

Local discontinuous Galerkin methods with explicit-implicit-null time discretizations for solving nonlinear diffusion problems

Haijin Wang¹, Qiang Zhang², Shiping Wang³ & Chi-Wang Shu^{4,*}

¹*School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China;*

²*Department of Mathematics, Nanjing University, Nanjing 210093, China;*

³*College of Shipbuilding Engineering, Harbin Engineering University, Harbin 150001, China;*

⁴*Division of Applied Mathematics, Brown University, Providence, RI 02912, U.S.A.*

Email: hjwang@njupt.edu.cn, qzh@nju.edu.cn, wangshiping@hrbeu.edu.cn, shu@dam.brown.edu

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Abstract In this paper we discuss the local discontinuous Galerkin methods coupled with two specific explicit-implicit-null time discretizations for solving one-dimensional nonlinear diffusion problems $U_t = (a(U)U_x)_x$. The basic idea is to add and subtract two equal terms $a_0 U_{xx}$ on the right hand side of the partial differential equation, then to treat the term $a_0 U_{xx}$ implicitly and the other terms $(a(U)U_x)_x - a_0 U_{xx}$ explicitly. We give stability analysis for the method on a simplified model by the aid of energy analysis, which gives a guidance for the choice of a_0 , i.e. $a_0 \geq \max\{a(u)\}/2$ to ensure the unconditional stability of the first order and second order schemes. The optimal error estimate is also derived for the simplified model, and numerical experiments are given to demonstrate the stability, accuracy and performance of the schemes for nonlinear diffusion equations.

Keywords local discontinuous Galerkin, explicit-implicit-null time discretization, nonlinear diffusion, stability, error estimates

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1 Introduction

Many partial differential equations (PDE) which arise in physics or engineering involve the computation of nonlinear diffusion, such as the miscible displacement in porous media [16] which is widely used in the exploration of underground water, oil, and gas, the carburizing model [6] which is derived in the chemical heat treatment in mechanical industry, the high-field model in semiconductor device simulations [7,8], and so on. It is well known that the time discretization is a very important issue for such problems containing complicated nonlinear diffusion coefficients. Explicit time marching always suffer from stringent time step restriction. Implicit time marching can overcome the constraint of small time step, however, this method becomes cumbersome if the diffusion coefficients vary in space or depend on the solution (quasi-linear or nonlinear cases), since a Newton iteration is required at each time step.

* Corresponding author

To cope with both the shortcomings of the explicit and implicit time marching methods, we notice that the implicit time discretization can be actually very efficient for solving diffusion equations with constant coefficients, since the inverse matrix is only needed to be solved once. This observation inspire us to add and subtract a term with constant diffusion coefficient $a_0 U_{xx}$ on the right hand side of the considered PDE

$$U_t = (a(U)U_x)_x, \quad x \in \Omega = [a, b], t \in (0, T] \quad (1.1)$$

where $a(U) \geq 0$ and $a(U)$ is bounded and smooth, and then apply the implicit-explicit (IMEX) time marching methods [2] to the equivalent PDE

$$U_t = \underbrace{(a(U)U_x)_x - a_0 U_{xx}}_{T_1} + \underbrace{a_0 U_{xx}}_{T_2}. \quad (1.2)$$

Namely, we treat the damping term T_2 implicitly and the remaining term T_1 explicitly.

Such idea had been adopted by Douglas and Dupont [14] to assure the stability for a nonlinear diffusion equation on a rectangle. The similar idea has also been adopted, for example, by Smereka [22] in the context of flow by mean curvature and surface diffusion, by Jin and Filbet [17] in the context of the Boltzmann equation of rarefied gas dynamics when the Knudsen number is very small, in the context of hyperbolic systems with diffusive relaxation [4], and for the solution of PDEs on surfaces [21]. In a recent study, Duchemin and Eggers [15] proposed to call this method as explicit-implicit-null (EIN) method.

In this paper, we exploit EIN method coupled with local discontinuous Galerkin (LDG) spatial discretization to solve the nonlinear diffusion equation (1.1). The LDG method was introduced by Cockburn and Shu in [12] for solving convection diffusion equations, motivated by the work of Bassi and Rebay [3] for the compressible Navier-Stokes equations. The idea of the LDG method is to rewrite the equations with higher order derivatives into an equivalent first order system, then apply the DG method [11] to the system, so the LDG scheme shares the advantages of the DG methods. It can easily handle meshes with hanging nodes, elements of general shapes and local spaces of different types, thus it is flexible for *hp*-adaptivity. Besides, a key advantage of the LDG scheme is the local solvability, that is, the auxiliary variables approximating the derivatives of the solution can be locally eliminated [5, 12].

Two EIN time marching schemes with LDG spatial discretization (EIN-LDG) will be analyzed in the present paper. The first order scheme is a combination of forward Euler discretization and backward Euler discretization for the explicit part and the implicit part, respectively, which was considered in our previous work [23, 24] for solving one-dimensional convection-diffusion problem and time-dependent fourth order problem. The second order scheme to be considered in this paper is different from the one we used in [23, 24], the new scheme is a modification of the second order scheme proposed by Cooper and Sayfy [13]. By the aid of the energy analysis, we show that the proposed schemes are unconditionally stable provided $a_0 \geq a/2$ for the simplified linear model $U_t = a U_{xx}$, where $a > 0$ is a constant. The optimal error estimates will also be given by energy analysis for the simplified model. We would like to point out that it is necessary to do energy analysis even for the linear model, since the spatial discretization may result in non-normal systems with a growing dimension, hence the spectral stability analysis based on scalar eigenvalues arguments may be misleading [18].

Based on the stability and error analysis for the simplified model, we propose a guidance for the choice of a_0 for the general model $U_t = (a(U)U_x)_x$, that is, $a_0 \geq \max\{a(u)\}/2$, where u is the numerical solution. It is worth pointing out that it is not necessary to scan the maximum of $a(u)$ and adjust a_0 at every time level, theoretically we can choose a_0 as a sufficiently large positive constant. However, too large a_0 may cause larger errors and may require a smaller time step from our numerical observation. So in practical computing, we adjust a_0 after certain number of time steps to alleviate numerical errors and to keep high efficiency in the meantime. We point out that the EIN-LDG schemes also work well for convection-diffusion problems with nonlinear diffusions. To verify the accuracy and performance of the proposed schemes, we present several numerical experiments, including the simulations for porous media equations and the high-field model in semiconductor device simulations.

The paper is organized as follows. In Section 2 we present the semi-discrete LDG scheme and the time-discretization methods. Sections 3 and 4 are devoted to the stability and error analysis of the EIN-LDG methods, respectively. In Section 5 we will present numerical results to verify the accuracy and the performance of the proposed schemes. The conclusion is given in Section 6.

2 The LDG scheme and time-discretization

In this section, we will present the discontinuous finite element space, the semi-discrete LDG scheme, and the implicit-explicit (IMEX) Runge-Kutta (RK) time-discretization methods.

2.1 The discontinuous finite element space

Let $\mathcal{T}_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=1}^N$ be a partition of Ω , where $x_{\frac{1}{2}} = a$ and $x_{N+\frac{1}{2}} = b$ are the two boundary endpoints. Denote the cell length as $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $j = 1, \dots, N$, and define $h = \max_j h_j$. We assume $\mathcal{T}_h$ is quasi-uniform in this paper, that is, there exists a positive constant ρ such that for all j there holds $h_j/h \geq \rho$, as h goes to zero.

Associated with this mesh, we define the discontinuous finite element space

$$V_h = \{v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j), \forall j = 1, \dots, N\}, \quad (2.1)$$

where $\mathcal{P}_k(I_j)$ denotes the space of polynomials in I_j of degree at most k . Note that functions in this space are allowed to have discontinuities across element interfaces. At each element interface point, for any piecewise function p , there are two traces along the right-hand and left-hand, denoted by p^+ and p^- , respectively. The jump is denoted by $\llbracket p \rrbracket = p^+ - p^-$.

2.2 The semi-discrete LDG scheme

We begin with equation (1.2) to define the LDG scheme. Denote by $b(U) = \sqrt{a(U)}$, by introducing $P = b(U)U_x$ and $Q = U_x$, the equation can be written as

$$U_t + (a_0 Q - b(U)P)_x = a_0 Q_x, \quad (2.2a)$$

$$P - B(U)_x = 0, \quad (2.2b)$$

$$Q - U_x = 0, \quad (2.2c)$$

where $B(U) = \int^U b(s)ds$. The semi-discrete LDG scheme is to find $u, q, p \in V_h$, such that for arbitrary $v, r, w \in V_h$ we have

$$(u_t, v) = \tilde{\mathcal{L}}(b(u)p, v) - a_0 \mathcal{L}(q, v) + a_0 \mathcal{L}(q, v), \quad (2.3a)$$

$$(q, r) = \mathcal{K}(u, r), \quad (2.3b)$$

$$(p, w) = \tilde{\mathcal{K}}(B(u), w), \quad (2.3c)$$

where

$$\mathcal{L}(q, v) = - \sum_{j=1}^N \left[(q, v_x)_j - \hat{q}_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^- + \hat{q}_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^+ \right], \quad (2.4a)$$

$$\mathcal{K}(u, r) = - \sum_{j=1}^N \left[(u, r_x)_j - \hat{u}_{j+\frac{1}{2}} r_{j+\frac{1}{2}}^- + \hat{u}_{j-\frac{1}{2}} r_{j-\frac{1}{2}}^+ \right], \quad (2.4b)$$

$$\tilde{\mathcal{L}}(b(u)p, v) = - \sum_{j=1}^N \left[(b(u)p, v_x)_j - (\widehat{b(u)p})_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^- + (\widehat{b(u)p})_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^+ \right], \quad (2.4c)$$

$$\tilde{\mathcal{K}}(B(u), w) = - \sum_{j=1}^N \left[(B(u), w_x)_j - \widehat{B(u)}_{j+\frac{1}{2}} w_{j+\frac{1}{2}}^- + \widehat{B(u)}_{j-\frac{1}{2}} w_{j-\frac{1}{2}}^+ \right]. \quad (2.4d)$$

The “hat” terms are numerical fluxes which are taken as in [12, 26, 27]

$$\hat{q} = q^+, \quad \hat{p} = p^+, \quad \hat{u} = u^-, \quad \widehat{B(u)} = B(u^-)$$

and

$$\widehat{b(u)} = \begin{cases} \llbracket B(u) \rrbracket / \llbracket u \rrbracket & \text{if } \llbracket u \rrbracket \neq 0 \\ b((u^+ + u^-)/2) & \text{otherwise} \end{cases},$$

where we omitted the subscripts $j - \frac{1}{2}$ and $j + \frac{1}{2}$. For simplicity of analysis, we consider the periodic boundary conditions, i.e., $w_{\frac{1}{2}}^- = w_{N+\frac{1}{2}}^-$ and $w_{N+\frac{1}{2}}^+ = w_{\frac{1}{2}}^+$ for $w = u, p, q$. For other boundary conditions, such as Dirichlet boundary condition problems, we refer the readers to [5, 25] for the setting of numerical fluxes.

The initial solution u^0 can be taken as any approximation of the initial condition $U(x, 0)$, for example the Gauss-Radau projection of $U(x, 0)$.

We have the following lemma which can be obtained easily by integrating by parts, so we omit the proof and refer the reader to [28].

Lemma 2.1. *For any pairs of (u_1, q_1) and (u_2, q_2) belonging to $V_h \times V_h$, we have*

$$\mathcal{L}(q_1, u_2) = -\mathcal{K}(u_2, q_1) = -(q_2, q_1), \quad (2.5)$$

and for any pairs of (u_1, p_1) and (u_2, p_2) belonging to $V_h \times V_h$, we have

$$\tilde{\mathcal{L}}(b(u_1)p_2, u_1) = -\tilde{\mathcal{K}}(B(u_1), p_2) = -(p_1, p_2). \quad (2.6)$$

We will discretize the operator $\tilde{\mathcal{L}}(b(u)p, v) - a_0\mathcal{L}(q, v)$ in (2.3a) explicitly and the other operator $a_0\mathcal{L}(q, v)$ implicitly. The fully discrete scheme will be referred to as EIN-LDG scheme in this paper. In the next subsection we will give a brief introduction of the IMEX RK time discretizations.

2.3 The IMEX RK time discretizations

For a detailed introduction to IMEX RK schemes, we refer the readers to [2] and [13]. To give a brief introduction of the scheme, let us consider the system of ordinary differential equations

$$\frac{d\mathbf{y}}{dt} = L(t, \mathbf{y}) + N(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad (2.7)$$

where $\mathbf{y} = [y_1, y_2, \dots, y_d]^\top$, $L(t, \mathbf{y})$ and $N(t, \mathbf{y})$ are derived from the spatial discretization of the two parts of the right hand side of PDEs. By applying the general s -stage IMEX RK time marching scheme, the solution of (2.7) advanced from time t^n to $t^{n+1} = t_n + \tau$ is given by:

$$\begin{aligned} \mathbf{Y}_1 &= \mathbf{y}_n, \\ \mathbf{Y}_i &= \mathbf{y}_n + \tau \sum_{j=1}^i a_{ij} L(t_n^j, \mathbf{Y}_j) + \tau \sum_{j=1}^{i-1} \hat{a}_{ij} N(t_n^j, \mathbf{Y}_j), \quad 2 \leq i \leq s+1, \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \tau \sum_{i=1}^{s+1} b_i L(t_n^i, \mathbf{Y}_i) + \tau \sum_{i=1}^{s+1} \hat{b}_i N(t_n^i, \mathbf{Y}_i), \end{aligned}$$

where τ is the time step, $\mathbf{Y}_i$ denotes the intermediate stages, $c_i = \sum_{j=1}^i a_{ij} = \sum_{j=1}^{i-1} \hat{a}_{ij}$, and $t_n^j = t_n + c_j\tau$. Denote $A = (a_{ij})$, $\hat{A} = (\hat{a}_{ij}) \in \mathbb{R}^{(s+1) \times (s+1)}$, $\mathbf{b}^\top = [b_1, \dots, b_{s+1}]$, $\hat{\mathbf{b}}^\top = [\hat{b}_1, \dots, \hat{b}_{s+1}]$ and $\mathbf{c}^\top = [0, c_2, \dots, c_{s+1}]$, then we can express the general s -stage IMEX RK scheme as the following Butcher tableau

$$\begin{array}{c|cc} \mathbf{c} & A & \hat{A} \\ \hline & \mathbf{b}^\top & \hat{\mathbf{b}}^\top \end{array} \quad (2.8)$$

In the above tableau, the pair $(A | \mathbf{b})$ determines an s -stage diagonally implicit RK method and $(\hat{A} | \hat{\mathbf{b}})$ defines an $(s + 1)$ -stage (s -stage if $\hat{b}_{s+1} = 0$) explicit RK method. The first order IMEX RK method is taking the forward Euler discretization for the explicit part and the backward Euler discretization for the implicit part, which is expressed in the Butcher tableau

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ \hline & 0 & 1 & 1 & 0 \end{array} \quad (2.9)$$

The second order IMEX RK method presented in this paper is

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 0 \\ \hline & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 0 \end{array} \quad (2.10)$$

which is a modification of the second order scheme

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu}{2} & \frac{\mu}{2} & 0 & 0 & \frac{\mu}{2} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{\mu-1}{\mu} & \frac{1}{\mu} & 0 \\ \hline & \frac{1}{2} & 0 & \frac{1}{2} & \frac{\mu-1}{\mu} & \frac{1}{\mu} & 0 \end{array} \quad (2.11)$$

given by [13], where $\mu \neq 0$. Notice that if we let $\mu = 1$, then (2.10) and (2.11) are only different in the discretization of $L(t, \mathbf{y})$ at the first intermediate stage, scheme (2.11) discretizes $L(t, \mathbf{y})$ explicitly at the first stage, while the modified scheme (2.10) discretize $L(t, \mathbf{y})$ implicitly at the first stage. Owing to the implicit discretization at the first stage, the stability of the modified scheme (2.10) is better than the original one (2.11), especially when adopting it for the convection-diffusion problems. This is why we consider the modified scheme (2.10) in this paper.

3 Stability analysis

In this section, we will present the stability analysis for the proposed EIN-LDG schemes. We would like to investigate how to choose a_0 such that the schemes are stable. For simplicity of analysis, we consider the simplified equation

$$U_t = aU_{xx}, \quad (3.1)$$

with constant diffusion coefficient $a > 0$. Adding and subtracting a term a_0U_{xx} we get

$$U_t = (a - a_0)U_{xx} + a_0U_{xx}. \quad (3.2)$$

Then the LDG scheme reads

$$(u_t, v) = (a - a_0)\mathcal{L}(q, v) + a_0\mathcal{L}(q, v), \quad (3.3a)$$

$$(q, r) = \mathcal{K}(u, r), \quad (3.3b)$$

where $\mathcal{L}$ and $\mathcal{K}$ have been defined in Section 2.

3.1 First order scheme

Now we consider the first order EIN-LDG scheme, which is the first order IMEX time discretization (2.9) coupled with (3.3), i.e.,

$$(u^{n+1}, v) = (u^n, v) + (a - a_0)\tau\mathcal{L}(q^n, v) + a_0\tau\mathcal{L}(q^{n+1}, v), \quad (3.4a)$$

$$(q^{n,\ell}, r) = \mathcal{K}(u^{n,\ell}, r), \quad \text{for } \ell = 0, 1, \quad (3.4b)$$

where $w^{n,0} = w^n$ and $w^{n,1} = w^{n+1}$ for $w = u, q$.

For the simplified linear model, if we let $a_0 = a$ then the scheme (3.4) degenerates to backward Euler scheme, which is unconditionally stable in the sense that

$$\|u^n\| \leq \|u^0\|, \quad \forall n. \quad (3.5)$$

So we only consider the case $a_0 \neq a$. We state the stability result in the following theorem.

Theorem 3.1. *If $a_0 \geq \frac{a}{2}$ and $a_0 \neq a$, then the first order EIN-LDG scheme (3.4) is unconditionally stable in the sense that*

$$\|u^n\|^2 + a_0\tau\|q^n\|^2 \leq \|u^0\|^2 + a_0\tau\|q^0\|^2. \quad (3.6)$$

Proof. Taking $v = u^{n+1}$ in (3.4a), and by the property (2.5) we have

$$\frac{1}{2}\|u^{n+1}\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 - \frac{1}{2}\|u^n\|^2 = -(a - a_0)\tau(q^n, q^{n+1}) - a_0\tau\|q^{n+1}\|^2. \quad (3.7)$$

Rearranging the terms yields

$$LHS = \frac{1}{2}\|u^{n+1}\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 - \frac{1}{2}\|u^n\|^2 + a_0\tau\|q^{n+1}\|^2 = (a_0 - a)\tau(q^n, q^{n+1}) = RHS.$$

By simple use of the Cauchy-Schwarz and the Young's inequalities we get

$$RHS \leq |a_0 - a|\tau\|q^n\|\|q^{n+1}\| \leq \frac{a_0}{2}\tau\|q^{n+1}\|^2 + \frac{(a_0 - a)^2}{2a_0}\tau\|q^n\|^2.$$

Hence, if we let $\frac{(a_0 - a)^2}{2a_0} \leq \frac{a_0}{2}$, i.e., $a_0 \geq \frac{a}{2}$, then

$$LHS \leq \frac{a_0}{2}\tau(\|q^n\|^2 + \|q^{n+1}\|^2).$$

As a result, we have

$$\frac{1}{2}\|u^{n+1}\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{a_0}{2}\tau(\|q^{n+1}\|^2 - \|q^n\|^2) \leq 0,$$

that is

$$\|u^{n+1}\|^2 + a_0\tau\|q^{n+1}\|^2 \leq \|u^n\|^2 + a_0\tau\|q^n\|^2.$$

And hence we are led to (3.6). $\square$

3.2 Second order scheme

The second order EIN-LDG scheme, which is the second order IMEX scheme (2.10) coupled with the LDG method (3.3), reads

$$(u^{n,1}, v) = (u^n, v) + \frac{1}{2}(a - a_0)\tau\mathcal{L}(q^n, v) + \frac{1}{2}a_0\tau\mathcal{L}(q^{n,1}, v), \quad (3.8a)$$

$$(u^{n+1}, v) = (u^n, v) + (a - a_0)\tau\mathcal{L}(q^{n,1}, v) + \frac{1}{2}a_0\tau[\mathcal{L}(q^n, v) + \mathcal{L}(q^{n+1}, v)], \quad (3.8b)$$

$$(q^{n,\ell}, r) = \mathcal{K}(u^{n,\ell}, r), \quad \text{for } \ell = 0, 1, 2, \quad (3.8c)$$

where $w^{n,0} = w^n$ and $w^{n,2} = w^{n+1}$ for $w = u, q$.

The same as in the first order scheme, we only consider the case $a_0 \neq a$, since in the case $a_0 = a$ we can also easily get (3.5) unconditionally. The stability result is given in the following theorem.

Theorem 3.2. *If $a_0 \geq \frac{a}{2}$ and $a_0 \neq a$, then the second order EIN-LDG scheme (3.8) satisfies*

$$\|u^n\|^2 + \frac{1}{2}a_0\tau\|q^n\|^2 \leq \|u^0\|^2 + \frac{1}{2}a_0\tau\|q^0\|^2. \quad (3.9)$$

Proof. Subtracting (3.8a) from (3.8b) we get

$$(u^{n+1} - u^{n,1}, v) = (a_0 - \frac{1}{2}a)\tau\mathcal{L}(q^n, v) + (a - \frac{3}{2}a_0)\tau\mathcal{L}(q^{n,1}, v) + \frac{a_0}{2}\tau\mathcal{L}(q^{n+1}, v). \quad (3.10)$$

Taking $v = u^{n,1}$ in (3.8a) we have

$$\frac{1}{2}\|u^{n,1}\|^2 + \frac{1}{2}\|u^{n,1} - u^n\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{1}{2}(a - a_0)\tau(q^n, q^{n,1}) + \frac{1}{2}a_0\tau\|q^{n,1}\|^2 = 0, \quad (3.11)$$

where we have used property (2.5). Taking $v = u^{n+1}$ in (3.10) we have

$$\begin{aligned} & \frac{1}{2}\|u^{n+1}\|^2 + \frac{1}{2}\|u^{n+1} - u^{n,1}\|^2 - \frac{1}{2}\|u^{n,1}\|^2 + (a_0 - \frac{a}{2})\tau(q^n, q^{n+1}) \\ & + (a - \frac{3}{2}a_0)\tau(q^{n,1}, q^{n+1}) + \frac{1}{2}a_0\tau\|q^{n+1}\|^2 = 0. \end{aligned} \quad (3.12)$$

Adding (3.11) and (3.12) together, and multiplying by 2, we get

$$\begin{aligned} & \|u^{n+1}\|^2 + \|u^{n,1} - u^n\|^2 + \|u^{n+1} - u^{n,1}\|^2 - \|u^n\|^2 + a_0\tau[\|q^{n,1}\|^2 + \|q^{n+1}\|^2] \\ & + \tau[(a - a_0)(q^n, q^{n,1}) + (2a_0 - a)(q^n, q^{n+1}) + (2a - 3a_0)(q^{n,1}, q^{n+1})] = 0. \end{aligned}$$

Then by adding and subtracting $\delta\tau\|q^n\|^2$ we obtain

$$\begin{aligned} & \|u^{n+1}\|^2 + \|u^{n,1} - u^n\|^2 + \|u^{n+1} - u^{n,1}\|^2 - \|u^n\|^2 \\ & + \delta\tau(\|q^{n+1}\|^2 - \|q^n\|^2) + \tau \int_{\Omega} \mathbf{q}^{\top} \mathbb{A} \mathbf{q} \, dx = 0, \end{aligned} \quad (3.13)$$

where $\mathbf{q} = (q^n, q^{n,1}, q^{n+1})^{\top}$, and

$$\mathbb{A} = \begin{pmatrix} \delta & \frac{1}{2}(a - a_0) & a_0 - \frac{a}{2} \\ \frac{1}{2}(a - a_0) & a_0 & a - \frac{3}{2}a_0 \\ a_0 - \frac{a}{2} & a - \frac{3}{2}a_0 & a_0 - \delta \end{pmatrix}. \quad (3.14)$$

On the other hand, taking $v = u^{n,1} - u^n$ in (3.8a) we have

$$\|u^{n,1} - u^n\|^2 + \frac{1}{2}(a - a_0)\tau(q^n, q^{n,1} - q^n) + \frac{1}{2}a_0\tau(q^{n,1}, q^{n,1} - q^n) = 0,$$

owing to (2.5). That is

$$\|u^{n,1} - u^n\|^2 + \tau \int_{\Omega} \mathbf{q}^{\top} \mathbb{B} \mathbf{q} \, dx = 0, \quad (3.15)$$

where

$$\mathbb{B} = \begin{pmatrix} \frac{1}{2}(a_0 - a) & \frac{a}{4} - \frac{a_0}{2} & 0 \\ \frac{a}{4} - \frac{a_0}{2} & \frac{1}{2}a_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.16)$$

Adding (3.13) and $\sigma \times (3.15)$ together leads to

$$\begin{aligned} & \|u^{n+1}\|^2 + (1 + \sigma)\|u^{n,1} - u^n\|^2 + \|u^{n+1} - u^{n,1}\|^2 - \|u^n\|^2 \\ & + \delta\tau(\|q^{n+1}\|^2 - \|q^n\|^2) + \tau \int_{\Omega} \mathbf{q}^{\top} (\mathbb{A} + \sigma\mathbb{B}) \mathbf{q} \, dx = 0. \end{aligned} \quad (3.17)$$

Here $0 \leq \delta \leq a_0$ and $\sigma > -1$ are free parameters. For convenience, we let $\delta = \frac{1}{2}a_0$. We claim that there exists $\sigma > -1$ such that the matrix $\mathbb{A} + \sigma\mathbb{B}$ is positive definite for any $a_0 > \frac{1}{2}a$, whose proof will be deferred to Lemma 3.3. We can also verify that $\mathbb{A} + \sigma\mathbb{B}$ is semi-positive definite for $a_0 = \frac{1}{2}a$ if $\sigma = 0$, since the eigenvalues of the matrix are $\frac{3}{4}a$, $\frac{1}{4}a$ and 0 in this situation. Thus we can get

$$\|u^{n+1}\|^2 + \frac{1}{2}a_0\tau\|q^{n+1}\|^2 \leq \|u^n\|^2 + \frac{1}{2}a_0\tau\|q^n\|^2. \quad (3.18)$$

And hence we obtain (3.9). $\square$

Lemma 3.3. Let $\delta = \frac{1}{2}a_0$, for any $a_0 > \frac{1}{2}a$ and $a_0 \neq a$, there exists $\sigma > -1$ such that $\mathbb{A} + \sigma\mathbb{B}$ is positive definite, where $\mathbb{A}$ and $\mathbb{B}$ are defined in (3.14) and (3.16) respectively.

Proof. Assume $a_0 = \theta a$, then

$$\mathbb{A} + \sigma\mathbb{B} = \frac{1}{2}a \begin{pmatrix} \theta + \sigma(\theta - 1) & 1 - \theta + \sigma(\frac{1}{2} - \theta) & 2\theta - 1 \\ 1 - \theta + \sigma(\frac{1}{2} - \theta) & (2 + \sigma)\theta & 2 - 3\theta \\ 2\theta - 1 & 2 - 3\theta & \theta \end{pmatrix}.$$

To ensure $\mathbb{A} + \sigma\mathbb{B}$ is positive definite, we require all the leading principle minors are positive, namely

$$\sigma(\theta - 1) + \theta > 0, \quad (3.19a)$$

$$-\frac{1}{4}\sigma^2 + (\theta^2 + \theta - 1)\sigma + \theta^2 + 2\theta - 1 > 0, \quad (3.19b)$$

$$-\frac{1}{4}\theta\sigma^2 + (6\theta^2 - 7\theta + 2)\sigma - (4\theta^3 + 4\theta^2 - 11\theta + 4) > 0. \quad (3.19c)$$

In what follows, we will prove the solution (σ) of (3.19) exists provided that $\theta > \frac{1}{2}$ and $\theta \neq 1$. From (3.19b), we get

$$2(\theta^2 + \theta - 1) - 2\sqrt{\Delta_1} < \sigma < 2(\theta^2 + \theta - 1) + 2\sqrt{\Delta_1}, \quad (3.20)$$

where $\Delta_1 = (\theta^2 + \theta - 1)^2 + \theta^2 + 2\theta - 1 = \theta^4 + 2\theta^3$, which is always positive if $\theta > \frac{1}{2}$. From (3.19c) we get

$$\frac{2(6\theta^2 - 7\theta + 2) - 2\sqrt{\Delta_2}}{\theta} < \sigma < \frac{2(6\theta^2 - 7\theta + 2) + 2\sqrt{\Delta_2}}{\theta}, \quad (3.21)$$

where $\Delta_2 = (6\theta^2 - 7\theta + 2)^2 - \theta(4\theta^3 + 4\theta^2 - 11\theta + 4) = 32\theta^4 - 88\theta^3 + 84\theta^2 - 32\theta + 4$ which is positive if $\theta > \frac{1}{2}$ and $\theta \neq 1$.

To simplify the notations, we denote $A_1 = 2(\theta^2 + \theta - 1) - 2\sqrt{\Delta_1}$, $A_2 = 2(\theta^2 + \theta - 1) + 2\sqrt{\Delta_1}$, $B_1 = \frac{2(6\theta^2 - 7\theta + 2) - 2\sqrt{\Delta_2}}{\theta}$, and $B_2 = \frac{2(6\theta^2 - 7\theta + 2) + 2\sqrt{\Delta_2}}{\theta}$. Since

$$(A_1 - B_2)(A_2 - B_1) = -\frac{4}{\theta} \left[2\sqrt{\Delta_1\Delta_2} + (12\theta^4 - 5\theta^3 - 24\theta^2 + 28\theta - 8) \right] < 0 \quad \text{if } \theta > \frac{1}{2},$$

we can conclude that the intersection of (3.20) and (3.21) is not empty.

In addition, from (3.19a) we get

$$\begin{cases} \sigma < \frac{\theta}{1-\theta} & \text{if } \frac{1}{2} < \theta < 1, \\ \sigma > -\frac{\theta}{\theta-1} & \text{if } \theta > 1. \end{cases} \quad (3.22)$$

So, if $\frac{1}{2} < \theta < 1$, then we require $\max\{A_1, B_1\} < \frac{\theta}{1-\theta}$. This condition can be verified by noticing that

$$2(\theta^2 + \theta - 1) - \frac{\theta}{1-\theta} < 0 \quad \text{and} \quad \frac{2(6\theta^2 - 7\theta + 2)}{\theta} - \frac{\theta}{1-\theta} < 0,$$

for $\frac{1}{2} < \theta < 1$. If $\theta > 1$, we require $\min\{A_2, B_2\} > -\frac{\theta}{\theta-1}$, it holds obviously since

$$2(\theta^2 + \theta - 1) + \frac{\theta}{\theta-1} > 0 \quad \text{and} \quad \frac{2(6\theta^2 - 7\theta + 2)}{\theta} + \frac{\theta}{\theta-1} > 0,$$

for $\theta > 1$. Thus we proved that the solution of (3.19) exists.

Furthermore, we can check that $\min\{A_2, B_2\} > -1$ in the case $\theta > 1$, and in the case $\frac{1}{2} < \theta < 1$, $\min\{A_2, B_2, \frac{\theta}{1-\theta}\} > -1$, so we complete the proof of this lemma. $\square$

Remark 3.4. In the above stability analysis for the linear model, it is required to study the positive definiteness of the matrix $\mathbb{A} + \sigma\mathbb{B}$ which is a constant matrix. The arguments, however, are not easy to extend to nonlinear problems, since the corresponding matrix will depend on the numerical solutions at different intermediate time stages, it will be more complicated to study the positive definiteness of the matrix. So we need to seek new techniques to overcome the difficulties, which will be left for future work. Even though the analysis for the nonlinear model is not available at present, the stability analysis for the linear model can provide us with some guidance in designing schemes for nonlinear diffusion problems.

4 Optimal error estimates

With the stability result in the previous section, it is conceptually straightforward to obtain error estimates for smooth solutions of the simplified model (3.2) with $a > 0$ being a constant. We will only give the error estimates for the second order EIN-LDG scheme (3.8) as an example. To this end, we would like to introduce two Gauss-Radau projections, from $H^1(\mathcal{T}_h) = \{\phi \in L^2(\Omega) : \phi|_{I_j} \in H^1(I_j), \forall j = 1, \dots, N\}$ to V_h , denoted by π_h^- and π_h^+ respectively. For any function $p \in H^1(\mathcal{T}_h)$, the projections $\pi_h^\pm p$ are defined as the unique element in V_h such that

$$(\pi_h^- p - p, v)_{I_j} = 0, \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (\pi_h^- p)_{j+\frac{1}{2}}^- = p_{j+\frac{1}{2}}^-, \quad (4.1a)$$

$$(\pi_h^+ p - p, v)_{I_j} = 0, \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (\pi_h^+ p)_{j-\frac{1}{2}}^+ = p_{j-\frac{1}{2}}^+, \quad (4.1b)$$

for any $j = 1, 2, \dots, N$. In view of the exact collocation on one endpoint of each element, the Gauss-Radau projections provide a great help to obtain the optimal error estimates.

Denote by $\eta = p - \pi_h^\pm p$ the projection error. By a standard scaling argument [9], it is easy to obtain the following approximation property

$$\|\eta\| \leq Ch^{\min(k+1, s)} \|p\|_{H^s}, \quad (4.2)$$

where the bounding constant $C > 0$ is independent of h . Furthermore, by the definition of the operators $\mathcal{L}$ and $\mathcal{K}$ we have

$$\mathcal{L}(p - \pi_h^+ p, v) = \mathcal{K}(p - \pi_h^- p, v) = 0 \quad (4.3)$$

for any $p \in H^1(\mathcal{T}_h)$ and $v \in V_h$, due to the periodic boundary condition.

Following [23], we introduce three “reference” functions, denoted by $\mathbf{W}^{(\ell)} = (U^{(\ell)}, Q^{(\ell)})$, $\ell = 0, 1, 2$, associated with the second order IMEX RK time discretization (2.10). In detail, $U^{(0)} = U$ is the exact solution of problem (3.2) and then we define

$$U^{(1)} = U^{(0)} + \frac{1}{2}(a - a_0)\tau Q_x^{(0)} + \frac{1}{2}a_0\tau Q_x^{(1)}, \quad (4.4a)$$

$$U^{(2)} = U^{(0)} + (a - a_0)\tau Q_x^{(1)} + \frac{1}{2}a_0\tau(Q_x^{(0)} + Q_x^{(2)}), \quad (4.4b)$$

where

$$Q^{(\ell)} = U_x^{(\ell)}, \quad \text{for } \ell = 0, 1, 2. \quad (4.4c)$$

For any indices n and ℓ under consideration, the reference function at each stage time level is defined as $\mathbf{W}^{n, \ell} = (U^{n, \ell}, Q^{n, \ell}) = \mathbf{W}^{(\ell)}(x, t^n)$. Here $\mathbf{W}^{n, 0} = \mathbf{W}^n$ and $\mathbf{W}^{n, 2} = \mathbf{W}^{n+1}$.

At each stage time, we denote the error between the exact (reference) solution and the numerical solution by $\mathbf{e}^{n, \ell} = (e_u^{n, \ell}, e_q^{n, \ell}) = (U^{n, \ell} - u^{n, \ell}, Q^{n, \ell} - q^{n, \ell})$. As the standard treatment in finite element analysis, we would like to divide the error in the form $\mathbf{e} = \boldsymbol{\xi} - \boldsymbol{\eta}$, where

$$\boldsymbol{\eta} = (\eta_u, \eta_q) = (\pi_h^- U - U, \pi_h^+ Q - Q), \quad \boldsymbol{\xi} = (\xi_u, \xi_q) = (\pi_h^- U - u, \pi_h^+ Q - q), \quad (4.5)$$

here we have dropped the superscripts n and ℓ for simplicity.

We would like to assume that the exact solution U satisfies the following smoothness

$$U \in L^\infty(0, T; H^{k+2}), \quad D_t U \in L^\infty(0, T; H^{k+1}), \quad \text{and} \quad D_t^3 U \in L^\infty(0, T; L^2), \quad (4.6)$$

where $D_t^\ell U$ is the ℓ -th order time derivative of U .

By the smoothness assumption (4.6), it follows from (4.2) that the stage projection errors satisfy

$$\|\eta_u^{n, \ell}\| + \|\eta_q^{n, \ell}\| \leq Ch^{k+1} (\|U^{n, \ell}\|_{H^{k+1}} + \|Q^{n, \ell}\|_{H^{k+1}}) \leq Ch^{k+1}, \quad (4.7a)$$

for any n and $\ell = 0, 1, 2$ under consideration. And owing to the linear structure of the Gauss-Radau projection, we have

$$\|\eta_u^{n, 1} - \eta_u^n\| \leq Ch^{k+1} \|U^{n, 1} - U^n\|_{H^{k+1}} \leq Ch^{k+1} \tau, \quad (4.7b)$$

$$\|\eta_u^{n+1} - \eta_u^{n,1}\| \leq Ch^{k+1} \|U^{n+1} - U^{n,1}\|_{H^{k+1}} \leq Ch^{k+1} \tau. \quad (4.7c)$$

Here the bounding constant $C > 0$ depends solely on the smoothness of the exact solution and is independent of n, h, τ .

In what follows we will focus on the estimate of the error ξ . Notice that the “reference” function satisfies the following variational forms

$$(U^{n,1}, v) = (U^n, v) + \frac{1}{2}(a - a_0)\tau \mathcal{L}(Q^n, v) + \frac{1}{2}a_0\tau \mathcal{L}(Q^{n,1}, v), \quad (4.8a)$$

$$(U^{n+1}, v) = (U^n, v) + (a - a_0)\tau \mathcal{L}(Q^{n,1}, v) + \frac{1}{2}a_0\tau [\mathcal{L}(Q^n, v) + \mathcal{L}(Q^{n+1}, v)] + (\zeta^n, v), \quad (4.8b)$$

$$(Q^{n,\ell}, r) = \mathcal{K}(U^{n,\ell}, r), \quad \text{for } \ell = 0, 1, 2, \quad (4.8c)$$

where $\zeta^n = \mathcal{O}(\tau^3)$ by the smoothness assumption (4.6).

Subtracting these variational forms from those in scheme (3.8), in the same order, we obtain the following error equations

$$(\xi_u^{n,1}, v) = (\xi_u^n, v) + (\eta_u^{n,1} - \eta_u^n, v) + \frac{1}{2}(a - a_0)\tau \mathcal{L}(\xi_q^n, v) + \frac{1}{2}a_0\tau \mathcal{L}(\xi_q^{n,1}, v), \quad (4.9a)$$

$$\begin{aligned} (\xi_u^{n+1}, v) &= (\xi_u^n, v) + (\eta_u^{n+1} - \eta_u^n, v) + (a - a_0)\tau \mathcal{L}(\xi_q^{n,1}, v) \\ &\quad + \frac{1}{2}a_0\tau [\mathcal{L}(\xi_q^n, v) + \mathcal{L}(\xi_q^{n+1}, v)] + (\zeta^n, v), \end{aligned} \quad (4.9b)$$

$$(\xi_q^{n,\ell}, r) = (\eta_q^{n,\ell}, r) + \mathcal{K}(\xi_u^{n,\ell}, r), \quad \text{for } \ell = 0, 1, 2, \quad (4.9c)$$

since $\mathcal{L}(\eta_q, v) = \mathcal{K}(\eta_u, r) = 0$ by property (4.3).

Subtracting (4.9a) from (4.9b) we get

$$\begin{aligned} (\xi_u^{n+1} - \xi_u^{n,1}, v) &= (a_0 - \frac{1}{2}a)\tau \mathcal{L}(\xi_q^n, v) + (a - \frac{3}{2}a_0)\tau \mathcal{L}(\xi_q^{n,1}, v) + \frac{a_0}{2}\tau \mathcal{L}(\xi_q^{n+1}, v) \\ &\quad + (\eta_u^{n+1} - \eta_u^{n,1}, v) + (\zeta^n, v). \end{aligned} \quad (4.10)$$

Taking $v = \xi_u^{n,1}$ in (4.9a), $v = \xi_u^{n+1}$ in (4.10), then proceeding along the similar line as the stability analysis in Subsection 3.2, we obtain

$$\begin{aligned} &\|\xi_u^{n+1}\|^2 + \|\xi_u^{n,1} - \xi_u^n\|^2 + \|\xi_u^{n+1} - \xi_u^{n,1}\|^2 - \|\xi_u^n\|^2 \\ &\quad + \delta\tau(\|\xi_q^{n+1}\|^2 - \|\xi_q^n\|^2) + \tau \int_{\Omega} \xi_q^\top \mathbb{A} \xi_q \, dx = T_1, \end{aligned} \quad (4.11)$$

where $\xi_q = (\xi_q^n, \xi_q^{n,1}, \xi_q^{n+1})^\top$, $\mathbb{A}$ is the same as in (3.14), and

$$\begin{aligned} T_1 &= 2(\eta_u^{n,1} - \eta_u^n, \xi_u^{n,1}) + (a - a_0)\tau(\eta_q^{n,1}, \xi_q^n) + a_0\tau(\eta_q^{n,1}, \xi_q^{n,1}) \\ &\quad + 2(\eta_u^{n+1} - \eta_u^{n,1}, \xi_u^{n+1}) + (2a_0 - a)\tau(\eta_q^{n+1}, \xi_q^n) + (2a - 3a_0)\tau(\eta_q^{n+1}, \xi_q^{n,1}) \\ &\quad + a_0\tau(\eta_q^{n+1}, \xi_q^{n+1}) + 2(\zeta^n, \xi_u^{n+1}). \end{aligned}$$

On the other hand, taking $v = \xi_u^{n,1} - \xi_u^n$ in (4.9a) we get

$$\|\xi_u^{n,1} - \xi_u^n\|^2 + \tau \int_{\Omega} \xi_q^\top \mathbb{B} \xi_q \, dx = T_2, \quad (4.12)$$

where $\mathbb{B}$ is the same as in (3.16) and

$$T_2 = (\eta_u^{n,1} - \eta_u^n, \xi_u^{n,1} - \xi_u^n) + \frac{1}{2}(a - a_0)\tau(\eta_q^{n,1} - \eta_q^n, \xi_q^n) + \frac{1}{2}a_0\tau(\eta_q^{n,1} - \eta_q^n, \xi_q^{n,1}). \quad (4.13)$$

A simple use of the Cauchy-Schwarz and Young's inequalities and the properties (4.7) we have

$$|T_1 + T_2| \leq \varepsilon\tau(\|\xi_u^n\|^2 + \|\xi_u^{n,1}\|^2 + \|\xi_u^{n+1}\|^2) + \varepsilon\tau(\|\xi_q^n\|^2 + \|\xi_q^{n,1}\|^2 + \|\xi_q^{n+1}\|^2) + C(h^{2k+2}\tau + \tau^5),$$

for arbitrary ε . So adding (4.11) and $\sigma \times (4.12)$ together leads to

$$\begin{aligned}
 & \|\xi_u^{n+1}\|^2 + (1 + \sigma)\|\xi_u^{n,1} - \xi_u^n\|^2 + \|\xi_u^{n+1} - \xi_u^{n,1}\|^2 - \|\xi_u^n\|^2 \\
 & + \delta\tau(\|\xi_q^{n+1}\|^2 - \|\xi_q^n\|^2) + \tau \int_{\Omega} \xi_q^\top (\mathbb{A} + \sigma\mathbb{B} - \varepsilon\mathbb{I}) \xi_q \, dx \\
 & \leq \varepsilon\tau(\|\xi_u^n\|^2 + \|\xi_u^{n,1}\|^2 + \|\xi_u^{n+1}\|^2) + C(h^{2k+2}\tau + \tau^5) \\
 & \leq \varepsilon\tau(\|\xi_u^n\|^2 + \|\xi_u^{n,1} - \xi_u^n\|^2 + \|\xi_u^{n+1} - \xi_u^{n,1}\|^2) + C(h^{2k+2}\tau + \tau^5).
 \end{aligned} \tag{4.14}$$

Here $0 \leq \delta \leq a_0$ and $\sigma > -1$ are free parameters, $\mathbb{I}$ is the identity matrix.

As in the stability analysis, we let $\delta = \frac{1}{2}a_0$. Since the matrix $\mathbb{A} + \sigma\mathbb{B}$ is symmetric, from Lemma 3.3 we conclude that, for $a_0 > \frac{1}{2}a$ there exists $\sigma > -1$ such that the matrix $\mathbb{A} + \sigma\mathbb{B} - \varepsilon\mathbb{I}$ is also positive definite, by choosing ε small enough such that $\varepsilon \leq \frac{1}{2}\lambda$, where λ is the smallest positive eigenvalue of the matrix $\mathbb{A} + \sigma\mathbb{B}$. Note that in the case $a_0 = \frac{1}{2}a$, we are not able to ensure the positive definiteness of the matrix $\mathbb{A} + \sigma\mathbb{B} - \varepsilon\mathbb{I}$, since in this case the matrix $\mathbb{A} + \sigma\mathbb{B}$ is only semi-positive definite. Thus for $a_0 > \frac{1}{2}a$, using the discrete Gronwall's inequality yields

$$\|\xi_u^{n+1}\|^2 + \frac{1}{2}a_0\tau\|\xi_q^{n+1}\|^2 \leq \|\xi_u^0\|^2 + \frac{1}{2}a_0\tau\|\xi_q^0\|^2 + C(h^{2k+2} + \tau^4). \tag{4.15}$$

Taking $u^0 = \pi_h^- U^0$ we get $\xi_u^0 = 0$ and hence from (4.9c) we get $\|\xi_q^0\| \leq \|\eta_q^0\| \leq Ch^{k+1}$, so we are led to

$$\|\xi_u^n\| \leq C(h^{k+1} + \tau^2). \tag{4.16}$$

Finally we obtain the following theorem by (4.7), (4.16) and the triangle inequality.

Theorem 4.1. *Let $U(x, t)$ be the exact solution of equation (3.2) satisfying the smoothness assumption (4.6), and let u^n be the solution of the second order EIN-LDG scheme (3.8). Then if $a_0 > \frac{a}{2}$ we have*

$$\max_{n\tau \leq T} \|U(x, t^n) - u^n\| \leq C(h^{k+1} + \tau^2), \tag{4.17}$$

where C is a bounding constant independent of n, h, τ .

5 Numerical Experiments

In this section, we will numerically validate the accuracy and performance of the LDG spatial discretization (2.3) coupled with the first and second order IMEX schemes (2.9) and (2.10). In addition, we would like to test for a third order IMEX scheme proposed in [2]

$$\begin{array}{c|cccccc|cccccc}
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
 \frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{11}{18} & \frac{1}{18} & 0 & 0 & 0 & 0 \\
 \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{5}{6} & -\frac{5}{6} & \frac{1}{2} & 0 & 0 & 0 \\
 1 & 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{7}{4} & \frac{3}{4} & -\frac{7}{4} & 0 & 0 \\
 \hline
 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{7}{4} & \frac{3}{4} & -\frac{7}{4} & 0 & 0
 \end{array} \tag{5.1}$$

5.1 The stability and accuracy test

In this subsection we test the stability and accuracy of the proposed schemes. We will consider two examples. In each example, the source term $f(x, t)$ is chosen properly such that the exact solution satisfies the given equation. The final computing time is $T = 10$ and uniform meshes are adopted for all tests in this subsection. In addition, we take piecewise constant, piecewise linear and piecewise quadratic polynomials in the LDG spatial discretization for the first order, the second order and the third order

IMEX time discretization, respectively, such that the orders accuracy of errors in space and time match if the time step $\tau = \mathcal{O}(h)$.

Example 1. The diffusion equation $u_t = (a(u)u_x)_x + f(x, t)$ with exact solution

$$u(x, t) = \sin(x - t)$$

defined on $[-\pi, \pi]$. We will consider three cases:

(i) $a(u) = \frac{1}{2}$, (ii) $a(u) = u^2 + 1$, (iii) $a(u) = \sin^2 u$.

For this example, the time step is $\tau = h$, where $h = 2\pi/N$ is the mesh size, N is the number of elements.

In Tables 1-3, we list the L^2 errors and orders of accuracy for the three cases. In each table, we display the numerical results of the three IMEX schemes (2.9), (2.10) and (5.1) coupled with the LDG method (2.3) with different a_0 . From these tables, we see that the first and second order EIN-LDG schemes are stable and can achieve optimal error accuracy in both space and time if $a_0 \geq \max\{a(u^0)\}/2$, where u^0 is the approximation of the initial solution. From the experiment we also find that the smallest a_0 to ensure the stability of the third order EIN-LDG scheme is about $0.54 \max\{a(u^0)\}$, and we observe optimal error accuracy in both space and time if $a_0 \geq 0.54 \max\{a(u^0)\}$. From the numerical results we can also find that larger a_0 may cause larger errors.

Table 1 The L^2 errors and orders of accuracy for Example 1: $a(u) = 1/2$.

schemes	N	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
		$a_0 = 0.24$		$a_0 = 0.25$		$a_0 = 1$		$a_0 = 10$	
(2.9) with $k = 0$	80	8.04E-02	-	8.09E-02	-	1.40E-01	-	7.94E-01	-
	160	4.02E-02	1.00	4.04E-02	1.00	7.13E-02	0.97	4.84E-01	0.71
	320	2.01E-02	1.00	2.02E-02	1.00	3.60E-02	0.98	2.73E-01	0.83
	640	1.73E+11	-42.97	1.01E-02	1.00	1.81E-02	0.99	1.46E-01	0.90
	1280	4.33E+46	-117.59	5.05E-03	1.00	9.08E-03	1.00	7.56E-02	0.95
		$a_0 = 0.24$		$a_0 = 0.25$		$a_0 = 1$		$a_0 = 10$	
(2.10) with $k = 1$	80	3.90E+09	-	8.80E-04	-	2.02E-03	-	1.24E-01	-
	160	3.09E+25	-52.82	2.19E-04	2.00	5.17E-04	1.97	4.13E-02	1.58
	320	7.67E+57	-107.61	5.48E-05	2.00	1.31E-04	1.98	1.22E-02	1.76
	640	1.53+123	-216.92	1.37E-05	2.00	3.30E-05	1.99	3.33E-03	1.87
	1280	Infinity	-Inf	3.42E-06	2.00	8.28E-06	1.99	8.73E-04	1.93
		$a_0 = 0.26$		$a_0 = 0.27$		$a_0 = 1$		$a_0 = 10$	
(5.1) with $k = 2$	80	6.37E+12	-	6.92E-06	-	5.10E-05	-	3.31E-02	-
	160	3.67E+31	-62.32	8.67E-07	3.00	6.58E-06	2.95	6.49E-03	2.35
	320	1.26E+70	-128.01	1.09E-07	2.99	8.41E-07	2.97	1.05E-03	2.63
	640	8.64+147	-258.56	1.36E-08	3.00	1.06E-07	2.99	1.51E-04	2.80
	1280	NaN	NaN	1.73E-09	2.98	1.31E-08	3.02	2.03E-05	2.90

Example 2. To test the efficiency of the proposed methods for problems with large variation of diffusion coefficients, we consider the diffusion equation

$$u_t = (a(x)u_x)_x + f(x, t)$$

with the same exact solution as Example 1. We will consider $a(x) = 1 + b \sin^2(x)$ for $b = 10, 100$ and 1000. Obviously, the diffusion coefficient varies from 1 to $1 + b$, and the variation is larger if b is larger.

The choice of a_0 and time step in different situations are given in Table 4. We see that the first and second order schemes are stable if $a_0 \geq 0.5(1 + b)$ and the third order scheme is stable if $a_0 \geq 0.54(1 + b)$. The numerical results are listed in Table 5, from which we can observe optimal orders of accuracy of

Table 2 The L^2 errors and orders of accuracy for Example 1: $a(u) = u^2 + 1$.

schemes	N	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
		$a_0 = 0.85$		$a_0 = 0.9$		$a_0 = 1$		$a_0 = 10$	
(2.9) with $k = 0$	80	9.73E-02	-	1.01E-01	-	1.08E-01	-	6.72E-01	-
	160	4.90E-02	0.99	5.09E-02	0.99	5.47E-02	0.98	3.91E-01	0.78
	320	2.46E-02	0.99	2.56E-02	0.99	2.75E-02	0.99	2.11E-01	0.89
	640	NaN	NaN	1.28E-02	1.00	1.38E-02	1.00	1.10E-01	0.94
	1280	NaN	NaN	6.42E-03	1.00	6.90E-03	1.00	5.61E-02	0.97
		$a_0 = 0.95$		$a_0 = 0.98$		$a_0 = 1$		$a_0 = 10$	
(2.10) with $k = 1$	80	1.04E-03	-	1.03E-03	-	1.02E-03	-	8.64E-02	-
	160	2.62E-04	1.99	2.59E-04	1.99	2.57E-04	1.98	2.83E-02	1.61
	320	6.65E-05	1.98	6.61E-05	1.97	6.56E-05	1.97	8.32E-03	1.76
	640	NaN	NaN	1.65E-05	2.00	1.64E-05	2.00	2.27E-03	1.88
	1280	NaN	NaN	4.12E-06	2.00	4.13E-06	1.99	5.95E-04	1.93
		$a_0 = 1$		$a_0 = 1.05$		$a_0 = 1.1$		$a_0 = 10$	
(5.1) with $k = 2$	80	3.56E-05	-	3.88E-05	-	4.23E-05	-	2.17E-02	-
	160	4.73E-06	2.91	5.23E-06	2.89	5.77E-06	2.87	4.25E-03	2.35
	320	NaN	NaN	7.45E-07	2.81	8.30E-07	2.80	6.97E-04	2.61
	640	NaN	NaN	9.52E-08	2.97	1.07E-07	2.96	1.01E-04	2.79
	1280	NaN	NaN	1.27E-08	2.91	1.40E-08	2.93	1.37E-05	2.88

Table 3 The L^2 errors and orders of accuracy for Example 1: $a(u) = \sin^2 u$.

schemes	N	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
		$a_0 = 0.5 \sin^2 1 - 0.1$		$a_0 = 0.5 \sin^2 1$		$a_0 = 1$		$a_0 = 10$	
(2.9) with $k = 0$	80	9.68E-02	-	1.05E-01	-	1.73E-01	-	1.06E+00	-
	160	4.89E-02	0.99	5.34E-02	0.98	8.99E-02	0.95	5.89E-01	0.85
	320	1.14E+00	-4.54	2.70E-02	0.99	4.60E-02	0.97	3.20E-01	0.88
	640	1.54E+00	-0.43	1.35E-02	0.99	2.34E-02	0.98	1.71E-01	0.91
	1280	1.44E+00	0.09	6.80E-03	1.00	1.18E-02	0.99	8.93E-02	0.93
		$a_0 = 0.5 \sin^2 1 - 0.1$		$a_0 = 0.5 \sin^2 1$		$a_0 = 1$		$a_0 = 10$	
(2.10) with $k = 1$	80	2.41E+01	-	1.01E-03	-	3.41E-03	-	1.49E-01	-
	160	3.67E+01	-0.61	2.58E-04	1.97	9.05E-04	1.91	5.05E-02	1.56
	320	7.41E+01	-1.01	6.55E-05	1.98	2.34E-04	1.95	1.52E-02	1.73
	640	1.52E+02	-1.04	1.65E-05	1.99	5.97E-05	1.97	4.22E-03	1.85
	1280	3.34E+02	-1.13	4.14E-06	1.99	1.51E-05	1.99	1.12E-03	1.92
		$a_0 = 0.5 \sin^2 1$		$a_0 = 0.54 \sin^2 1$		$a_0 = 1$		$a_0 = 10$	
(5.1) with $k = 2$	80	4.01E-05	-	4.27E-05	-	1.87E-04	-	4.11E-02	-
	160	5.81E-06	2.79	6.26E-06	2.77	2.98E-05	2.65	8.28E-03	2.31
	320	1.01E+03	-27.37	8.78E-07	2.83	4.39E-06	2.76	1.38E-03	2.59
	640	1.88E+03	-0.90	1.16E-07	2.92	6.01E-07	2.87	2.03E-04	2.77
	1280	3.25E+03	-0.79	1.49E-08	2.95	7.92E-08	2.93	2.78E-05	2.87

Table 4 The constant a_0 and time step taken in the experiments.

$\begin{array}{c} \text{scheme} \\ \backslash \\ b \end{array}$	10	100	1000
(2.9) with $k = 0$	$a_0 = 6, \tau = 0.1h$	$a_0 = 51, \tau = 0.1h$	$a_0 = 501, \tau = 0.01h$
(2.10) with $k = 1$	$a_0 = 6, \tau = 0.1h$	$a_0 = 51, \tau = 0.1h$	$a_0 = 501, \tau = 0.01h$
(5.1) with $k = 2$	$a_0 = 6, \tau = 0.1h$	$a_0 = 55, \tau = 0.05h$	$a_0 = 540, \tau = 0.01h$

Table 5 The L^2 errors and orders of accuracy for Example 2: $a(x) = 1 + b \sin^2(x)$.

schemes	N	$b = 10$		$b = 100$		$b = 1000$	
		L^2 error	order	L^2 error	order	L^2 error	order
(2.9) with $k = 0$	80	4.66E-02	-	8.66E-02	-	5.19E-02	-
	160	2.33E-02	1.00	4.73E-02	0.87	2.63E-02	0.98
	320	1.17E-02	1.00	2.48E-02	0.93	1.34E-02	0.97
	640	5.84E-03	1.00	1.27E-02	0.96	6.76E-03	0.99
	1280	2.92E-03	1.00	6.46E-03	0.98	3.40E-03	0.99
(2.10) with $k = 1$	80	6.92E-04	-	5.02E-03	-	2.06E-03	-
	160	1.74E-04	1.99	1.56E-03	1.69	6.40E-04	1.69
	320	4.36E-05	2.00	4.49E-04	1.79	1.84E-04	1.80
	640	1.09E-05	2.00	1.23E-04	1.87	5.01E-05	1.88
	1280	2.73E-06	2.00	3.24E-05	1.92	1.32E-05	1.93
(5.1) with $k = 2$	80	5.60E-06	-	1.58E-04	-	3.65E-04	-
	160	7.40E-07	2.92	2.79E-05	2.50	7.41E-05	2.30
	320	9.66E-08	2.94	4.36E-06	2.68	1.30E-05	2.51
	640	1.23E-08	2.97	6.28E-07	2.80	2.04E-06	2.67
	1280	1.46E-09	3.08	8.35E-08	2.91	2.88E-07	2.82

the proposed schemes. We also note that, small mesh size and small time step are required to observe optimal error accuracy for large b .

From the stability analysis in Section 3 and the numerical experiments in this subsection, we propose a guidance for the choice of a_0 for general model $U_t = (a(U)U_x)_x$, that is, $a_0 \geq \max\{a(u)\}/2$ for the first and second order schemes, and $a_0 \geq 0.54 \max\{a(u)\}$ for the third order scheme, where u is the numerical solution at the corresponding time level. In our experiments, the LU factorization is used as the linear solver, it will cost more computation to solve a linear system with different coefficient matrix at each time level. Actually in practical computing, it is not necessary to scan the maximum of $a(u)$ and adjust a_0 at every time step. In the next two subsections, we will simulate the porous medium equation and the high-field model, where we adjust a_0 after every 100 time steps.

5.2 Numerical simulation to the porous medium equation

To further validate the performance of the proposed schemes, we consider the porous medium equation (PME)

$$u_t = (u^m)_{xx}, \quad (5.2)$$

in which m is a constant greater than one. This equation often occurs in nonlinear problems of heat and mass transfer, combustion theory, and flow in porous media, where u is either a concentration or a temperature required to be non negative. We assume the initial solution $u_0(x)$ is a bounded non negative continuous function, then (5.2) can be written as

$$u_t = (a(u)u_x)_x, \quad (5.3)$$

with $a(u) = mu^{m-1}$. It is a degenerate parabolic equation since u may be 0 at some points. The LDG schemes coupled with the explicit third order RK time marching for solving this kind of problems were studied in [29], where a slope limiter was introduced to ensure the non negativity of the numerical solutions.

In this subsection, we present the numerical results given by the EIN-LDG schemes. In all the following experiments, we adopt $k = 2$ for the spatial discretization, and the second order scheme (2.10) for the time discretization. The same slope limiter as in [29] is adopted at each intermediate stage. Thanks to the limiter, we can ensure the non negativity of numerical solutions, and thus can ensure the diffusion coefficient $a(u)$ is non negative. Moreover, the physical meaning of u can be maintained, and the possible numerical oscillation near discontinuous interfaces can be eliminated. All the experiments are tested on uniform mesh with mesh size $h = 0.02$, the time step is $\tau = \mathcal{O}(h)$. In the experiments of this subsection, we adjust a_0 after every 100 time steps, according to the maximum of $a(u)$. We take $a_0 = \max a(u)/2$ at the corresponding time levels.

Test 1. Equation (5.3) with the Barenblatt solution

$$B_m(x, t) = t^{-s} \left[\left(1 - \frac{s(m-1)}{2m} \frac{|x|^2}{t^{2s}} \right)_+ \right]^{1/(m-1)}, \quad (5.4)$$

where $u_+ = \max\{u, 0\}$ and $s = 1/(m+1)$. We begin the computation from $t = 1$ in order to avoid the singularity of the Barenblatt solution near $t = 0$. The boundary condition is $u(\pm 6, t) = 0$ for $t \geq 1$. We plot in Figure 1 the numerical results for $m = 2, 3, 5, 8$ at $t = 2$. From this figure, we see that our scheme can simulate the Barenblatt solution accurately and sharply, without noticeable oscillations near the interface.

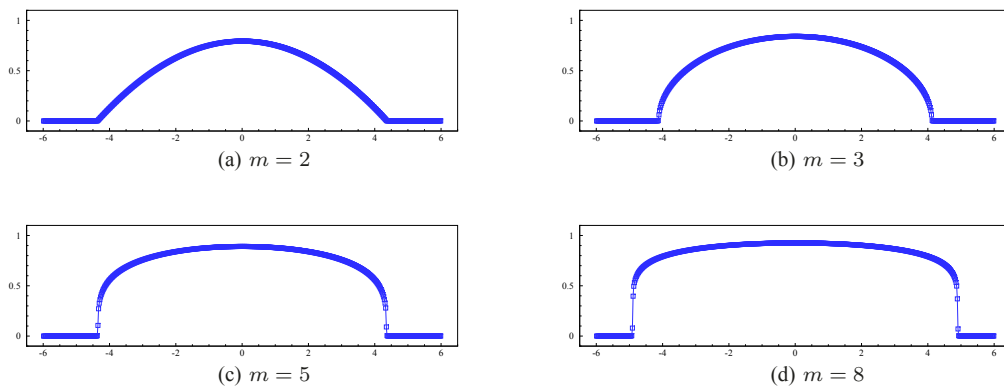


Figure 1 Numerical results for the Barenblatt solution: $t = 2$.

Test 2. The collision of two-Box solutions with the same or different heights. If the variable u is regarded as the temperature, this model can be used to describe how the temperature changes when two hot spots are suddenly put in the computation domain. In Figure 2 we plot the evolution of the numerical solution for the PME with $m = 5$. The initial condition is the two-Box solution with the same height, namely

$$u_0(x) = \begin{cases} 1, & \text{if } x \in (-3.7, -0.7) \cup (0.7, 3.7) \\ 0, & \text{otherwise} \end{cases} \quad (5.5)$$

with the boundary condition $u(\pm 5.5, t) = 0$ for all $t > 0$.

In Figure 3 we plot the evolution of the numerical solution for the PME with parameter $m = 8$. The initial condition is defined as

$$u_0(x) = \begin{cases} 1, & \text{if } x \in (-4, -1), \\ 1.5, & \text{if } x \in (0, 3), \\ 0, & \text{otherwise} \end{cases} \quad (5.6)$$

with the boundary condition $u(\pm 6, t) = 0$ for all $t > 0$.

From these simulations, we can see an analogous evolution whether the heights of the two boxes in the initial condition are the same or not. Two-Box solutions first move outward independently before the collision, then they join each other to make the temperature smooth, and finally the solution becomes almost constant in the common support.

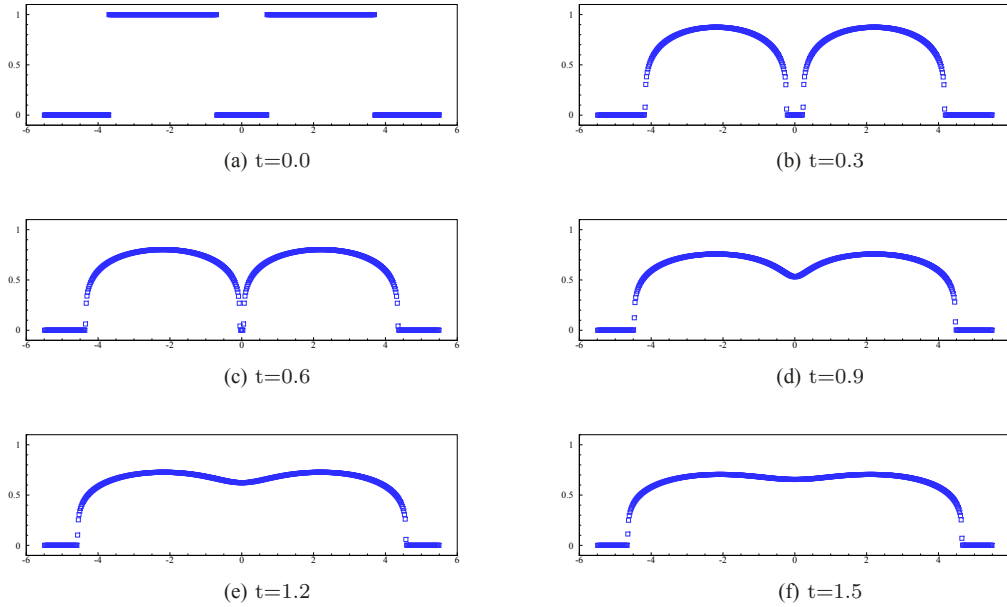


Figure 2 Collision of the two-Box solution with the same height.

Test 3. To test the waiting time phenomenon [1], i.e., the interface of the support does not move outward until the waiting time, we consider the PME with $m = 8$. The initial condition is defined as a fast-varying solution, namely,

$$u_0(x) = \begin{cases} \cos x, & \text{if } x \in (-\pi/2, \pi/2) \\ 0, & \text{otherwise} \end{cases} \quad (5.7)$$

with the boundary condition $u(\pm\pi, t) = 0$ for all $t > 0$. We plot in Figure 4 the evolution of the numerical solutions. We observe that the interface begins to move outward around $t = 1.4$, before that, the interface does not move outward, which verifies the waiting-time phenomenon.

From the above experiments, we see that our scheme can simulate the PME accurately. The main advantage is the fact that larger time steps can be chosen compared with the explicit time discretization methods, where $\tau = \mathcal{O}(h^2)$ is required.

5.3 Numerical simulation to the high-field model

In this subsection, we apply the proposed scheme to the one-dimensional high-field (HF) model [7, 8] in semiconductor device simulations, which is a convection-diffusion system coupled with a Poisson potential equation. The notations for the model are only valid in this subsection. The HF model is described by the following equation

$$n_t + J_x = 0, \quad x \in (0, 0.6) \quad (5.8)$$

where

$$J = J_{hyp} + J_{vis},$$

and

$$J_{hyp} = -\mu n E + \gamma \mu \left(\frac{e}{\varepsilon} \right) n (-\mu n E + \omega),$$

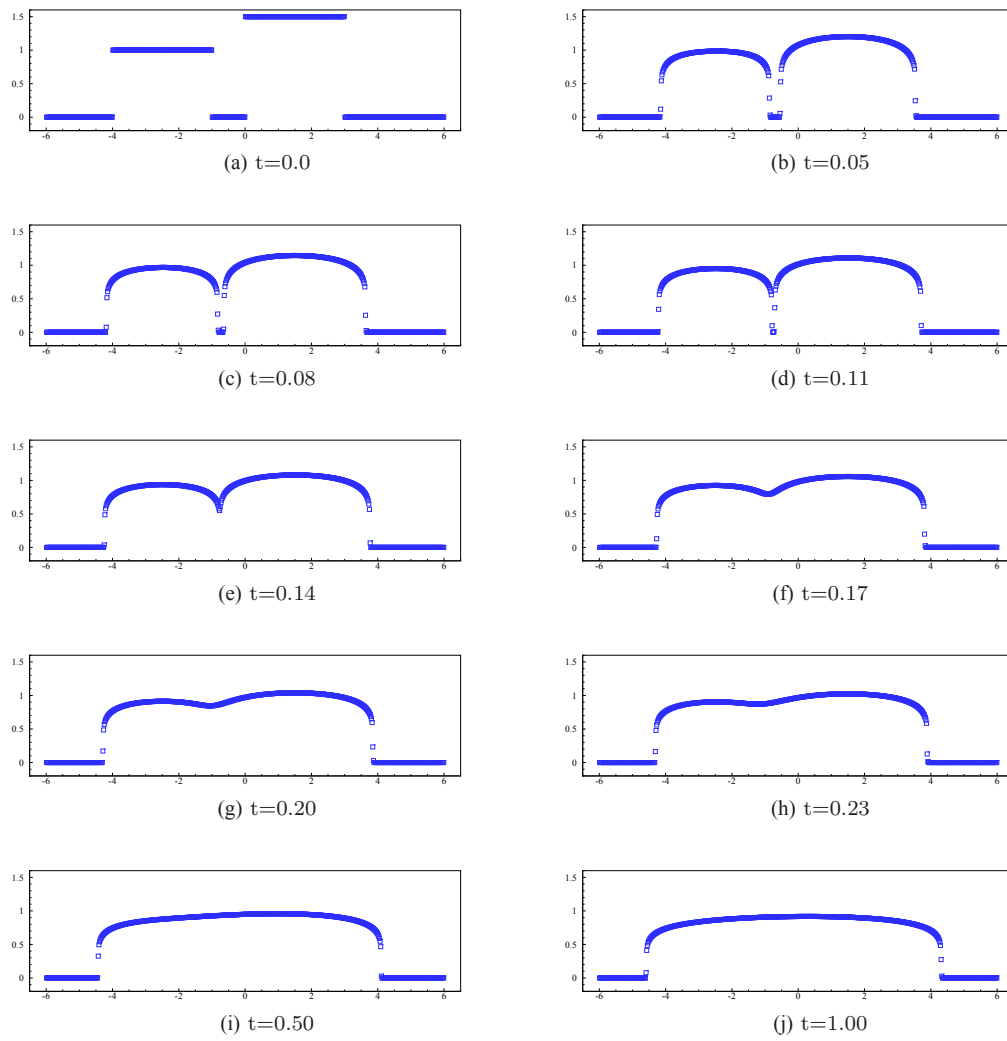
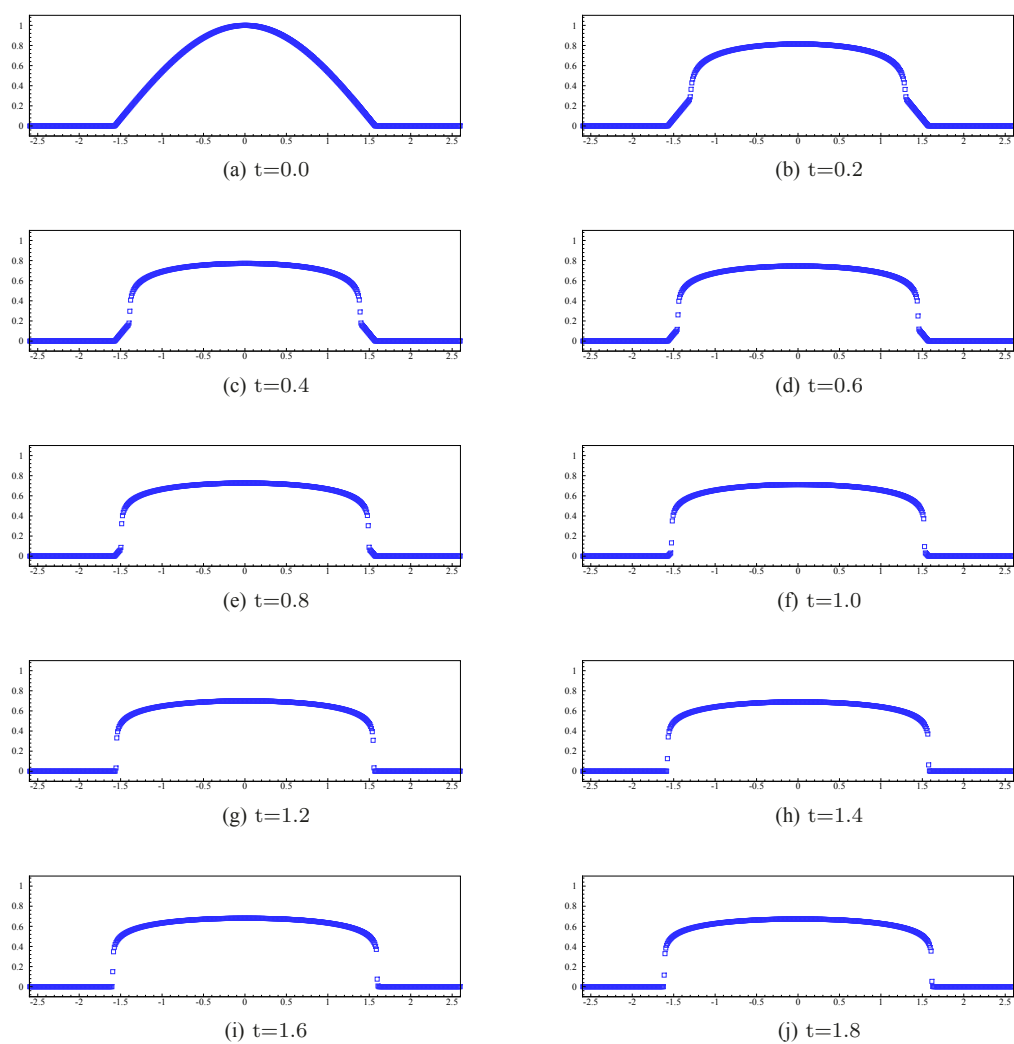


Figure 3 Collision of the two-Box solution with different heights.

**Figure 4** Waiting-time phenomenon.

$$J_{vis} = -\gamma[n(\theta + 2\mu^2 E^2)]_x + \gamma\mu E(\mu n E)_x.$$

In the HF model, the unknown variable n is the electron concentration, $E = -\phi_x$ is the electric field, and ϕ is the electric potential which is given by the Poisson equation

$$\phi_{xx} = \frac{e}{\varepsilon}(n - n_d), \quad (5.9)$$

with n_d being a given doping (also the initial condition for n). The boundary conditions for n and E are periodic, and for ϕ is Dirichlet boundary condition which will be given later.

In the above model, the parameter μ is the mobility, e is the electron charge, ε is the dielectric permittivity, $\omega = (\mu n E)|_{x=0}$ is taken to be a constant, $\gamma = \frac{m\mu}{e}$ is the relaxation parameter, with m being the electron effective mass, and $\theta = \frac{k}{m}T_0$, where k is the Boltzmann constant and T_0 is the lattice temperature.

The LDG method has been applied to solve problem (5.8) in [19] by Liu and Shu, where they used the third order explicit RK method in the time discretization. In their later work [20], an IMEX-LDG method was adopted to solve the drift-diffusion model, for which the coefficient of diffusion is constant. In [20], the IMEX-LDG method shows good efficiency compared with explicit methods. Here the diffusion of HF model is nonlinear, we will use the proposed EIN-LDG scheme.

For the convenience of adopting EIN-LDG scheme, we rewrite the HF model (5.8) as

$$n_t + \left(-\mu n E - \gamma\mu^2 \frac{e}{\varepsilon} n^2 E + \gamma\mu \frac{e}{\varepsilon} \omega n - 3\gamma\mu E n(\mu E)_x\right)_x - [(\gamma\theta + \gamma\mu^2 E^2)n_x]_x = 0. \quad (5.10)$$

Using $E_x = -\frac{e}{\varepsilon}(n - n_d)$, we can write the equation as

$$n_t + f(n, E)_x - (a(E)n_x)_x = 0, \quad (5.11)$$

where

$$\begin{aligned} f(n, E) &= \gamma\mu^2 \frac{e}{\varepsilon} n E (2n - 3n_d) - \mu n E (1 + 3\gamma E \mu_x) + \gamma\mu \frac{e}{\varepsilon} \omega n, \\ a(E) &= \gamma\theta + \gamma\mu^2 E^2. \end{aligned}$$

Then by adding a term $a_0 n_{xx}$ on both sides of (5.11) we get

$$n_t + \underbrace{f(n, E)_x - [(a(E) - a_0)n_x]_x}_{\text{explicit}} = \underbrace{a_0 n_{xx}}_{\text{implicit}}, \quad (5.12)$$

with periodic boundary conditions for n and E , where a_0 is a properly chosen positive constant. We solve (5.12) by the standard LDG scheme with the third order IMEX scheme (5.1), where piecewise quadratic polynomials space is adopted in spatial discretization, Lax-Friedrichs numerical flux and alternating numerical flux are used for the convection and diffusion parts, respectively. We treat the part on the left hand side explicitly, and the part on the right hand side implicitly.

We point out that the potential equation (5.9) is also solved by the LDG method, i.e, finding $(\phi_h, \psi_h) \in V_h \times V_h$, such that for any $(v, r) \in V_h \times V_h$, there holds

$$\left(\frac{e}{\varepsilon}(n - n_d), v\right)_j = -(\psi_h, v_x)_j + \widehat{\psi}_{h, j+\frac{1}{2}} v_{j+\frac{1}{2}}^- - \widehat{\psi}_{h, j-\frac{1}{2}} v_{j-\frac{1}{2}}^+, \quad (5.13a)$$

$$(\psi_h, r)_j = -(\phi_h, r_x)_j + \widehat{\phi}_{h, j+\frac{1}{2}} r_{j+\frac{1}{2}}^- - \widehat{\phi}_{h, j-\frac{1}{2}} r_{j-\frac{1}{2}}^+, \quad (5.13b)$$

for $j = 1, 2, \dots, N$, where we take the minimal dissipation numerical flux as in [10], specifically

$$\widehat{\phi}_{h, j+\frac{1}{2}} = \begin{cases} \phi_a, & j = 0, \\ (\phi_h)_{j+\frac{1}{2}}^-, & j = 1, 2, \dots, N-1, \\ \phi_b, & j = N. \end{cases}$$

$$\widehat{\psi}_{h,j+\frac{1}{2}} = \begin{cases} (\psi_h)_{j+\frac{1}{2}}^+, & j = 0, 1, \dots, N-1, \\ (\psi_h)_{N+\frac{1}{2}}^- - \frac{1}{h} \left((\phi_h)_{N+\frac{1}{2}}^- - \phi_b \right), & j = N. \end{cases}$$

Here ϕ_a and ϕ_b are the given Dirichlet boundary conditions, h is the mesh size. The numerical approximation of electric field is given by $E_h = -\psi_h$.

Next we simulate the HF model with the same parameters as in [19]. The doping n_d is a piecewise-defined function in $[0, 0.6]$, $n_d = 5 \times 10^{17} \text{cm}^{-3}$ in $[0, 0.1]$ and $[0.5, 0.6]$, $n_d = 2 \times 10^{15} \text{cm}^{-3}$ in $[0.15, 0.45]$, and a smooth transition in between. The lattice temperature is $T_0 = 300\text{K}$. The constants $k = 0.138 \times 10^{-4}$, $\varepsilon = 11.7 \times 8.85418$, $e = 0.1602$, $m = 0.26 \times 0.9109 \times 10^{-31} \text{kg}$, and the mobility $\mu = 0.0088 \left(1 + \frac{14.2273}{1 + \frac{n_d}{143200}} \right)$ in our units. The boundary conditions are given as follows: $\phi_a = \frac{kT}{e} \ln(\frac{n_d}{n_i})$ at the left boundary, with $n_i = 1.4 \times 10^{10} \text{cm}^{-3}$, $\phi_b = \phi_a + v_{bias}$ with the voltage drop $v_{bias} = 1.5$ at the right boundary for the potential; $T = 300\text{K}$ at both boundaries for the temperature; and $n = 5 \times 10^{17} \text{cm}^{-3}$ at both boundaries for the concentration.

In the simulations, we let $a_0 = \max\{a(E_h)\}$ in (5.12) and adjust it after every 100 steps, here $E_h = -\psi_h$ is solved from (5.13). The code runs until the numerical solution converges to the steady state, we use $\|n_h^{nt} - n_h^{nt-1}\|_{L^1} < 10^{-6}$ as the criterion for stopping computation, where n_h is the numerical solution of the electron concentration n , and nt is the number of time steps. The positivity limiter [29] is not necessary for this example, since the minimum value of n_h will not be below 0 due to the initial setting of n_d defined above.

Table 6 and Table 7 show the time step, the number of time steps, the numerical steady time, and the CPU time to reach the steady state for the third order explicit RK LDG (EX-RK-LDG) and the third order EIN-LDG methods when we use 100 mesh cells and 200 mesh cells in $[0, 0.6]$, respectively. From these tables, we see that the proposed EIN-LDG scheme can take much larger time steps compared with the explicit method, and hence it saves in CPU time significantly. On the other hand, due to the larger time step, the numerical steady time for EIN-LDG scheme is greater than that for the EX-RK-LDG scheme. Figure 5 plots the simulation results of the HF model with 200 mesh cells, for both the EX-RK-LDG method and the EIN-LDG method. It shows that the EIN-LDG method gives the same convergent results as the explicit method. The EIN-LDG scheme is thus a reliable and efficient tool for the study of models such as the HF model to describe the correct physics.

Table 6 The time step τ , the number of time steps nt , the numerical steady time t , and the CPU time to reach the steady state for third order EX-RK-LDG and third order EIN-LDG methods with 100 mesh cells in $[0, 0.6]$.

	3rd order EX-RK-LDG	3rd order EIN-LDG				
τ	4.604E-6	1.2E-4	1.8E-4	2.4E-4	3.0E-4	3.6E-4
nt	265231	13517	9253	7069	5735	4834
t	1.272	1.622	1.666	1.697	1.720	1.740
CPU time	506	59.51	41.39	32.25	27.27	22.99

Table 7 The time step τ , the number of time steps nt , the numerical steady time t , and the CPU time to reach the steady state for third order EX-RK-LDG and third order EIN-LDG methods with 200 mesh cells in $[0, 0.6]$.

	3rd order EX-RK-LDG	3rd order EIN-LDG				
τ	1.151E-6	1.2E-4	1.8E-4	2.4E-4	3.0E-4	3.6E-4
nt	930776	13508	9248	7065	5732	4831
t	1.122	1.621	1.665	1.696	1.720	1.739
CPU time	5434.47	205.41	140.76	108.22	85.27	72.04

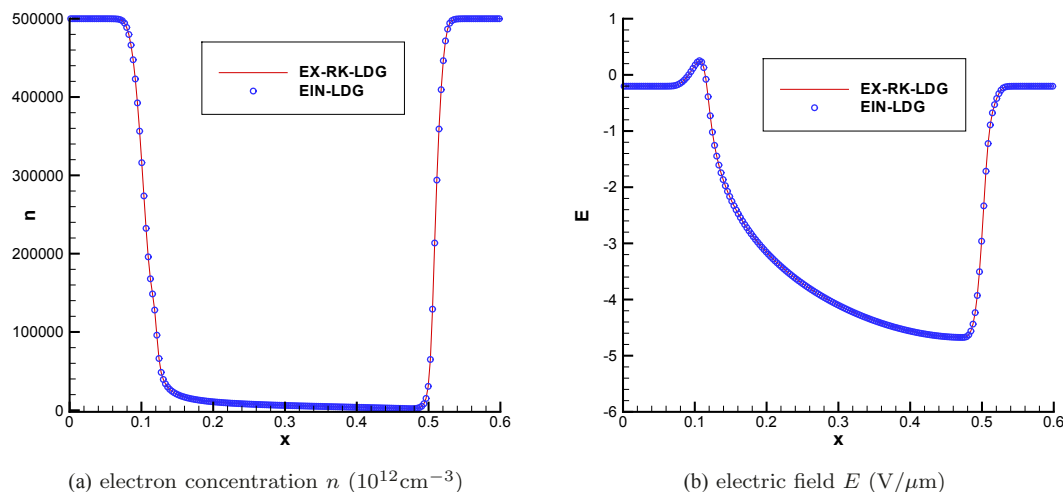


Figure 5 The simulation results of HF model in $[0, 0.6]$ with 200 mesh cells, for third order EX-RK-LDG and third order EIN-LDG methods, $\tau=3.6\text{E-}4$ in EIN-LDG method.

6 Conclusion

We have developed a class of EIN-LDG schemes for solving one-dimensional nonlinear diffusion problems, where a constant diffusion term is added and subtracted to the original equation, and then one of the terms is treated implicitly and the remaining terms are treated explicitly. We have presented the stability and error analysis of the first and second order EIN-LDG schemes for a simplified model, and based on the stability result we have provided a guidance for the choice of a_0 to ensure the unconditional stability of the schemes. Numerical experiments show that the proposed first and second order schemes are stable and can achieve optimal orders of accuracy when $a_0 \geq \max\{a(u)\}/2$. A third order time discretization is also considered numerically. The schemes have good performance and high efficiency for the PME and the high-field model in semiconductor device simulations. The application of the EIN-LDG schemes to solve two and higher spatial dimensional problems is straightforward, for which the proposed schemes will be more efficient compared with explicit or standard implicit schemes. This will be left for our future work.

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References

- 1 Angenent S. Analyticity of the interface of the porous medium equation after waiting time. *Proc Am Math Soc*, 1998, 102: 329–336
- 2 Ascher U M, Ruuth S J, Spiteri R J. Implicit-explicit Runge-Kutta methods for time-dependent partial differential equations. *Appl Numer Math*, 1997, 25: 151–167
- 3 Bassi F, Rebay S. A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations. *J Comput Phys*, 1997, 131: 267–279
- 4 Boscarino S, Russo G. On a class of uniformly accurate IMEX Runge-Kutta schemes and application to hyperbolic systems with relaxation. *SIAM J Sci Comput*, 2009, 31: 1926–1945
- 5 Castillo P, Cockburn B, Schötzau D, Schwab C. Optimal a priori error estimates for the *hp*-version of the local discontinuous Galerkin method for convection-diffusion problems. *Math Comp*, 2001, 71: 455–478
- 6 Cavaliere P, Zavarise G, Perillo M. Modeling of the carburizing and nitriding processes. *Comp Mater Sci*, 2009, 46: 26–35
- 7 Cercignani C, Gamba I M, Jerome J W, Shu C-W. Device benchmark comparisons via kinetic, hydrodynamic, and

- high-field models. *Comput Methods Appl Mech Engrg*, 2000, 181: 381–392
- 8 Cercignani C, Gamba I M, Levermore C D. High field approximations to Boltzmann-Poisson system boundary conditions in a semiconductor. *Appl Math Lett*, 1997, 10: 111–117
 - 9 Ciarlet P G. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, New York, 1978.
 - 10 Cockburn B, Dong B. An analysis of the minimal dissipation local discontinuous Galerkin method for convection-diffusion problems. *J Sci Comput*, 2007, 32: 233–262
 - 11 Cockburn B, Shu C-W. Runge-Kutta discontinuous Galerkin methods for convection-dominated problems. *J Sci Comput*, 2001, 16: 173–261
 - 12 Cockburn B, Shu C-W. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J Numer Anal*, 1998, 35: 2440–2463
 - 13 Cooper G J, Sayfy A. Additive Runge-Kutta methods for stiff ordinary differential equations. *Math Comp*, 1983, 40: 207–218
 - 14 Douglas Jr J, Dupont T F. Alternating-direction Galerkin methods on rectangles, in: B. Hubbard (Ed.). *Numerical Solution of Partial Differential Equations II*, Academic Press, 1971, 133–214.
 - 15 Duchemin L, Eggers J. The explicit-implicit-null method: removing the numerical instability of PDEs. *J Comput Phys*, 2014, 263: 37–52
 - 16 Ewing R E, Wheeler M F. Galerkin methods for miscible displacement problems in porous media. *SIAM J Numer Anal*, 1980, 17: 351–365
 - 17 Filbet F, Jin S. A class of asymptotic-preserving schemes for kinetic equations and related problems with stiff sources. *J Comput Phys*, 2010, 229: 7625–7648
 - 18 Levy D, Tadmor E. From semidiscrete to fully discrete: stability of Runge-Kutta schemes by the energy method. *SIAM Rev*, 1998, 40: 40–73
 - 19 Liu Y, Shu C-W. Error analysis of the semi-discrete local discontinuous Galerkin method for semiconductor device simulation models. *Sci China Math*, 2010, 53: 3255–3278
 - 20 Liu Y, Shu C-W. Analysis of the local discontinuous Galerkin method for the drift-diffusion model of semiconductor devices. *Sci China Math*, 2016, 59: 115–140
 - 21 Macdonald C B, Ruuth S J. The implicit closest point method for the numerical solution of partial differential equations on surfaces. *SIAM J Sci Comput*, 2009, 31: 4330–4350
 - 22 Smereka P. Semi-implicit level set methods for curvature and surface diffusion motion. *J Sci Comput*, 2003, 19: 439–456
 - 23 Wang H J, Shu C-W, Zhang Q. Stability and error estimates of the local discontinuous Galerkin method with implicit-explicit time-marching for advection-diffusion problems. *SIAM J Numer Anal*, 2015, 53: 206–227
 - 24 Wang H J, Zhang Q, Shu C-W. Stability analysis and error estimates of local discontinuous Galerkin methods with implicit-explicit time-marching for the time-dependent fourth order PDEs. *ESAIM: M^2AN* , 2017, 51: 1931–1955
 - 25 Wang H J, Zhang Q, Shu C-W. Third order implicit-explicit Runge-Kutta local discontinuous Galerkin methods with suitable boundary treatment for convection-diffusion problems with Dirichlet boundary conditions. *J Comput Appl Math*, 2018, 342: 164–179
 - 26 Xu Y, Shu C-W. Local discontinuous Galerkin methods for high-order time-dependent partial differential equations. *Commun Comput Phys*, 2010, 7: 1–46
 - 27 Yan J, Shu C-W. Local discontinuous Galerkin methods for partial differential equations with higher order derivatives. *J Sci Comput*, 2002, 17: 27–47
 - 28 Zhang Q, Gao F Z. A fully-discrete local discontinuous Galerkin method for convection-dominated Sobolev equation. *J Sci Comput*, 2012, 51: 107–134
 - 29 Zhang Q, Wu Z. Numerical simulation for porous medium equation by local discontinuous Galerkin finite element method. *J Sci Comput*, 2009, 38: 127–148