{p_i + e_i : i = 1, 2..., l\}$ is a set of mutually orthogonal projections, applying (2) of 3.1,

$$x + x' = \sum_{i=1}^{l} (x_i + x'_i) \in_{\epsilon_1} \operatorname{conv}(\mathcal{U}_{p+e}(\sum_{i=1}^{l} x_i)) = \operatorname{conv}(\mathcal{U}_{p+e}(x)),$$

where $\epsilon_1 = \max\{|\lambda_i| : i = 1, 2, ..., l\}/K = \|x\|/K$.

(2) There are unitaries $u_j \in A$ and mutually orthogonal projections $q_j \in qAq$ such that $u_j^* e u_j = q_j$, $j = 1, 2, ..., K$. Let $t_j = \frac{1}{K+1}$, $j = 0, 1, 2, ..., K$, define

$$y_0 = t_0 y' + \sum_{j=1}^{K} t_j u_j^* y' u_j.$$

Then $y_0 \in \operatorname{conv}(\mathcal{U}_{q+e}(y'))$. Moreover $\|y_0\| \leq \frac{\|y'\|}{K+1}$. It follows that

$$0 \in_{\epsilon_2} \operatorname{conv}(\mathcal{U}_{q+e}(y')).$$

Since $y \in \operatorname{conv}(\mathcal{U}_p(y))$, by (2) of 3.1, we have

$$y \in_{\epsilon_2} \operatorname{conv}(\mathcal{U}(y+y')).$$

(3) First, we consider the case $K[e_i] =_u [p_i]$.

If $0 \in \operatorname{sp}_{pAp}(x)$, without lost generality, we may assume $\lambda_1 = 0$. Applying (2) to $\sum_{i=2}^{l} p_i$ (in place of p), e (in place of e), p_1 (in place of q), x (in place of y) and as well as y' (in place of y'), we get

$$x = \sum_{i=2}^{l} \lambda_i p_i \in_{\eta_1} \operatorname{conv}(\mathcal{U}(x+y')), \text{ where } \eta_1 = \frac{\|y'\|}{K+1}.$$

Apply (1) to x and x', which are viewed as elements in pAp and eAe, respectively,

$$x + x' \in_{\eta_2} \operatorname{conv}(\mathcal{U}(x)), \text{ where } \eta_2 = \frac{\|x\|}{K}.$$

By (1) of 3.1,

$$x + x' \in_{\eta_3} \operatorname{conv}(\mathcal{U}(x+y')), \text{ where } \eta_3 = \frac{\|y'\| + \|x\|}{K}.$$

In case $0 \notin \operatorname{sp}_{pAp}(x)$, let $\lambda_1 \in \operatorname{sp}_{pAp}(x)$, we consider $x - \lambda_1 p, x' - \lambda_1 e$, and $y' - \lambda_1 e$. Then $0 \in \operatorname{sp}_{pAp}(x - \lambda_1 p)$. Replacing x, x', y' by $x - \lambda_1 p, x' - \lambda_1 e, y' - \lambda_1 e$ respectively, by the proof above, we have

$$x - \lambda_1 p + x' - \lambda_1 e \in_{\eta_3} \operatorname{conv}(\mathcal{U}(x - \lambda_1 p + y' - \lambda_1 e)),$$

where $\eta_3 = \frac{\|y'-\lambda_1 e\| + \|x - \lambda_1 p\|}{K}$. That is

$$x + x' - \lambda_1 1_A \in_{\eta_3} \operatorname{conv}(\mathcal{U}(x + y' - \lambda_1 1_A)).$$

Therefore,

$$x + x' \in_{\epsilon_3} \operatorname{conv}(\mathcal{U}(x+y')), \quad \text{where } \epsilon_3 = \frac{\|y'\| + 3\|x\|}{K}.$$

In general, $K[e_i] \leq_u [p_i]$ implies that there is a projection $p_i' \leq p_i$ such that $K[e_i] =_u [p_i'], i = 1, 2..., l$. Define $x_1 = \sum_{i=1}^{l} \lambda_i p_i'$ and $p' = \sum_{i=1}^{l} p_i'$. It follows that

$$x_1 + x' \in_{\frac{\|y'\|+3\|x_1\|}{K}} \operatorname{conv}(\mathcal{U}_{p'+e}(x_1 + y')).$$

Note that $\|x_1\| \leq \|x\|$. Note also that $x + x' = (x_1 + x') + (x - x_1)$ and $x - x_1 \in \mathrm{conv}(\mathcal{U}_{p-p'}(x - x_1))$. Since $x_1 + x' \in (p' + e)A(p' + e)$, $x - x_1 \in (p - p')A(p - p')$ and $(p' + e)(p - p') = 0$, by (2) of 3.1, we conclude that

$$x + x' \in_{\epsilon_3} \mathrm{conv}(\mathcal{U}((x_1 + y') + (x - x_1))) = \mathrm{conv}(\mathcal{U}(x + y')). \quad \square$$

Proposition 3.7. *Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, x and y be two normal elements. Suppose that $x \in \overline{\mathrm{conv}(\mathcal{U}(y))}$. Then there exists a sequence of trace preserving completely positive contractive linear maps $\Phi_n : A \to A$ such that $\lim_{n\to\infty} \|\Phi_n(y) - x\| = 0$.*

Proof. There are $0 \leq \lambda_{i,n} \leq 1$ with $\sum_{i=1}^{r(n)} \lambda_{i,n} = 1$ and unitaries $u_{i,n} \in A$ such that

$$\lim_{n\to\infty} \|x - \sum_{i=1}^{r(n)} \lambda_{i,n}(u_{i,n}^* y u_{i,n})\| = 0.$$

Define $\Phi_n : A \to A$ by $\Phi_n(a) = \sum_{i=1}^{r(n)} \lambda_{i,n} u_{i,n}^* a u_{i,n}$ for all $a \in A$. Then ψ_n is a unital completely positive contractive linear map and

$$\tau(\Phi_n(a)) = \sum_{i=1}^{r(n)} \lambda_{i,n} \tau(u_{i,n}^* a u_{i,n}) = \sum_{i=1}^{r(n)} \lambda_{i,n} \tau(a) = \tau(a)$$

for all $a \in A$, and all $\tau \in T(A)$. $\square$

The following is known. We state here for reader's convenience.

Lemma 3.8. *Let A be a unital simple C^*-algebra with $T(A) \neq \emptyset$. Let $a, b \in A_+$ with $\|a\|, \|b\| \leq 1$ such that $\tau(a) < \tau(b)$ for all $\tau \in T(A)$. Then there is $0 \leq b_0 \leq b$ such that $\tau(b_0) = \tau(a)$ for all $\tau \in T(A)$. Moreover, there are $y_n \in A$ such that*

$$\sum_{n=1}^{\infty} y_n^* y_n = a \ \text{ and } \ \sum_{n=1}^{\infty} y_n y_n^* = b_0 \leq b,$$

where the sums converge in norm.

Proof. Let $f \in \mathrm{Aff}(T(A))_+$ be such that $f(\tau) = \tau(b - a)$ for all $\tau \in T(A)$. Let $1 > \epsilon > 0$. It follows from 9.3 of [24] that there exists $0 \leq b' \leq 1 + \epsilon$ in A such that $\tau(b') = f(\tau)$ for all $\tau \in T(A)$. Put $b_1 = \mathrm{diag}(a, b')$ in $M_2(A)$ and put $B = M_2(A)$. We view A as the upper left corner of $M_2(A)$. Then $\tau(b_1) = \tau(b)$ for all $\tau \in T(B)$. By Theorem 2.9 of [6], $b_1 - b \in A_0$ (notation in [6]). In other words, there are $x_1, x_2, ...,$ in B such that

$$\sum_{n=1}^{\infty} x_n^* x_n = b_1 \ \text{ and } \ \sum_{n=1}^{\infty} x_n x_n^* = b.$$

Let $e_1 = \mathrm{diag}(1_A, 0)$. Then $e_1 a = a e_1 = a$. Put $y_n = x_n e_1$. Then

$$\sum_{n=1}^{\infty} y_n^* y_n = e_1(\sum_{n=1}^{\infty} x_n^* x_n) e_1 = a \text{ and}$$
$$\sum_{n=1}^{\infty} y_n y_n^* = \sum_{n=1}^{\infty} x_n e_1 x_n^* \le \sum_{n=1}^{\infty} x_n x_n^* = b.$$

Choose $b_0 = \sum_{n=1}^{\infty} y_n y_n^*$. Then $0 \le b_0 \le b$ and $\tau(b_0) = \tau(a)$ for all $\tau \in T(A)$. □

4. Simple C^*-algebras with tracial rank zero

Definition 4.1 *([21]).* A unital simple C^*-algebra has tracial rank zero if for any $\epsilon > 0$, any non-zero $r \in A_+$, any $\mathcal{F} \subset A$, there exists a finite dimensional C^*-algebra $B \subset A$ with unit $p \in A$, and such that

$$x \approx_\epsilon x' + x'', \text{ where } x' \in B,$$
$$x'' \in (1-p)A(1-p), \text{ for all } x \in \mathcal{F},$$
$$1 - p \lesssim r.$$

If A has tracial rank zero, we write $TR(A) = 0$.

The above definition is equivalent to the following: For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$ and any $r \in A_+ \setminus \{0\}$, there exists a projection $p \in A$, a finite dimensional C^*-subalgebra F of A with $1_F = p$, and a unital $\mathcal{F}$-ϵ-multiplicative completely positive contractive linear map $L : A \to F$ such that

$$\|px - xp\| < \epsilon \text{ for all } x \in \mathcal{F},$$
$$\mathrm{dist}(pxp, F) < \epsilon \text{ for all } x \in \mathcal{F},$$
$$\|x - ((1-p)x(1-p) + L(x))\| < \epsilon \text{ for all } x \in \mathcal{F},$$
$$1 - p \lesssim r.$$

If $TR(A) = 0$, then A has stable rank one and real rank zero. Moreover, A has the comparison property: if $p, q \in M_n(A)$ are two projections for some integer $n \ge 1$ and $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, one has $[p] \le [q]$ (see [21]).

Suppose that A is an infinite dimensional unital simple C^*-algebra. Fix a non-zero $r \in A_+$ and an integer $K \ge 1$. Note that $\overline{rAr}$ contains a positive element with infinite spectrum. It follows that there are $K + 1$ many non-zero mutually orthogonal elements $a_1, a_2, ..., a_{K+1}$ in $\overline{rAr}$. It follows (Lemma 2.3 of [15], for example) that there is a non-zero $r' \in \overline{rAr}$ such that $r' \lesssim a_i$, $i = 1, 2, ..., K + 1$. In the above definition, using r' instead of r, one may have $1 - p \lesssim r'$. Therefore $K[1 - p] \le [r]$.

Recall the following known fact:

Lemma 4.2. *Let x be a normal element and $\lambda \in \mathbb{C}$. Suppose that* $\mathrm{dist}(\lambda, \mathrm{sp}(x)) = d_0 > 0$ *and, for some $y \in A$ with $\|x - y\| < d_0$. Then $\lambda \notin \mathrm{sp}(y)$.*

Proof. Since x is normal element and $\lambda \notin \mathrm{sp}(x)$, $\|(x-\lambda)^{-1}\| = \frac{1}{\mathrm{dist}(\lambda,\mathrm{sp}(x))}$. Then

$$\|(x-\lambda)^{-1}(y-\lambda) - 1\| \le \|(x-\lambda)^{-1}\|\|(y-\lambda)-(x-\lambda)\| = \frac{\|x-y\|}{\mathrm{dist}(\lambda,\mathrm{sp}(x))} < 1.$$

Consequently, $(x-\lambda)^{-1}(y-\lambda)$ is invertible, and $\lambda \notin \mathrm{sp}(y)$. □

Lemma 4.3. *Suppose A is a unital simple C^*-algebra with tracial rank zero. If $x, y \in \mathcal{N}(A)$, then for any $\epsilon > 0$, any integer $K \ge 1$, and any non-zero projection $r \in A$, there exists a finite dimensional C^*-algebra B of A with unit p such that*

$x \approx_\epsilon x' + x'', y \approx_\epsilon y' + y''$, where $x', y' \in \mathcal{N}(B), x'', y'' \in \mathcal{N}((1-p)A(1-p))$,

$$x' = \sum_{i=1}^{l} \lambda_i p_j \quad y' = \sum_{i=1}^{m} \mu_i q_i, \tag{e 4.1}$$

where $\{\lambda_1, \lambda_2, ..., \lambda_l\}$ and $\{\mu_1, \mu_2, ..., \mu_m\}$ are distinct and ϵ-dense in $\mathrm{sp}(x)$ *and in* $\mathrm{sp}(y)$ *respectively, and $\{p_1, p_2, ..., p_l\}$, $\{q_1, q_2, ..., q_m\}$ are projections in A with $\sum_{i=1}^{l} p_i = \sum_{j=1}^{m} q_j = p$, and*

$$K[1-p] \le [r], \;\; K[1-p] \le [p_i] \;\; \text{and} \;\; K[1-p] \le [q_j], \;\; 1 \le i \le l, \;\; 1 \le j \le m. \tag{e 4.2}$$

Moreover, we may also assume that $\mathrm{sp}(x'')$ *(in $(1-p)A(1-p)$) is ϵ-dense in* $\mathrm{sp}(x)$ *and* $\mathrm{sp}(y'')$ *(in $(1-p)A(1-p)$) is ϵ-dense in* $\mathrm{sp}(y)$.

Proof. Without loss of generality, we may assume that $\|x\|, \|y\| \le 1$. Fix $\epsilon > 0$, let $0 < \epsilon_0 < \epsilon/4$. Let $\{\lambda_1, \lambda_2, ..., \lambda_l\} \subset \mathrm{sp}(x)$ and let $\{\mu_1, \mu_2, ..., \mu_m\} \subset \mathrm{sp}(y)$ be such that both sets are distinct and $\epsilon/2$-dense in $\mathrm{sp}(x)$ and $\mathrm{sp}(y)$, respectively, and

$$\zeta = \min\{\epsilon_0/2, |\lambda_i - \lambda_{i'}|, |\mu_j - \mu_{j'}| : i \ne i', j \ne j'\} > 0.$$

Since $\|x\|, \|y\| \le 1$, there are $f_i \in C(\mathbb{D})$ and $g_j \in C(\mathbb{D})$, where $\mathbb{D}$ is the closed unit disc of $\mathbb{C}$, such that $f_i(t) = 1$ for $|t - \lambda_i| < \zeta/4$, $f_i(t) = 0$ for $|t - \lambda_i| > \zeta/2$, $1 \le i \le l$ and $0 \le f_i(t) \le 1$; $g_j(t) = 1$ for $|t - \mu_j| < \zeta/4$, and $g_j(t) = 0$ for $|t - \mu_j| > \zeta/2$, $1 \le j \le m$. Note that for any i, j, $f_i(x) \ne 0$ and $g_j(y) \ne 0$.

Since A is simple, there are $a_{ik}, b_{jk} \in A$ such that

$$\sum_{k=1}^{n(i)} a_{ik}^* f_i(x) a_{ik} = 1_A \;\; \text{and} \;\; \sum_{k=1}^{n'(j)} b_{jk}^* g_j(y) b_{jk} = 1_A. \tag{e 4.3}$$

Let

$$N_0 = \max\{n(i), n'(j) : 1 \le i \le l; 1 \le j, \le m\} \text{ and } M = \max\{\|a_{ik}\|, \|b'_{jk}\| : i, j, k\}.$$

Choose $\delta_1 > 0$ such that, in any unital C^*-algebra W, if $h \in W_+$ and $\|1_W - h\| < \delta_1$, then h is invertible in W and $\|1_W - h^{-1/2}\| < \epsilon_0/2^9N_0(M+1)^2$. For convenience, we may assume that $\delta_1 < \epsilon_0$.

Note that f_i, g_j are continuous on $\mathbb{D}$, $i = 1, 2, ..., l$ and $j = 1, 2, ..., m$. There is $\delta_2 > 0$ such that, for any $S, T \in \mathcal{N}(A)$ with $\|S\| \le 1, \|T\| \le 1$, $\|S - T\| < \delta_2$ implies

$$\|f_i(S) - f_i(T)\| < \delta_1/2^9N_0(M+1)^2, i = 1, 2, ..., l \tag{e 4.4}$$

$$\|g_j(S) - g_j(T)\| < \delta_1/2^9N_0(M+1)^2, j = 1, 2, ..., m. \tag{e 4.5}$$

Without loss of generality, we may assume that $\delta_2 < \epsilon/4$.

Note that every unital hereditary C^*-subalgebra of A has stable rank one [22]. In particular, they have (IR) (see [7]). By 4.4 of [7], there exists $\delta_3 > 0$ such that for any z in a C^*-algebra with (IR) with $\|z\| \le 1$ and with the property that

$$\|z^*z - zz^*\| < \delta_3,$$

then there exists a normal element z' in that C^*-algebra such that

$$\|z - z'\| < \delta_2/2.$$

Choose

$$\eta = \min\{\delta_1, \delta_2, \delta_3\}/2^9N_0(M+1)^2.$$

Put $\mathcal{G} = \{1_A, x, y, a_{ik}, b_{jk}, f_i(x), g_j(x) : i, j, k\}$.

Since $TR(A) = 0$, for any integer $K \ge 1$, there is a finite dimensional C^*-subalgebra $B' \subset A$ with $p' = 1_{B'}$ such that

$$B' = M_{r_1}(\mathbb{C}) \bigoplus M_{r_2}(\mathbb{C}) \bigoplus \cdots \bigoplus M_{r_N}(\mathbb{C}), \tag{e 4.6}$$

$$\|z - (z' + z'')\| < \eta \text{ for all } z \in \mathcal{G}, \tag{e 4.7}$$

$$\|p'zp' - z'\| < \eta, \ \|(1-p')z(1-p') - z''\| < \eta \text{ for all } z \in \mathcal{G}, \tag{e 4.8}$$

$$\|x - (x_1 + x_2)\| < \eta, \ \|y - (y_1 + y_2)\| < \eta \text{ and} \tag{e 4.9}$$

$$K[1 - p'] \le [r], \tag{e 4.10}$$

where $x_1, y_1, z' \in B'$, $x_2, y_2, z'' \in (1-p')A(1-p')$.

Moreover, we may assume, without loss of generality, that $\|x_i\| \le 1$ and $\|y_i\| \le 1$, $i = 1, 2$. Note that, since x and y are normal,

$$\|x_i^*x_i - x_ix_i^*\| < 2\eta < \delta_3 \text{ and}$$
$$\|y_i^*y_i - y_iy_i^*\| < 2\eta < \delta_3, \quad i = 1, 2.$$

It follows from 4.4 of [7] that there are $x_{0,1}, y_{0,1} \in \mathcal{N}(B')$ and $x_{0,2}, y_{0,2} \in \mathcal{N}((1-p')A(1-p'))$ such that

$$\|x_1 - x_{0,1}\| < \delta_2/2 \text{ and } \|x_2 - x_{0,2}\| < \delta_2/2.$$

Then

$$\|x - (x_{0,1} + x_{0,2})\| < \delta_2 \text{ and } \|y - (y_{0,1} + y_{0,2})\| < \delta_2. \tag{e 4.11}$$

Therefore, by the choice of δ_2 (see (e 4.4) and (e 4.5)),

$$\|f_i(x) - (f_i(x_{0,1}) + f_i(x_{0,2}))\| < \delta_1/2^9N_0(M+1)^2, \ i = 1, 2, ..., l$$
$$\|g_j(y) - (g_j(y_{0,1}) + g_j(y_{0,2}))\| < \delta_1/2^9N_0(M+1)^2, \ j = 1, 2, ..., m.$$

Moreover,

$$\|p'f_i(x)p' - f_i(x_{0,1})\| < \delta_1/2^9N_0(M+1)^2, i = 1, 2, ..., l$$
$$\|p'g_j(y)p' - g_j(y_{0,1})\| < \delta_1/2^9N_0(M+1)^2, \ j = 1, 2, ..., m.$$

For any i and j, by (e 4.8), (e 4.3) (recall $1_A, a_{ik}, b_{jk}, x, y, f_i(x), g_j(y) \in \mathcal{G}$), there are $a'_{ik}, b'_{jk} \in B'$ and $a''_{ik}, b''_{j,k} \in (1-p')A(1-p')_+$ such that

$$\|p' - \sum_{k=1}^{n(i)}(a'_{ik})^*f_i(x_{0,1})a'_{ik}\| < 2^7N_0(M+1)^2\eta + \delta_1/2^9N_0(M+1)^2 < \delta_1 \text{ and}$$
$$\|p' - \sum_{k=1}^{n'(j)}(b'_{jk})^*g_j(y_{0,1})b'_{jk}\| < 2^7N_0(M+1)^2\eta + \delta_1/2^9N_0(M+1)^2 < \delta_1,$$
$$\|(1-p') - \sum_{k=1}^{n(i)}(a''_{ik})^*f_i(x_{0,2})a''_{ik}\| < \delta_1, \text{ and } \|(1-p') - \sum_{k=1}^{n'(j)}(b''_{jk})^*g_j(y_{0,2})b''_{jk}\| < \delta_1.$$

Then, by the choose of δ_1, for any i, there is $h_i \in B'_+$ such that

$$p' = \sum_{k=1}^{n(i)} h_i(a'_{ik})^*f_i(x_{0,1})a'_{ik}h_i.$$

Let $c'_{ik} = a'_{ik}h_i$, then

$$p' = \sum_{k=1}^{n(i)} (c'_{ik})^* f_i(x_{0,1}) c'_{ik}. \tag{e 4.12}$$

Similarly, for any i, j, we have $d'_{jk} \in B'$, $c''_{ik}, d''_{jk} \in (1-p')A(1-p')$ such that

$$1 - p' = \sum_{k=1}^{n(i)} (c''_{ik})^* f_i(x_{0,2}) c''_{ik}, \tag{e 4.13}$$

$$p' = \sum_{k=1}^{n'(j)} (d'_{jk})^* g_j(y_{0,1}) d'_{jk}, \text{ and} \tag{e 4.14}$$

$$1 - p' = \sum_{k=1}^{n'(j)} (d''_{jk})^* g_j(y_{0,2}) d''_{jk}. \tag{e 4.15}$$

Choose projections p'_i, q'_j in B', and $\lambda'_i, \mu'_j \in \mathbb{C}$ such that $\sum_{i=1}^{l'} p'_i = p'$ and $\sum_{j=1}^{m'} q'_j = p'$, and $x_{0,1} = \sum_{i=1}^{l'} \lambda'_i p'_i$, and $y_{0,1} = \sum_{j=1}^{m'} \mu'_j q'_j$.

Since $\|x - (x_{0,1} + x_{0,2})\| < \delta_2$, by 4.2, $\mathrm{sp}(x_{0,1} + x_{0,2}) \subset \{\lambda \in \mathbb{C} : \mathrm{dist}(\lambda, \mathrm{sp}(x)) < \delta_2\}$. It follows that $\mathrm{sp}(x_{0,1})$ (in $p'Ap'$) and $\mathrm{sp}(x_{0,2})$ (in $(1-p')A(1-p')$) are subsets of $\{\lambda \in \mathbb{C} : \mathrm{dist}(\lambda, \mathrm{sp}(x)) < \delta_2\}$.

Recall that $\{\lambda_1, \lambda_2, ..., \lambda_l\}$ is $\epsilon_0/2$-dense in $\mathrm{sp}(x)$. Choose $S_1 = \{\lambda_{i'} : |\lambda_{i'} - \lambda_1| < \epsilon_0\}$. Suppose that $S_1, S_2, ..., S_k$ $(k < l)$ are chosen, let $S_{k+1} = \{\lambda_{i'} : |\lambda_{i'} - \lambda_{k+1}| < \epsilon_0\} \setminus (\cup_{i=1}^k S_i)$. Note, by the choice of ζ, $S_{k+1} \supset \{\lambda'_{i'} : |\lambda'_{i'} - \lambda_i| \le \zeta\}$. By the induction, we obtain mutually disjoint subsets $S_1, S_2, ..., S_l$ such that $\mathrm{sp}(x_{0,1}) \subset \cup_{i=1}^l S_i$,

$$\{\lambda'_{i'} : |\lambda'_{i'} - \lambda_i| \le \zeta\} \subset S_i \subset \{|\lambda'_{i'} - \lambda_i| < \epsilon_0\}, \ i = 1, 2, ..., l.$$

Put $\hat{p}_i = \sum_{\lambda_{i'} \in S_i} p'_{i'}$. Then $f_i(x_{0,1}) \le \hat{p}_i, i = 1, 2, ..., l$. By (e 4.12), $f_i(x_{0,1}) \ne 0$, whence $\hat{p}_i \ne 0$. Set $x'_{0,1} = \sum_{i=1}^l \lambda_i \hat{p}_i$. Then $\sum_{i=1}^l \hat{p}_i = p'$ and

$$\|x'_{0,1} - x_{0,1}\| < \epsilon_0. \tag{e 4.16}$$

Similarly, there is $y'_{0,1} \in \mathcal{N}(B')$ such that $y'_{0,1} = \sum_{i=1}^m \mu_j \hat{q}_j$, $\hat{q}_j \ne 0$, $\sum_{j=1}^m \hat{q}_j = p'$ and

$$\|y'_{0,1} - y_{0,1}\| < \epsilon_0. \tag{e 4.17}$$

By (e 4.13), for any i, $f_i(x_{0,2}) \ne 0$. It follows that $\{z : |z - \lambda_i| < \zeta\} \cap \mathrm{sp}(x_{0,2}) \ne \emptyset$. So $\mathrm{sp}(x_{0,2})$ is δ_2-dense in $\mathrm{sp}(x)$. Similarly, $\mathrm{sp}(y_{0,2})$ is δ_2-dense in $\mathrm{sp}(y)$.

Since A is a simple C^*-algebra of real rank zero, it is easy to find a non-zero projection $e_0 \lesssim \hat{p}_i$ and $e_0 \lesssim \hat{q}_j$ for $1 \le i \le l$ and $1 \le j \le m$.

Since $TR((1-p')A(1-p')) = 0$, by repeating the above process, one obtains a finite dimensional C^* subalgebra B'' with unit $p''(\le 1 - p')$ such that

(i) $x_{0,2} \approx_{\delta_2} x''_{0,1} + x''_{0,2}, y_{0,2} \approx_{\delta_2} y''_{0,1} + y''_{0,2}$, where $x''_{0,1}, y''_{0,1} \in \mathcal{N}(B''), x''_{0,2}, y''_{0,2} \in \mathcal{N}(((1-p'-p'')A(1-p'-p'')))$,

(ii)

$$\|x''_{0,1} - \sum_{i=1}^{l} \lambda_i p''_i\| < \epsilon_0, \ \|y''_{0,1} - \sum_{j=1}^{m} \mu_j q''_j\| < \epsilon_0, \tag{e 4.18}$$

(iii) (in $(1-p'-p'')A(1-p'-p'')$), $\mathrm{sp}(x''_{0,2})$, $\mathrm{sp}(y''_{0,2})$ are δ_2-dense in $\mathrm{sp}(x_{0,2}), \mathrm{sp}(y_{0,2})$ respectively. Then it follows they are ϵ-dense in $\mathrm{sp}(x), \mathrm{sp}(y)$ respectively.

(iv) $K[1-p'-p''] \le [e_0]$.

Consequently (as $\delta_2 < \epsilon$), $\mathrm{sp}(x''_{0,2})$ (in $(1-p'-p'')A(1-p'-p'')$) is ϵ-dense in $\mathrm{sp}(x)$ and $\mathrm{sp}(y''_{0,2})$ (in $(1-p'-p'')A(1-p'-p'')$) is ϵ-dense in $\mathrm{sp}(y)$, respectively.

Let $B := B' \oplus B''$ and $p := 1_B = p' + p''$. Set $x' = x'_{0,1} + \sum_{i=1}^{l} \lambda_i p''_i = \sum_{i=1}^{l} \lambda_i (\hat{p}_i + p''_i)$, $x'' = x''_{0,2}$, $y' = y'_{0,1} + \sum_{j=1}^{m} \mu_j q''_j = \sum_{j=1}^{m} \mu_j (\hat{q}_j + q''_j)$ and $y'' = y''_{0,2}$. Then $x', y' \in B$. By (e 4.11), (e 4.16), (i) above and (e 4.18),

$$x \approx_{\delta_2} x_{0,1} + x_{0,2} \approx_{\epsilon_0 + \delta_2} x'_{0,1} + (x''_{0,1} + x''_{0,2}) \approx_{\epsilon_0} x' + x''_{0,2} = x' + x''.$$

In other words ($\delta_2 + \epsilon_0 + \delta_2 + \epsilon_0 < \epsilon$),

$$\|x - (x' + x'')\| < \epsilon.$$

Similarly,

$$\|y - (y' + y'')\| < \epsilon.$$

Define $p_i = \hat{p}_i + p''_i$, $i = 1, 2, ..., l$, and $q_j = \hat{q}_j + q''_j$, $j = 1, 2, ..., m$. Then $e_0 \lesssim \hat{p}_i \lesssim p_i$ and $e_0 \lesssim \hat{q}_j \lesssim q_j$, $1 \le i \le l$ and $1 \le j \le m$. It follows, for $i = 1, 2, ..., l$, $j = 1, 2, ..., m$

$$K[1-p] \le [r], K[1-p] \le [p_i], \text{ and } K[1-p] \le [q_j]. \quad \square$$

Lemma 4.4. *Let A be a unital C^*-algebra with a unique tracial state τ and let ϵ_1 and ϵ_2 be two positive numbers. Suppose $\{p_1, p_2, ..., p_l\}$ and $\{q_1, q_2, ..., q_l\}$ are two sets of mutually orthogonal and mutually unitarly equivalent projections with $\sum_{i=1}^{l} p_i = 1_A = \sum_{i=1}^{l} q_i$, such that there is a unitary $u \in A$ with $u^* p_i u = q_i$, $i = 1, 2, ..., l$.*

Let $x = \sum_{i=1}^{l} \lambda_i p_i$ and $y = \sum_{i=1}^{l} \mu_i q_i$ be normal in A, where $\lambda_i, \mu_i \in \mathbb{C}$. Suppose that $S_1, S_2, ..., S_N$ are mutually disjoint subsets of $\{1, 2, ..., l\}$ such that $\sqcup S_k = \{1, 2, ..., l\}$ and, $\mu_i = \mu_j$ for all $i, j \in S_k$, and, $y = \sum_{k=1}^{N} \mu'_k Q_k$ where $\mu'_k = \mu_j$ for some $j \in S_k$ and $Q_k = \sum_{i \in S_k} q_i$. Suppose that $\varphi : A \to A$ is a unital completely positive linear map such that $\varphi(y) \approx_{\epsilon_1} x$ and

$$|\tau(\varphi(Q_k)) - \tau(Q_k)| \le s\epsilon_2, \ \ k = 1, ..., N, \tag{e 4.19}$$

where

$$s = \inf\{\tau(Q_k) : k = 1, 2, ..., N; \tau \in T(A)\},$$

then there is a trace preserving unital completely positive map $\psi : A \to A$ such that

$$\|\psi(y) - x\| \le 2\epsilon_2\|y\| + 2\epsilon_1 \text{ and } x \in_{2\epsilon_2\|y\|+2\epsilon_1} \mathrm{conv}(\mathcal{U}(y)).$$

Proof. We first consider the case that $p_i = q_i$, $1 \le i \le l$. Let $\varphi' : A \to A$ be defined by

$$\varphi'(a) = \sum_{i=1}^{l} p_i\varphi(a)p_i \text{ for all } a \in A.$$

Then φ' is a unital completely positive linear map from A to A. Moreover, one checks that $\tau(\varphi'(a)) = \tau(\varphi(a))$ for any $a \in A$.

Define, for $j \in S_k$,

$$d_{ij} = \tau(p_i\varphi(Q_k)p_i)/\tau(Q_k), \ i = 1, 2, ..., l, k = 1, 2, ..., N.$$

Note that

$$\sum_{j\in S_k} d_{ij} = l\tau(p_i\varphi(Q_k)), \ i = 1, 2, ..., l, k = 1, ..., N.$$

Since φ is unital, for any $i \in \{1, 2, ..., l\}$,

$$\sum_{j=1}^{l} d_{ij} = l\sum_{k=1}^{N}\sum_{j\in S_k} \tau(p_i\varphi(Q_k))/\tau(Q_k) = 1. \tag{e 4.20}$$

Since $\|\varphi(y) - x\| < \epsilon_1$ (note that we have assumed that $q_i = p_i$, $i = 1, 2, ..., l$), for any $i \in \{1, 2, ..., l\}$,

$$\|\lambda_i p_i - \sum_{j=1}^{l} \mu_j p_i\varphi(p_j)p_i\| = \|p_i(x - \varphi(y))p_i\| < \epsilon_1. \tag{e 4.21}$$

This also implies that

$$\|x - \varphi'(y)\| < \epsilon_1.$$

It follows that for any $i = 1, 2, ..., l$,

$$|\lambda_i - \sum_{j=1}^{l} \mu_j d_{ij}| = l|\tau(\lambda_i p_i - \sum_{j=1}^{l} \mu_j p_i(\varphi(p_j)p_i))| < \epsilon_1. \tag{e 4.22}$$

We also have, for any $j \in S_k$,

$$\sum_{i=1}^{l} d_{ij} = \sum_{i=1}^{l} \tau(p_i\varphi(Q_k))/\tau(Q_k) = \tau(\varphi(Q_k))/\tau(Q_k). \tag{e 4.23}$$

By (e 4.19), for any $k \in \{1, 2, ..., N\}$,

$$|\tau(\varphi(Q_k))/\tau(Q_k) - 1| < s\epsilon_2/\tau(Q_k) \le \epsilon_2.$$

Then there are $\epsilon'_j \in \mathbb{R}$ with $|\epsilon'_j| \le \epsilon_2$ such that

$$\sum_{i=1}^{l} d_{ij} = 1 + \epsilon'_j, j = 1, ..., l. \tag{e 4.24}$$

Let $\Lambda_+ = \{j : \epsilon_j \ge 0\}$ and $\Lambda_- = \{j : \epsilon_j < 0\}$. Note that

$$\sum_{j\in\Lambda_+} (1 + \epsilon'_j) + \sum_{j\in\Lambda_-} (1 + \epsilon'_j) = \sum_{j=1}^{l}\sum_{i=1}^{l} d_{ij} = \sum_{i=1}^{l}\sum_{j=1}^{l} d_{ij} = l. \tag{e 4.25}$$

Therefore

$$\sum_{j\in\Lambda_+} \epsilon'_j + \sum_{j\in\Lambda_-} \epsilon'_j = 0. \tag{e 4.26}$$

Let

$$\epsilon_{ij} = \frac{d_{ij}\epsilon'_j}{1 + \epsilon'_j},\ j \in \Lambda_+,\ i = 1, ..., l,$$

then

$$0 \le \epsilon_{ij} \le d_{ij}, j \in \Lambda_+;\ i = 1, ..., l, \tag{e 4.27}$$

$$\sum_{i=1}^{l} \epsilon_{ij} = \epsilon'_j,\ \ j \in \Lambda_+, \tag{e 4.28}$$

and, by (e 4.20), for all i,

$$\sum_{j\in\Lambda_+} \epsilon_{ij} = \sum_{j\in\Lambda_+} \frac{d_{ij}\epsilon'_j}{1+\epsilon'_j} \le \max_{j\in\Lambda_+} \frac{\epsilon'_j}{1+\epsilon'_j}\Big(\sum_{j\in\Lambda_+} d_{ij}\Big) \tag{e 4.29}$$

$$\le \max_{j\in\Lambda_+} \frac{\epsilon'_j}{1+\epsilon'_j} < \epsilon_2. \tag{e 4.30}$$

Let $a_j = -\epsilon'_j$, $j \in \Lambda_-$ and $b_i = \sum_{j\in\Lambda_+} \epsilon_{ij}$, $i \in \{1, 2, ..., l\}$. Then, by (e 4.26) and (e 4.28),

$$\sum_{j\in\Lambda_-} a_j = \sum_{j\in\Lambda_-}(-\epsilon_j') = \sum_{j\in\Lambda_+}\epsilon_j' = \sum_{i=1}^{l} b_i.$$

By the Reisz interpolation, (see page 85 of [1]), there are
$\{\epsilon_{ij} : j \in \Lambda_-;\ i = 1, 2, ..., l\}$ with $\epsilon_{ij} \ge 0$ such that

$$\sum_{i}^{l} \epsilon_{ij} = -\epsilon_j' = a_j,\ j \in \Lambda_- \text{ and} \tag{e 4.31}$$

$$\sum_{j\in\Lambda_-} \epsilon_{ij} = b_i,\ \ i \in \{1, 2, ..., l\}. \tag{e 4.32}$$

Note that

$$\sum_{i=1}^{l} \epsilon_{ij} = \epsilon_j' < \epsilon_2,\ \ j \in \Lambda_+. \tag{e 4.33}$$

Put

$$d_{ij}' = d_{ij} - \epsilon_{ij},\ j \in \Lambda_+;\ i = 1, 2, ..., l,\ \ d_{ij}' = d_{ij} + \epsilon_{ij}, j \in \Lambda_-;\ i = 1, 2, ..., l.$$

By (e 4.27), $d_{ij}' \ge 0$. Then by (e 4.20) and (e 4.24),

$$\sum_{j=1}^{l} d_{ij}' = 1 - \sum_{j\in\Lambda_+}\epsilon_{ij} + \sum_{j\in\Lambda_-}\epsilon_{ij} = 1,\ \text{ for all }\ i = 1, 2, ..., l \qquad \text{(using (e 4.32))}$$

$$\sum_{i=1}^{l} d_{ij}' = 1 + \epsilon_j' - \sum_{i=1}^{l}\epsilon_{ij} = 1 + \epsilon_j' - \epsilon_j' = 1,\ \text{ for all }\ j \in \Lambda_+ \qquad \text{(using (e 4.28))}$$

$$\sum_{i=1}^{l} d_{ij}' = 1 + \epsilon_j' + \sum_{i=1}^{l}\epsilon_{ij} = 1 + \epsilon_j' - \epsilon_j' = 1,\ \text{ for all }\ j \in \Lambda_- \qquad \text{(using (e 4.31))}$$

In other words, for any i, j,

$$\sum_{i=1}^{l} d_{ij}' = \sum_{j=1}^{l} d_{ij}' = 1.$$

Let $D' = (d_{ij}')_{l\times l}$ and let $\varphi_{D'} = \sum_{\sigma\in\Sigma_l} t_\sigma \mathrm{Ad}\, u_\sigma$ be induced trace preserving completely positive linear mapping in part (2) in 3.5. View each u_σ as a unitary matrix in $M_l(\mathbb{C}\cdot 1_A) \subset M_l(A)$, and define

$$\varphi_{D'}\Big((c_{ij})\Big) = \sum_{\sigma\in\Sigma_l} t_\sigma u_\sigma^*(c_{ij})u_\sigma$$

for all $(c_{ij}) \in M_l(A)$. Note that,

$$\varphi_{D'}\Big(\sum_{i=1}^{l}\mu_i p_i\Big)=\sum_{i=1}^{l}\lambda_i' p_i,$$

where $(\lambda_1',\lambda_2',...,\lambda_l')$ is given by

$$(\lambda_1',\lambda_2',...,\lambda_l')^T=D'(\mu_1,\mu_2,...,\mu_l)^T$$

It follows from (e 4.22), (e 4.33) that for any $i=1,2,...,l$,

$$|\lambda_i-\lambda_i'|\le\epsilon_1+\Big|\sum_{j=1}^{l}\mu_j d_{ij}-\sum_{j=1}^{l}\mu_j d_{ij}'\Big|\le\epsilon_1+\max_{1\le j\le l}\{|\mu_j|\}\Big(\sum_{i=1}^{l}|\epsilon_{ij}|\Big)$$
$$=\epsilon_1+\max_{1\le j\le l}\{|\mu_j|\}\Big(\sum_{i\in\Lambda_+}\epsilon_{ij}+\sum_{i\in\Lambda_-}|\epsilon_{ij}|\Big)\le\epsilon_1+2\epsilon_2\|y\|.$$

Consequently,

$$\|\varphi_{D'}(y)-x\|\le\|\varphi'(y)-x\|+\|\varphi'(y)-\varphi_{D'}(y)\|<2\epsilon_1+2\epsilon_2\|y\|.$$

In general case, there is a unitary $u\in A$ such that $u^*q_iu=p_i$, $i=1,2,...,l$. Put $\varphi'=\mathrm{Ad}\,u\circ\varphi$ and $x'=u^*xu$. Then we can apply the above to obtain a trace preserving completely positive contractive linear map $\varphi_{D'}':A\to A$ such that

$$\|\varphi_{D'}'(y)-x'\|<2\epsilon_1+2\epsilon_2\|y\|.$$

Let $\psi=\mathrm{Ad}\,u^*\circ\varphi_{D'}'$. Then ψ is trace preserving completely positive contractive linear map and

$$\|\psi(y)-x\|<2\epsilon_1+2\epsilon_2\|y\|.\quad\Box$$

Corollary 4.5. *Let F be a finite dimensional C^*-algebra. Let x, $y\in F$ be two normal elements such that $y=\sum_{k=1}^N\mu_kq_k$, where $q_1,q_2,...,q_k\in F$ are projections such that $\sum_{k=1}^Nq_k=1_F$, and $\mu_k\in\mathbb{C}$, $k=1,2,...,N$. Suppose that there is a unital completely positive linear map $\varphi:F\to F$ such that $\varphi(y)\approx_{\epsilon_1}x$ and*

$$|\tau(\varphi(q_k))-\tau(q_k)|\le s\epsilon_2,\ \ k=1,...,N,\quad for\ all\ \tau\in T(F),\tag{e 4.34}$$

where

$$s=\inf\{\tau(q_k):k=1,2,...,N;\tau\in T(F)\},\tag{e 4.35}$$

then there is a trace preserving unital completely positive map $\psi:F\to F$ such that

$$\|\psi(y)-x\|\le2\epsilon_2\|y\|+2\epsilon_1\ \ and\ \ x\in_{2\epsilon_2\|y\|+2\epsilon_1}\mathrm{conv}(\mathcal{U}(y)).\tag{e 4.36}$$

Proof. Suppose $F = M_{r_1}(\mathbb{C}) \bigoplus M_{r_2}(\mathbb{C}) \bigoplus \cdots \bigoplus M_{r_K}(\mathbb{C})$. It is clear that the general case can be reduced to the case that $F = M_n(\mathbb{C})$. Then we may write $x = \sum_{i=1}^n \lambda_i p_i$, where $p_1, p_2, ..., p_n$ are mutually orthogonal rank one projections and $\lambda_i \in \mathbb{C}$, $i = 1, 2, ..., n$. The corollary then follows immediately from 4.4. $\square$

Lemma 4.6. *Let A be a unital infinite dimensional simple C^*-algebra with real rank zero and stable rank one and let $p_1, p_2, ..., p_l$ and e be mutually orthogonal projections in A with $\sum_{i=1}^l p_i + e = 1_A$. Let $x = x_1 + x_2$, where $x_1 = \sum_{j=1}^l \lambda_j p_j$, and $x_2 \in eAe$ be normal elements. Suppose that $K > 1$ is an integer such that $(2K+1)[e] \le [p_j]$ $(1 \le j \le l)$ and $\{\lambda_1, \lambda_2, ..., \lambda_l\}$ is η-dense in $\mathrm{sp}(x)$. Then, for any $\epsilon > 0$,*

$$\mathrm{dist}(x, \mathrm{conv}(\mathcal{U}(x_1))) \le \frac{8\|x\|}{K} + \epsilon + \eta.$$

Proof. Choose a projection $p_0 \le p_1$ such that $[p_0] = [e]$. Note that, by 3.4 of [7], since A has stable rank one, there exists a normal element $x_2' \in eAe$ such that $\mathrm{sp}(x_2') \subset \Gamma$, where Γ is an one-dimensional finite CW complex in the plane ($\mathrm{sp}(x_2')$ is the spectrum of x_2' in eAe) and $\|x_2' - x_2\| < \epsilon/8$. Then, $\mathrm{sp}(x_2') = X_0 \cup X_1$, where $X_0 \subset \mathbb{C}$ is a compact subset with covering dimension no more than 1 such that $K_1(C(X_0)) = \{0\}$ and where X_1 is an one-dimensional finite CW complex in the plane. Note, by 4.2, $\mathrm{sp}(x_2') \subset \{\lambda \in \mathbb{C} : \mathrm{dist}(\lambda, \mathrm{sp}(x)) < \epsilon/8\}$. It follows from Lemma 3 of [19] that there exists $x_0 \in \mathcal{N}(p_0 A p_0)$ with $\mathrm{sp}(x_0) = X_1$ (spectrum in $p_0 A p_0$), such that, for any $\lambda \notin \mathrm{sp}(x_2')$,

$$[\lambda p_0 - x_0] = -[\lambda e - x_2'] \text{ in } K_1(A).$$

Choose any mutually orthogonal projections $\{e_i : i = 1, ..., l\} \subset eAe$ such that $\sum_{i=1}^l e_i = e$. Choose $p_i' \le p_i$ such that $[p_i'] = [e_i]$, $i = 1, 2, ..., l$.

Define $x_3' = \sum_{j=1}^l \lambda_j p_j'$ and $x_3 = \sum_{j=1}^l \lambda_j (p_j - p_j')$. We may write

$$x = x_3' + x_3 + x_2.$$

Note that, since A has stable rank one, one computes that

$$[p_j - p_j'] = [p_j] - [p_j'] \ge (2K+1)[e] - [e] = 2K[e] \ge 2K[p_j'], \tag{e 4.37}$$

$j = 1, 2, ..., l$. It follows from (3) of 3.6 and 3.1 that

$$x \in_{4\|x\|/2K} \mathrm{conv}(\mathcal{U}(x_0 + x_3 + x_2)). \tag{e 4.38}$$

Since $\lambda(p_0 + e) - (x_0 + x_2') \in \mathrm{Inv}_0((p_0 + e)A(p_0 + e))$ for all $\lambda \notin \mathrm{sp}(x_2')$, by [19],

$$\|(x_0 + x_2') - x_4\| < \epsilon/16, \tag{e 4.39}$$

for some $x_4' \in (p_0+e)A(p_0+e)$ with finite spectrum $\mathrm{sp}(x_4') \subset \mathrm{sp}(x_2') \subset \{\lambda \in \mathbb{C} : \mathrm{dist}(\lambda, \mathrm{sp}(x)) < \epsilon/8\}$ (as normal elements in $(p_0+e)A(p_0+e)$). Therefore, there is a normal element $x_4 = \sum_{j=1}^{l} \lambda_j q_j$ (recall that $\{\lambda_1, \lambda_2, ..., \lambda_l\}$ is η-dense in $\mathrm{sp}(x)$) such that

$$\|(x_0 + x_2) - x_4\| < (\epsilon/16 + \epsilon/8) + (\epsilon/8 + \eta) < \epsilon/2 + \eta, \tag{e 4.40}$$

where $\{q_1, q_2, ..., q_l\}$ is a set of mutually orthogonal projections in $(p_0+e)A(p_0+e)$.

By (e 4.37),

$$K[q_i] \le K[p_0 + e] = 2K[e] \le [p_i - p_i'], i = 1, ..., l.$$

Thus, by part (3) of 3.6 (note $x_1 = x_3' + x_3$),

$$x_4 + x_3 \in_{\frac{4\|x\|}{K}} \mathrm{conv}(\mathcal{U}(x_1)). \tag{e 4.41}$$

By (e 4.38), (e 4.40) and (e 4.41), we conclude that

$$x \in_{\frac{8\|x\|}{K}+\epsilon+\eta} \mathrm{conv}(\mathcal{U}(x_1)). \quad \square$$

Theorem 4.7. *Let A be a unital separable simple C^*-algebra with tracial rank zero. Then for any normal elements $x, y \in A$, $x \in \overline{\mathrm{conv}(\mathcal{U}(y))}$ if and only if there exists a sequence of unital completely positive linear maps $\psi_n : A \to A$ such that*

$$\lim_{n\to\infty} \|\psi_n(y) - x\| = 0 \ \ and \tag{e 4.42}$$

$$\lim_{n\to\infty} \sup\{|\tau(\psi_n(a)) - \tau(a)| : \tau \in T(A)\} = 0 \ \ for\ all\ a \in A. \tag{e 4.43}$$

Proof. Let $x, y \in \mathcal{N}(A)$ and let $\{\psi_n\} : A \to A$ be a sequence of unital completely positive contractive linear maps which satisfies (e 4.42) and (e 4.43).

Without loss of generality, we may assume that $\|x\|, \|y\| \le 1$. By replacing y by $y - \lambda 1_A$ for some $\lambda \in \mathrm{sp}(y)$ and x by $x - \lambda 1_A$, without loss of generality, we may assume that $0 \in \mathrm{sp}(y)$.

Fix $\epsilon > 0$. Put $\epsilon_0 = \epsilon/2^{14}$ and an integer K such that $0 < 2^8/K < \epsilon_0$.

It follows from 4.3 that there exists a finite dimensional C^*-subalgebra B of A with $p_0 = 1_B$ and there are $x_0, y_0 \in \mathcal{N}(B)$, $x_0', y_0' \in \mathcal{N}((1-p_0)A(1-p_0))$ such

$$x \approx_{\epsilon_0} x_0 + x_0', \quad y \approx_{\epsilon_0} y_0 + y_0', \tag{e 4.44}$$

where

$$x_0 = \sum_{i=1}^{h} \lambda_i p_{i0}, \quad y_0 = \sum_{k=1}^{N} \mu_k q_{k0},$$

where $\{\lambda_1, ..., \lambda_h\}$ and $\{\mu_1, ..., \mu_N\}$ are distinct and ϵ_0-dense in $\mathrm{sp}(x)$ and $\mathrm{sp}(y)$ respectively. Moreover, we may assume $\mu_1 = 0$, and $p_{i0}, q_{k0} \in B$ are projections satisfying condition:

$$\sum_{i=1}^{h} p_{i0} = \sum_{k=1}^{N} q_{k0} = p_0 \text{ and} \tag{e 4.45}$$

$$(2K+1)[1-p_0] \le [p_{i0}], \quad i = 1, 2, ..., h \text{ and} \tag{e 4.46}$$

$$(2K+1)[1-p_0] \le [q_{k0}], k = 1, 2, ..., N. \tag{e 4.47}$$

Put $q_{00} = 1 - p_0$ and

$$s_0 = \inf\{\tau(q_{k0}) : \tau \in T(A), \quad k = 0, 1, ..., N\} > 0. \tag{e 4.48}$$

It follows from (2) of 3.6 and the fact $1/((2K+1)+1) < \epsilon_0$ that

$$y_0 \in_{\epsilon_0} \mathrm{conv}(\mathcal{U}(y_0 + y_0')), \text{ and } y_0 \in_{2\epsilon_0} \mathrm{conv}(\mathcal{U}(y)). \tag{e 4.49}$$

Then there is a trace preserving unital completely positive contractive linear map $\varphi_1 : A \to A$ such that

$$\|\varphi_1(y) - y_0\| < 2\epsilon_0. \tag{e 4.50}$$

It follows from 4.6 that there is a second trace preserving unital completely positive contractive linear map $\varphi_2 : A \to A$ such that

$$\|\varphi_2(y_0) - y\| < \frac{8\|y\|}{K} + \epsilon_0 + \epsilon_0 < 3\epsilon_0. \tag{e 4.51}$$

Put $B_0 = \mathbb{C}(1 - p_0) \oplus B$. Let $\mathcal{P}_{B_0}$ be the set of all non-zero projections in B_0. There are only finitely many unitary equivalence classes of projections in $\mathcal{P}_{B_0}$. Put

$$s_{00} = \inf\{\tau(e) : \tau \in T(A) \text{ and } e \in \mathcal{P}_{B_0}\} > 0.$$

Put

$$\epsilon_1 = (s_{00}/64NK)\epsilon_0.$$

Note that the unit ball of B_0 is compact.

Choose large n_0 so that

$$\|\psi_{n_0} \circ \varphi_2(y_0) - x\| < 4\epsilon_0 \text{ and} \tag{e 4.52}$$

and for all $b \in B_0$,

$$\sup\{|\tau(\psi_{n_0}\circ\varphi_2(b))-\tau(\varphi_2(b))|:\tau\in T(A)\}\le(\epsilon_1/4)\|b\|. \tag{e 4.53}$$

Put $\varphi_3=\psi_{n_0}\circ\varphi_2$. Since φ_2 is trace preserving, by (e 4.53),

$$\sup\{|\tau(\varphi_3(b))-\tau(b)|:\tau\in T(A)\}\le(\epsilon_1/4)\|b\| \text{ for all } b\in B_0. \tag{e 4.54}$$

We also have

$$\|\varphi_3(y_0)-x\|<4\epsilon_0 \tag{e 4.55}$$

There are $x_{ik}, y_{ik}\in A$ such that (see 2.9 of [6] and Lemma 3.8 above)

$$\Big\|\varphi_3(q_{k0})-\sum_{i=1}^{m(k)}x_{ik}^*x_{ik}\Big\|<\epsilon_1,\quad \Big\|\sum_{i=1}^{m(k)}x_{ik}x_{ik}^*-q_{k0}\Big\|<\epsilon_1, \tag{e 4.56}$$

$$\Big\|\sum_{i=1}^{m(k)'}y_{ik}^*y_{ik}-q_{k0}\Big\|<\epsilon_1 \text{ and } \sum_{i=1}^{m(k)'}y_{ik}y_{ik}^*\ge s_0\cdot 1_A, \tag{e 4.57}$$

$k=0,1,2,...,N$.

Note that we may assume, without loss of generality, (by splitting x_{ik}, y_{ik} into more terms), that $\|x_{ik}\|$, $\|y_{ik}\|\le 1$ for all i and k.

Put

$$\mathcal{F}_1=\{1_A,y_0,p_0,p_{10},...,p_{h0};q_{00},q_{10},...,q_{N0}\}\cup\mathcal{F}_0\cup\{\varphi_3(y_0)\},$$

where

$$\mathcal{F}_0=\{x_{ik},x_{ik}^*:1\le i\le m(k),0\le k\le N\}\cup\{y_{ij},y_{ik}^*:1\le i\le m(k)',0\le k\le N\}.$$

Then put $M_{00}=\max\{m(k),m(k)':k=0,1,...,N\}$.

Let $\epsilon_2=\epsilon_1/(2^{10}(M_{00}+1)NK)$. Choose a finite subset $\mathcal{F}_{B_0}$ of the unit ball of B_0 which is $\epsilon_2/4$-dense. Since B_0 is projective, choose $\delta_0>0$ and a finite subset $\mathcal{G}_{B_0}$ such that, for any unital $\mathcal{G}_{B_0}$-δ_0-multiplicative completely positive contractive linear map L' from B_0 to a unital C^*-algebra A', there exists a unital homomorphism $h':B_0\to A'$ such that

$$\|h'(b)-L'(b)\|<\epsilon_2/4 \text{ for all } b\in\mathcal{F}_{B_0}.$$

Consequently,

$$\|h'(b)-L'(b)\|\le(\epsilon_2/2)\|b\| \text{ for all } b\in B_0.$$

We may assume that $\mathcal{G}_{B_0}$ contains a generating set of B_0.

Put

$$\mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{G}_{B_0} \cup \{x, y_0\}.$$

Choose a large finite subset $\mathcal{F}_3$ of A such that for any non-zero element $z \in \mathcal{F}_2$, there are $a_{zi}, b_{zi} \in \mathcal{F}_3$ such that

$$\sum_{i=1}^{r(z)} a_{zi} z b_{zi} = 1_A. \tag{e 4.58}$$

Put $M_1 = 2\max\{r(z)\max\{\|a_{zi}\| + \|b_{zi}\| : 1 \le i \le r(z)\} : z \in \mathcal{F}_2 \setminus \{0\}\}$. Let $\epsilon_3 = \min\{\epsilon_2/M_1, \delta_0/M_1\}$.

By the virtue of 4.3, there is a finite dimensional C^*-subalgebra B_1 with $1_{B_1} = p$ such that

$$\|pz - zp\| < \epsilon_3 \text{ for all } z \in \mathcal{F}_3, \tag{e 4.59}$$

$$\mathrm{dist}(pzp, B_1) < \epsilon_3 \text{ for all } z \in \mathcal{F}_3 \text{ and} \tag{e 4.60}$$

$$\sup\{\tau(1-p) : \tau \in T(A)\} < \epsilon_3, \tag{e 4.61}$$

as well as $\bar{x}_1, y_1 \in \mathcal{N}(B_1)$ $\bar{x}_2, y_2 \in \mathcal{N}((1-p)A(1-p))$, projections $q_i, p_i \in B_1$ and $q_i' \in (1-p)A(1-p)$ such that

$$pxp \approx_{\epsilon_3} \bar{x}_1,\ (1-p)x(1-p) \approx_{\epsilon_3} \bar{x}_2, \tag{e 4.62}$$

$$py_0p \approx_{\epsilon_3} y_1,\ (1-p)y_0(1-p) \approx_{\epsilon_3} y_2, \tag{e 4.63}$$

$$\bar{x}_1 = \sum_{j=1}^{N'} \lambda_j' p_j,\ y_1 = \sum_{i=1}^{N} \mu_i q_i,\ y_2 = \sum_{i=1}^{N} \mu_i q_i' \tag{e 4.64}$$

$$q_{i0} \approx_{\epsilon_3} q_i + q_i',\ \ i = 0, 1, ..., N \tag{e 4.65}$$

$$q_0 + \sum_{i=1}^{N} q_i = \sum_{i=1}^{N'} p_i = p,\ \sum_{i=1}^{N} q_i' = 1 - p, \tag{e 4.66}$$

$$(2K+1)[1-p] \le [q_i] \text{ and } (2K+1)[1-p] \le [p_j] \tag{e 4.67}$$

$$\text{for all } 0 \le i \le N,\ 1 \le j \le N', \tag{e 4.68}$$

where $\{\lambda_1', \lambda_2', ..., \lambda_{N'}'\}$ is ϵ_3-dense in $\mathrm{sp}(x)$. We may write

$$B_1 = M_{R(1)}(\mathbb{C}) \bigoplus M_{R(2)}(\mathbb{C}) \bigoplus \cdots \bigoplus M_{R(k_0)}(\mathbb{C})$$

and let $\pi_r : B_1 \to M_{R(r)}(\mathbb{C})$ be the quotient map, $r = 1, 2, ..., k_0$.

Moreover, by the choice of δ_0, we may assume that there exists a homomorphism $h_0 : B_0 \to B_1$ such that

$$\|h_0(b) - pbp\| \le (\epsilon_2/2)\|b\| < \epsilon_2\|b\| \text{ for all } b \in B_0 \setminus \{0\}. \tag{e 4.69}$$

Moreover, we may assume $h_0(1_{B_0}) = p = 1_{B_1}$. Also, by (e 4.58), the choice of $\mathcal{G}_{B_0}$ and ϵ_3, we have

$$\sum_{i=1}^{r(z)} h_0(a_{zi})h_0(z)h_0(b_{zi}) \approx_{4M_1\epsilon_3} 1_{B_1}. \tag{e 4.70}$$

Since $4M_1\epsilon_3 < 1/2$, $h_0(z) \ne 0$ for all $z \in \mathcal{G}$. Thus h_0 is unital and injective.

Furthermore, we may assume that $y_1 = h_0(y_0)$, $q_k = h_0(q_{k0})$, $k = 1, 2, ..., N$.

Define $L = h_0 \circ \varphi_3 \circ (h_0|_{h_0(B_0)})^{-1}$. Note that

$$L(q_k) = h_0 \circ \varphi_3(h_0^{-1}(q_k)) = h_0 \circ \varphi_3(q_{k0}), \tag{e 4.71}$$

$$L(y_1) = h_0 \circ \varphi_3(h_0^{-1}(y_1)) = h_0 \circ \varphi_3(y_0), \text{ and} \tag{e 4.72}$$

$$L(p) = h_0\varphi_3(h_0^{-1}(p)) = h_0 \circ \varphi_3(1_{B_0}) = p. \tag{e 4.73}$$

Since both B_1 and $h_0(B_0)$ are finite dimensional, there is a conditional expectation $E : B_1 \to h_0(B_0)$. By replacing L by $L \circ E$, then L is extended to a unital completely positive linear map $B_1 \to B_1$.

We have, by (e 4.69), (e 4.55) and (e 4.62),

$$\|L(y_1) - \bar{x}_1\| \le \|h_0(\varphi_3(y_0)) - p\varphi_3(y_0)p\| + \|p\varphi_3(y_0)p - pxp\| \tag{e 4.74}$$

$$+\|pxp - \bar{x}_1\| < \epsilon_2 + 4\epsilon_0 + \epsilon_3 < \epsilon/16. \tag{e 4.75}$$

Put $\bar{x} = \bar{x}_1 + \bar{x}_2$. By (e 4.67), and applying 4.6,

$$\bar{x} \in_{\frac{8\|x\|}{K} + \epsilon_3 + \epsilon_3} \operatorname{conv}(\mathcal{U}(\bar{x}_1)). \tag{e 4.76}$$

Therefore (by (e 4.59) and (e 4.62)), with $\eta_0 = \frac{8\|x\|}{K} + 2\epsilon_3 + 4\epsilon_3 < \epsilon/4$,

$$x \in_{\eta_0} \operatorname{conv}(\mathcal{U}(\bar{x}_1)). \tag{e 4.77}$$

For each $z \in \mathcal{F}_3$, by (e 4.60), there is $L(z) \in B_1$ such that

$$\|pzp - L(z)\| < \epsilon_3 \text{ and } \|pz^*p - L(z)^*\| < \epsilon_3. \tag{e 4.78}$$

By (e 4.71), (e 4.69), (e 4.56), (e 4.57) and (e 4.78),

$$L(q_k) = h_0 \circ \varphi_3(q_{k0}) \approx_{\epsilon_2} p\varphi_3(q_{k,0})p \approx_{\epsilon_1} \sum_{i=1}^{m(k)} px_{ik}^* x_{ik}p \tag{e 4.79}$$

$$\approx_{m(k)\epsilon_3} \sum_{i=1}^{m(k)} px_{ik}^* px_{ik}p \approx_{4m(k)\epsilon_3} \sum_{i=1}^{m(k)} L(x_{ik})^* L(x_{ik}). \tag{e 4.80}$$

Similarly

$$q_k \approx_{\epsilon_3} pq_{k,0}p \approx_{\epsilon_1} \sum_{i=1}^{m(k)} px_{ik}x_{ik}^*p \approx_{5m(k)\epsilon_3} \sum_{i=1}^{m(k)} L(x_{ik})L(x_{ik})^*, \tag{e 4.81}$$

$$pq_{k,0}p \approx_{\epsilon_1} \sum_{i=1}^{m(k)'} py_{ik}^*y_{ik}p \approx_{5m(k)'\epsilon_3} \sum_{i=1}^{m(k)'} L(y_{ik})^*L(y_{ik}) \text{ and} \tag{e 4.82}$$

$$\sum_{i=1}^{m(k)'} L(y_{ik})L(y_{ik})^* \approx_{5m(k)'\epsilon_3} \sum_{i=1}^{m(k)'} py_{j,k}y_{ik}^*p \geq s_0p. \tag{e 4.83}$$

Note that

$$10m(k)\epsilon_3 + 10m(k)'\epsilon_3 + \epsilon_2 + 3\epsilon_1 < s_0/8.$$

By (e 4.80), (e 4.81), (e 4.82) and (e 4.83), we have

$$t(L(q_k)) \geq s_0/2 \text{ for all } t \in T(B_1), k = 0, 1, ..., N, \tag{e 4.84}$$

$$|t(L(q_k)) - t(q_k)| < (s_0/4)\epsilon_0, \text{ for all } t \in T(B_1), k = 0, 1, ..., N. \tag{e 4.85}$$

Let $Q_1 = q_1 + q_0, Q_i = q_i, i = 2, 3, ..., N$. Then, for $k = 1, 2, ..., N$,

$$t(L(Q_k)) \geq s_0/2 \text{ for all } t \in T(B_0), \tag{e 4.86}$$

$$|t(L(Q_k)) - t(Q_k)| < (s_0/2)\epsilon_0 < (s_0/2)(\epsilon/16) \text{ for all } t \in T(B_0). \tag{e 4.87}$$

Note (e 4.75), (e 4.86) and (e 4.87), by the choice s_0, applying 4.5 to $L, \bar{x}_1, y_1, \epsilon/16, \epsilon/16$ (in place of $\varphi, x, y, \epsilon_1, \epsilon_2$, respectively), we obtain a trace preserving completely positive contractive linear map $\Phi : B_1 \to B_1$ such that

$$\|\Phi(y_1) - \bar{x}_1\| < 2(\epsilon/16) + 2(\epsilon/16) \text{ and } \bar{x}_1 \in_{\epsilon/4} \text{conv}(\mathcal{U}(y_1)). \tag{e 4.88}$$

By (e 4.67),

$$K[1-p] \leq (2K+1)[1-p] \leq [q_k] \leq [Q_k], k = 1, 2, ..., N.$$

Since $\mu_1 = 0$, applying part (2) of 3.6,

$$y_1 \in_{\eta_2} \text{conv}(\mathcal{U}(y_1 + y_2)), \text{ where } \eta_2 = \frac{\|y_2\|}{K+1} < \epsilon_0. \tag{e 4.89}$$

By (e 4.59), (e 4.63), (e 4.64), ($y_0 \approx_{4\epsilon_3} y_1 + y_2$), and (e 4.49)

$$y_1 \in_{\eta_2 + 4\epsilon_3} \text{conv}(\mathcal{U}(y_0)). \tag{e 4.90}$$

It follows (by also (e 4.50)) that

$$y_1 \in_{\eta_2+4\epsilon_3+2\epsilon_0} \mathrm{conv}(\mathcal{U}(y)). \tag{e 4.91}$$

Since $\eta_2 + 4\epsilon_3 + 2\epsilon_0 < \epsilon/2$, it follows from (e 4.91), (e 4.88) and (e 4.77), that

$$x \in_\epsilon \mathrm{conv}(\mathcal{U}(y)). \quad \square$$

Corollary 4.8. *Let A be a unital separable simple C^*-algebra with tracial rank zero and let x, $y \in A$ be two normal elements. Then the following are equivalent:*

(1) $x \in \overline{\mathrm{conv}(\mathcal{U}(y))}$;

(2) *there exists a sequence of unital completely positive maps $\psi_n : A \to A$ such that*

$$\lim_{n\to\infty} \|\psi_n(y) - x\| = 0 \ \text{ and}$$
$$\tau(\psi_n(a)) = \tau(a) \ \text{ for all } a \in A \text{ and } \text{ for all } \tau \in T(A); \ \text{ and}$$

(3) *there exists a sequence of unital completely positive maps $\psi_n : A \to A$ such that*

$$\lim_{n\to\infty} \|\psi_n(y) - x\| = 0 \ \text{ and} \tag{e 4.92}$$
$$\lim_{n\to\infty} \sup\{|\tau(\psi_n(a)) - \tau(a)| : \tau \in T(A)\} = 0 \ \text{ for all } a \in A. \tag{e 4.93}$$

Proof. We have established that (1) and (3) are equivalent. It is also clear that (2) implies (3), and (1) implies (2) follows from 3.7. $\square$

5. Normal elements with small boundaries

The following follows from [18].

Lemma 5.1 *([18]). Let X be a compact subset of the plane, $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There is $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: If $\varphi : C(X) \to F$ is a $\mathcal{G}$-δ-multiplicative completely positive contractive linear map, where F is a finite dimensional C^*-algebra, then there exists unital homomorphism $h : C(X) \to F$ such that*

$$\|\varphi(f) - h(f)\| < \epsilon \ \text{ for all } f \in \mathcal{F}.$$

Definition 5.2. Let X be a compact metric space, let A be a C^*-algebra with $QT(A) \neq \emptyset$, and let $T \subset QT(A)$ be a subset. Suppose that $\varphi : C(X) \to A$ is a unital homomorphism. We shall say φ has the (SB) property with respect to T, if, for any $\delta > 0$, there is a finite open cover $\{O_1, O_2, ..., O_m\}$ of X with $\max\{\mathrm{diam}(O_i) : i = 1, ..., m\} < \delta$ such that

$\mu_\tau(\partial(O_j)) = \mu_\tau(\overline{O}_j \setminus O_j) = 0$, for all $\tau \in T$, $i = 1, 2, ..., m$, where μ_τ is the probability Borel measure induced by the state $\tau \circ \varphi$.

If $T = QT(A)$, we shall simplify say that φ has the (SB) property.

Note that every quasi-trace on a commutative C^*-algebra is a trace.

The following is easily proved (see the proof of Lemma 3 of [16]).

Proposition 5.3. *Let A be a unital simple C^*-algebra with $T(A) \neq \emptyset$, and let X be a compact metric space. Suppose that $\varphi : C(X) \to A$ is a unital homomorphism with the (SB) property. Then the following holds: For any $\delta_1 > 0$ and $\eta > 0$, there exists $\delta_2 > 0$ with $\delta_2 < \delta_1$ and there exists a compact subset K of X such that*

(i) *$X \setminus K$ is a finite disjoint union of open subsets $\{O_j : 1 \leq j \leq m\}$ with* $\max\{\mathrm{diam}(O_j) : 1 \leq j \leq m\} < \delta_1$,
(ii) *$Y_i' \cap Y_j' = \emptyset$, if $i \neq j$, where $Y_j' = \{t \in X : \mathrm{dist}(t, O_j) \leq \delta_2/16\}$, $j = 1, 2, ..., m$,*
(iii) *$\cup_{j=1}^m Y_j \supset X$, $Y_j = \{t \in X : \mathrm{dist}(t, O_j) < \delta_2\}$, $j = 1, 2, ..., m$,*
(iv) *$\mu_\tau(K) < \min\{\eta, \inf\{\mu_\tau(O_j) : \tau \in T(A), 1 \leq j \leq m\}\}/16(m + 1)$ for all $\tau \in T(A)$.*

Proposition 5.4. *In 5.3, if we replace Y_i by open subsets $Y_i \setminus \cup_{j \neq i} Y_j'$, $i = 1, 2, ..., m$, then, we have*

(iii)′ *$\cup_{j=1}^m Y_j \supset X$, $Y_i' \subset Y_i \subset \{t \in X : \mathrm{dist}(t, O_i) < \delta_2\}, i = 1, 2, ..., m$, and*
(v) *$Y_i \cap Y_j' = \emptyset$, when $i \neq j$.*

Proof. Set $\tilde{Y}_i = Y_i \setminus \cup_{j \neq i} Y_j'$, $i = 1, 2, ..., m$. Then $\tilde{Y}_i$ is open and $Y_i' \subset \tilde{Y}_i$ (recall that $Y_i' \cap Y_j' = \emptyset$, $i \neq j$), and $\tilde{Y}_i \cap Y_j' = \emptyset$, whenever $i \neq j$. To see $\cup_{i=1}^m \tilde{Y}_i = X$, let $t \in X$. If $t \in Y_j'$ for some j, then $t \in \tilde{Y}_j \subset \cup_{i=1}^m \tilde{Y}_i$. Otherwise, $t \notin \cup_{j=1}^m Y_j'$. However, $t \in Y_i$ for some i. Therefore $t \in Y_i \setminus \cup_{j=1}^m Y_j' \subset Y_i \setminus \cup_{j \neq i} Y_j' = \tilde{Y}_i$. Replacing Y_i by $\tilde{Y}_j$ in 5.3, we obtain (iii)′, (v). □

Lemma 5.5 *(see the proof of Lemma 2 of [16] and that of Lemma 4.8 of [17]). Let A be a unital simple C^*-algebra $T(A) \neq \emptyset$ such that its extremal points $\partial_e(T(A))$ has countably many points. Suppose that X is a compact metric space and $\varphi : C(X) \to A$ is a unital homomorphism. Then φ has the (SB) property.*

Proof. Let $\delta > 0$. For each $\xi \in X$, consider $S_{\xi,r} = \{t \in X : \mathrm{dist}(t, \xi) = r\}$, where $0 < r < \delta/2$. Since $\partial_e(T(A))$ is countable, there is $0 < r_\xi < \delta/2$ such that

$$\mu_\tau(S_{\xi,r_\xi}) = 0 \text{ for all } \tau \in \partial_e(T(A)).$$

It follows that

$$\mu_\tau(S_{\xi,r_\xi}) = 0 \text{ for all } \tau \in T(A).$$

Let $O_\xi = \{t \in X : \mathrm{dist}(\xi, t) < r_\xi\}$. Then $\cup_{\xi \in X} O_\xi = X$. There are $\xi_1, \xi_2, ..., \xi_m \in X$ such that

$$\cup_{i=1}^m O_{\xi_i} \supset X.$$

Note that $\mu_\tau(\partial(O_{\xi_i})) = 0$, $i = 1, 2, ..., m$. $\square$

Let A be a unital C^*-algebra and let $x, y \in A$ be two normal elements with $X = sp(x)$ and $Y = sp(y)$. *For the rest of this paper, we will denote by* $j_x : C(X) \to A$ *(*$j_y : C(Y) \to A$*) the embedding defined by* $j_x(f) = f(x)$ *(*$j_y(g) = g(y)$*) for all* $f \in C(X)$ *(for all* $g \in C(Y)$*). Moreover* $z_x \in C(X)$ *and* $z_y \in C(Y)$ *is the function defined by the identity map on* X *(and* Y*). In particular,* $j_x(z_x) = x$ *and* $j_y(z_y) = y$.

Lemma 5.6. *Let* A *be a unital simple* C^**-algebra with tracial rank zero, and let* $x \in A$ *be a normal element with* $\mathrm{sp}(x) = X$*. Suppose that the induced monomorphism* $j_x : C(X) \to A$ *has the (SB) property. Then, for any* $\epsilon > 0$*, any* $\sigma > 0$*, and any finite subset* $\mathcal{F} \subset C(X)$*, there exists a finite subset* $\mathcal{G} \subset C(X)$ *and* $\eta > 0$ *satisfying the following condition: If* $\varphi : C(X) \to F$ *is a unital homomorphism, where* F *is a finite dimensional* C^**-subalgebra of* A *such that*

$$\sup\{|\tau(\varphi(g)) - \tau(j_x(g))| : \tau \in T(A)\} < \eta \ \textit{for all}\ g \in \mathcal{G}, \tag{e 5.1}$$

then there exists a unital homomorphism $\varphi_1 : C(X) \to F$ *and a unital completely positive linear map* $L : F \to A$ *such that* $L \circ \varphi_1(C(X)) \subset j_x(C(X))$

$$\|\varphi(f) - \varphi_1(f)\| < \epsilon \ \textit{for all}\ f \in \mathcal{F},$$
$$\|L \circ \varphi_1(f) - j_x(f)\| < \epsilon \ \ \textit{for all}\ f \in \mathcal{F}\ \textit{and}$$
$$\sup\{|\tau \circ L(a) - \tau(a)| : \tau \in T(A)\} \le \sigma\|a\| \ \textit{for all}\ a \in F.$$

Proof. Choose $\delta_1 > 0$ such that

$$|f(t) - f(t')| < \min\{\epsilon/64, \sigma/4\} \ \text{for all}\ f \in \mathcal{F}, \tag{e 5.2}$$

if $\mathrm{dist}(t, t') < 2\delta_1$. Choose $\eta_0 = \min\{\epsilon/64, \sigma/4\}$. Suppose that $j_x : C(X) \to A$ is a unital homomorphism with the (SB) property. Without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of $C(X)$.

There is $\delta_2 > 0$ with $\delta_2 < \delta_1/4$ and there are a compact subset $K \subset X$, open subsets $O_1, O_2, ..., O_N$, $Y_1', Y_2', ..., Y_N'$, and $Y_1, Y_2, ..., Y_N$ of X satisfy the condition (i), (ii), (iv) in 5.3 and (iii)', (v) in 5.4. Let $K = X \setminus \cup_{i=1}^N O_j$ be as in 5.3 associated with η_0 (in place of η) and j_x (in place of φ).

Let $f_1, f_2, ..., f_N$ be a partition of unity with compact support $\mathrm{supp}(f_j) \subset Y_j$, $j = 1, 2, ..., N$. Let $g_K \in C(X)$ be a function such that $g_K(t) = 0$ if $\mathrm{dist}(t, \cup_{j=1}^N O_j) \le \delta_2/64$

and $g_K(t) = 1$ if $\mathrm{dist}(t, \cup_{j=1}^N O_j) \ge \delta_2/16$ and $0 \le g_K(t) \le 1$. Then $\mathrm{supp}(g_K) \subset K$. By (iv),

$$\tau(j_x(g_K)) < \min\{\eta_0, \inf\{\mu_\tau(O_j) : \tau \in T(A),\ 1 \le j \le N\}\}/16(N+1) \qquad \text{(e 5.3)}$$

for all $\tau \in T(A)$.

Put $\xi_j \in O_j$, $j = 1, 2, ..., N$. Without loss of generality, we may assume that

$$\|\sum_{i=1}^N f(\xi_j) f_j - f\| < \min\{\epsilon/16, \sigma/4\} \text{ for all } f \in \mathcal{F}. \qquad \text{(e 5.4)}$$

Define $h_j \in C(X)_+$ such that $0 \le h_j \le 1$, $h_j|_{O_j} = 1$ and $h_j(redt) = 0$ if $t \notin Y_j'$, $j = 1, 2, ..., N$. Let $\mathcal{G} = \mathcal{F} \cup \{f_j, h_j : 1 \le j \le N\} \cup \{g_K\}$.

$$s_0 = \inf\{\tau(j_x(h_j)) : \tau \in T(A),\ 1 \le j \le N\} > 0. \qquad \text{(e 5.5)}$$

Let $\varphi : C(X) \to F$ be a unital homomorphism for some finite dimensional C^*-subalgebra F of A which satisfies (e 5.1) for $\eta = \min\{\eta_0/8(N+1), s_0/2\}$.

Write $\varphi(f) = \sum_{k=1}^n f(t_k)p_k$ for all $f \in C(X)$, where $p_1, p_2, ..., p_n$ are mutually orthogonal projections and $t_1, t_2, ..., t_n$ are distinct points. By the choice of s_0 and η_0, and we have that $\{t_1, t_2, ..., t_n\} \cap Y_j' \ne \emptyset$, $j = 1, 2, ..., N$. To see this, we note that for all $\tau \in T(A)$ and $j = 1, 2, ..., N$,

$$\tau(\varphi(h_j)) \ge \tau(j_x(h_j)) - \eta \ge s_0/2$$

But $\varphi(h_j) = \sum_{t_k \in Y_j'} h_j(t_k)p_k$. Therefore $\{t_1, t_2, ..., t_n\} \cap Y_j' \ne \emptyset$ for all $j \in \{1, 2, ..., N\}$.

Note that

$$\varphi(\sum_{i=1}^N f(\xi_i) f_i) = \sum_{i=1}^N f(\xi_i)\varphi(f_i). \qquad \text{(e 5.6)}$$

Note also that, by (e 5.1) and (iv) of 5.3,

$$\tau(\varphi(g_K)) < \eta + \eta_0/16(N+1) < \eta_0/8(N+1) \text{ for all } \tau \in T(A). \qquad \text{(e 5.7)}$$

Put $K_j = \{t \in X : \mathrm{dist}(x, O_j) \ge \delta_2/16\} \cap K \subset K$, $j = 1, 2, ..., N$. It follows that

$$\sum_{t_i \in K_j} \tau(p_i) \le \tau(\varphi(g_K)) < \eta_0/8(N+1) \text{ for all } \tau \in T(A). \qquad \text{(e 5.8)}$$

Let $q_1 = \sum_{t_j \in Y_1} p_j$, $q_2 = (1_F - q_1)\sum_{t_j \in Y_2} p_i$,..., $q_N = (1 - \sum_{i=1}^{N-1} q_i)(\sum_{t_j \in Y_N} p_i)$. For any i, let $S_i = \{k : p_k \le q_i\}$, then $S_1 \sqcup S_2 \sqcup \cdots \sqcup S_N = \{1, 2, ..., n\}$. Note that, if $j \in S_i$, then

$t_j \in Y_i$, and, by (v) of 5.4, if $t_j \in Y_i'$, then $j \in S_i$. Moreover, $q_i \neq 0$ for all $i \in \{1, 2, ..., N\}$ and $q_i q_j = 0$ if $i \neq j$ and $\sum_{i=1}^N q_i = 1_F$.

We may also write $q_i = \sum_{k \in S_i} p_k$.

Since $\sum_{j=1}^N f_j(t) = 1$ for all $t \in X$, by (v) of 5.4, $f_j(t_i) = 1$ if $t_i \in Y_j'$. We also have

$$\varphi(f_j) = \sum_{t_i \in Y_j} f_j(t_i)p_i = (\sum_{t_i \in Y_j'} f_j(t_i)p_i) + (\sum_{t_i \in Y_j \cap K_j} f_j(t_i)p_i)$$

It follows from (e 5.7) that

$$|\tau(q_j) - \tau(\varphi(f_j))| \leq \tau(\varphi(g_K)) < \eta_0/8(N+1) \text{ for all } \tau \in T(A). \tag{e 5.9}$$

By the assumption of (e 5.1) for f_j,

$$|\tau(q_j) - \tau(j_x(f_j))| \tag{e 5.10}$$

$$\leq |\tau(q_j) - \tau(\varphi(f_j))| + |\tau(\varphi(f_j)) - \tau(j_x(f_j))| < \eta_0/4(N+1). \tag{e 5.11}$$

Let C be the C^*-subalgebra of F generated by $q_1, q_2, ..., q_N$. Write

$$F = M_{r(1)} \bigoplus M_{r(2)} \bigoplus \cdots \bigoplus M_{r(m)}$$

and $\pi_j : F \to M_{r(j)}$ is the quotient map. Define $C_j = \pi_j(C)$, $j = 1, 2, ..., m$.

Put $q_{ij} = \pi_j(q_i)$ and $p_{k,j} = \pi_j(p_k)$ which we also view them as projections in F as well as projections in A. Put $g_i = f_i + (\eta_0/4N) \cdot 1$, $i = 1, 2, ..., N$. Then, for fixed i, by (e 5.10), for all $\tau \in T(A)$,

$$\tau(j_x(g_i)) - \eta_0/2N < \sum_{j=1}^m \tau(q_{ij}) = \tau(q_i) < \tau(j_x(g_i)).$$

Thus, by 3.8 we obtain $a_{ij} \in A_+$ such that (again, viewing q_{ij} as projection in A),

$$a_{ij} \leq j_x(g_i), \ \sum_{j=1}^m a_{ij} = j_x(g_i) \text{ and } \tau(a_{ij}) = \tau(q_{ij}), \ 1 \leq j \leq m-1, \tag{e 5.12}$$

$$\tau(a_{im}) = \tau(j_x(g_i)) - \sum_{j=1}^{m-1} \tau(a_{ij}) \text{ and} \tag{e 5.13}$$

$$|\tau(a_{im}) - \tau(q_{im})| = |\tau(q_i) - \tau(j_x(g_i))| \leq \eta_0/2N, \ 1 \leq i \leq N \tag{e 5.14}$$

for all $\tau \in T(A)$.

Define $\varphi_1 : C(X) \to C \subset F$ by $\varphi_1(f) = \sum_{i=1}^N f(\xi_i)q_i$ for all $f \in C(X)$. Then, by (e 5.2),

$$\|\varphi_1(f) - \varphi(f)\| < \min\{\epsilon/16, \sigma/4\} \text{ for all } f \in \mathcal{F}. \tag{e 5.15}$$

Define $L_1 : \bigoplus_{j=1}^m C_j \to A$ by

$$L_1(\sum_{i=1}^N(\sum_{j=1}^m \lambda_{ij}q_{ij})) = \sum_{i=1}^N(\sum_{j=1}^m \lambda_{ij}a_{ij}) \text{ for } \lambda_{ij} \in \mathbb{C}.$$

Then, by (e 5.12) and (e 5.14), for all $c \in C$,

$$|\tau \circ L_1(c) - \tau(c)| = |\sum_{i=1}^N \sum_{j=1}^m \lambda_{ij}(\tau(a_{ij}) - \tau(q_{ij}))| \tag{e 5.16}$$

$$= |\sum_{i=1}^N \lambda_{im}\tau(a_{im} - q_{im})| \le (\eta_0/2)\max\{|\lambda_{im}| : i = 1, 2, ..., N\} \tag{e 5.17}$$

$$\le (\sigma/2)\|c\| \text{ for all } \tau \in T(A). \tag{e 5.18}$$

We also note that

$$L_1(\sum_{i=1}^N \lambda_i q_i) = \sum_{i=1}^N \lambda_i j_x(g_i) \in j_x(C(X)). \tag{e 5.19}$$

In other words, L_1 maps C into $j_x(C(X))$. Moreover, by (e 5.4),

$$\|L_1 \circ \varphi_1(f) - j_x(f)\| = \|L_1(\sum_{i=1}^N f(\xi_i)q_i) - j_x(f)\| \tag{e 5.20}$$

$$\le \|\sum_{i=1}^N f(\xi_i)j_x(g_i) - \sum_{i=1}^N f(\xi_i)j_x(f_i)\| \tag{e 5.21}$$

$$+\|\sum_{i=1}^N f(\xi_i)j_x(f_i) - j_x(f)\| \tag{e 5.22}$$

$$< \sum_{i=1}^N |f(\xi_i)|(\eta_0/4N) + \epsilon/16 < \epsilon/4 \text{ for all } f \in \mathcal{F}. \tag{e 5.23}$$

Since $M_{r(j)}$ and C_j are von-Neumann algebras, there exists a conditional expectation $E_j : M_{r(j)} \to C_j$ such that

$$t(E_j(a)) = t(a) \text{ for all } a \in M_{r(j)}, \text{ where } t \in T(M_{r(j)}), \; j = 1, 2, ..., m.$$

Consequently,

$$\tau(E_j(a)) = \tau(a) \text{ for all } a \in M_{r(j)} \text{ and } \tau \in T(A). \tag{e 5.24}$$

Define $E : F \to \bigoplus_{i=1}^m C_i$ by $E = \oplus_{i=1}^m E_i \circ \pi_i$. Note $E^2 = E$. Hence we also have, by (e 5.15),

$$\|E \circ \varphi(f) - \varphi_1(f)\| = \|E \circ \varphi(f) - E \circ \varphi_1(f)\| < \epsilon/4 \text{ for all } f \in \mathcal{F}. \tag{e 5.25}$$

Put $L = L_1 \circ E : F \to A$. Then $L(\varphi_1(C(X))) \subset j_x((C(X))$. Moreover, for any $f \in C(X)$, by (e 5.25) and (e 5.23),

$$\|L \circ \varphi(f) - j_x(f)\| \tag{e 5.26}$$
$$\leq \|L_1 \circ E \circ \varphi(f) - L_1 \circ E \circ \varphi_1(f)\| + \|L_1 \circ E \circ \varphi_1(f) - j_x(f)\| \tag{e 5.27}$$
$$\leq \epsilon/4 + \|L_1 \circ \varphi_1(f) - j_x(f)\| < \epsilon/4 + \epsilon/4 \text{ for all } f \in \mathcal{F}. \tag{e 5.28}$$

Furthermore, by (e 5.24) and (e 5.17),

$$|\tau \circ L(a) - \tau(a)| = |\tau \circ L(a) - \tau(E(a))| = |\tau(L_1(E(a)) - \tau(E(a))|$$
$$\leq (\sigma/2)\|E(a)\| \leq (\sigma/2)\|a\| \text{ for all } a \in F.$$

The lemma follows from the above inequality, (e 5.15) and (e 5.28). □

Lemma 5.7. *Let A be a unital simple C^*-algebra with tracial rank zero and with a unique tracial state. Suppose that $x \in A$ is a normal element with $\mathrm{sp}(x) = X$. Then, for any $\epsilon > 0$, any $\sigma > 0$, and any finite subset $\mathcal{F} \subset C(X)$, there exists a finite subset $\mathcal{G} \subset C(X)$ and $\eta > 0$ satisfying the following condition: If $\varphi : C(X) \to F$ is a unital homomorphism, where F is a finite dimensional C^*-subalgebra of A such that*

$$|\tau(\varphi(g)) - \tau(j_x(g))| < \eta \text{ for all } g \in \mathcal{G}, \tag{e 5.29}$$

then there exists a unital completely positive contractive linear map $L : F \to j_x(C(X))$ such that

$$\|L \circ \varphi(f) - j_x(f)\| < \epsilon \quad \text{for all } f \in \mathcal{F} \text{ and}$$
$$|\tau \circ L(a) - \tau(a)| \leq \sigma\|a\| \text{ for all } a \in F,$$

Proof. By the assumption and 5.5, j_x has the (SB) property. The proof is a simplification of that of 5.6. We keep all lines of the proof of 5.6 until C is defined. Let τ be the only tracial state of A. Then since both F and C are von Neumann algebras, there is a conditional expectation $E : F \to C$ such that

$$\tau(E(a)) = \tau(a) \text{ for all } a \in F. \tag{e 5.30}$$

Define $L_1 : C \to j_x(C(X))$ by $L_1(q_i) = j_x(f_i),\ \ i = 1, 2, ..., N$. This implies, by (e 5.17), that

$$|\tau(L_1(c)) - \tau(c)| \le (\sigma/4)\|c\| \text{ for all } c \in C. \tag{e 5.31}$$

Let φ_1 be as the same as defined in the proof of 5.6. Note that $\varphi_1(C(X)) = C$ since $q_i \in C$, $i = 1, 2, ..., N$. Define $L = L_1 \circ E$. Then

$$\|L(\varphi_1(f)) - j_x(f)\| = \|L(\sum_{i=1}^{N} f(\xi_i)q_i - j_x(f))\| \tag{e 5.32}$$

$$= \|\sum_{i=1}^{N} f(\xi_i)j_x(f_i) - j_x(f)\| < \min\{\epsilon/16, \sigma/4\} \text{ for all } f \in \mathcal{F}. \tag{e 5.33}$$

It follows that, as $\|\varphi_1(f) - \varphi(f)\| < \min\{\epsilon/16, \sigma/4\}$ for all $f \in \mathcal{F}$ (by (e 5.15)),

$$\|L(\varphi(f)) - j_x(f)\| \le \|L(\varphi(f)) - L(\varphi_1(f))\| \tag{e 5.34}$$

$$+ \|L(\varphi_1(f) - j_x(f))\| < \min\{\epsilon/8, \sigma/2\} \text{ for all } f \in \mathcal{F}. \tag{e 5.35}$$

We also have, by (e 5.31),

$$\begin{aligned} |\tau \circ L(a) - \tau(a)| &= |\tau \circ L_1 \circ E(a) - \tau(a)| \\ &\le |\tau(L_1(E(a))) - \tau(E(a))| + |\tau(E(a)) - \tau(a)| \\ &\le (\sigma/2)\|E(a)\| \le (\sigma/2)\|a\| \text{ for all } a \in F. \quad \square \end{aligned}$$

Theorem 5.8. *Let A be a unital separable simple C^*-algebra with tracial rank zero. Suppose that $x, y \in A$ are two normal elements with $\mathrm{sp}(x) = X$ and $\mathrm{sp}(y) = Y$. Suppose that j_x has the (SB) property and suppose that there exists a sequence of unital positive linear maps $\Phi_n : C(X) \to C(Y)$ such that*

$$\lim_{n\to\infty} \|\Phi_n(z_x) - z_y\| = 0 \text{ and} \tag{e 5.36}$$

$$\lim_{n\to\infty} \sup\{|\tau(\Phi_n(f)(y)) - \tau(f(x))| : \tau \in T(A)\} = 0 \text{ for all } f \in C(X). \tag{e 5.37}$$

Then

$$y \in \overline{\mathrm{conv}(\mathcal{U}(x))}.$$

Proof. Without loss of generality, we may assume that $\|x\|, \|y\| \le 1$. To show $y \in \overline{\mathrm{conv}(\mathcal{U}(x))}$, without loss of generality, we may assume that $0 \in \mathrm{sp}(x)$, as in the beginning of the proof of 4.7.

Let $\epsilon > 0$ and $\sigma > 0$. Let $\mathcal{F} = \{1, z_x\} \subset C(X)$ be a finite subset.

Choose $\delta_0 > 0$ such that

$$|f(t) - f(t')| < \epsilon/64 \text{ for all } f \in \mathcal{F}, \quad \text{if } |t - t'| < \delta_0. \tag{e 5.38}$$

Choose $\delta_{00} = \min\{\delta_0, \epsilon/2^{10}\}$. One may write

$$X = \cup_{i=1}^N \bar{O}_i,$$

where O_i is an open subset of X with diameter no more than $\delta_{00}/3$ and $O_i \cap O_j = \emptyset$ if $i \neq j$. Choose $\xi_i \in O_i$ and $d_i > 0$ with

$$O(\xi_i, d_i) = \{t \in X : \mathrm{dist}(\xi_i, t) < d_i\} \subset O_i, i = 1, 2, ..., m.$$

Since $0 \in X$, as we assumed, without loss of generality, we may also assume that $\xi_1 = 0$. Since A is a unital simple C^*-algebra, for each $\xi \in X$,

$$\inf\{\mu_\tau(O(\xi, d_i/2)) : \tau \in T(A)\} > 0, \tag{e 5.39}$$

where μ_τ is the Borel probability measure induced by $\tau \circ j_x$. Then

$$s_0 = \inf\{\inf\{\mu_\tau(O(\xi_i, d_i/2)) : \tau \in T(A)\} : 1 \le i \le N\} > 0. \tag{e 5.40}$$

For each $i \in \{1, 2, ..., N\}$, choose $g_i \in C(X)$ with $0 \le g_i \le 1$ such that $g_i(t) = 1$ if $t \in O(\xi_i, d_i/2)$ and $g_i(t) = 0$ if $t \notin O_i$, $i = 1, 2, ..., N$. Note that $g_i g_j = 0$ if $i \neq j$. Put $\mathcal{F}_1 = \mathcal{F} \cup \{g_i : 1 \le i \le N\}$. Note $\{\xi_1, \xi_2, ..., \xi_N\}$ is δ_{00}-dense in X.

Choose an integer $K > 0$ such that $12 \cdot 2^8/K < \epsilon$. Let

$$\epsilon_1 = \min\{\epsilon/2^{10}(N+1), \sigma/8(K+1)(N+1), s_0\epsilon/2^{10}(K+1)(N+1)\}. \tag{e 5.41}$$

Let $\eta > 0$ and finite subset set $\mathcal{G} \subset C(X)$ be given by 5.6 for $\epsilon_1/4$ (in place of ϵ), for $\epsilon_1/2$ (in place of σ), and for $\mathcal{F}_1$ (in place of $\mathcal{F}$). Without loss of generality, we may also assume that $\|g\| \le 1$, if $g \in \mathcal{G}$.

Choose a finite subset $\mathcal{G}_X \subset C(X)$ (in place of $\mathcal{G}$), $\delta_1 > 0$ (in place of δ), $\mathcal{F}_1$ (in place of $\mathcal{G}$) be given by 5.1 and $\epsilon_2 = \min\{\eta/2, \epsilon_1/4\}$ (in place of ϵ) as well as X.

Put $\epsilon_3 = \min\{\epsilon_2, \delta_1/2\}$.

Fix a finite subset $\mathcal{F}_A \subset A$. Let us assume that $\|a\| \le 1$ if $a \in \mathcal{F}_A$ and

$$\mathcal{F}_A \supset j_x(\mathcal{G}) \cup j_x(\mathcal{G}_X) \cup \{y\}.$$

Since A has tracial rank zero, there is a finite dimensional C^*-subalgebra $F_1 \subset A$ with $1_{F_1} = p$ and an $\mathcal{F}_A$-ϵ_3-multiplicative completely positive contractive linear map $\psi : A \to F_1$ such that

$$\|ap - pa\| < \epsilon_3 \ \text{ for all } \ a \in \mathcal{F}_A, \tag{e 5.42}$$

$$\|a - ((1-p)a(1-p) \oplus \psi(a))\| < \epsilon_3 \ \text{ for all } \ a \in \mathcal{F}_A, \tag{e 5.43}$$

$$\tau(1-p) < \epsilon_3/16 \ \text{ for all } \ \tau \in T(A). \tag{e 5.44}$$

By applying 5.1, there is a unital homomorphism $\varphi' : C(X) \to F_1$ such that

$$\|\varphi'(g) - \psi(j_x(g))\| < \min\{\eta/2, \epsilon_1/4\} \text{ for all } g \in \mathcal{G}. \tag{e 5.45}$$

Moreover, by (e 5.43) and (e 5.45),

$$\sup\{|\tau \circ \varphi'(g) - \tau(j_x(g))| : \tau \in T(A)\} < \eta \text{ for all } g \in \mathcal{G} \text{ and} \tag{e 5.46}$$

$$\tau(\varphi'(g_i)) > 63s_0/64 \text{ for all } \tau \in T(A),\ i = 1, 2, ..., N. \tag{e 5.47}$$

It follows from 5.6 that there exists a unital completely positive contractive linear map $L_1 : F_1 \to A$ and a unital homomorphism $\varphi_1 : C(X) \to F_1$ such that

$$L_1(\varphi_1(C(X))) \subset j_x(C(X)), \tag{e 5.48}$$

$$\|\varphi_1(f) - \varphi'(f)\| < \epsilon_1/4 \text{ and} \tag{e 5.49}$$

$$\|L_1 \circ \varphi'(f) - j_x(f)\| < \epsilon_1/4 \text{ for all } f \in \mathcal{G}, \text{ and} \tag{e 5.50}$$

$$|\tau \circ L_1(c) - \tau(c)| \le (\epsilon_1/2)\|c\| \text{ for all } \tau \in T(A) \text{ and } c \in F_1. \tag{e 5.51}$$

By (e 5.47) and (e 5.49), we have

$$\tau(\varphi_1(g_i)) \ge 15s_0/16 \text{ for all } \tau \in T(A). \tag{e 5.52}$$

Write $\varphi_1(f) = \sum_{i=1}^{m'} f(t_i)p_i$ for all $f \in C(X)$, where $t_i \in X$ and $p_1, p_2, ..., p_{m'}$ are mutually orthogonal projections in F_1. We may also write

$$\varphi_1(f) = \sum_{k=1}^{N}(\sum_{t_i \in O_k} f(t_i)p_i) + (\sum_{t_i \in X \setminus \cup_{k=1}^N O_k} f(t_i)p_i) \tag{e 5.53}$$

for all $f \in C(X)$. Note that

$$\sum_{t_i \in O_k} p_i \ge \varphi_1(g_k),\ \ k = 1, 2, ..., N. \tag{e 5.54}$$

Define $q_1 = \sum_{t_i \in \bar{O}_1} p_i$, $q_2 = (1 - q_1)(\sum_{t_i \in \bar{O}_2} p_i), ..., q_N = (1 - \sum_{i=1}^{N-1} q_i)(\sum_{t_i \in \bar{O}_N} p_i)$. Then $\sum_{i=1}^N q_i = p$, and

$$q_k \ge \sum_{t_i \in O_k} p_i \ge \varphi_1(g_k),\ \ k = 1, 2, ..., N. \tag{e 5.55}$$

Define the homomorphism $\varphi_2 : C(X) \to F_1$ by

$$\varphi_2(f) = \sum_{i=1}^{N} f(\xi_i)q_i \text{ for all } f \in C(X). \tag{e 5.56}$$

We have (using (e 5.38))

$$\|\varphi_2(f) - \varphi_1(f)\| < \epsilon/64 \text{ for all } f \in \mathcal{F}, \tag{e 5.57}$$

and (using (e 5.55) and (e 5.52))

$$\tau(q_i) > 15s_0/16 \text{ for all } \tau \in T(A), \ \ i = 1, 2, ..., N. \tag{e 5.58}$$

Note that $\varphi_2(C(X)) \subset \varphi_1(C(X))$. Therefore $L_1(\varphi_2(C(X))) \subset j_x(C(X))$, in particular,

$$q_k \in \varphi_2(C(X)) \text{ and } L_1(q_k) \in j_x(C(X)), \ \ k = 1, 2, ..., N. \tag{e 5.59}$$

By (e 5.43), (e 5.45), (e 5.49), and (e 5.57),

$$x \approx_{\epsilon_3} (1-p)x(1-p) + \psi(x) \approx_{\epsilon_1/4} (1-p)x(1-p) + \varphi'(z_x)$$
$$\approx_{\epsilon_1/4} (1-p)x(1-p) + \varphi_1(z_x) \approx_{\epsilon/64} (1-p)x(1-p) + \varphi_2(z_x).$$

Note $\epsilon_3 + \epsilon_1/4 + \epsilon_1/4 + \epsilon/64 < \epsilon/2$. By (e 5.55), (e 5.52), and (e 5.44), for any $\tau \in T(A), \tau(q_i) > 15s_0/16 > K\tau(1-p)$. Applying the comparison property, we have $K[1-p] \le [q_i]$ for all i.

Since $\varphi_2(z_x) = \sum_{i=1}^N \xi_i q_i$ with $\xi_1 = 0$, applying (2) of 3.6,

$$\varphi_2(z_x) \in_{1/(K+1)+\epsilon/2} \mathrm{conv}(\mathcal{U}(x)). \tag{e 5.60}$$

Suppose that $\Phi(= \Phi_{n_0}) : C(X) \to C(Y)$ (for some large n_0) is a unital positive linear map such that

$$\|\Phi(z_x) - z_y\| < \epsilon_1/4 \text{ and} \tag{e 5.61}$$

$$\sup\{|\tau(\Phi(g)(y)) - \tau(j_x(g))| : \tau \in T(A)\} < \epsilon_1/4 \tag{e 5.62}$$

for all $g \in \mathcal{G} \cup j_x^{-1}(L_1(\varphi_2(\mathcal{G} \cup \mathcal{G}_X)) \cup \{j_x^{-1}(L_1(q_i)) : 1 \le i \le N\}$, where j_x^{-1} is the inverse of $j_x : C(X) \to j_x(C(X))$.

Note that, by (e 5.62) and (e 5.51), for all $\tau \in T(A)$,

$$|\tau(\Phi(j_x^{-1}(L_1(q_i)))(y)) - \tau(q_i)| \tag{e 5.63}$$

$$\le |\tau(\Phi(j_x^{-1}(L_1(q_i)))(y)) - \tau(L_1(q_i))| + |\tau(L_1(q_i)) - \tau(q_i)| \tag{e 5.64}$$

$$< \epsilon_1/4 + \epsilon_1/2 < \epsilon_1. \tag{e 5.65}$$

Then, by (e 5.58),

$$\tau(\Phi(j_x^{-1} \circ L_1(q_i))(y)) > 3s_0/4 \text{ for all } \tau \in T(A), i = 1, 2, ..., N. \tag{e 5.66}$$

Put $\mathcal{F}_Y = \Phi(j^{-1}(L_1(\mathcal{G}))) \cup \{\Phi(j_x^{-1}(L_1(q_i))) : 1 \le i \le N\} \cup \{1, z_y\}$. Then $\mathcal{F}_Y \subset C(Y)$.

Let $\mathcal{G}_Y \subset C(Y)$ and $\delta_Y > 0$ be given by 5.1 for $\mathcal{F}_Y$ (in place of $\mathcal{F}$) and ϵ_3 (in place of ϵ) as well as Y (in place of X). Let $\epsilon_4 = \min\{\epsilon_3, \delta_Y/2\}$ and $F_0 = \mathbb{C}(1-p) \oplus F_1$.

Choose a large finite subset $\mathcal{F}'_A \subset A$ such that $j_y(\mathcal{F}_Y) \cup \{\varphi_2(z_x)\} \subset \mathcal{F}'_A$. We may also assume that $\mathcal{F}'_A$ contains an ϵ_3-dense subset of the unit ball of F_0. By choosing even smaller ϵ_4 and larger $\mathcal{F}'_A$, since F_0 is semiprojective, we may assume that there is a homomorphism h' from F_0 such that

$$\|h'(a) - L'(a)\| < \epsilon_3 \|a\| \text{ for all } a \in F_0 \backslash \{0\} \text{ and} \tag{e 5.67}$$

for any $\mathcal{F}'_A$-ϵ_4-multiplicative completely positive contractive linear map L' from F_0 to any C^*-algebra.

Since A has tracial rank zero, by applying 4.3, there are finite dimensional C^*-subalgebra F_2 with $e = 1_{F_2}$ and a $\mathcal{F}'_A$-ϵ_4-multiplicative completely positive contractive linear map $\psi_1 : A \to F_2$ such that

$$\|ae - ea\| < \epsilon_4 \text{ for all } a \in \mathcal{F}'_A, \tag{e 5.68}$$

$$\|a - ((1-e)a(1-e) \oplus \psi_1(a))\| < \epsilon_4 \text{ for all } a \in \mathcal{F}'_A, \tag{e 5.69}$$

$$y \approx_{\epsilon_4} y_0 + y_1, \;\; y_0 = \sum_{i=1}^{l} \lambda_i e_i \text{ and } y_1 \in \mathcal{N}((1-e)A(1-e)), \tag{e 5.70}$$

$$\tau(1-e) < \epsilon_4/16 \,(\text{ and } \tau(e) \ge (1-\epsilon_4/16)) \text{ for all } \tau \in T(A) \tag{e 5.71}$$

$$\text{and } (2K+1)[1-e] \le [e_i], \;\; i = 1, 2, ..., l. \tag{e 5.72}$$

We also assume that $\{\lambda_1, \lambda_2, ..., \lambda_l\}$ is ϵ_4-dense in Y. By (e 5.65) and (e 5.66), as in the proof of 4.7, by choosing sufficiently large $\mathcal{F}'_A$, we may assume that

$$|t(\psi_1(\Phi \circ j_x^{-1} \circ L_1(q_k))) - t(\psi_1(q_k))| < 2\epsilon_1 \text{ for all } t \in T(F_2) \text{ and} \tag{e 5.73}$$

$$t(\psi_1(q_k)) \ge s_0/2 \text{ for all } t \in T(F_2), \;\; 1 \le k \le N. \tag{e 5.74}$$

Moreover

$$t(\psi_1(1-e)) < \epsilon_3 \text{ for all } t \in T(F_2). \tag{e 5.75}$$

We may further assume that $\psi_1(b) \neq 0$ for any b in the ϵ_3-dense subset of the unit ball of F_0.

By 5.1, there is a unital homomorphism $h_Y : C(Y) \to F_2$ such that

$$\|h_Y(b) - \psi_1(j_y(b))\| < \epsilon_3 \text{ for all } b \in \mathcal{F}_Y. \tag{e 5.76}$$

Note, that $h_Y(z_y) \approx_{\epsilon_3} \psi_1(z_y) \approx_{\epsilon_4} eye \approx_{\epsilon_4} y_0$,

$$\|h_Y(z_y) - y_0\| < \epsilon_4 + \epsilon_3 + \epsilon_4 < 3\epsilon_3/2. \tag{e 5.77}$$

Moreover, there is a homomorphism $h_F : F_0 \to F_2$ such that

$$\|h_F(a) - \psi_1(a)\| < \epsilon_3 \|a\| \text{ for all } a \in F_0 \setminus \{0\}. \tag{e 5.78}$$

In particular,

$$\|h_F(\varphi_2(z_x)) - \psi_1(\varphi_2(z_x))\| \le \epsilon_3, \tag{e 5.79}$$

$$h_F(1_{F_0}) = \psi_1(1_{F_0}) = 1_{F_2} = e, \tag{e 5.80}$$

$$\varphi_2(z_x) \approx_{2\epsilon_3} (1-p)\varphi_2(z_x)(1-p) + h_F(\varphi_2(z_x)). \tag{e 5.81}$$

Since $\psi_1(b) \ne 0$ for those b in an ϵ_3-dense subset of the unit ball of F_0, h_F is injective and unital. Define $L_F : h_F(\varphi_2(C(X))) \to F_2$ by $L_F = h_Y \circ \Phi \circ j_x^{-1} \circ L_1 \circ (h_F|_{h_F(F_0)})^{-1}$. L_F is a unital completely positive contractive linear map. We note that

$$L_F(h_F(q_k)) = h_Y \circ \Phi \circ j_x^{-1} \circ L_1(q_k) \text{ and} \tag{e 5.82}$$

$$L_F(h_F(\varphi_2(z_x)) = h_Y \circ \Phi \circ j_x^{-1} \circ L_1(\varphi_2(z_x)). \tag{e 5.83}$$

It follows from (e 5.76), (e 5.73) and (e 5.78) that, for all $t \in T(F_2)$,

$$|t(L_F(h_F(q_k))) - t(h_F(q_k))| \tag{e 5.84}$$

$$\le |t(h_Y(\Phi \circ j_x^{-1} \circ L_1(q_k))) - t(\psi_1(\Phi \circ j_x^{-1} \circ L_1(q_k)))| \tag{e 5.85}$$

$$+ |t(\psi_1(\Phi \circ j_x^{-1} \circ L_1(q_k))) - t(\psi_1(q_k))| + |t(\psi_1(q_k)) - t(h_F(q_k))| \tag{e 5.86}$$

$$< \epsilon_3 + 2\epsilon_1 + \epsilon_3 < 3\epsilon_1 < (s_0/4)(\epsilon/16), \quad k = 1, 2, ..., N. \tag{e 5.87}$$

As in the proof of 4.7, since both $h_F(\varphi_2(C(X)))(\subset F_2)$ and F_2 are finite dimensional, there is a conditional expectation $E : F_2 \to h_F(\varphi_2(C(X)))$. By replacing L_F by $L_F \circ E$, we can extend L_F to a unital completely positive linear map $F_2 \to F_2$. Put $\bar{q}_1 = e - \sum_{i=1}^N h_F(q_i) + h_F(q_1) = e - \sum_{i=2}^N h_F(q_i)$. Then, by (e 5.75) and by (e 5.78),

$$|t(L_F(\bar{q}_1)) - t(\bar{q}_1)| \le \sum_{i=2}^N |t(L_F(h_F(q_i)) - t(h_F(q_i))| \tag{e 5.88}$$

$$< 3N\epsilon_1 < (s_0/4)(\epsilon/16) \text{ for all } t \in T(F_2). \tag{e 5.89}$$

By (e 5.78) and (e 5.74), for all $k = 2, 3, ..., N$

$$t(h_F(q_k)) \ge s_0/2 - \epsilon_3 \ge s_0/4 \text{ for all } t \in T(F_2). \tag{e 5.90}$$

Also

$$t(\bar{q}_1) \ge t(q_1) \ge s_0/4 \text{ for all } t \in T(F_2). \tag{e 5.91}$$

By (e 5.82), (e 5.49), (e 5.50), (e 5.57), (e 5.76), and (e 5.61)

$$L_F(h_F \circ \varphi_2(z_x)) = h_Y \circ \Phi \circ j_x^{-1} \circ L_1(\varphi_2(z_x)) \tag{e 5.92}$$

$$\approx_{\epsilon_1/4+\epsilon_1/4} h_Y \circ \Phi \circ j_x^{-1} \circ L_1(\varphi'(z_x)) \tag{e 5.93}$$

$$\approx_{\epsilon_1/4} h_Y \circ \Phi \circ j_x^{-1}((j_x(z_x))) \tag{e 5.94}$$

$$= h_Y \circ \Phi(z_x) \approx_{\epsilon_1/4} h_Y(z_y) \approx_{3\epsilon_3/4} y_0. \tag{e 5.95}$$

Let $\epsilon_5 = \epsilon_1/2 + \epsilon_1/4 + 3\epsilon_3/4$. Then $\epsilon_5 < \epsilon/16$.

Note that $h_F \circ \varphi_2(z_x) = \sum_{i=2}^N \xi_i h_F(q_i) + 0 \cdot \bar{q}_1$ and L_F now is defined on F_2.

By the choice of s_0, applying 4.5 to $L_F, y_0, h_F \circ \varphi_2(z_x), \epsilon_5, \epsilon/16$ (in place of $\varphi, x, y, \epsilon_1, \epsilon_2$ respectively), we obtain

$$y_0 \in_{2(\epsilon/16)+2\epsilon_5} \operatorname{conv}(\mathcal{U}(h_F \circ \varphi_2(z_x))). \tag{e 5.96}$$

By (e 5.72) and 4.6,

$$y \in_{8/K+\epsilon_4+\epsilon_4} \operatorname{conv}(\mathcal{U}(y_0)). \tag{e 5.97}$$

For any $\tau \in T(A)$, let $t_\tau := \tau(e)^{-1}\tau|_{F_2} \in T(F_2)$. By (e 5.71), (e 5.41), (e 5.90) and (e 5.91),

$$K(\tau(1-e)) < K\epsilon_4/16 < \tau(e)s_0/4 \le \tau(e)t_\tau(h_F(q_k)) = \tau(h_F(q_k)), k = 2, ..., N.$$

Applying the comparison property, $K[1-e] \le [h_F(q_k)], k = 2, ..., N$. Similarly, $K[1-e] \le [\bar{q}_1]$.

By the fact $\xi_1 = 0$ and (e 5.81), applying (2) of 3.6,

$$h_F(\varphi_2(z_x)) \in_{4/(K+1)+2\epsilon_3} \operatorname{conv}(\mathcal{U}(\varphi_2(z_x)) \tag{e 5.98}$$

By (e 5.60), (e 5.98), (e 5.96) and (e 5.97), we obtain

$$y \in_\epsilon \operatorname{conv}(\mathcal{U}(x)). \quad \square$$

Corollary 5.9. *Let A be a unital separable simple C^*-algebra with tracial rank zero and with countably many extremal tracial states. Suppose that $x, y \in A$ are two normal elements with $\mathrm{sp}(x) = X$ and $\mathrm{sp}(y) = Y$. Suppose that there exists a sequence of unital positive linear maps $\Phi_n : C(X) \to C(Y)$ such that*

$$\lim_{n\to\infty} \|\Phi_n(z_x) - z_y\| = 0 \;\; \textit{and} \tag{e 5.99}$$

$$\lim_{n\to\infty} \sup\{|\tau(\Phi_n(f)(y)) - \tau(f(x))| : \tau \in T(A)\} = 0 \;\; \textit{for all } f \in C(X). \tag{e 5.100}$$

Then $y \in \overline{\operatorname{conv}(\mathcal{U}(x))}$.

Theorem 5.10. *Let A be a unital separable simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$, and let $x, y \in A$ be two normal elements with $sp(x) = X$ and $sp(y) = Y$. Suppose either the embedding j_x has the (SB) property (or $QT(A)$ has countably many extremal points), and suppose that exists a sequence of unital positive linear maps $\Phi_n : C(X) \to C(Y)$ such that*

$$\lim_{n\to\infty} \|\Phi_n(x) - y\| = 0 \text{ and } \quad (\mathrm{e}\,5.101)$$

$$\lim_{n\to\infty} \sup\{|\tau(\Phi_n(f)(y)) - \tau(f(x))| : \tau \in T(A)\} = 0 \text{ for all } f \in C(X). \quad (\mathrm{e}\,5.102)$$

Then $y \in \overline{\mathrm{conv}(\mathcal{U}(x))}$.

Proof. It follows from Theorem 4.5 of [20] that there exists a unital simple AH-algebra B with real rank zero and no dimension growth such that there is a unital monomorphism $H : B \to A$ which induces the following identification:

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Since both A and B have real rank zero, by [5], $\rho_A(K_0(A))$ is dense in $\mathrm{Aff}(QT(A))$ and $\rho_B(K_0(B))$ is dense in $\mathrm{Aff}(T(B))$. It follows that H induces an affine isomorphism $H_\sharp$ from $\mathrm{Aff}(T(B))$ onto $\mathrm{Aff}(QT(A))$.

Fix $x, y \in \mathcal{N}(A)$. Let $\gamma : C(X)_{s.a.} \to \mathrm{Aff}(QT(A))$ be given by $\gamma(f)(\tau) = \tau(j_x(f))$ for all $\tau \in QT(A)$. Note that $[j_x]$ and γ are compatible. Note also that $\tau \circ j_x$ is a tracial state on $C(X)$. It follows from 5.3 of [25] that there is a normal element $x_1 \in H(B)$ with $\mathrm{sp}(x_1) = X$ and $[j_{x_1}] = [j_x]$ and $\tau(j_{x_1}(f)) = H_\sharp^{-1} \circ \gamma(f)(\tau)$ for all $\tau \in QT(A)$, where $j_{x_1} : C(X) \to H(B) \subset A$ is induced by x_1. Then, by Theorem 5.6 of [12] ($T(A)$ there should be $QT(A)$), x_1 and x are approximately unitarily equivalent.

Exactly the same argument shows that there is $y_1 \in \mathcal{N}(B)$ such that y_1 and y are approximately unitarily equivalent, and there exists a unital injective homomorphism $j_{y_1} : C(Y) \to H(B) \subset A$ induced by y_1. Note that, by [21], B has tracial rank zero. Let $\{u_n\}$ be a sequence of unitaries of A such that $x_1 = \lim_{n\to\infty} u_n^* x u_n$. Then, for any $\tau \in QT(A)$, $\tau(x_1) = \tau \circ H(x_1) = \tau(x)$. Since $H_\sharp$ is an affine isomorphism, the embedding $j_{x_1} : C(X) \to H(B)$ has the (SB) property under the hypothesis.

By 5.8 or 5.9, $x_1 \in \overline{\mathrm{conv}(\mathcal{U}(y_1))}$. It follows that $x_1 \in \overline{\mathrm{conv}(\mathcal{U}(y))}$, whence $x \in \overline{\mathrm{conv}(\mathcal{U}(y))}$. □

Theorem 5.11. *Let A be a unital separable simple C^*-algebra with tracial rank zero and with a unique tracial state. Suppose that $x, y \in A$ are two normal elements with $\mathrm{sp}(x) = X$ and $\mathrm{sp}(y) = Y$. Suppose that there exists a sequence of unital completely positive linear maps $\Phi_n : A \to A$ such that*

$$\lim_{n\to\infty} \|\Phi_n(x) - y\| = 0 \text{ and } \quad (\mathrm{e}\,5.103)$$

$$\lim_{n\to\infty} |\tau(\Phi_n(a)) - \tau(a)| = 0 \text{ for all } a \in A. \quad (\mathrm{e}\,5.104)$$

Then there exists a sequence of unital positive linear maps $\Psi_n : C(X) \to C(Y)$ *such that*

$$\lim_{n\to\infty} \|\Psi_n(z_x) - z_y\| = 0 \ \textit{and}$$
$$\lim_{n\to\infty} |\tau(\Psi_n(f)(y))) - \tau(f(x))| = 0 \ \textit{for all}\ f \in C(X).$$

Proof. Recall (see (7) of Section 2) that every quasi-trace of A is a trace. Let $\epsilon > 0$, $\sigma > 0$ and let $\mathcal{F}_X \subset C(X)$ be a finite subset in the unit ball. Put $\mathcal{F}_Y = \{1, z_y\} \subset C(Y)$. Let $\eta > 0$ and $\mathcal{G}_Y \subset C(Y)$ be finite subset given in 5.6 by for $\epsilon/2$ (in place of ϵ) and $\sigma/2$ (in place of σ), and Y in place of X. We may assume that $\mathcal{F}_Y \subset \mathcal{G}_Y$. Put $\epsilon_1 = \min\{\epsilon/16, \sigma/16, \eta/16\}$. Let τ be the unique tracial state of A.

Let $\Phi : A \to A$ be a unital completely positive linear map such that

$$\|\Phi(x) - y\| < \epsilon_1 \ \text{and} \tag{e 5.105}$$
$$|\tau(\Phi(j_x(f))) - \tau(j_x(f))| < \epsilon_1 \ \text{for all}\ f \in \mathcal{F}_X. \tag{e 5.106}$$

Let $\mathcal{F}_A = \mathcal{F}_X \cup \mathcal{G}_Y \cup \Phi(\mathcal{F}_X)$. Without loss of generality, we may assume that $\mathcal{F}_A$ is in the unit ball of A. Since A has tracial rank zero, there is a finite dimensional C^*-subalgebra $F \subset A$ with $1_F = p$ and a completely positive contractive linear map $\psi : A \to F$ such that

$$\|pa - ap\| < \epsilon_1 \ \text{for all}\ a \in \mathcal{F}_A, \tag{e 5.107}$$
$$\|a - ((1-p)a(1-p) + \psi(a))\| < \epsilon_1 \ \text{for all}\ a \in \mathcal{F}_A \ \text{and} \tag{e 5.108}$$
$$\tau(1-p) < \epsilon_1. \tag{e 5.109}$$

Without loss of generality, by 5.1, we may assume that there exists a unital homomorphism $\varphi : C(Y) \to F$ such that

$$\|\varphi(j_y(g) - \psi(g)\| < \epsilon_1 \ \text{for all}\ g \in \mathcal{G}_Y. \tag{e 5.110}$$

We also have

$$|\tau(\psi(a)) - \tau(a)| < 3\epsilon_1 \ \text{for all}\ a \in \mathcal{F}_A \ \text{and} \tag{e 5.111}$$
$$|\tau(\varphi(g)) - \tau(j_y(g))| < 4\epsilon_1 \ \text{for all}\ g \in \mathcal{G}_Y. \tag{e 5.112}$$

Let $C = \varphi(C(Y))$ be the C^*-subalgebra of F. By applying Lemma 5.7, we obtain a unital completely positive contractive linear map $L : F \to C(Y)$ such that

$$\|L \circ \varphi(f) - j_y(f)\| < \epsilon/2 \ \text{for all}\ f \in \mathcal{G}_Y \ \text{and} \tag{e 5.113}$$
$$|\tau \circ L(b) - \tau(b)| \le (\sigma/2)\|b\| \ \text{for all}\ b \in F. \tag{e 5.114}$$

Let $S \subset F$ be a finite subset which is ϵ_1-dense in the unit ball of F. Define $\Psi : C(X) \to C(Y)$ by $\Psi(f) = L \circ \psi \circ \Phi(j_x(f))$ for $f \in C(X)$. Then, by (e 5.105), (e 5.110) and (e 5.113),

$$\|\Psi(z_x) - y\| \le \|L \circ \psi \circ \Phi(x) - L \circ \psi(y)\| \tag{e 5.115}$$

$$+\|L \circ \psi(y) - L \circ \varphi(z_y)\| + \|L \circ \varphi(z_y) - y\| \tag{e 5.116}$$

$$< \|\Phi(x) - y\| + \epsilon_1 + \epsilon/2 < 2\epsilon_1 + \epsilon/2 < \epsilon. \tag{e 5.117}$$

Moreover, by (e 5.114), (e 5.111) and (e 5.106)

$$|\tau \circ \Psi(f) - \tau(j_x(f))| \le |\tau(L \circ \psi \circ \Phi(j_x(f))) - \tau(\psi \circ \Phi(j_x(f)))|$$
$$+|\tau(\psi \circ \Phi(j_x(f))) - \tau(\Phi(j_x(f)))| + |\tau(\Phi(j_x(f))) - \tau(j_x(f))|$$
$$\le \sigma/2 + 3\epsilon_1 + \epsilon_1 < \sigma \text{ for all } f \in \mathcal{F}_X. \quad \square$$

Let X be a compact metric space. Denote by $M(X)^{\mathbf{1}}$ the set of all probability Borel measures.

Let A be a unital simple C^*-algebra with $T(A) \neq \emptyset$, and let $x \in A$ be normal element. For each $\tau \in T(A)$, denote by $\mu_{\tau,X}$ (or just μ_τ) the probability Borel measure induced by $\tau \circ j_x$. Define $T_X = \{\mu_{\tau,X} : \tau \in T(A)\}$.

Theorem 5.12. *Let A be a unital separable simple C^*-algebra with tracial rank zero and with unique tracial state τ. Suppose that x and y are two normal elements with $X = \mathrm{sp}(x)$ and $Y = \mathrm{sp}(y)$. Then the following are equivalent:*

(1) $y \in \overline{\mathrm{conv}(\mathcal{U}(x))}$;
(2) *There exists a sequence of unital trace preserving completely positive linear maps $\Phi_n : A \to A$ such that*

$$\lim_{n\to\infty} \|\Phi_n(x) - y\| = 0;$$

(3) *There exists a sequence of unital completely positive linear maps $\Phi_n : A \to A$ such that*

$$\lim_{n\to\infty} \|\Phi_n(x) - y\| = 0 \textit{ and } \lim_{n\to\infty} |\tau(\Phi_n(a)) - \tau(a)| = 0 \textit{ for all } a \in A;$$

(4) *There exists a sequence of unital completely positive linear maps $\Psi_n : C(X) \to C(Y)$ such that*

$$\lim_{n\to\infty} \|\Psi_n(z_x) - z_y\| = 0 \textit{ and} \tag{e 5.118}$$

$$\lim_{n\to\infty} \Big|\tau(\Psi_n(f)(y))) - \tau(f(x))\Big| = 0 \textit{ for all } f \in C(X); \tag{e 5.119}$$

(5) *There exists a sequence of affine continuous maps* $\gamma_n : M(Y)^{\mathbf{1}} \to M(X)^{\mathbf{1}}$ *such that*

$$\lim_{n\to\infty} \sup\{|\int_X x d(\gamma_n(\mu)) - \int_Y y d\mu| : \mu \in M(Y)^{\mathbf{1}}\} = 0 \ \text{and} \tag{e 5.120}$$

$$\lim_{n\to\infty} \Big|\int_X f d\mu_{\tau,X} - \int_X f d\gamma_n(\mu_{\tau,Y})\Big| = 0 \tag{e 5.121}$$

for all $f \in C(X)$.

Proof. That (1), (2) and (3) are equivalent follows from 4.8. That (3) implies (4) follows from 5.11.

Suppose that (4) holds. Define $\gamma_n : M(Y)^{\mathbf{1}} \to M(X)^{\mathbf{1}}$ by $\int_X f d(\gamma_n(\mu)) = \int_Y \Psi_n(f) d\mu$ for all $\mu \in M(Y)^{\mathbf{1}}$. Clearly γ_n is a continuous affine map. Denote by $S(C(Y))$ the state space of $C(Y)$. Then, by (e 5.118),

$$\lim_{n\to\infty} \sup\{|s(\Psi_n(z_x)) - s(z_y)| : s \in S(C(Y))\} = 0.$$

Since one may identify $S(C(Y))$ with $M(Y)^{\mathbf{1}}$, (e 5.120) follows. It is also clear that (e 5.121) follows from (e 5.119). Thus (5) holds.

We now show that (5) implies (4). If (5) holds, for any $s \in Y$, and any $f \in C(X)$, define $\Psi_n(f)(s) = \gamma_n(\delta_s)(f)$, where $\delta_s \in S(C(Y)) = M(Y)^{\mathbf{1}}$ is the Dirac measure at s. Then (e 5.120) implies that

$$\lim_{n\to\infty} \sup\{|\Psi_n(z_x)(s) - z_y(s)| : s \in Y\} = 0.$$

However,

$$\sup\{|\Psi_n(z_x)(s) - z_y(s)| : s \in Y\} = \|\Psi_n(z_x) - z_y\|.$$

It follows that (e 5.118) holds. Also (e 5.119) follows from (e 5.121).

It remains to show that (4) implies (1) which follows from 5.9. □

Corollary 5.13. *Let A be a unital separable simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and with unique quasi-trace τ such that $\tau(1_A) = 1$. Suppose that x and y are two normal elements with $X = \mathrm{sp}(x)$ and $Y = \mathrm{sp}(y)$. Then (1), (4) and (5) in 5.12 are also equivalent, by replacing the tracial state by the quasi-trace.*

Proof. Let τ be the quasi-trace. Then, by II.4.3 of [5], τ is 2-quasi-trace. Note that any quasi-trace restricted on a commutative C^*-algebra is a trace. We also note that (4) and (5) are equivalent. It remains to show (1) and (4) are equivalent. We deploy the argument of the proof of 5.10. We keep all notation there.

If (1) holds, then $x_1 \in \overline{\mathrm{conv}}(\mathcal{U}(y_1))$. Thus 5.12 can apply to x_1 (in place x) and y_1 (in place of y) (in B). Since $\mathrm{sp}(x) = \mathrm{sp}(x_1)$ and $\mathrm{sp}(y) = \mathrm{sp}(y_1)$, as functions in $C(X)$ and $C(Y)$, $z_x = z_{x_1}$ and $z_y = z_{y_1}$, respectively. Therefore (e 5.118) holds. Since x_1 and x are approximately unitarily equivalent, $\tau(f(x_1)) = \tau(f(x))$ for all $f \in C(X)$, and $\tau(g(y)) = \tau(g(y_1))$ for all $g \in C(Y)$. Hence (e 5.119) also holds.

Suppose (4) holds. Then there exists a sequence of unital completely positive linear maps $\Psi_n : C(X) \to C(Y)$ such that

$$\lim_{n\to\infty} \|\Psi_n(z_x) - z_y\| = 0 \text{ and } \tag{e 5.122}$$

$$\lim_{n\to\infty} \Big|\tau(\Psi_n(f)(y)) - \tau(f(x))\Big| = 0 \text{ for all } f \in C(X). \tag{e 5.123}$$

The same reason given above shows (4) holds for x_1 (in place of x) and y_1 (in place of y) in B. Therefore $y_1 \in \overline{\mathrm{conv}(\mathcal{U}(x_1))}$. It follows that $y \in \overline{\mathrm{conv}(\mathcal{U}(x))}$. So (1) holds. □

6. Approximate unitary equivalence

by $j_x : C(\mathrm{sp}(x)) \to A$ the injective homomorphism defined by

Let A be a unital simple C^*-algebra with tracial rank zero and $x, y \in A$ be two normal elements. By [23], x and y are approximately unitarily equivalent, i.e., there exists a sequence of unitaries $\{u_n\}$ of A such that

$$\lim_{n\to\infty} \|u_n^* x u_n - y\| = 0,$$

if and only if $(j_x)_{*i} = (j_y)_{*i}$, $i = 0, 1$, and, $\tau \circ j_x = \tau \circ j_y$ for all $\tau \in T(A)$ (see also 5.6 of [12] for a slightly more general setting of this statement).

Theorem 6.1. *Let A be a unital separable simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and with a unique quasi-trace τ with $\tau(1_A) = 1$ and $QT(A) = T(A)$. Let $x, y \in A$ be two normal elements. Then the following are equivalent:*

(1) $x \in \overline{\mathrm{conv}(\mathcal{U}(y))}$ *and* $y \in \overline{\mathrm{conv}(\mathcal{U}(x))}$;
(2) $\mathrm{sp}(x) = \mathrm{sp}(y)$ *and* $\mu_x = \mu_y$, *where* μ_x *and* μ_y *are Borel probability measures induced by* $\tau \circ j_x$ *and* $\tau \circ j_y$, *respectively,*

Proof. Suppose that (2) holds. Let $\varphi : C(\mathrm{sp}(x)) \to C(\mathrm{sp}(y))$ be defined by $\varphi(f) = f(y)$ for all $f \in C(\mathrm{sp}(x))$. Then (1) follows by 5.13.

Suppose that (1) holds. Let $\pi_\tau : A \to B(H_\tau)$ be the representation of A given by the tracial state τ. Let $M = \pi_\tau(A)''$. Then M is a type II_1 factor. Note that since A is simple, π_τ is faithful. Note (1) implies that

$$\pi_\tau(x) \in \overline{\mathrm{conv}(\mathcal{U}(\pi_\tau(y)))} \text{ and } \pi_\tau(y) \in \overline{\mathrm{conv}(\mathcal{U}(\pi_\tau(x)))}.$$

By (vi) of Theorem 2.2 of [9], $\tau(g(x)) = \tau(g(y))$ for all continuous convex function on $\mathbb{R}^2$. It follows from Proposition I.1.1 of [1] that

$$\tau(f(x)) = \tau(f(\pi_\tau(x))) = \tau(f(\pi_\tau(y))) = \tau(f(y))$$

for all $f \in C(\mathrm{sp}(x))$. Consequently $\mathrm{sp}(x) = \mathrm{sp}(\pi_\tau(x)) = \mathrm{sp}(\pi_\tau(y)) = \mathrm{sp}(y)$. Thus (2) holds. □

Let A be a C^*-algebra. Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the usual order preserving homomorphism.

Corollary 6.2. *Let A be a unital separable simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and with a unique quasi-trace τ such that $\tau(1_A) = 1$ and $QT(A) = T(A)$, and let $x, y \in A$ be two normal elements. Suppose that $K_1(A) = \{0\}$ and $\ker\rho_A = \{0\}$. Then the following are equivalent:*

(1) $x \in \overline{\mathrm{conv}(\mathcal{U}(y))}$ *and* $y \in \overline{\mathrm{conv}(\mathcal{U}(x))}$;
(2) $\mathrm{sp}(x) = \mathrm{sp}(y)$ *and* $\tau(f(x)) = \tau(f(y))$ *for all* $f \in C(\mathrm{sp}(x))$.
(3) *x and y are approximately unitarily equivalent in A.*

Proof. It is clear that (3) implies (1). Thus, by 6.1, it remains to show that (2) implies (3). Assume that (2) holds. By the assumption, $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ is an order preserving injective homomorphism. Therefore, (2), together with the assumption that $K_1(A) = \{0\}$, implies that $(j_x)_{*i} = (j_y)_{*i}$, $i = 0, 1$, where j_x and j_y are embedding from $C(\mathrm{sp}(x))$ to A induced by x and y, respectively. Since $K_0(C(\mathrm{sp}(x))) = C(\mathrm{sp}(x), \mathbb{Z})$ is a free abelian group (see [30]), it follows from the Universal Coefficient Theorem that $[j_x] = [j_y]$ in $KL(C(\mathrm{sp}(x)), A)$. Then, by 5.6 of [12], j_x and j_y are approximately unitarily equivalent, whence (3) holds. □

Let A be a unital simple C^*-algebra of tracial rank zero such that $K_1(A) \neq \{0\}$. It follows from Theorem 6.11 of [27] that there are two normal elements x, y such that (2) of 6.2 holds but $(j_x)_{*1} \neq (j_y)_{*1}$. Then x and y are not approximately unitarily equivalent. However, by 6.2, $x \in \overline{\mathrm{conv}}(\mathcal{U}(y))$ and $y \in \overline{\mathrm{conv}}(\mathcal{U}(x))$. Suppose that $K_1(A) = \{0\}$ but $\ker\rho_A \neq \{0\}$. Suppose $X \subset \mathbb{C}$ is a compact subset which is not connected. Then, by 6.11 of [27] again, there are normal elements $x, y \in A$ with $\mathrm{sp}(x) = \mathrm{sp}(y) = X$ such that (2) of 6.2 holds but $(j_x)_{*0} \neq (j_y)_{*0}$. Then x and y are not approximately unitarily equivalent. However, by 6.2 again, $x \in \overline{\mathrm{conv}}(\mathcal{U}(y))$ and $y \in \overline{\mathrm{conv}}(\mathcal{U}(x))$. Nevertheless, we have the following:

Corollary 6.3. *Let A be a unital separable simple AF-algebra with a unique tracial state and let x, $y \in A$ be two normal elements with connected spectrum.*

Then the following are equivalent:

(1) $x \in \overline{\operatorname{conv}(\mathcal{U}(y))}$ *and* $y \in \overline{\operatorname{conv}(\mathcal{U}(x))}$;

(2) $\operatorname{sp}(x) = \operatorname{sp}(y)$ *and* $\tau(f(x)) = \tau(f(y))$ *for all* $f \in C(\operatorname{sp}(x))$.

(3) x *and* y *are approximately unitarily equivalent in* A.

Proof. Again, it remains to show (2) implies (3). Since both $\operatorname{sp}(x)$ and $\operatorname{sp}(y)$ are connected, (2) implies that $(j_x)_{*i} = (j_y)_{*i}$, $i = 0, 1$. Then, by 3.4 of [23], as in the proof of 6.2, (3) holds. □

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