Implicit-explicit local discontinuous Galerkin methods with generalized alternating numerical fluxes for convection-diffusion problems

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Abstract

Local discontinuous Galerkin methods with generalized alternating numerical fluxes coupled with implicit-explicit time marching for solving convection-diffusion problems is analyzed in this paper, where the explicit part is treated by a strong-stability-preserving Runge-Kutta scheme, and the implicit part is treated by an L-stable diagonally implicit Runge-Kutta method. Based on the generalized alternating numerical flux, we establish the important relationship between the gradient and interface jump of the numerical solution with the independent numerical solution of the gradient, which plays a key role in obtaining the unconditional stability of the proposed schemes. Also by the aid of the generalized Gauss-Radau projection, optimal error estimates can be shown. Numerical experiments are given to verify the stability and accuracy of the proposed schemes with different numerical fluxes.

Keywords. Implicit-explicit scheme, local discontinuous Galerkin method, generalized alternating numerical flux, convection-diffusion equation.

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1 Introduction

Local discontinuous Galerkin (LDG) method is one of the widely used numerical methods during the last two decades. Since it was introduced by Cockburn and Shu [11], motivated

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by the work of Bassi and Rebay [3] for solving the compressible Navier-Stokes equations, the LDG method has been rapidly developed and widely applied to many practical problems, such as nonlinear wave equations with higher order derivatives [35, 34], semiconductor device simulations [22, 23], the incompressible fluid flows [10], porous medium equations [39], the miscible displacement in porous media [14], wave equation in heterogeneous media [9], the Keller-Segel chemotaxis model [21], and so on.

The idea of the LDG method is to rewrite equations with higher order derivatives into an equivalent first order system, then apply the DG method [27] to the system. The choice of the numerical flux is an important ingredient in the design of LDG method. In the pioneering paper [11], the authors presented a general form of numerical fluxes and showed suboptimal error estimate. In later works, e.g. [6, 32], optimal error estimates were given for semi-discrete and fully-discrete LDG methods with purely alternating numerical fluxes (PANF). Compared with PANF, the generalized alternating numerical fluxes (GANF) are easier to define for linear equations with varying-coefficients or nonlinear equations [7], so they gained attention by researchers. Recently, the optimal error estimate of LDG methods with GANF was derived in [7] by carefully defining the generalized Gauss-Radau (GGR) projection, the corresponding local analysis for singularly perturbed problems was also carried out [8], these works are all in the framework of semi-discrete LDG methods.

An important issue we shall consider in this paper is the time discretization. In our previous work [29, 30, 31], a class of stiffly accurate implicit-explicit (IMEX) Runge-Kutta (RK) time discretization schemes [2, 5] were considered for convection-diffusion equations, where the convection part is treated explicitly and the diffusion part is treated implicitly. Those IMEX schemes coupled with LDG spatial discretization with PANF were shown to be unconditionally stable, in the sense that the time step is only required to be upper bounded by a positive constant which is independent of mesh size but depends on the coefficients of convection and diffusion. In this paper, we pay attention to a class of IMEX RK formulas which are not stiffly accurate but have strong-stability-preserving (SSP) property [13]. These formulas were proposed in [25] for hyperbolic systems of conservation laws with stiff relaxation, where the explicit parts are total variation diminishing (TVD) schemes [28], and the implicit parts are L-stable diagonally implicit Runge-Kutta (DIRK) methods. We call these methods as IMEX SSP methods in this paper.

The main difference between stiffly accurate schemes and IMEX SSP methods lies in the following aspect: for stiffly accurate schemes, the solution at the end of the time step is identified with the solution at the last internal stage [16, 2], while for IMEX SSP methods, an additional quadrature is used at the end of the time step. So the IMEX SSP methods require more storage than stiffly accurate schemes and it seems that they are less efficient. However, for some problems such as semiconvection problems in astrophysics, the SSP property is necessary to suppress spurious oscillations in the spatial discretization [19], IMEX SSP methods can enhance the stability and accuracy of the simulations [18]. Based on the computational advantages, IMEX SSP methods have been adopted in many applications, such as BGK kinetic equations [26], compressible Navier-Stokes equations [20], optimal control problems [17], highly nonlinear PDEs [15], and so on.

The objective of this paper is to study the L^2-norm stability and optimal error analysis for LDG methods with GANF, coupled with two specific second and third order IMEX SSP time discretizations proposed in [25]. Compared with the stiffly accurate schemes considered in [29], an additional quadrature is used at the end of the time step in the IMEX SSP schemes, this makes the construction of energy equation and the corresponding energy analysis much more complicated than what we have done in [29]. We will establish energy equation along the similar line as those established in [37, 38] for explicit Runge-Kutta discontinuous Galerkin (RKDG) methods.

Besides the construction of energy equations, the crucial step is to build up the important relationship between the gradient and interface jump of the numerical solution with the independent numerical solution of the gradient, in the LDG methods with GANF, just as what we did in [29] for PANF. Different from [29], where the relationship can be built up locally (i.e, it holds in each cell), for GANF the relationship has to be established globally; see Lemma 2.3 and its proof. By the aid of this important relationship and the energy analysis, we can derive similar stability results for the LDG methods with GANF coupled with the second and third order IMEX SSP schemes as that in [29]. Also the optimal error estimates will be obtained by the aid of the GGR projection.

The remaining part of this paper is organized as follows. In Section 2 we present the semi-discrete LDG method and the IMEX SSP time discretization schemes. Sections 3 and 4 are devoted to the stability and optimal error estimates of the proposed fully-discrete LDG schemes. In Section 5, numerical experiments are given to verify the theoretical results and to illustrate the effects of different choices of the numerical fluxes. Concluding remarks and proof of some of the technical lemmas are given in Section 6 and the Appendix respectively.

2 The LDG method and IMEX SSP schemes

2.1 The semi-discrete LDG scheme

In this subsection we present the definition of semi-discrete LDG schemes for the linear convection-diffusion problem

$$U_t + cU_x - dU_{xx} = 0, \qquad (x,t) \in Q_T = (a,b) \times (0,T], \tag{2.1a}$$

$$U(x,0) = U_0(x), \qquad x \in \Omega = (a,b), \tag{2.1b}$$

where $d > 0$ is the diffusion coefficient and c is the velocity of the flow field. Without loss of generality, we assume that both d and c are constants and $c > 0$. The initial solution $U_0(x)$ is assumed to be in $L^2(\Omega)$. For the simplicity of analysis, we only consider periodic boundary condition in this paper. The analysis for other boundary conditions is much more complicated, one can refer to [33] for the discussion of Dirichlet boundary condition.

Let $Q = \sqrt{d}U_x$, the LDG scheme starts from the following equivalent first-order differential system

$$U_t + cU_x - \sqrt{d}Q_x = 0, \quad Q - \sqrt{d}U_x = 0, \qquad (x,t) \in Q_T, \tag{2.2}$$

with the same initial condition (2.1b) and boundary condition.

Let $\mathcal{T}_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=1}^N$ be the partition of Ω, where $x_{\frac{1}{2}} = a$ and $x_{N+\frac{1}{2}} = b$ are the boundary endpoints. Denote the cell length as $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $j = 1, \ldots, N$, and define $h = \max_j h_j$. We assume $\mathcal{T}_h$ is quasi-uniform in this paper, that is, there exists a positive constant ν such that for all j there holds $h_j/h \geq \nu$, as h goes to zero.

Associated with this mesh, we define the discontinuous finite element space

$$V_h = \left\{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j), \, \forall j = 1, \ldots, N \right\}, \tag{2.3}$$

where $\mathcal{P}_k(I_j)$ denotes the space of polynomials in I_j of degree at most $k \geq 0$. Note that the functions in this space are allowed to have discontinuities across element interfaces. At each element interface point, for any piecewise function v, there are two traces along the right-hand and left-hand, denoted by v^+ and v^-, respectively, and the jump is denoted by $[\![v]\!] = v^+ - v^-$.

Multiplying (2.2) by test functions v and r, integrating over each cell I_j and integrating by parts, then restricting unknown functions and test functions in finite element space V_h and taking proper numerical fluxes, one can define the LDG scheme, please refer to [11] for more details. In this paper, we would like to adopt the "upwind-biased" numerical flux [24] for the convection, and the GANF [7] for the diffusion, then we can define the semi-discrete LDG scheme as follows: for any $t > 0$, find the numerical solution $u(t), q(t) \in V_h$ (where the argument x is omitted), such that the variational forms

$$(u_t, v)_j = c\mathcal{Z}_j^{\vartheta}(u, v) - \sqrt{d}\mathcal{Z}_j^{\tilde{\theta}}(q, v), \tag{2.4a}$$

$$(q, r)_j = -\sqrt{d}\mathcal{Z}_j^{\theta}(u, r), \tag{2.4b}$$

hold in each cell I_j, $j = 1, 2, \ldots, N$, for any test functions $v, r \in V_h$. Here $\vartheta \geq \frac{1}{2}$ and θ are parameters associated with convection and diffusion respectively, and $\tilde{\theta} = 1 - \theta$. Since the central numerical flux ($\theta = \frac{1}{2}$) will affect the accuracy of the LDG scheme in the case of odd polynomial degree k [34], we will mainly consider $\theta \neq \frac{1}{2}$ in this paper. Note that $\vartheta = 1$

gives the upwind numerical flux, and $\theta = 0, 1$ give the PANF. The notation $(\cdot,\cdot)_j$ means the inner product in $L^2(I_j)$ and

$$\mathcal{Z}_j^{\beta}(w,v) = (w, v_x)_j - w^{(\beta)}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + w^{(\beta)}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}, \tag{2.5}$$

for any functions w and v. Here and below, $w^{(\beta)} = \beta w^- + \tilde{\beta} w^+$, and $\tilde{\beta} = 1 - \beta$ for any parameter β. $w^-_{\frac{1}{2}} = w^-_{N+\frac{1}{2}}$ and $w^+_{N+\frac{1}{2}} = w^+_{\frac{1}{2}}$ due to the periodic boundary condition. One can refer to [8] for the definition of numerical fluxes for Dirichlet boundary conditions.

The initial condition $u(x,0)$ can be taken as any approximation of the given initial solution $U_0(x)$, for example, the standard L^2 projection of $U_0(x)$. We have now defined the semi-discrete LDG scheme.

For the convenience of analysis, we denote by $(q,r) = \sum_{j=1}^{N}(q,r)_j$ the inner product in $L^2(\Omega)$. Summing up the variational formulations (2.4) over $j = 1, 2, \ldots, N$, and letting $\mathcal{Z}^{\beta} = \sum_{j=1}^{N} \mathcal{Z}_j^{\beta}$, $\mathcal{H} = c\mathcal{Z}^{\vartheta}, \mathcal{L} = -\sqrt{d}\mathcal{Z}^{\tilde{\theta}}, \mathcal{K} = -\sqrt{d}\mathcal{Z}^{\theta}$, we can write the above semi-discrete LDG scheme in the global form: for any $t > 0$, find the numerical solution $u, q \in V_h$ such that the variation equations

$$(u_t, v) = \mathcal{H}(u,v) + \mathcal{L}(q,v), \tag{2.6a}$$

$$(q, r) = \mathcal{K}(u,r), \tag{2.6b}$$

hold for any $v, r \in V_h$.

2.2 The properties of the LDG spatial discretization

We present some properties of the LDG spatial discretizaiton in this subsection. To this end, let us first introduce some notations and the inverse inequality.

We use the standard notations and norms in Sobolev spaces, for example, $H^{\ell}(D)$ $(\ell \geq 1)$ denotes the space where the function itself and its derivatives up to ℓ-th order are all square-integrable in domain D. And we define the (mesh-dependent) broken Sobolev space

$$H^{\ell}(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v|_{I_j} \in H^{\ell}(I_j),\ \forall j = 1, \ldots, N \right\}, \tag{2.7}$$

which contains the discontinuous finite element space V_h. Associated with the space $H^{\ell}(\mathcal{T}_h)$, we would like to define the following semi-norms

$$[\![v]\!]^2 = \sum_{j=1}^{N} [\![v]\!]^2_{j-\frac{1}{2}}, \qquad \|v\|^2_{\Gamma_h} = \sum_{j=1}^{N} \|v\|^2_{\partial I_j}$$

for arbitrary $v \in H^{\ell}(\mathcal{T}_h)$, where $\|v\|_{\partial I_j} = \sqrt{(v^+_{j-\frac{1}{2}})^2 + (v^-_{j+\frac{1}{2}})^2}$ is the L^2-norm on the boundary of I_j. In addition,

$$\|v\|^2 = \sum_{j=1}^{N} \|v\|^2_j, \qquad \|v\|^2_{H^{\ell}(\mathcal{T}_h)} = \sum_{j=1}^{N} \|v\|^2_{H^{\ell}(I_j)},$$

where $\|v\|_j$ and $\|v\|_{H^\ell(I_j)}$ are the L^2-norm and H^ℓ-norm of v in cell I_j, respectively.

For any function $v \in V_h$, there exists an inverse constant $\mu > 0$ independent of v, h and j such that [1]

$$\|v\|_{\partial I_j} \leq \sqrt{\mu h^{-1}} \|v\|_j. \tag{2.8}$$

Integrating by parts and using the periodic boundary condition, we can easily get the following properties, similar results for upwind numerical flux and PANF can be found in [36, 29]. We omit the details here to save space.

Lemma 2.1. *For any $w, v \in V_h$, there hold the equalities*

$$\mathcal{Z}^\beta(v, v) = -(\beta - \frac{1}{2})[\![v]\!]^2, \tag{2.9}$$

$$\mathcal{Z}^\beta(w, v) = -\mathcal{Z}^{\tilde{\beta}}(v, w). \tag{2.10}$$

Corollary 2.1. *For $u, q, \tilde{u}, \tilde{q} \in V_h$, suppose $(q, r) = \mathcal{K}(u, r)$ and $(\tilde{q}, r) = \mathcal{K}(\tilde{u}, r)$ for any $r \in V_h$, then we have*

$$\mathcal{L}(\tilde{q}, u) = \mathcal{L}(q, \tilde{u}) = -(q, \tilde{q}). \tag{2.11}$$

Proof. By the definition of $\mathcal{L}$ and $\mathcal{K}$, and owing to (2.10) we can easily get

$$\mathcal{L}(\tilde{q}, u) = -\mathcal{K}(u, \tilde{q}) = -(q, \tilde{q}).$$

And similarly, $\mathcal{L}(q, \tilde{u}) = -\mathcal{K}(\tilde{u}, q) = -(\tilde{q}, q)$. □

Lemma 2.2. *For any $w, v \in V_h$, there exists a positive constant C_β depending on β such that*

$$|\mathcal{Z}^\beta(w, v)| \leq C_\beta \left(\|w_x\| + \sqrt{\mu h^{-1}} [\![w]\!] \right) \|v\|, \tag{2.12a}$$

$$|\mathcal{Z}^\beta(w, v)| \leq C_\beta \left(\|v_x\| + \sqrt{\mu h^{-1}} [\![v]\!] \right) \|w\|. \tag{2.12b}$$

In [29] we presented an important relationship (Lemma 2.4 in [29]) between the gradient and interface jump of the numerical solution with the independent numerical solution of the gradient, in the LDG scheme with PANF, which plays a key role in obtaining the unconditional stability of the fully discrete LDG schemes. In the following lemma we will show that the same relationship also holds for GANF excluding the central flux.

Lemma 2.3. *Let $\theta \neq \frac{1}{2}$. Suppose $u, q \in V_h$ satisfy (2.6b), then there exists a positive constant $C_\star$, which is independent of h and d but may depend on μ, θ and k, such that*

$$\|u_x\| + \sqrt{\mu h^{-1}} [\![u]\!] \leq \sqrt{\frac{C_\star}{d}} \|q\|. \tag{2.13}$$

Proof. From (2.4b) and (2.5) we have

$$\begin{aligned}(q,r)_j =& -\sqrt{d}\left[(u,r_x)_j - u^{(\theta)}_{j+\frac12} r^-_{j+\frac12} + u^{(\theta)}_{j-\frac12} r^+_{j-\frac12}\right]\\ =&\sqrt{d}\left[(u_x,r)_j - (u^-_{j+\frac12} - u^{(\theta)}_{j+\frac12}) r^-_{j+\frac12} + (u^+_{j-\frac12} - u^{(\theta)}_{j-\frac12}) r^+_{j-\frac12}\right]\\ =&\sqrt{d}\left[(u_x,r)_j + \tilde{\theta}[\![u]\!]_{j+\frac12} r^-_{j+\frac12} + \theta[\![u]\!]_{j-\frac12} r^+_{j-\frac12}\right],\end{aligned} \tag{2.14}$$

where integration by parts is used in the second step. Thus, owing to the periodic boundary conditions, we have

$$(q,r) = \sqrt{d}\sum_{j=1}^{N}\left[(u_x,r)_j + [\![u]\!]_{j+\frac12} r^{(\tilde{\theta})}_{j+\frac12}\right]. \tag{2.15}$$

In what follows we first show that there exists positive constant C_1 such that

$$\|u_x\|^2 \le \frac{C_1}{d}\|q\|^2. \tag{2.16}$$

In the special cases $\theta = 0$ or $\theta = 1$, we can get (2.16) by showing it holds in each cell, i.e, $\|u_x\|_j^2 \le \frac{C_1}{d}\|q\|_j^2$. Since in these cases, there is only one boundary term in (2.14), suitable test function r can be taken to eliminate the boundary term; see [29] for more details. However, it is difficult to find such test function so that the two boundary terms in (2.14) can be eliminated simultaneously for general θ. So for general θ, we begin with the global formulation (2.15). The basic idea is to take suitable test function r such that $(u_x,r)_j = \|u_x\|_j^2$ and $r^{(\tilde{\theta})}_{j+\frac12} = 0$ for every $j = 1,2,\cdots,N$. To this end, we take r as piecewise-defined function in the form

$$r(x) = u_x(x) - \left[\lambda_1 u_x(x^-_{j+\frac12}) + \lambda_2 u_x(x^+_{j+\frac12})\right] P_k^j(x), \quad \text{for} \quad x \in I_j, \tag{2.17}$$

where λ_1, λ_2 are θ-dependent constants to be determined later, and $P_k^j(x) = L_k(\frac{2(x-x_j)}{h_j})$ for $x \in I_j$, with $L_k(\cdot)$ being the standard Legendre polynomial of degree k in $[-1,1]$, so we have $(u_x, P_k^j)_j = 0$, $P_k^j(x_{j+\frac12}) = 1$ and $P_k^j(x_{j-\frac12}) = (-1)^k$.

From (2.17), we obtain

$$\begin{aligned} r^-_{j+\frac12} =& (1-\lambda_1)u_x(x^-_{j+\frac12}) - \lambda_2 u_x(x^+_{j+\frac12}),\\ r^+_{j+\frac12} =& -(-1)^k\lambda_1 u_x(x^-_{j+\frac12}) + [1-(-1)^k\lambda_2]u_x(x^+_{j+\frac12}).\end{aligned}$$

To ensure $r^{(\tilde{\theta})}_{j+\frac12} = 0$, we need to choose different λ_1, λ_2 for different cases when k is even or odd. After a simple manipulation, we get

$$\lambda_1 = \tilde{\theta}, \quad \lambda_2 = \theta$$

when k is even, and

$$\lambda_1 = \frac{-\tilde{\theta}}{2\theta - 1}, \quad \lambda_2 = \frac{-\theta}{2\theta - 1}$$

when k is odd.

Now taking r as (2.17) with the above choice of λ_1 and λ_2 in (2.15), we get

$$\|u_x\|^2 = \sum_{j=1}^{N} (u_x, r)_j = \frac{1}{\sqrt{d}} \sum_{j=1}^{N} (q, r)_j \le \varepsilon \|r\|^2 + \frac{1}{4\varepsilon d}\|q\|^2, \tag{2.18}$$

for arbitrary $\varepsilon > 0$. Notice that

$$\begin{aligned}
\|r\|_j^2 &= \|u_x\|_j^2 + |\lambda_1 u_x(x^-_{j+\frac{1}{2}}) + \lambda_2 u_x(x^+_{j+\frac{1}{2}})|^2 \|P_k^j\|_j^2 \\
&\le \|u_x\|_j^2 + 2\max\{\lambda_1^2, \lambda_2^2\}(\|u_x\|^2_{\partial I_j} + \|u_x\|^2_{\partial I_{j+1}})\|P_k^j\|^2 \\
&\le \|u_x\|_j^2 + \frac{2\mu \max\{\lambda_1^2, \lambda_2^2\}}{2k+1}(\|u_x\|_j^2 + \|u_x\|_{j+1}^2),
\end{aligned}$$

by the inverse inequality (2.8) and the fact that $\|P_k^j\|^2 = \frac{h_j}{2k+1}$. Thus (2.18) becomes

$$\|u_x\|^2 \le C_1 \varepsilon \|u_x\|^2 + \frac{1}{4\varepsilon d}\|q\|^2, \tag{2.19}$$

where $C_1 = 1 + \frac{4\mu \max\{\lambda_1^2, \lambda_2^2\}}{2k+1}$. Hence taking $\varepsilon = \frac{1}{2C_1}$ yields (2.16).

Next we show there exists positive constant C_2 such that

$$[\![u]\!]^2 \le \frac{C_2 h}{d}\|q\|^2. \tag{2.20}$$

Taking $r = 1$ in (2.14), we get

$$\tilde{\theta}[\![u]\!]_{j+\frac{1}{2}} + \theta[\![u]\!]_{j-\frac{1}{2}} = \frac{1}{\sqrt{d}}(q, 1)_j - (u_x, 1)_j \doteq b_j, \quad \forall j. \tag{2.21}$$

In the special cases $\theta = 0$ or $\theta = 1$, the system (2.21) is decoupled, $[\![u]\!]_{j-\frac{1}{2}}$ can be solved locally, and thus (2.20) is very easy to get. So here we only consider the general case $\theta \neq 0, 1$. Owing to the periodic boundary condition, it forms a linear system

$$\mathcal{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{x} = ([\![u]\!]_{1/2}, \cdots, [\![u]\!]_{N-1/2})^\top$, $\boldsymbol{b} = (b_1, \cdots, b_N)^\top$, and $\mathcal{A}$ is an $N \times N$ circulant matrix in the form

$$\mathcal{A} = \begin{pmatrix} \theta & \tilde{\theta} & & & \\ & \theta & \tilde{\theta} & & \\ & & \ddots & \ddots & \\ & & & \theta & \tilde{\theta} \\ \tilde{\theta} & & & & \theta \end{pmatrix}.$$

Notice that $det(\mathcal{A}) = \theta^N(1-\varsigma^N)$, where $\varsigma = -\tilde{\theta}/\theta$, so $\mathcal{A}$ is invertible when $\theta \neq \frac{1}{2}$, and the inverse matrix of $\mathcal{A}$ is also a circulant matrix in the form [24]

$$\mathcal{A}^{-1} = \frac{1}{\theta(1-\varsigma^N)} \begin{pmatrix} 1 & \varsigma & \varsigma^2 & \cdots & \varsigma^{N-1} \\ \varsigma^{N-1} & 1 & \varsigma & \cdots & \varsigma^{N-2} \\ & & \ddots & \ddots & \\ \varsigma^2 & \varsigma^3 & \cdots & 1 & \varsigma \\ \varsigma & \varsigma^2 & \cdots & \varsigma^{N-1} & 1 \end{pmatrix}.$$

Notice that the row-norm and column-norm of $\mathcal{A}^{-1}$ are equal and satisfy

$$\|\mathcal{A}^{-1}\|_1 = \|\mathcal{A}^{-1}\|_\infty = \frac{1}{|\theta(1-\varsigma^N)|}\sum_{i=0}^{N-1}|\varsigma^i| = \frac{1}{|\theta(1-\varsigma^N)|}\frac{1-|\varsigma|^N}{1-|\varsigma|} \leq \frac{1}{|\theta||1-|\varsigma||},$$

so from [12] we get the spectral norm

$$\|\mathcal{A}^{-1}\|_2 \leq \|\mathcal{A}^{-1}\|_1^{\frac{1}{2}}\|\mathcal{A}^{-1}\|_\infty^{\frac{1}{2}} \leq \frac{1}{|\theta||1-|\varsigma||}.$$

Moreover, from the definition of b_j in (2.21), we get the l_2 norm of vector $\boldsymbol{b}$ which satisfies

$$|\boldsymbol{b}|_2^2 = \sum_{j=1}^N b_j^2 \leq \sum_{j=1}^N 2h\left[\frac{1}{d}\|q\|_j^2 + \|u_x\|_j^2\right] \leq \frac{2(1+C_1)h}{d}\|q\|^2,$$

due to Cauchy-Schwarz inequality and (2.16). Hence we can get

$$[\![u]\!]^2 = \sum_{j=1}^N [\![u]\!]_{j-\frac{1}{2}}^2 = |\boldsymbol{x}|_2^2 \leq \|\mathcal{A}^{-1}\|_2^2|\boldsymbol{b}|_2^2 \leq \frac{C_2 h}{d}\|q\|^2,$$

where $C_2 = \frac{2(1+C_1)}{|\theta|^2|1-|\varsigma||^2}$.

Finally taking $C_\star = (\sqrt{C_1}+\sqrt{\mu C_2})^2$ we complete the proof of this lemma. □

This lemma does not hold for the central flux, i.e, $\theta = \frac{1}{2}$. For example, in the case when k is odd, we let $u(x)|_{I_j} = P_k^j(x)$, where $P_k^j(x) = L_k(\frac{2(x-x_j)}{h_j})$ has been defined in the above proof, then u satisfies $(u, r_x)_j = 0$ for any $r \in V_h$, and for any j, $u_{j+\frac{1}{2}}^{(\frac{1}{2})} = \frac{1}{2}(u_{j+\frac{1}{2}}^- + u_{j+\frac{1}{2}}^+) = \frac{1}{2}[1+(-1)^k] = 0$. So $q = 0$ satisfies (2.14) and thus this special choice of u and $q = 0$ satisfy (2.6b), but obviously they do not satisfy the relationship (2.13). Similarly, for even k, let $u(x)|_{I_j} = (-1)^j P_k^j(x)$ for $j = 1, \cdots, N$ (where N is even), then $q = 0$ satisfies (2.14), but u, q do not satisfy the relationship (2.13).

By applying Lemmas 2.2 and 2.3, we can easily get the estimate for the convection terms, which is given in the following lemma.

Lemma 2.4. *Let $\theta \neq \frac{1}{2}$. Suppose $u, q \in V_h$ satisfy $(q, v) = \mathcal{K}(u, v)$ for any $v \in V_h$, then we have*

$$|\mathcal{H}(u, v)| \leq C \frac{c}{\sqrt{d}} \|q\| \|v\|, \tag{2.22}$$

where $C > 0$ is independent of c, d and h.

Remark 2.1. *Even though Lemma 2.3 is invalid in the case $\theta = \frac{1}{2}$, Lemma 2.4 also holds in the special case $\vartheta = \theta = \frac{1}{2}$. Since by the definition of $\mathcal{H}$ and $\mathcal{K}$ we have*

$$\mathcal{H}(u, v) = -\frac{c}{\sqrt{d}} \mathcal{K}(u, v) = -\frac{c}{\sqrt{d}} (q, v) \leq \frac{c}{\sqrt{d}} \|q\| \|v\|.$$

2.3 The IMEX SSP schemes

To give a brief introduction of the IMEX SSP scheme, let us consider the system of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = N(\mathbf{y}) + L(\mathbf{y}), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{2.23}$$

where $\mathbf{y} = [y_1, y_2, \cdots, y_d]^\top$. By applying explicit and implicit discretization for $N(\mathbf{y})$ and $L(\mathbf{y})$, respectively, the solution of (2.23) advanced from time t^n to $t^{n+1} = t^n + \tau$ is given by:

$$\begin{aligned} \mathbf{Y}_i &= \mathbf{y}_n + \tau \sum_{j=1}^{i-1} \tilde{a}_{ij} N(\mathbf{Y}_j) + \tau \sum_{j=1}^{s} a_{ij} L(\mathbf{Y}_j), \quad 1 \leq i \leq s, \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \tau \sum_{i=1}^{s} \tilde{b}_i N(\mathbf{Y}_i) + \tau \sum_{i=1}^{s} b_i L(\mathbf{Y}_i), \end{aligned} \tag{2.24}$$

where $\mathbf{Y}_i$ denotes the intermediate stages. Let

$$\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}, \quad c_i = \sum_{j=1}^{s} a_{ij}.$$

Denote $\tilde{A} = (\tilde{a}_{ij}), A = (a_{ij}) \in \mathbb{R}^{s \times s}$, $\tilde{\boldsymbol{b}}^\top = [\tilde{b}_1, \cdots, \tilde{b}_s], \boldsymbol{b}^\top = [b_1, \cdots, b_s]$ and $\tilde{\boldsymbol{c}}^\top = [\tilde{c}_1, \cdots, \tilde{c}_s], \boldsymbol{c}^\top = [c_1, \cdots, c_s]$, then we can represent the above formula as a double tableau in the Butcher notation

$$\begin{array}{c|c} \tilde{\boldsymbol{c}} & \tilde{A} \\ \hline & \tilde{\boldsymbol{b}}^\top \end{array} \qquad \begin{array}{c|c} \boldsymbol{c} & A \\ \hline & \boldsymbol{b}^\top \end{array} \tag{2.25}$$

Formally, it is a little different from that considered in [29], where the vectors $\tilde{\boldsymbol{c}} = \boldsymbol{c}$ and thus the formulas can be expressed in a single Butcher tableau. Moreover, the IMEX formulas considered in [29] are stiffly accurate, i.e, in the implicit part of the above tableau, $\boldsymbol{b}^\top$ equals the last row of the matrix A. Here we consider a class of IMEX formulas whose vector $\boldsymbol{b}^\top$ is not equal to the last row of the matrix A, which means an additional quadrature is used at the end of the time step. We will take two specific formulas proposed in [25] as examples, where the SSP schemes [13] are taken for the explicit discretization, and L-stable [16] diagonally implicit RK schemes ($a_{ij} = 0$ for $j > i$) are taken for the implicit discretization.

Second order IMEX SSP scheme:

$$
\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}
\qquad
\begin{array}{c|cc} \gamma & \gamma & 0 \\ 1-\gamma & 1-2\gamma & \gamma \\ \hline & 1/2 & 1/2 \end{array}
\tag{2.26}
$$

where $\gamma = 1 - \frac{\sqrt{2}}{2}$ was considered in [25]. In this paper, we consider γ as a parameter in certain range, which will be discussed later.

Third order IMEX SSP scheme:

$$
\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 & 0 \\ \hline & 0 & 1/6 & 1/6 & 2/3 \end{array}
\qquad
\begin{array}{c|cccc} \alpha & \alpha & 0 & 0 & 0 \\ 0 & -\alpha & \alpha & 0 & 0 \\ 1 & 0 & 1-\alpha & \alpha & 0 \\ 1/2 & \varphi & \phi & \rho & \alpha \\ \hline & 0 & 1/6 & 1/6 & 2/3 \end{array}
\tag{2.27}
$$

where $\alpha = \frac{3}{2} - \frac{\sqrt{57}}{6} \approx 0.241694261$ is the smallest root of $6\alpha^3 - 21\alpha^2 + 13\alpha - 2 = 0$, φ, ϕ and ρ are positive constants satisfying the following relationship:

$$
\alpha = 4\varphi, \quad 2\varphi + \phi = \frac{1}{4}, \quad \text{and} \quad \rho = \frac{1}{2} - \alpha - \varphi - \phi. \tag{2.28}
$$

3 The stability analysis for the IMEX-LDG schemes

In this section we would like to study the stability of the above two IMEX SSP schemes with the LDG spatial discretization (2.6), the corresponding fully discrete schemes are denoted as IMEX-LDG(k, s), where k is the degree of piecewise polynomials used in the LDG spatial discretization, and s is the order of IMEX SSP time discretization. Let $\{t^n = n\tau\}_{n=0}^{M}$ be an uniform partition of the time interval $[0, T]$, with time step τ. The time step could actually change from step to step, but in this paper we take the time step as a constant for simplicity. Given u^n, hence q^n, we would like to find the numerical solution at the next time level t^{n+1}, maybe through several intermediate stages $t^{n,\ell}$, by the above IMEX SSP methods (2.26) and (2.27).

The IMEX-LDG(k, 2) scheme reads: for any $v \in V_h$

$$
(u^{n,1}, v) = (u^n, v) + \gamma\tau\mathcal{L}(q^{n,1}, v), \tag{3.1a}
$$

$$
(u^{n,2}, v) = (u^n, v) + \tau\mathcal{H}(u^{n,1}, v) + (1-2\gamma)\tau\mathcal{L}(q^{n,1}, v) + \gamma\tau\mathcal{L}(q^{n,2}, v), \tag{3.1b}
$$

$$
(u^{n+1}, v) = (u^n, v) + \frac{\tau}{2}\left[\mathcal{H}(u^{n,1}, v) + \mathcal{H}(u^{n,2}, v)\right] + \frac{\tau}{2}\left[\mathcal{L}(q^{n,1}, v) + \mathcal{L}(q^{n,2}, v)\right], \tag{3.1c}
$$

and the auxiliary solutions $q^{n,\ell} \in V_h$ are determined by the variation form

$$(q^{n,\ell}, r) = \mathcal{K}(u^{n,\ell}, r), \quad \forall r \in V_h, \quad \text{for} \quad \ell = 1, 2. \tag{3.1d}$$

The IMEX-LDG$(k, 3)$ scheme reads: for any $v \in V_h$

$$(u^{n,1}, v) = (u^n, v) + \alpha\tau\mathcal{L}(q^{n,1}, v), \tag{3.2a}$$

$$(u^{n,2}, v) = (u^n, v) - \alpha\tau\mathcal{L}(q^{n,1}, v) + \alpha\tau\mathcal{L}(q^{n,2}, v), \tag{3.2b}$$

$$(u^{n,3}, v) = (u^n, v) + \tau\mathcal{H}(u^{n,2}, v) + (1-\alpha)\tau\mathcal{L}(q^{n,2}, v) + \alpha\tau\mathcal{L}(q^{n,3}, v), \tag{3.2c}$$

$$\begin{aligned}(u^{n,4}, v) = (u^n, v) &+ \frac{\tau}{4}\left[\mathcal{H}(u^{n,2}, v) + \mathcal{H}(u^{n,3}, v)\right] + \varphi\tau\mathcal{L}(q^{n,1}, v) + \phi\tau\mathcal{L}(q^{n,2}, v) \\ &+ \rho\tau\mathcal{L}(q^{n,3}, v) + \alpha\tau\mathcal{L}(q^{n,4}, v),\end{aligned} \tag{3.2d}$$

$$\begin{aligned}(u^{n+1}, v) = (u^n, v) &+ \frac{\tau}{6}\left[\mathcal{H}(u^{n,2}, v) + \mathcal{H}(u^{n,3}, v) + 4\mathcal{H}(u^{n,4}, v)\right] \\ &+ \frac{\tau}{6}\left[\mathcal{L}(q^{n,2}, v) + \mathcal{L}(q^{n,3}, v) + 4\mathcal{L}(q^{n,4}, v)\right],\end{aligned} \tag{3.2e}$$

and $q^{n,\ell} \in V_h$ satisfy (3.1d) for $\ell = 1, 2, 3, 4$.

Theorem 3.1. *Let $\vartheta \geq \frac{1}{2}$ and $\theta \neq \frac{1}{2}$. There exists positive constant τ_0 which is independent of h, such that if $\tau \leq \tau_0$, then the solution of schemes (3.1) and (3.2) satisfy*

$$\|u^n\| \leq \|u^0\|, \qquad \forall n. \tag{3.3}$$

In what follows, we will present the proof for Theorem 3.1. Since the explicit parts are the same as the SSP schemes analyzed in [37, 38], we can imitate [37, 38] to build up energy equations. In the following we use C to denote a generic constant independent of c, d and n, h, τ, which may have different values in different occurrences.

3.1 Proof for the IMEX-LDG$(k, 2)$ scheme

3.1.1 Energy equation

Let $\{w^{n,\ell}\}_{\forall n}^{\ell=0,1,2}$ be a series of functions defined at every stage time levels, $w^{n,0} = w^n$. For the convenience of analysis, we would like to adopt two series of simplified notations

$$\mathbb{R}_1 w^n = w^{n,1} - w^n, \quad \mathbb{R}_2 w^n = w^{n,2} - w^{n,1}, \quad \mathbb{R}_3 w^n = w^{n+1} - \frac{1}{2}(w^{n,1} + w^{n,2}), \tag{3.4}$$

and

$$\mathbb{S}_1 w^n = \mathbb{R}_2 w^n, \quad \mathbb{S}_2 w^n = \mathbb{R}_3 w^n - \frac{1}{2}\mathbb{R}_2 w^n. \tag{3.5}$$

Furthermore, we would like to introduce another two series of notations $\tilde{\mathbb{R}}$ and $\tilde{\mathbb{S}}$ corresponding to $\mathbb{R}$ and $\mathbb{S}$, respectively, which are related to the implicit discretization of the diffusion part.

$$\tilde{\mathbb{R}}_1 w^n = \gamma w^{n,1}, \quad \tilde{\mathbb{R}}_2 w^n = (1-3\gamma) w^{n,1} + \gamma w^{n,2}, \quad \tilde{\mathbb{R}}_3 w^n = \frac{\gamma}{2} w^{n,1} + \frac{1-\gamma}{2} w^{n,2}, \tag{3.6}$$

and

$$\tilde{\mathbb{S}}_1 w^n = \tilde{\mathbb{R}}_2 w^n, \quad \tilde{\mathbb{S}}_2 w^n = \tilde{\mathbb{R}}_3 w^n - \frac{1}{2}\tilde{\mathbb{R}}_2 w^n. \tag{3.7}$$

Then we get

$$(\mathbb{R}_\ell u^n, v) = b_\ell \tau \mathcal{H}(u^{n,\ell-1}, v) + \tau \mathcal{L}(\tilde{\mathbb{R}}_\ell q^n, v), \qquad \text{for} \quad \ell = 1,2,3, \tag{3.8}$$

where $b_\ell = 0$ for $\ell = 1$ and $b_\ell = \frac{1}{\ell-1}$ for $\ell = 2,3$. And

$$(\mathbb{S}_\ell u^n, v) = \frac{1}{\ell}\tau \mathcal{H}(\mathbb{S}_{\ell-1} u^n, v) + \tau \mathcal{L}(\tilde{\mathbb{S}}_\ell q^n, v), \qquad \text{for} \quad \ell = 1,2, \tag{3.9}$$

where $\mathbb{S}_0 u^n = u^{n,1}$.

Let $v_1 = 2u^{n,1}$, $v_2 = u^{n,1}$, $v_3 = 2u^{n,2}$. Taking $v = v_\ell$ in (3.8) for $\ell = 1,2,3$, respectively, adding them together, we obtain the energy equation

$$\|u^{n+1}\|^2 - \|u^n\|^2 + \|\mathbb{R}_1 u^n\|^2 = V_1 + V_2 + V_3, \tag{3.10a}$$

where

$$V_1 = \tau \sum_{\ell=1}^{3} b_\ell \mathcal{H}(u^{n,\ell-1}, v_\ell), \tag{3.10b}$$

$$V_2 = \tau \sum_{\ell=1}^{3} \mathcal{L}(\tilde{\mathbb{R}}_\ell q^n, v_\ell), \tag{3.10c}$$

$$V_3 = \|\mathbb{S}_2 u^n\|^2. \tag{3.10d}$$

3.1.2 Energy estimate

From (2.9) we have

$$V_1 = -(\vartheta - \frac{1}{2})c\tau \left([\![u^{n,1}]\!]^2 + [\![u^{n,2}]\!]^2 \right) \leq 0. \tag{3.11}$$

Owing to Corollary 2.1 we can obtain

$$V_2 = -\tau \left[2(q^{n,1}, \tilde{\mathbb{R}}_1 q^n) + (q^{n,1}, \tilde{\mathbb{R}}_2 q^n) + 2(q^{n,2}, \tilde{\mathbb{R}}_3 q^n) \right] = -\tau \int_\Omega \boldsymbol{q}^{n\top} \mathbb{A}_1 \boldsymbol{q}^n \mathrm{d}x, \tag{3.12}$$

where $\boldsymbol{q}^n = (q^{n,1}, q^{n,2})^\top$ and

$$\mathbb{A}_1 = \begin{pmatrix} 1-\gamma & \gamma \\ \gamma & 1-\gamma \end{pmatrix}, \tag{3.13}$$

which is positive definite if $0 < \gamma < \frac{1}{2}$.

Next we present the estimate for V_3. To this end, we adopt the similar argument as in [4] and give the following lemma firstly.

Lemma 3.1. *For any $v \in V_h$, we have*

$$(\mathbb{S}_2 u^n, v) = \omega_1(\mathbb{R}_1 u^n, v) + \omega_2(\mathbb{R}_2 u^n, v) + \frac{\tau}{2}\mathcal{H}(\mathbb{S}_1 u^n, v) - \omega_2 \tau \mathcal{H}(u^{n,1}, v), \tag{3.14}$$

where $\omega_1 = \frac{-1+4\gamma-2\gamma^2}{2\gamma^2}$, $\omega_2 = \frac{1-2\gamma}{2\gamma}$.

Proof. From (3.9) and the notations from (3.4) to (3.7), and owing to the linear structure of the operators $\mathcal{H}$ and $\mathcal{L}$, we have

$$(\mathbb{S}_2 u^n, v) = \frac{1}{2}\tau\mathcal{H}(\mathbb{S}_1 u^n, v) + \frac{4\gamma - 1}{2}\tau\mathcal{L}(q^{n,1}, v) + \frac{1-2\gamma}{2}\tau\mathcal{L}(q^{n,2}, v). \tag{3.15}$$

And from (3.8) we get

$$\tau\mathcal{L}(q^{n,1}, v) = \frac{1}{\gamma}(\mathbb{R}_1 u^n, v), \tag{3.16}$$

$$\begin{aligned} \tau\mathcal{L}(q^{n,2}, v) &= \frac{1}{\gamma}\left[(\mathbb{R}_2 u^n, v) - \tau\mathcal{H}(u^{n,1}, v) - (1-3\gamma)\tau\mathcal{L}(q^{n,1}, v)\right] \\ &= \frac{1}{\gamma}(\mathbb{R}_2 u^n, v) - \frac{1-3\gamma}{\gamma^2}(\mathbb{R}_1 u^n, v) - \frac{1}{\gamma}\tau\mathcal{H}(u^{n,1}, v). \end{aligned} \tag{3.17}$$

Substituting (3.16), (3.17) into (3.15) yields (3.14). □

By taking $v = \mathbb{S}_2 u^n$ in (3.14), and using Cauchy-Schwarz inequality for the first two terms and applying Lemma 2.4 for the last two terms, we have

$$\|\mathbb{S}_2 u^n\| \le |\omega_1|\|\mathbb{R}_1 u^n\| + |\omega_2|\|\mathbb{R}_2 u^n\| + V, \tag{3.18}$$

where

$$V = C\frac{c}{\sqrt{d}}\tau(\|q^{n,1}\| + \|q^{n,2}\|).$$

As a consequence, using Young's inequality leads to

$$\begin{aligned} \|\mathbb{S}_2 u^n\|^2 &\le (1+\hat{\varepsilon})\left(|\omega_1|\|\mathbb{R}_1 u^n\| + |\omega_2|\|\mathbb{R}_2 u^n\|\right)^2 + (1+\hat{\varepsilon}^{-1})V^2 \\ &\le (1+\hat{\varepsilon})(1+\tilde{\varepsilon})|\omega_1|^2\|\mathbb{R}_1 u^n\|^2 + (1+\hat{\varepsilon})(1+\tilde{\varepsilon}^{-1})|\omega_2|^2\|\mathbb{R}_2 u^n\|^2 + (1+\hat{\varepsilon}^{-1})V^2, \end{aligned}$$

for arbitrary positive constants $\hat{\varepsilon}$ and $\tilde{\varepsilon}$. In order to ensure the stability, we would like to require that there exists a positive constant $\sigma_0 \in (0,1)$, such that for a given $\hat{\varepsilon}$

$$(1+\hat{\varepsilon})(1+\tilde{\varepsilon})|\omega_1|^2 \le 1 \quad \text{and} \quad (1+\hat{\varepsilon})(1+\tilde{\varepsilon}^{-1})|\omega_2|^2 \le \sigma_0 \tag{3.19}$$

hold for some $\tilde{\varepsilon}$. That is to say

$$0 < \frac{1}{\frac{\sigma_0}{(1+\hat{\varepsilon})\omega_2^2} - 1} \le \tilde{\varepsilon} \le \frac{1}{(1+\hat{\varepsilon})\omega_1^2} - 1.$$

To ensure (3.19) holds for some $\tilde{\varepsilon}$, we need to solve the inequality

$$0 < \frac{1}{\frac{\sigma_0}{(1+\hat{\varepsilon})\omega_2^2} - 1} \le \frac{1}{(1+\hat{\varepsilon})\omega_1^2} - 1, \tag{3.20}$$

which will give a range for the parameter γ. Apparently the range depends on the choice of $\hat{\varepsilon}$ and σ_0. For the convenience of discussion, we take $\hat{\varepsilon} = \frac{1}{3}$ and $\sigma_0 = \frac{8}{9}$. In this case, solving (3.20) by Maple we get $\gamma \in [\gamma_1, \gamma_2] \approx [0.27998, 0.37932]$. Thus

$$\|\mathbb{S}_2 u^n\|^2 \leq \|\mathbb{R}_1 u^n\|^2 + \sigma_0 \|\mathbb{R}_2 u^n\|^2 + 4V^2. \tag{3.21}$$

Now the only remaining thing is to estimate $\|\mathbb{R}_2 u^n\|$. Taking $v = \mathbb{R}_2 u^n$ in (3.8) for $\ell = 2$, applying Corollary 2.1, Lemma 2.4 and Young's inequality we get

$$\begin{aligned} \|\mathbb{R}_2 u^n\|^2 =& \tau \mathcal{H}(u^{n,1}, \mathbb{R}_2 u^n) - \tau(\mathbb{R}_2 q^n, \tilde{\mathbb{R}}_2 q^n) \\ \leq& C\frac{c}{\sqrt{d}} \tau \|q^{n,1}\| \|\mathbb{R}_2 u^n\| - \tau \int_\Omega \boldsymbol{q}^{n\top} \mathbb{A}_2 \boldsymbol{q}^n \mathrm{d}x \\ \leq& \varepsilon \|\mathbb{R}_2 u^n\|^2 + \frac{Cc^2\tau}{4\varepsilon d} \tau \|q^{n,1}\|^2 - \tau \int_\Omega \boldsymbol{q}^{n\top} \mathbb{A}_2 \boldsymbol{q}^n \mathrm{d}x, \end{aligned} \tag{3.22}$$

for arbitrary ε, where

$$\mathbb{A}_2 = \begin{pmatrix} 3\gamma - 1 & \frac{1}{2} - 2\gamma \\ \frac{1}{2} - 2\gamma & \gamma \end{pmatrix}. \tag{3.23}$$

Taking $\varepsilon = 1 - \sigma_0 = \frac{1}{9}$, we get

$$\sigma_0 \|\mathbb{R}_2 u^n\|^2 \leq \frac{9Cc^2\tau}{4d} \tau \|q^{n,1}\|^2 - \tau \int_\Omega \boldsymbol{q}^{n\top} \mathbb{A}_2 \boldsymbol{q}^n \mathrm{d}x. \tag{3.24}$$

Thus

$$\|\mathbb{S}_2 u^n\|^2 \leq \|\mathbb{R}_1 u^n\|^2 + \frac{9Cc^2\tau}{4d} \tau \|q^{n,1}\|^2 - \tau \int_\Omega \boldsymbol{q}^{n\top} \mathbb{A}_2 \boldsymbol{q}^n \mathrm{d}x + 4V^2. \tag{3.25}$$

Consequently, from (3.10), (3.11), (3.12) and (3.25), we get

$$\|u^{n+1}\|^2 - \|u^n\|^2 \leq -\tau \int_\Omega \boldsymbol{q}^{n\top} (\mathbb{A}_1 + \mathbb{A}_2) \boldsymbol{q}^n \mathrm{d}x + \frac{41}{4} \frac{Cc^2\tau}{d} \tau (\|q^{n,1}\|^2 + \|q^{n,2}\|^2). \tag{3.26}$$

It can be verified that $\mathbb{A}_1 + \mathbb{A}_2 - \gamma \mathbb{I}$ is positive definite if $\gamma \in [\frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{1}{2} + \frac{\sqrt{2}}{4}] \supset [\gamma_1, \gamma_2]$. Hence, if $\tau \leq \tau_0$ such that $\frac{41}{4} \frac{Cc^2}{d} \tau \leq \gamma$, i.e, $\tau \leq \frac{4\gamma d}{41Cc^2}$, then

$$\|u^{n+1}\| \leq \|u^n\| \leq \cdots \leq \|u^0\|. \tag{3.27}$$

Remark 3.1. *In Theorem 3.1, we require $\vartheta \geq \frac{1}{2}$ to ensure $V_1 \leq 0$ in (3.11). Actually $\vartheta < \frac{1}{2}$ also works, since in this case*

$$V_1 \leq (\frac{1}{2} - \vartheta) \frac{C_\star ch}{d} \tau \left(\|q^{n,1}\|^2 + \|q^{n,2}\|^2 \right), \tag{3.28}$$

according to Lemma 2.3. So if h is small enough such that V_1 can be bounded by the stability term $\tau \int_\Omega \boldsymbol{q}^{n\top} (\mathbb{A}_1 + \mathbb{A}_2) \boldsymbol{q}^n \mathrm{d}x$, then the theorem can also be proven.

Remark 3.2. *Owing to Remark 2.1, Theorem 3.1 also holds in the special case $\vartheta = \theta = \frac{1}{2}$.*

3.2 Proof for the IMEX-LDG$(k, 3)$ scheme

The line of proof for the third order scheme is similar as but more complicated than that for the second order scheme. The main difficulty is the construction of the energy equation.

3.2.1 Energy equation

Let $\{w^{n,\ell}\}_{\forall n}^{\ell=0,1,2,3,4}$ be a series of functions defined at the different stage time levels, $w^{n,0} = w^n$. Following [38], we define two series of simplified notations

$$\begin{aligned} &\mathbb{E}_1 w^n = w^{n,1} - w^n, \quad \mathbb{E}_2 w^n = w^{n,2} - w^{n,1}, \quad \mathbb{E}_3 w^n = w^{n,3} - w^{n,2}, \\ &\mathbb{E}_4 w^n = 4w^{n,4} - 3w^{n,2} - w^{n,3}, \quad \mathbb{E}_5 w^n = \frac{3}{2}w^{n+1} - w^{n,4} - \frac{1}{2}w^{n,2}, \end{aligned} \tag{3.29}$$

and

$$\mathbb{D}_1 w^n = \mathbb{E}_3 w^n, \quad \mathbb{D}_2 w^n = \frac{1}{2}(\mathbb{E}_4 w^n - \mathbb{E}_3 w^n), \quad \mathbb{D}_3 w^n = \frac{1}{3}(2\mathbb{E}_5 w^n - \mathbb{E}_4 w^n - \mathbb{E}_3 w^n). \tag{3.30}$$

For the convenience of expression, we define another two series of notations $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{D}}$ corresponding to $\mathbb{E}$ and $\mathbb{D}$, respectively, which are used to simplify notations about the implicit discretization of the diffusion part.

$$\tilde{\mathbb{E}}_1 w^n = \alpha w^{n,1}, \tag{3.31a}$$

$$\tilde{\mathbb{E}}_2 w^n = -2\alpha w^{n,1} + \alpha w^{n,2}, \tag{3.31b}$$

$$\tilde{\mathbb{E}}_3 w^n = \alpha w^{n,1} + (1-2\alpha) w^{n,2} + \alpha w^{n,3}, \tag{3.31c}$$

$$\begin{aligned} \tilde{\mathbb{E}}_4 w^n &= (3\alpha + 4\varphi) w^{n,1} + (4\phi - 2\alpha - 1) w^{n,2} + (4\rho - \alpha) w^{n,3} + 4\alpha w^{n,4} \\ &= 4\alpha w^{n,1} - 4\alpha w^{n,2} + (1-4\alpha) w^{n,3} + 4\alpha w^{n,4}, \end{aligned} \tag{3.31d}$$

$$\begin{aligned} \tilde{\mathbb{E}}_5 w^n &= (\frac{\alpha}{2} - \varphi) w^{n,1} + (\frac{1}{4} - \frac{\alpha}{2} - \phi) w^{n,2} + (\frac{1}{4} - \rho) w^{n,3} + (1-\alpha) w^{n,4} \\ &= \frac{\alpha}{4} w^{n,1} + \frac{3\alpha}{4} w^{n,3} + (1-\alpha) w^{n,4}, \end{aligned} \tag{3.31e}$$

and

$$\tilde{\mathbb{D}}_1 w^n = \tilde{\mathbb{E}}_3 w^n, \quad \tilde{\mathbb{D}}_2 w^n = \frac{1}{2}(\tilde{\mathbb{E}}_4 w^n - \tilde{\mathbb{E}}_3 w^n), \quad \tilde{\mathbb{D}}_3 w^n = \frac{1}{3}(2\tilde{\mathbb{E}}_5 w^n - \tilde{\mathbb{E}}_4 w^n - \tilde{\mathbb{E}}_3 w^n). \tag{3.32}$$

Then we get

$$(\mathbb{E}_\ell u^n, v) = d_\ell \tau \mathcal{H}(u^{n,\ell-1}, v) + \tau \mathcal{L}(\tilde{\mathbb{E}}_\ell q^n, v), \qquad \text{for} \quad \ell = 1,2,3,4,5, \tag{3.33}$$

where $d_\ell = 0$ for $\ell = 1, 2$ and $d_\ell = 1$ for $\ell = 3, 4, 5$. And

$$(\mathbb{D}_\ell u^n, v) = \frac{1}{\ell}\tau \mathcal{H}(\mathbb{D}_{\ell-1} u^n, v) + \tau \mathcal{L}(\tilde{\mathbb{D}}_\ell q^n, v), \qquad \text{for} \quad \ell = 1,2,3, \tag{3.34}$$

where $\mathbb{D}_0 u^n = u^{n,2}$.

Let $v_1 = 6u^{n,1}$, $v_2 = 6u^{n,2}$, $v_3 = u^{n,2}$, $v_4 = u^{n,3}$ and $v_5 = 4u^{n,4}$. Taking $v = v_\ell$ in (3.33) for $\ell = 1, 2, 3, 4, 5$, respectively, adding them together, we obtain

$$3\|u^{n+1}\|^2 - 3\|u^n\|^2 + 3\|\mathbb{E}_1 u^n\|^2 + 3\|\mathbb{E}_2 u^n\|^2 = T_1 + T_2 + T_3, \tag{3.35a}$$

where

$$T_1 = \tau \sum_{\ell=1}^{5} d_\ell \mathcal{H}(u^{n,\ell-1}, v_\ell), \tag{3.35b}$$

$$T_2 = \tau \sum_{\ell=1}^{5} \mathcal{L}(\tilde{\mathbb{E}}_\ell q^n, v_\ell), \tag{3.35c}$$

$$T_3 = \|\mathbb{D}_2 u^n\|^2 + 3(\mathbb{D}_3 u^n, \mathbb{D}_1 u^n) + 3(\mathbb{D}_3 u^n, \mathbb{D}_2 u^n) + 3\|\mathbb{D}_3 u^n\|^2. \tag{3.35d}$$

3.2.2 Energy estimate

By the definition of d_ℓ and v_ℓ and according to (2.9), we have

$$T_1 = -(\vartheta - \frac{1}{2})\tau \left([\![u^{n,2}]\!]^2 + [\![u^{n,3}]\!]^2 + 4[\![u^{n,4}]\!]^2 \right) \leq 0. \tag{3.36}$$

Owing to Corollary 2.1, we can get

$$\begin{aligned} T_2 =& -\tau \left[6(q^{n,1}, \tilde{\mathbb{E}}_1 q^n) + 6(q^{n,2}, \tilde{\mathbb{E}}_2 q^n) + (q^{n,2}, \tilde{\mathbb{E}}_3 q^n) + (q^{n,3}, \tilde{\mathbb{E}}_4 q^n) + 4(q^{n,4}, \tilde{\mathbb{E}}_5 q^n) \right] \\ =& -\tau \int_\Omega \boldsymbol{q}^{n\top} \mathbb{B}_1 \boldsymbol{q}^n \mathrm{d}x, \end{aligned} \tag{3.37}$$

where $\boldsymbol{q}^n = (q^{n,1}, q^{n,2}, q^{n,3}, q^{n,4})^\top$ and

$$\mathbb{B}_1 = \begin{pmatrix} 6\alpha & -\frac{11}{2}\alpha & 2\alpha & \frac{1}{2}\alpha \\ -\frac{11}{2}\alpha & 1+4\alpha & -\frac{3}{2}\alpha & 0 \\ 2\alpha & -\frac{3}{2}\alpha & 1-4\alpha & \frac{7}{2}\alpha \\ \frac{1}{2}\alpha & 0 & \frac{7}{2}\alpha & 4-4\alpha \end{pmatrix}. \tag{3.38}$$

Next we estimate T_3 following the trick adopted in [38], we rewrite T_3 as

$$T_3 = -\|\mathbb{D}_2 u^n\|^2 + \underbrace{2\|\mathbb{D}_2 u^n\|^2 + 3(\mathbb{D}_3 u^n, \mathbb{D}_1 u^n)}_{R_1} + \underbrace{3(\mathbb{D}_3 u^n, \mathbb{D}_2 u^n)}_{R_2} + \underbrace{3\|\mathbb{D}_3 u^n\|^2}_{R_3}. \tag{3.39}$$

To estimate R_1, we take $v = 2\mathbb{D}_2 u^n$ and $v = 3\mathbb{D}_1 u^n$ in (3.34) for $\ell = 2$ and $\ell = 3$, respectively, adding them together and using Corollary 2.1 we get

$$\begin{aligned} R_1 =& \underbrace{\tau \left[\mathcal{H}(\mathbb{D}_1 u^n, \mathbb{D}_2 u^n) + \mathcal{H}(\mathbb{D}_2 u^n, \mathbb{D}_1 u^n) \right]}_{R_4} + \tau \left[2\mathcal{L}(\tilde{\mathbb{D}}_2 q^n, \mathbb{D}_2 u^n) + 3\mathcal{L}(\tilde{\mathbb{D}}_3 q^n, \mathbb{D}_1 u^n) \right] \\ =& R_4 - \tau \left[2(\mathbb{D}_2 q^n, \tilde{\mathbb{D}}_2 q^n) + 3(\mathbb{D}_1 q^n, \tilde{\mathbb{D}}_3 q^n) \right] = R_4 - \tau \int_\Omega \boldsymbol{q}^{n\top} \mathbb{B}_2 \boldsymbol{q}^n \mathrm{d}x, \end{aligned} \tag{3.40}$$

where

$$\mathbb{B}_2 = \begin{pmatrix} 0 & \frac{3}{4}\alpha & -\frac{15}{4}\alpha & 3\alpha \\ \frac{3}{4}\alpha & 2-4\alpha & \frac{17}{4}\alpha & -2-\alpha \\ -\frac{15}{4}\alpha & \frac{17}{4}\alpha & \frac{19}{2}\alpha - 2 & 2-10\alpha \\ 3\alpha & -2-\alpha & 2-10\alpha & 8\alpha \end{pmatrix}. \tag{3.41}$$

To estimate R_2, we take $v = 3\mathbb{D}_2 u^n$ in (3.34) for $\ell = 3$, using (2.9) and Corollary 2.1 yields

$$\begin{aligned} R_2 =& \tau\mathcal{H}(\mathbb{D}_2 u^n, \mathbb{D}_2 u^n) + 3\tau\mathcal{L}(\tilde{\mathbb{D}}_3 q^n, \mathbb{D}_2 u^n) \\ =& -(\vartheta - \frac{1}{2})c\tau[\![\mathbb{D}_2 u^n]\!]^2 - 3\tau(\mathbb{D}_2 q^n, \tilde{\mathbb{D}}_3 q^n) \leq -\tau \int_\Omega \boldsymbol{q}^{n\top}\mathbb{B}_3 \boldsymbol{q}^n \mathrm{d}x, \end{aligned} \tag{3.42}$$

where

$$\mathbb{B}_3 = \begin{pmatrix} 0 & \frac{9}{4}\alpha & \frac{9}{4}\alpha & -\frac{9}{2}\alpha \\ \frac{9}{4}\alpha & 1-6\alpha & 1-\frac{21}{4}\alpha & 9\alpha - 2 \\ \frac{9}{4}\alpha & 1-\frac{21}{4}\alpha & 1-\frac{9}{2}\alpha & \frac{15}{2}\alpha - 2 \\ -\frac{9}{2}\alpha & 9\alpha - 2 & \frac{15}{2}\alpha - 2 & 4-12\alpha \end{pmatrix}. \tag{3.43}$$

Combining the above estimates, we have

$$\begin{aligned} &3\|u^{n+1}\|^2 - 3\|u^n\|^2 + 3\|\mathbb{E}_1 u^n\|^2 + 3\|\mathbb{E}_2 u^n\|^2 + \tau \int_\Omega \boldsymbol{q}^{n\top} \sum_{i=1}^{3} \mathbb{B}_i \boldsymbol{q}^n \mathrm{d}x \\ \leq& -\|\mathbb{D}_2 u^n\|^2 + R_3 + R_4. \end{aligned} \tag{3.44}$$

It can be verified that the matrix $\sum_{i=1}^3 \mathbb{B}_i$ is positive definite by verifying all the leading principle minors are positive, and along the same way, $\sum_{i=1}^3 \mathbb{B}_i - \sigma\mathbb{I}$ is also positive definite if $0 < \sigma \leq \frac{1}{12}\alpha$.

In what follows we use the stability terms $\|\mathbb{D}_2 u^n\|^2$ and $\tau\|q^{n,\ell}\|^2$ to estimate R_3 and R_4. Owing to Lemma 2.4 we get

$$|R_4| \leq C\frac{c}{\sqrt{d}}\tau\|\mathbb{D}_1 q^n\|\|\mathbb{D}_2 u^n\| \leq \frac{1}{4}\|\mathbb{D}_2 u^n\|^2 + 2\frac{Cc^2\tau}{d}\tau(\|q^{n,2}\|^2 + \|q^{n,3}\|^2), \tag{3.45}$$

where Young's inequality is used in the last step.

The most technical term is R_3. To estimate it, we adopt the technique used in estimating V_3 in Subsection 3.1. Eliminating the $\mathcal{L}(\tilde{\mathbb{D}}_3 q^n, v)$ terms (using (3.33)) in (3.34) for $\ell = 3$, we can represent $(\mathbb{D}_3 u^n, v)$ as

$$\begin{aligned} (\mathbb{D}_3 u^n, v) =& \frac{1}{3}\tau\mathcal{H}(\mathbb{D}_2 u^n, v) + (\frac{1}{2} - \frac{1}{6\alpha})\tau\mathcal{H}(u^{n,3}, v) + (\frac{1}{2} - \frac{5}{6\alpha} + \frac{1}{6\alpha^2})\tau\mathcal{H}(u^{n,2}, v) \\ &+ \kappa_1(\mathbb{D}_2 u^n, v) + \kappa_2(\widetilde{\mathbb{D}_0} u^n, v), \end{aligned} \tag{3.46}$$

where

$$\kappa_1 = \frac{1}{3\alpha} - 1 \approx 0.379152868,$$
$$\kappa_2 = \frac{1}{\alpha} - \frac{1}{6\alpha^2} - 1 \approx 0.284364653,$$
$$\widetilde{\mathbb{D}_0} u^n = u^{n,3} + (\frac{1}{\alpha} - 1)u^{n,2} - \frac{1}{\alpha}u^{n,1}.$$

Here we have used the relationship $6\alpha^3 - 21\alpha^2 + 13\alpha - 2 = 0$.

Taking $v = \mathbb{D}_3 u^n$ in (3.46), owing to Lemma 2.4, and Cauchy-Schwarz inequality we have

$$\|\mathbb{D}_3 u^n\| \leq \kappa_1 \|\mathbb{D}_2 u^n\| + \kappa_2 \|\widetilde{\mathbb{D}_0} u^n\| + R, \tag{3.47}$$

where

$$R = C\frac{c}{\sqrt{d}}\tau(\|q^{n,2}\| + \|q^{n,3}\| + \|q^{n,4}\|).$$

As a result,

$$\begin{aligned} R_3 =& 3\|\mathbb{D}_3 u^n\|^2 \leq 3\left\{(1+\hat{\varepsilon})[\kappa_1\|\mathbb{D}_2 u^n\| + \kappa_2\|\widetilde{\mathbb{D}_0} u^n\|]^2 + (1+\hat{\varepsilon}^{-1})R^2\right\} \\ \leq& 3(1+\hat{\varepsilon})\left[(1+\tilde{\varepsilon})\kappa_1^2\|\mathbb{D}_2 u^n\|^2 + (1+\tilde{\varepsilon}^{-1})\kappa_2^2\|\widetilde{\mathbb{D}_0} u^n\|^2\right] + 3(1+\hat{\varepsilon}^{-1})R^2, \end{aligned} \tag{3.48}$$

where $\hat{\varepsilon}$ and $\tilde{\varepsilon}$ are arbitrary positive constants. Taking $\hat{\varepsilon} = 1/9$ and $\tilde{\varepsilon} = 1/2$ we get

$$\begin{aligned} R_3 \leq& 5\kappa_1^2\|\mathbb{D}_2 u^n\|^2 + 10\kappa_2^2\|\widetilde{\mathbb{D}_0} u^n\|^2 + 30R^2 \\ \leq& \frac{3}{4}\|\mathbb{D}_2 u^n\|^2 + \frac{7}{8}\|\widetilde{\mathbb{D}_0} u^n\|^2 + 30R^2, \end{aligned} \tag{3.49}$$

since $\kappa_1^2 \approx 0.1437568973$, $\kappa_2^2 \approx 0.08086325588$.

To estimate $\|\widetilde{\mathbb{D}_0} u^n\|$, we notice that

$$(\widetilde{\mathbb{D}_0} u^n, v) = \tau\mathcal{H}(u^{n,2}, v) + (\alpha-2)\tau\mathcal{L}(q^{n,1}, v) + (2-2\alpha)\tau\mathcal{L}(q^{n,2}, v) + \alpha\tau\mathcal{L}(q^{n,3}, v), \tag{3.50}$$

from (3.2). Taking $v = \widetilde{\mathbb{D}_0} u^n$ in (3.50) we get

$$\|\widetilde{\mathbb{D}_0} u^n\|^2 = \tau\mathcal{H}(u^{n,2}, \widetilde{\mathbb{D}_0} u^n) - \tau\left((\alpha-2)q^{n,1} + (2-2\alpha)q^{n,2} + \alpha q^{n,3}, \widetilde{\mathbb{D}_0} q^n\right). \tag{3.51}$$

Applying Lemma 2.4 on the first term and rearranging the second term lead to

$$\begin{aligned} \|\widetilde{\mathbb{D}_0} u^n\|^2 \leq& C\frac{c}{\sqrt{d}}\tau\|q^{n,2}\|\|\widetilde{\mathbb{D}_0} u^n\| - \tau\int_\Omega \boldsymbol{q}^{n\top}\mathbb{B}_4\boldsymbol{q}^n \mathrm{d}x \\ \leq& \varepsilon\|\widetilde{\mathbb{D}_0} u^n\|^2 + \frac{Cc^2\tau}{4\varepsilon d}\tau\|q^{n,2}\|^2 - \tau\int_\Omega \boldsymbol{q}^{n\top}\mathbb{B}_4\boldsymbol{q}^n \mathrm{d}x, \end{aligned} \tag{3.52}$$

for arbitrary ε, where

$$\mathbb{B}_4 = \begin{pmatrix} \frac{2}{\alpha} - 1 & \frac{5}{2} - \frac{\alpha}{2} - \frac{2}{\alpha} & \frac{\alpha-3}{2} & 0 \\ \frac{5}{2} - \frac{\alpha}{2} - \frac{2}{\alpha} & 2\alpha - 4 + \frac{2}{\alpha} & \frac{3-3\alpha}{2} & 0 \\ \frac{\alpha-3}{2} & \frac{3-3\alpha}{2} & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.53}$$

Taking $\varepsilon = \frac{1}{8}$, we get

$$\frac{7}{8}\|\widetilde{\mathbb{D}_0}u^n\|^2 \leq 2\frac{Cc^2\tau}{d}\tau\|q^{n,2}\|^2 - \tau\int_\Omega \boldsymbol{q}^{n\top}\mathbb{B}_4\boldsymbol{q}^n \mathrm{d}x. \tag{3.54}$$

So

$$R_3 \leq \frac{3}{4}\|\mathbb{D}_2u^n\|^2 + 30R^2 + 2\frac{Cc^2\tau}{d}\tau\|q^{n,2}\|^2 - \tau\int_\Omega \boldsymbol{q}^{n\top}\mathbb{B}_4\boldsymbol{q}^n \mathrm{d}x. \tag{3.55}$$

Hence, from (3.44), (3.45) and (3.55) we obtain

$$\begin{aligned} &3\|u^{n+1}\|^2 - 3\|u^n\|^2 + 3\|\mathbb{E}_1u^n\|^2 + 3\|\mathbb{E}_2u^n\|^2 + \tau\int_\Omega \boldsymbol{q}^{n\top}\sum_{i=1}^{4}\mathbb{B}_i\boldsymbol{q}^n \mathrm{d}x \\ \leq& 30R^2 + 2\frac{Cc^2\tau}{d}\tau(2\|q^{n,2}\|^2 + \|q^{n,3}\|^2) \leq 94\frac{Cc^2\tau}{d}\tau\int_\Omega \boldsymbol{q}^{n\top}\boldsymbol{q}^n \mathrm{d}x. \end{aligned} \tag{3.56}$$

It can be verified that the matrix $\sum_{i=1}^4 \mathbb{B}_i$ is positive definite and $\sum_{i=1}^4 \mathbb{B}_i - \frac{\alpha}{18}\mathbb{I}$ is also positive definite, thus if $\tau \leq \tau_0$ such that $94\frac{Cc^2}{d}\tau \leq \frac{\alpha}{18}$, then

$$\|u^{n+1}\| \leq \|u^n\| \leq \cdots \leq \|u^0\|. \tag{3.57}$$

4 Error estimates

In this section, we would like to take the IMEX-LDG$(k,2)$ scheme (3.1) as an example to present the error estimates, the line of proof for the IMEX-LDG$(k,3)$ scheme (3.2) is similar. The standard approach of error estimates is to introduce a suitable projection and to divide the error e into two parts, one is the projection error η, the other is the error ξ in the finite element space, then to estimate ξ by η. Hence the projection is a key ingredient in the error estimate. The principle of choosing the projection in the DG analysis is to eliminate the projection errors in the inner product and element interfaces as much as possible. Thus we usually choose the projection according to the very choice of the numerical fluxes, for example, the well-known Gauss-Radau projection for purely upwind numerical flux or PANF. Here for GANF, we adopt the GGR projection proposed in [24]. To simplify the analysis, we consider the simple case when the parameters $\vartheta = \theta \neq \frac{1}{2}$.

4.1 GGR projection

Following [24, 7], we define the GGR projection. For any periodic function $z \in H^1(\mathcal{T}_h)$, the projection $P_\beta z \in V_h$ satisfies

$$(P_\beta z - z, v)_j = 0, \quad \forall v \in P_{k-1}(I_j), \tag{4.1a}$$

$$(P_\beta z)^{(\beta)}_{j+\frac{1}{2}} = z^{(\beta)}_{j+\frac{1}{2}}, \tag{4.1b}$$

for any $j = 1, \cdots, N$ and any parameter β. Here $z^{(\beta)} = \beta z^- + \tilde{\beta} z^+$ with $\tilde{\beta} = 1 - \beta$. Obviously, this projection degenerates to the local Gauss-Radau projection if the parameter β is taken as 0 or 1. Hence it can be viewed as an extension of the local Gauss-Radau projections.

According to [7], we have the following lemma.

Lemma 4.1. *Assume $z \in H^\ell(\mathcal{T}_h)$ with $\ell \geq 1$. For $\beta \neq \frac{1}{2}$, the projection $P_\beta z$ is well-defined and the projection error $\eta = z - P_\beta z$ satisfies*

$$\|\eta\| + h^{1/2}\|\eta\|_{\Gamma_h} \leq C h^{\min\{k+1,\ell\}} \|z\|_{H^\ell(\mathcal{T}_h)}, \tag{4.2}$$

where the bounding constant $C > 0$ is independent of h and z.

4.2 Reference functions and energy equation

In this paper, we assume the exact solution U satisfies the following smoothness

$$U \in L^\infty(0, T; H^{k+2}), \quad U_t \in L^\infty(0, T; H^{k+1} \cap H^2), \tag{4.3a}$$

$$U_{tt} \in L^\infty(0, T; H^1), \quad U_{ttt} \in L^\infty(0, T; L^2), \tag{4.3b}$$

where $L^\infty(0, T; H^\ell)$ represents the set of functions v such that $\max_{0 \leq t \leq T} \|v(\cdot, t)\|_{H^\ell(\Omega)} < \infty$.

To proceed with error estimates, we introduce several reference functions, denoted by $U^{(\ell)}, Q^{(\ell)}$ for $\ell = 0, 1, 2$, associated with the second order IMEX SSP time discretization (2.26). In detail, $U^{(0)} = U$ is the exact solution of the problem (2.1) and then we define

$$U^{(1)} = U^{(0)} + \gamma\tau\sqrt{d} Q^{(1)}_x, \tag{4.4a}$$

$$U^{(2)} = U^{(0)} - \tau c U^{(1)}_x + (1 - 2\gamma)\tau\sqrt{d} Q^{(1)}_x + \gamma\tau\sqrt{d} Q^{(2)}_x, \tag{4.4b}$$

where

$$Q^{(\ell)} = \sqrt{d} U^{(\ell)}_x, \quad \text{for} \quad \ell = 1, 2. \tag{4.5}$$

For any indices n and ℓ under consideration, the reference functions at each stage time level are defined as $U^{n,\ell} = U^{(\ell)}(x, t^n), Q^{n,\ell} = Q^{(\ell)}(x, t^n)$.

Under the smoothness assumption (4.3) we can verify that the reference functions satisfy

$$U^{n+1} = U^n - \frac{\tau}{2} c(U^{n,1}_x + U^{n,2}_x) + \frac{\tau}{2}\sqrt{d}(Q^{n,1}_x + Q^{n,2}_x) + \zeta^n, \tag{4.6}$$

where ζ^n is the local truncation error, which satisfies

$$\|\zeta^n\| \le C\tau^3, \tag{4.7}$$

where C depends on the regularity of U_t, U_{tt} and U_{ttt}, the detailed proof will be given in the Appendix. As a consequence, the reference functions satisfy the following variational form

$$(U^{n,1}, v) = (U^n, v) + \gamma\tau\mathcal{L}(Q^{n,1}, v), \tag{4.8a}$$

$$(U^{n,2}, v) = (U^n, v) + \tau\mathcal{H}(U^{n,1}, v) + (1-2\gamma)\tau\mathcal{L}(Q^{n,1}, v) + \gamma\tau\mathcal{L}(Q^{n,2}, v), \tag{4.8b}$$

$$\begin{aligned}(U^{n+1}, v) = (U^n, v) &+ \frac{\tau}{2}\left[\mathcal{H}(U^{n,1}, v) + \mathcal{H}(U^{n,2}, v)\right] \\ &+ \frac{\tau}{2}\left[\mathcal{L}(Q^{n,1}, v) + \mathcal{L}(Q^{n,2}, v)\right] + (\zeta^n, v),\end{aligned} \tag{4.8c}$$

for any $v \in V_h$, and

$$(Q^{n,\ell}, r) = \mathcal{K}(U^{n,\ell}, r), \quad \forall r \in V_h, \quad \text{for} \quad \ell = 1, 2. \tag{4.8d}$$

At each stage time, we denote the error between the exact (reference) solution and the numerical solution by $\boldsymbol{e}^{n,\ell} = (e_u^{n,\ell}, e_q^{n,\ell}) = (U^{n,\ell} - u^{n,\ell}, Q^{n,\ell} - q^{n,\ell})$. As the standard treatment in finite element analysis, we would like to divide the error in the form $\boldsymbol{e} = \boldsymbol{\xi} - \boldsymbol{\eta}$, where

$$\begin{aligned}\boldsymbol{\eta} &= (\eta_u, \eta_q) = (P_\theta U - U, P_{\tilde\theta} Q - Q), \\ \boldsymbol{\xi} &= (\xi_u, \xi_q) = (P_\theta U - u, P_{\tilde\theta} Q - q),\end{aligned} \tag{4.9}$$

here we have dropped the superscripts n and ℓ for simplicity.

By the definition of the projections P_θ and $P_{\tilde\theta}$ we can verify that

$$\mathcal{H}(\eta_u, v) = 0, \qquad \mathcal{L}(\eta_q, v) = 0, \qquad \mathcal{K}(\eta_u, r) = 0, \tag{4.10}$$

for any $v, r \in V_h$, since we assume $\vartheta = \theta$ at present. In addition, by the smoothness assumption (4.3a), it follows from Lemma 4.1 and the linearity of the projections P_θ and $P_{\tilde\theta}$ that the stage projection errors and their evolutions satisfy

$$\|\eta_u^{n,\ell}\| + \|\eta_q^{n,\ell}\| + h^{1/2}\|\eta_u^{n,\ell}\|_{\Gamma_h} \le Ch^{k+1}, \tag{4.11a}$$

and

$$\|\mathbb{R}_{\ell+1}\eta_u^n\| \le Ch^{k+1}\tau, \tag{4.11b}$$

for any n and $\ell = 0, 1, 2$ under consideration. Here (4.11b) is obtained by the regularity $U_x, U_t \in L^\infty(0, T; H^{k+1})$.

In what follows we will focus on the estimate of the error in the finite element space, say, $\boldsymbol{\xi} \in V_h \times V_h$. To this end, we need to set up the error equations about $\boldsymbol{\xi}^{n,\ell}$. Subtracting those variational forms in (4.8) from those in the scheme (3.1), in the same order, we get

$$
\begin{aligned}
(\xi_u^{n,1}, v) &= (\xi_u^n, v) + \gamma\tau\mathcal{L}(\xi_q^{n,1}, v) + (\eta_u^{n,1} - \eta_u^n, v), &\text{(4.12a)}\\
(\xi_u^{n,2}, v) &= (\xi_u^n, v) + \tau\mathcal{H}(\xi_u^{n,1}, v)\\
&\quad + (1-2\gamma)\tau\mathcal{L}(\xi_q^{n,1}, v) + \gamma\tau\mathcal{L}(\xi_q^{n,2}, v) + (\eta_u^{n,2} - \eta_u^n, v), &\text{(4.12b)}\\
(\xi_u^{n+1}, v) &= (\xi_u^n, v) + \frac{\tau}{2}\left[\mathcal{H}(\xi_u^{n,1}, v) + \mathcal{H}(\xi_u^{n,2}, v)\right]\\
&\quad + \frac{\tau}{2}\left[\mathcal{L}(\xi_q^{n,1}, v) + \mathcal{L}(\xi_q^{n,2}, v)\right] + (\eta_u^{n+1} - \eta_u^n, v) + (\zeta^n, v). &\text{(4.12c)}
\end{aligned}
$$

Adopting the notations $\mathbb{R}$ and $\tilde{\mathbb{R}}$ as in Subsection 3.1, we obtain the following error equations

$$
\begin{aligned}
(\mathbb{R}_\ell \xi_u^n, v) &= b_\ell\tau\mathcal{H}(\xi_u^{n,\ell-1}, v) + \tau\mathcal{L}(\tilde{\mathbb{R}}_\ell\xi_q^n, v) + (\mathbb{R}_\ell\eta_u^n + \delta_{3\ell}\zeta^n, v), \quad \text{for} \quad \ell = 1,2,3, &\text{(4.13a)}\\
(\xi_q^{n,\ell}, r) &= \mathcal{K}(\xi_u^{n,\ell}, r) + (\eta_q^{n,\ell}, r), \quad \text{for} \quad \ell = 1,2, &\text{(4.13b)}
\end{aligned}
$$

where we have used (4.10). Here $b_\ell = 0$ for $\ell = 1$ and $b_\ell = \frac{1}{\ell-1}$ for $\ell = 2,3$, $\delta_{3\ell}$ is the Kronecker symbol which equals 1 if $\ell = 3$ and equals 0 otherwise.

Let $\tilde{v}_1 = 2\xi_u^{n,1}$, $\tilde{v}_2 = \xi_u^{n,1}$, $\tilde{v}_3 = 2\xi_u^{n,2}$. Taking $v = \tilde{v}_\ell$ in (4.13a) for $\ell = 1,2,3$, respectively, adding them together, we can obtain the energy equation

$$
\|\xi_u^{n+1}\|^2 - \|\xi_u^n\|^2 + \|\mathbb{R}_1\xi_u^n\|^2 = \sum_{i=1}^{4}\tilde{V}_i, \tag{4.14a}
$$

where

$$
\begin{aligned}
\tilde{V}_1 &= \tau\sum_{\ell=1}^{3} b_\ell\mathcal{H}(\xi_u^{n,\ell-1}, \tilde{v}_\ell), &\text{(4.14b)}\\
\tilde{V}_2 &= \tau\sum_{\ell=1}^{3}\mathcal{L}(\tilde{\mathbb{R}}_\ell\xi_q^n, \tilde{v}_\ell), &\text{(4.14c)}\\
\tilde{V}_3 &= \|\mathbb{S}_2\xi_u^n\|^2, &\text{(4.14d)}\\
\tilde{V}_4 &= \sum_{\ell=1}^{3} b_\ell(\mathbb{R}_\ell\eta_u^n + \delta_{3\ell}\zeta^n, \tilde{v}_\ell). &\text{(4.14e)}
\end{aligned}
$$

4.3 Energy estimate

Before proceeding with the energy estimate, we present the following important relationship

$$
\|(\xi_u)_x\| + \sqrt{\mu h^{-1}}[\![\xi_u]\!] \le \sqrt{\frac{C_\star}{d}}(\|\xi_q\| + \|\eta_q\|), \tag{4.15}
$$

which can be derived similarly as in Lemma 2.3, and hence

$$|\mathcal{H}(\xi_u, v)| \le C\frac{c}{\sqrt{d}}(\|\xi_q\| + h^{k+1})\|v\|, \tag{4.16}$$

for any $v \in V_h$.

In what follows, we give the estimate of $\tilde{V}_i$ for $i = 1, 2, 3, 4$. We will first consider the case $\vartheta > \frac{1}{2}$. By (2.9) we have

$$\tilde{V}_1 = -(\vartheta - \frac{1}{2})c\tau\left([\![\xi_u^{n,1}]\!]^2 + [\![\xi_u^{n,2}]\!]^2\right) \le 0. \tag{4.17}$$

Owing to (2.10) and the relationship (4.13b), we get

$$\tilde{V}_2 = -\tau\int_\Omega \boldsymbol{\xi}_{\boldsymbol{q}}^{n\top}\mathbb{A}_1\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x + \tau\int_\Omega \boldsymbol{\eta}_{\boldsymbol{q}}^{n\top}\mathbb{A}_1\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x, \tag{4.18}$$

where $\boldsymbol{\xi}_{\boldsymbol{q}}^n = (\xi_q^{n,1}, \xi_q^{n,2})^\top$, $\boldsymbol{\eta}_{\boldsymbol{q}}^n = (\eta_q^{n,1}, \eta_q^{n,2})^\top$ and $\mathbb{A}_1$ is defined in (3.13).

To estimate $\tilde{V}_3$, we notice that

$$\begin{aligned}(\mathbb{S}_2\xi_u^n, v) =& \frac{\tau}{2}\mathcal{H}(\mathbb{S}_1\xi_u^n, v) + \tau\mathcal{L}(\tilde{\mathbb{S}}_1\xi_q^n, v) + (\mathbb{S}_2\eta_u^n + \zeta^n, v)\\ =& \omega_1(\mathbb{R}_1\xi_u^n, v) + \omega_2(\mathbb{R}_2\xi_u^n, v) + \frac{\tau}{2}\mathcal{H}(\mathbb{S}_1\xi_u^n, v) - \omega_2\tau\mathcal{H}(\xi_u^{n,1}, v)\\ &+ (\mathbb{S}_2\eta_u^n - \omega_1\mathbb{R}_1\eta_u^n - \omega_2\mathbb{R}_2\eta_u^n + \zeta^n, v),\end{aligned} \tag{4.19}$$

where ω_1 and ω_2 are the same as before. Thus taking $v = \mathbb{S}_2\xi_u^n$ in (4.19) we get

$$\|\mathbb{S}_2\xi_u^n\| \le |\omega_1|\|\mathbb{R}_1\xi_u^n\| + |\omega_2|\|\mathbb{R}_2\xi_u^n\| + \tilde{V}, \tag{4.20}$$

where

$$\tilde{V} = C\frac{c}{\sqrt{d}}\tau(\|\xi_q^{n,1}\| + \|\xi_q^{n,2}\| + h^{k+1}) + C(h^{k+1}\tau + \tau^3).$$

Here (4.16) and (4.11), (4.7) are used. Along the same line as the estimate of $\|\mathbb{S}_2 u^n\|$ in Subsection 3.1.2, we can derive

$$\|\mathbb{S}_2\xi_u^n\|^2 \le \|\mathbb{R}_1\xi_u^n\|^2 + \sigma_0\|\mathbb{R}_2\xi_u^n\|^2 + 4\tilde{V}^2, \tag{4.21}$$

where σ_0 is taken as $\frac{8}{9}$ as before. Taking $v = \mathbb{R}_2\xi_u^n$ in (4.13a) for $\ell = 2$ yields

$$\|\mathbb{R}_2\xi_u^n\|^2 = \tau\mathcal{H}(\xi_u^{n,1}, \mathbb{R}_2\xi_u^n) + \tau\mathcal{L}(\tilde{\mathbb{R}}_2\xi_q^n, \mathbb{R}_2\xi_u^n) + (\mathbb{R}_2\eta_u^n, \mathbb{R}_2\xi_u^n).$$

Applying (4.16) for the first term, using (2.10) and (4.13b) for the second term and the Cauchy-Schwarz inequality for the last term, we get

$$\begin{aligned}\|\mathbb{R}_2\xi_u^n\|^2 \le& C\frac{c}{\sqrt{d}}\tau(\|\xi_q^{n,1}\| + h^{k+1})\|\mathbb{R}_2\xi_u^n\|\\ &- \tau\int_\Omega \boldsymbol{\xi}_{\boldsymbol{q}}^{n\top}\mathbb{A}_2\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x + \tau\int_\Omega \boldsymbol{\eta}_{\boldsymbol{q}}^{n\top}\mathbb{A}_2\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x + \|\mathbb{R}_2\eta_u^n\|\|\mathbb{R}_2\xi_u^n\|\\ \le& \varepsilon\|\mathbb{R}_2\xi_u^n\|^2 + \frac{Cc^2\tau}{\varepsilon d}\tau\|\xi_q^{n,1}\|^2 + Ch^{2k+2}\tau\\ &- \tau\int_\Omega \boldsymbol{\xi}_{\boldsymbol{q}}^{n\top}\mathbb{A}_2\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x + \tau\int_\Omega \boldsymbol{\eta}_{\boldsymbol{q}}^{n\top}\mathbb{A}_2\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x,\end{aligned} \tag{4.22}$$

for arbitrary $\varepsilon > 0$, where $\mathbb{A}_2$ is defined in (3.23). Taking $\varepsilon = 1 - \sigma_0 = \frac{1}{9}$ leads to

$$\tilde{V}_3 = \|\mathbb{S}_2 \xi_u^n\|^2 \le \|\mathbb{R}_1 \xi_u^n\|^2 + \frac{9Cc^2\tau}{d}\tau\|\xi_q^{n,1}\|^2 + Ch^{2k+2}\tau - \tau\int_\Omega \boldsymbol{\xi}_{\boldsymbol{q}}^{n\top}\mathbb{A}_2\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x + \tau\int_\Omega \boldsymbol{\eta}_{\boldsymbol{q}}^{n\top}\mathbb{A}_2\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x + 4\tilde{V}^2 \tag{4.23}$$

Finally, using the Cauchy-Schwarz inequality, (4.11), (4.7) and Young's inequality directly leads to

$$\tilde{V}_4 \le \tau(\|\xi_u^{n,1}\|^2 + \|\xi_u^{n,2}\|^2) + C(h^{2k+2}\tau + \tau^5). \tag{4.24}$$

The estimate of $\xi_u^{n,1}, \xi_u^{n,2}$ are presented in the following lemma, whose proof will be given in the Appendix.

Lemma 4.2. *Under the condition of Theorem 3.1, we have*

$$\|\xi_u^{n,\ell}\|^2 \le C\|\xi_u^n\|^2 + Ch^{2k+2}\tau, \quad \textit{for} \quad \ell = 1, 2, \tag{4.25}$$

where the bounding constant C is independent of h and τ.

As a result

$$\tilde{V}_4 \le C\tau\|\xi_u^n\|^2 + C(h^{2k+2}\tau + \tau^5). \tag{4.26}$$

Combining (4.17), (4.18), (4.23) and (4.26), we get

$$\begin{aligned}
\|\xi_u^{n+1}\|^2 - \|\xi_u^n\|^2 \le & -\tau\int_\Omega \boldsymbol{\xi}_{\boldsymbol{q}}^{n\top}(\mathbb{A}_1 + \mathbb{A}_2)\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x + \tau\int_\Omega \boldsymbol{\eta}_{\boldsymbol{q}}^{n\top}(\mathbb{A}_1 + \mathbb{A}_2)\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x \\
& + \frac{Cc^2\tau}{d}\tau\left(\|\xi_q^{n,1}\|^2 + \|\xi_q^{n,2}\|^2\right) + C\tau\|\xi_u^n\|^2 + C(h^{2k+2}\tau + \tau^5) \\
\le & \, C\tau\|\xi_u^n\|^2 + C(h^{2k+2}\tau + \tau^5) - \tau\int_\Omega \boldsymbol{\xi}_{\boldsymbol{q}}^{n\top}(\mathbb{A}_1 + \mathbb{A}_2)\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x \\
& + \left(\frac{Cc^2\tau}{d} + \varepsilon\right)\tau\int_\Omega \boldsymbol{\xi}_{\boldsymbol{q}}^{n\top}\boldsymbol{\xi}_{\boldsymbol{q}}^n \mathrm{d}x,
\end{aligned} \tag{4.27}$$

for arbitrary $\varepsilon > 0$. Since $\mathbb{A}_1 + \mathbb{A}_2 - \gamma\mathbb{I}$ is positive definite for $\gamma \in [\gamma_1, \gamma_2]$, taking ε small enough and letting $\tau \le \tau_0$ such that $\frac{Cc^2}{d}\tau + \varepsilon \le \gamma$, by the discrete Gronwall inequality, and noting that $\|\xi_u^0\| \le Ch^{k+1}$ (refer to [7]), we obtain

$$\|\xi_u^n\| \le C(h^{k+1} + \tau^2). \tag{4.28}$$

Remark 4.1. *In the case $\vartheta < \frac{1}{2}$, $\tilde{V}_1 \le 0$ does not hold. But we can estimate $\tilde{V}_1$ similarly as the estimate for V_1 in (3.28). According to (4.16), $\tilde{V}_1$ can be controlled by the stability terms provided by $\tilde{V}_2$, so we will get the same results as that for $\vartheta > \frac{1}{2}$.*

Owing to (4.28), (4.11) and the triangle inequality, we can obtain the final error estimate, which is summarized in the following theorem.

Theorem 4.1. *Let U be the exact solution of problem (2.1) satisfying the smoothness assumption (4.3) and let u be the numerical solution of scheme (3.1). Let $\vartheta = \theta \neq \frac{1}{2}$, there exists a positive constant τ_0 which is independent of the mesh size h, such that if $\tau \leq \tau_0$ then*

$$\max_{n\tau \leq T} \|U(t^n) - u^n\| \leq C(h^{k+1} + \tau^2), \tag{4.29}$$

where T is the final computing time and the bounding constant $C > 0$ is independent of n, h and τ.

Remark 4.2. *The optimal error estimate is not easy to get when $\vartheta \neq \theta$. In this situation,*

$$\mathcal{H}(\eta_u, v) = -c \sum_{j=1}^{N} (\eta_u^{(\vartheta)})_{j-\frac{1}{2}} [\![v]\!]_{j-\frac{1}{2}} \neq 0,$$

if we adopt the same projections $P_\theta U$ and $P_{\tilde{\theta}} Q$ as above. As a result, there will be some extra terms

$$W(v) = -\frac{\tau}{2}\mathcal{H}(\mathbb{S}_1 \eta_u^n, v) + \omega_2 \tau \mathcal{H}(\eta_u^{n,1}, v)$$

on the right hand side of (4.19). Even if we use the important relationship (4.15), we would still be unable to get the expected estimate to $W(\mathbb{S}_2 \xi_u^n)$, due to the loss of stability for terms like $\|\xi_q^{n+1}\|$. We could use the inverse inequality and get

$$W(\mathbb{S}_2 \xi_u^n) \leq C h^{k+1/2} \tau [\![\mathbb{S}_2 \xi_u^n]\!] \leq C h^k \tau \|\mathbb{S}_2 \xi_u^n\|,$$

which cannot lead to the same estimate to $\|\mathbb{S}_2 \xi_u^n\|$ as (4.20), but only to the sub-optimal error estimate $\mathcal{O}(h^k + \tau^2)$. However, numerical experiments do indicate optimal convergence rates in this case. In future work we will try to find different techniques to obtain optimal error estimates in this case.

5 Numerical experiments

We will present numerical experiments to illustrate the stability and error estimates of the proposed schemes, for different parameters ϑ, θ in the numerical fluxes. In all the following numerical experiments, piecewise polynomials of degree 1 and 2 are adopted respectively with the second order and third order IMEX SSP schemes, such that the orders of accuracy match in space and time if $\tau = \mathcal{O}(h)$.

To test the stability of the schemes, we consider problem (2.1) defined in $[-\pi, \pi]$ with the exact solution $U(x,t) = e^{-dt}\sin(x - ct)$. Different pairs of parameters $(\vartheta, \theta) = (\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{4}, \frac{3}{4})$, $(1, 1)$, $(\frac{5}{4}, \frac{5}{4})$, $(\frac{3}{2}, \frac{3}{2})$ and $(\vartheta, \theta) = (\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{2}, 1)$, $(\frac{3}{4}, \frac{1}{4})$, $(1, \frac{3}{2})$, $(\frac{5}{4}, \frac{3}{4})$ are tested on uniform meshes, with mesh size $h = 2\pi/N$, where N is the number of cells. Somewhat surprisingly, we find that the maximum time step τ_0 to ensure the stability of the schemes (in the sense that the L^2-norm decreases with time) are the same for different pairs of parameters. Table 1 lists the maximum time step τ_0. In the test, we take $N = 1280$. The final computing time is $T = 5000$. The result shows that $\tau_0 \approx \varpi d/c^2$ for some constant ϖ, and τ_0 is independent of h, because if we take $N = 640$, we can get the same results. For the second order scheme (3.1), we verify the stability for $\gamma \in [\gamma_1, \gamma_2]$, it seems that the ratio ϖ is larger for larger γ, we list the results for $\gamma = 0.28, 1-\frac{\sqrt{2}}{2}, 0.38$ as examples.

Table 1: The maximum time step τ_0 to ensure that the L^2-norm decreases with time for the IMEX-LDG(1,2) scheme (3.1) and the IMEX-LDG(2,3) scheme (3.2).

scheme	$d = 0.01$			$c = 0.1$			ϖ
	$c = 0.05$	$c = 0.1$	$c = 0.2$	$d = 0.01$	$d = 0.02$	$d = 0.04$	
(3.1), $\gamma = 0.28$	5.034	1.258	0.313	1.258	2.517	5.064	1.258
(3.1), $\gamma = 1-\frac{\sqrt{2}}{2}$	5.540	1.385	0.346	1.385	2.770	5.540	1.385
(3.1), $\gamma = 0.38$	7.402	1.848	0.461	1.848	3.701	7.399	1.848
(3.2)	2.632	0.657	0.164	0.657	1.316	2.632	0.657

To verify the error accuracy of the schemes, we first test the model equation (2.1) with $c = d = 1$. The computing time is $T = 1$ and uniform meshes are adopted. In Tables 2-5, we list the L^2-norm errors and orders of accuracy for the IMEX-LDG(1,2) and IMEX-LDG(2,3) schemes, for different pairs of parameters (ϑ, θ) $(\theta \neq \frac{1}{2})$. Optimal orders of accuracy can be observed from these tables. For the IMEX-LDG(1,2) scheme, we only list the results for $\gamma = 1-\frac{\sqrt{2}}{2}$ to save space, the orders of accuracy for other γ are almost the same, but the errors will be a little larger for larger γ.

Table 2: L^2-norm errors and orders of accuracy for the IMEX-LDG(1,2) scheme. $\vartheta = \theta$. $\tau = h$.

N	$\theta = \frac{1}{4}$		$\theta = \frac{3}{4}$		$\theta = 1$		$\theta = \frac{5}{4}$		$\theta = \frac{3}{2}$	
	error	order	error	order	error	order	error	order	error	order
40	4.99E-03	-	5.10E-03	-	4.89E-03	-	4.85E-03	-	4.83E-03	-
80	1.26E-03	1.99	1.27E-03	2.00	1.22E-03	2.00	1.21E-03	2.00	1.21E-03	2.00
160	3.16E-04	2.00	3.17E-04	2.00	3.06E-04	2.00	3.03E-04	2.00	3.03E-04	2.00
320	7.95E-05	1.99	7.97E-05	1.99	7.68E-05	1.99	7.63E-05	1.99	7.61E-05	1.99
640	1.98E-05	2.00	1.99E-05	2.00	1.92E-05	2.00	1.90E-05	2.00	1.90E-05	2.00

Table 3: L^2-norm errors and orders of accuracy for the IMEX-LDG(1,2) scheme. $\vartheta \neq \theta$. $\tau = 0.75h$.

N	$(\vartheta,\theta)=(\frac{1}{4},\frac{3}{4})$		$(\vartheta,\theta)=(\frac{1}{2},1)$		$(\vartheta,\theta)=(\frac{3}{4},\frac{1}{4})$		$(\vartheta,\theta)=(1,\frac{3}{2})$		$(\vartheta,\theta)=(\frac{5}{4},\frac{3}{4})$	
	error	order	error	order	error	order	error	order	error	order
40	2.88E-03	-	2.85E-03	-	2.75E-03	-	2.75E-03	-	1.39E-02	-
80	8.78E-04	1.71	8.43E-04	1.76	8.19E-04	1.75	7.38E-04	1.90	2.60E-03	2.42
160	2.34E-04	1.91	2.06E-04	2.04	2.00E-04	2.03	1.82E-04	2.02	5.09E-04	2.35
320	5.91E-05	1.99	5.02E-05	2.03	4.90E-05	2.03	4.49E-05	2.02	1.12E-04	2.18
640	1.40E-05	2.08	1.21E-05	2.05	1.18E-05	2.05	1.11E-05	2.02	2.52E-05	2.16

Table 4: L^2-norm errors and orders of accuracy for the IMEX-LDG(2,3) scheme. $\vartheta = \theta$. $\tau = h$.

N	$\theta=\frac{1}{4}$		$\theta=\frac{3}{4}$		$\theta=1$		$\theta=\frac{5}{4}$		$\theta=\frac{3}{2}$	
	error	order	error	order	error	order	error	order	error	order
40	1.86E-04	-	1.85E-04	-	1.86E-04	-	1.86E-04	-	1.86E-04	-
80	2.28E-05	3.02	2.28E-05	3.02	2.28E-05	3.02	2.29E-05	3.02	2.29E-05	3.02
160	2.85E-06	3.00	2.85E-06	3.00	2.85E-06	3.00	2.86E-06	3.00	2.86E-06	3.00
320	3.58E-07	2.99	3.58E-07	2.99	3.58E-07	2.99	3.58E-07	3.00	3.59E-07	3.00
640	4.45E-08	3.01	4.45E-08	3.01	4.46E-08	3.01	4.46E-08	3.01	4.47E-08	3.01

It is worth pointing out that in the special case $\theta = \frac{1}{2}$, if $\vartheta = \theta$ then the stability is almost the same as other parameter pairs, according to Remark 3.2. But if $\vartheta \neq \theta$, the stability results are very interesting. Our experiments indicate that the schemes are not stable if $\vartheta < \frac{1}{2}$, and if $\vartheta > \frac{1}{2}$ then the schemes are stable under the constraint $\tau \leq \lambda h$ for some constant λ, which is the standard CFL condition of RKDG methods for solving hyperbolic problems [37, 38]. The L^2-norm errors and orders of accuracy in the special case $\theta = \frac{1}{2}$ are shown in Table 6, from which we observe optimal accuracy except for the second order scheme with $k = 1$ in the case $\vartheta = \theta = \frac{1}{2}$, where sub-optimal accuracy is observed, which coincides with the conclusion given in [34] that only k-th order accuracy can be obtained for odd k if central numerical flux is adopted in the LDG scheme. Notice that in this test, we require smaller time step to ensure the stability of the schemes in the

Table 5: L^2-norm errors and orders of accuracy for the IMEX-LDG(2,3) scheme. $\vartheta \neq \theta$. $\tau = h$.

N	$(\vartheta,\theta)=(\frac{1}{4},\frac{3}{4})$		$(\vartheta,\theta)=(\frac{1}{2},1)$		$(\vartheta,\theta)=(\frac{3}{4},\frac{1}{4})$		$(\vartheta,\theta)=(1,\frac{3}{2})$		$(\vartheta,\theta)=(\frac{5}{4},\frac{3}{4})$	
	error	order	error	order	error	order	error	order	error	order
40	1.86E-04	-	1.86E-04	-	1.86E-04	-	1.87E-04	-	1.86E-04	-
80	2.29E-05	3.02	2.29E-05	3.02	2.29E-05	3.02	2.30E-05	3.02	2.29E-05	3.02
160	2.86E-06	3.00	2.86E-06	3.00	2.86E-06	3.00	2.87E-06	3.00	2.85E-06	3.00
320	3.60E-07	2.99	3.60E-07	2.99	3.60E-07	2.99	3.62E-07	2.99	3.60E-07	2.99
640	4.47E-08	3.01	4.48E-08	3.01	4.47E-08	3.01	4.50E-08	3.01	4.47E-08	3.01

case $\vartheta \neq \theta$ for $\theta = \frac{1}{2}$, while larger time step can be taken when $\vartheta = \theta = \frac{1}{2}$.

Next we consider the viscous Burgers' equation

$$U_t + UU_x = dU_{xx} + g(x,t), \tag{5.1}$$

in $[-\pi,\pi]$, where $g(x,t) = \frac{1}{2}e^{-2dt}\sin(2x)$. The exact solution is

$$U(x,t) = e^{-dt}\sin(x). \tag{5.2}$$

The numerical flux for the convection term is taken as $\frac{1}{2}[\vartheta(u^-)^2 + \tilde{\vartheta}(u^+)^2]$, and the numerical flux for the diffusion term is the GANF with parameter θ. For $d = 1, 0.2, 0.05$, the L^2-norm errors and orders of accuracy for IMEX-LDG(1,2) and IMEX-LDG(2,3) schemes with different pairs of parameter (ϑ,θ) are listed in Table 7 and Table 8, respectively. In this test, the computing time is $T = 1$ and uniform meshes are adopted. Optimal orders of accuracy are observed except for the case $(\vartheta,\theta) = (\frac{1}{2},\frac{1}{2})$ for the IMEX-LDG(1,2) scheme.

6 Concluding remarks

The LDG methods with generalized alternating numerical fluxes coupled with two specific IMEX SSP time discretizations for convection-diffusion problems have been shown to be unconditionally stable, in the sense that the time step is only required to be upper bounded by a positive constant which is independent of the mesh size. The key is the important relationship established between the gradient and interface jump of the numerical solution with the independent numerical solution of the gradient. The energy equations have been built up following those constructed for the explicit RKDG methods. By the aid of the generalized Gauss-Radau projection, we have also obtained optimal error estimates for the

Table 6: L^2-norm errors and orders of accuracy for IMEX-LDG(1,2) and IMEX-LDG(2,3) schemes. $\theta = \frac{1}{2}$. The time step for each column is $\tau = h, 0.1h, 0.1h, 0.1h$ respectively.

scheme	N	$\vartheta = \frac{1}{2}$		$\vartheta = \frac{3}{4}$		$\vartheta = 1$		$\vartheta = \frac{5}{4}$	
		error	order	error	order	error	order	error	order
IMEX-LDG(1,2)	40	9.23E-03	-	2.77E-03	-	2.04E-03	-	1.82E-03	-
	80	3.95E-03	1.22	8.02E-04	1.79	5.47E-04	1.90	4.77E-04	1.93
	160	1.88E-03	1.07	2.18E-04	1.88	1.42E-04	1.95	1.22E-04	1.97
	320	9.31E-04	1.02	5.70E-05	1.94	3.62E-05	1.97	3.09E-05	1.98
	640	4.64E-04	1.00	1.46E-05	1.97	9.13E-06	1.99	7.73E-06	2.00
IMEX-LDG(2,3)	40	1.85E-04	-	8.29E-06	-	8.35E-06	-	8.40E-06	-
	80	2.28E-05	3.02	1.03E-06	3.01	1.03E-06	3.01	1.04E-06	3.02
	160	2.85E-06	3.00	1.29E-07	3.00	1.31E-07	2.98	1.32E-07	2.97
	320	3.58E-07	2.99	1.61E-08	3.01	1.62E-08	3.02	1.62E-08	3.03
	640	4.45E-08	3.01	2.02E-09	3.00	2.04E-09	2.99	2.07E-09	2.97

proposed schemes. Numerical experiments have verified the theoretical results as well as illustrated the effect of different choices of the numerical fluxes. The results of this paper can also be extended to multi-dimensional and nonlinear convection-diffusion problems, which will be left for our future work.

7 Appendix

Proof of (4.7). Firstly, by Taylor's expansion

$$U^{n+1} = U^n + \tau U_t^n + \frac{\tau^2}{2} U_{tt}^n + \frac{\tau^3}{6} U_{ttt}(t_\xi),$$

where $t_\xi \in (t^n, t^{n+1})$ and we omit the argument x for simplicity. Secondly, notice that

$$\begin{aligned} U^{n,1} &= U^n + \gamma\tau\sqrt{d} Q_x^{n,1}, \\ U^{n,2} &= U^n + \tau U_t^{n,1} - 2\gamma\tau\sqrt{d} Q_x^{n,1} + \gamma\tau\sqrt{d} Q_x^{n,2}, \end{aligned}$$

Table 7: Burgers' equation: L^2-norm errors and orders of accuracy for IMEX-LDG(1,2) scheme. $\tau = 0.75h, 0.25h, 0.1h$ for $d = 1, 0.2, 0.05$ respectively.

d	N	$(\vartheta,\theta)=(\frac{1}{2},\frac{1}{2})$		$(\vartheta,\theta)=(\frac{1}{4},\frac{3}{4})$		$(\vartheta,\theta)=(\frac{3}{4},\frac{3}{4})$		$(\vartheta,\theta)=(\frac{3}{4},\frac{5}{4})$		$(\vartheta,\theta)=(\frac{5}{4},1)$	
		error	order	error	order	error	order	error	order	error	order
1	40	1.90E-02	-	1.89E-03	-	1.78E-03	-	1.08E-03	-	1.23E-03	-
	80	9.49E-03	1.00	5.99E-04	1.66	4.51E-04	1.98	3.00E-04	1.84	3.23E-04	1.93
	160	4.74E-03	1.00	1.47E-04	2.03	1.12E-04	2.00	7.33E-05	2.04	7.91E-05	2.03
	320	2.37E-03	1.00	3.59E-05	2.03	2.81E-05	2.00	1.80E-05	2.03	1.95E-05	2.02
	640	1.19E-03	1.00	8.60E-06	2.06	7.01E-06	2.00	4.40E-06	2.03	4.82E-06	2.02
0.2	40	4.37E-02	-	4.11E-03	-	3.67E-03	-	1.76E-03	-	2.26E-03	-
	80	2.18E-02	1.00	9.96E-04	2.04	9.25E-04	1.99	4.64E-04	1.92	5.90E-04	1.94
	160	1.09E-02	1.00	2.49E-04	2.00	2.32E-04	2.00	1.17E-04	1.99	1.45E-04	2.03
	320	5.46E-03	1.00	6.16E-05	2.01	5.79E-05	2.00	2.90E-05	2.01	3.55E-05	2.03
	640	2.73E-03	1.00	1.49E-05	2.05	1.45E-05	2.00	7.05E-06	2.04	8.69E-06	2.03
0.05	40	5.49E-02	-	3.38E-02	-	4.00E-03	-	2.06E-03	-	3.17E-03	-
	80	2.74E-02	1.00	1.87E-03	4.17	1.07E-03	1.90	5.15E-04	2.00	6.60E-04	2.26
	160	1.37E-02	1.00	3.14E-04	2.58	2.69E-04	2.00	1.28E-04	2.00	1.61E-04	2.03
	320	6.86E-03	1.00	7.00E-05	2.16	6.73E-05	2.00	3.22E-05	2.00	3.99E-05	2.01
	640	3.43E-03	1.00	1.70E-05	2.04	1.68E-05	2.00	8.05E-06	2.00	1.00E-05	1.99

and then

$$\begin{aligned}
&U^n - \frac{\tau}{2}c(U_x^{n,1} + U_x^{n,2}) + \frac{\tau}{2}\sqrt{d}(Q_x^{n,1} + Q_x^{n,2}) \\
=&U^n + \frac{\tau}{2}U_t^{n,1} + \frac{\tau}{2}U_t^{n,2} \\
=&U^n + \tau U_t^n + \frac{\tau^2}{2}U_{tt}^{n,1} + \frac{\gamma\tau^2}{2}(Q_{xt}^{n,2} - Q_{xt}^{n,1}) \\
=&U^n + \tau U_t^n + \frac{\tau^2}{2}U_{tt}^{n} + \frac{\gamma\tau^3}{2}\sqrt{d}Q_{xtt}^{n,1} + \frac{\gamma\tau^2}{2}\sqrt{d}(Q_{xt}^{n,2} - Q_{xt}^{n,1}).
\end{aligned}$$

As a result

$$\begin{aligned}
\zeta^n =& \frac{\tau^3}{6}U_{ttt}(x,t_\xi) - \frac{\gamma\tau^3}{2}\sqrt{d}Q_{xtt}^{n,1} - \frac{\gamma\tau^2}{2}\sqrt{d}(Q_{xt}^{n,2} - Q_{xt}^{n,1}) \\
=& \frac{\tau^3}{6}U_{ttt}(x,t_\xi) - \frac{\gamma\tau^3}{2}(U_{ttt}^{n,1} + cU_{xtt}^{n,1}) - \frac{\gamma\tau^2}{2}[(U^{n,2} - U^{n,1})_{tt} + c(U^{n,2} - U^{n,1})_{xt}).
\end{aligned}$$

Table 8: Burgers' equation: L^2-norm errors and orders of accuracy for IMEX-LDG(2,3) scheme. $\tau = h, 0.5h, 0.1h$ for $d = 1, 0.2, 0.05$ respectively.

d	N	$(\vartheta,\theta)=(\frac{1}{2},\frac{1}{2})$		$(\vartheta,\theta)=(\frac{1}{4},\frac{3}{4})$		$(\vartheta,\theta)=(\frac{3}{4},\frac{3}{4})$		$(\vartheta,\theta)=(\frac{3}{4},\frac{5}{4})$		$(\vartheta,\theta)=(\frac{5}{4},1)$	
		error	order	error	order	error	order	error	order	error	order
1	40	1.08E-04	-	1.08E-04	-	1.08E-04	-	1.09E-04	-	1.09E-04	-
	80	1.33E-05	3.02	1.33E-05	3.02	1.33E-05	3.02	1.34E-05	3.02	1.34E-05	3.02
	160	1.68E-06	2.98	1.68E-06	2.98	1.68E-06	2.98	1.70E-06	2.98	1.69E-06	2.98
	320	2.14E-07	2.98	2.14E-07	2.98	2.14E-07	2.98	2.15E-07	2.98	2.14E-07	2.98
	640	2.65E-08	3.01	2.66E-08	3.01	2.66E-08	3.01	2.68E-08	3.01	2.66E-08	3.01
0.2	40	1.84E-05	-	2.41E-05	-	2.13E-05	-	3.86E-05	-	2.75E-05	-
	80	2.30E-06	3.00	2.83E-06	3.09	2.65E-06	3.00	4.74E-06	3.03	3.45E-06	2.99
	160	2.87E-07	3.00	3.70E-07	2.94	3.32E-07	3.00	6.07E-07	2.97	4.36E-07	2.99
	320	3.59E-08	3.00	4.51E-08	3.04	4.15E-08	3.00	7.47E-08	3.02	5.49E-08	2.99
	640	4.49E-09	3.00	5.49E-09	3.04	5.18E-09	3.00	9.20E-09	3.02	6.87E-09	3.00
0.05	40	2.13E-05	-	2.48E-05	-	2.47E-05	-	4.20E-05	-	3.34E-05	-
	80	2.66E-06	3.00	3.08E-06	3.01	3.08E-06	3.00	5.32E-06	2.98	4.09E-06	3.03
	160	3.33E-07	3.00	3.85E-07	3.00	3.84E-07	3.00	6.67E-07	3.00	5.10E-07	3.01
	320	4.16E-08	3.00	4.81E-08	3.00	4.80E-08	3.00	8.34E-08	3.00	6.36E-08	3.00
	640	5.20E-09	3.00	6.03E-09	2.99	6.01E-09	3.00	1.04E-08	3.00	7.95E-09	3.00

Since

$$\begin{aligned} U^{n,2} - U^{n,1} &= \tau U_t^{n,1} - 3\gamma\tau\sqrt{d}Q_x^{n,1} + \gamma\tau\sqrt{d}Q_x^{n,2} \\ &= \tau U_t^{n,1} - 3\gamma\tau(U_t^{n,1} + cU_x^{n,1}) + \gamma\tau(U_t^{n,2} + cU_x^{n,2}). \end{aligned}$$

We get $\|\zeta^n\| = \mathcal{O}(\tau^3)$ if $U_{ttt}, U_{xtt}, U_{xxt} \in L^\infty(0,T;L^2)$.

Proof of Lemma 4.2. By taking $v = 2\xi_u^{n,1}$ in (4.12a) and $v = 2\xi_u^{n,2}$ in (4.12b), we get from (2.10) and (4.13b) that

$$\|\xi_u^{n,1}\|^2 + \|\xi_u^{n,1} - \xi_u^n\|^2 - \|\xi_u^n\|^2 = -2\gamma\tau\|\xi_q^{n,1}\|^2 + 2(\eta_u^{n,1} - \eta_u^n, \xi_u^{n,1}), \tag{7.1}$$

$$\begin{aligned} \|\xi_u^{n,2}\|^2 + \|\xi_u^{n,2} - \xi_u^n\|^2 - \|\xi_u^n\|^2 =& 2\tau\mathcal{H}(\xi_u^{n,1}, \xi_u^{n,2}) + 2(\eta_u^{n,2} - \eta_u^n, \xi_u^{n,2}) \\ &- 2(1-2\gamma)\tau(\xi_q^{n,1}, \xi_q^{n,2}) - 2\gamma\tau\|\xi_q^{n,2}\|^2 \\ &+ 2(1-2\gamma)\tau(\eta_q^{n,1}, \xi_q^{n,2}) + 2\gamma\tau(\eta_q^{n,2}, \xi_q^{n,2}). \end{aligned} \tag{7.2}$$

Hence applying Young's inequality yields

$$\|\xi_u^{n,1}\|^2 \leq 2(\|\xi_u^n\|^2 - 2\gamma\tau\|\xi_q^{n,1}\|^2) + Ch^{2k+2}\tau^2. \tag{7.3}$$

Using (4.16) for the term $2\tau\mathcal{H}(\xi_u^{n,1}, \xi_u^{n,2})$, applying Cauchy-Schwarz inequality and Young's inequality for the remaining terms, we get

$$\begin{aligned}
\|\xi_u^{n,2}\|^2 \leq\, & \|\xi_u^n\|^2 + C\frac{c}{\sqrt{d}}\tau(\|\xi_q^{n,2}\| + h^{k+1})\|\xi_u^{n,1}\| + Ch^{k+1}\tau\|\xi_u^{n,2}\| \\
& - 2(1-2\gamma)\tau(\xi_q^{n,1}, \xi_q^{n,2}) - 2\gamma\tau\|\xi_q^{n,2}\|^2 + Ch^{k+1}\tau\|\xi_q^{n,2}\| \\
\leq\, & \|\xi_u^n\|^2 + \|\xi_u^{n,1}\|^2 + \frac{Cc^2\tau}{2d}\tau\|\xi_q^{n,2}\|^2 + \frac{1}{2}\|\xi_u^{n,2}\|^2 + Ch^{2k+2}\tau \\
& - 2(1-2\gamma)\tau(\xi_q^{n,1}, \xi_q^{n,2}) - (2\gamma - \varepsilon)\tau\|\xi_q^{n,2}\|^2.
\end{aligned} \tag{7.4}$$

So taking $\varepsilon = \frac{\gamma}{2}$ and letting $\frac{Cc^2\tau}{2d} \leq \frac{\gamma}{2}$, we get

$$\begin{aligned}
\|\xi_u^{n,2}\|^2 \leq\, & 6\|\xi_u^n\|^2 - 2\tau[4\gamma\|\xi_q^{n,1}\|^2 + 2(1-2\gamma)(\xi_q^{n,1}, \xi_q^{n,2}) + \gamma\|\xi_q^{n,2}\|^2] + Ch^{2k+2}\tau \\
\leq\, & 6\|\xi_u^n\|^2 + Ch^{2k+2}\tau,
\end{aligned} \tag{7.5}$$

if $\gamma \in [\gamma_1, \gamma_2]$. Thus the lemma is proved. □

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