Optimal error estimates of the semidiscrete discontinuous Galerkin methods for two dimensional hyperbolic equations on Cartesian meshes using P^k elements

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Abstract

In this paper, we study the optimal error estimates of the classical discontinuous Galerkin method for time-dependent 2-D hyperbolic equations using P^k elements on uniform Cartesian meshes, and prove that the error in the L^2 norm achieves optimal $(k+1)$-th order convergence when upwind fluxes are used. For the linear constant coefficient case, the results hold true for arbitrary piecewise polynomials of degree $k \geq 0$. For variable coefficient and nonlinear cases, we give the proof for piecewise polynomials of degree $k = 0, 1, 2, 3$ and $k = 2, 3$, respectively, under the condition that the wind direction does not change. The theoretical results are verified by numerical examples.

Key Words: Optimal error estimate; discontinuous Galerkin method; upwind fluxes

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1 Introduction

In this paper, we study the semi-discrete discontinuous Galerkin (DG) method for solving 2-D hyperbolic equations on Cartesian meshes. The optimal error estimates can be obtained based on tensor-product polynomials for solving hyperbolic conservation laws in previous analysis [4]. We prove optimal error estimates of the DG approximation based on P^k, the piecewise polynomials of degree at most k under suitable restrictions. We consider the hyperbolic conservation laws

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \ (x,y) \in \Omega, \ t \geq 0 \\ u(x,y,0) = u_0(x,y), \ (x,y) \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a rectangular domain in $\mathbb{R}^2$ and periodic boundary condition or inflow-outflow boundary conditions. The initial condition $u_0(x,y)$ is a given smooth function. For simplicity, in the following we will only discuss the case with periodic boundary condition, although this is not essential for the analysis; inflow-outflow boundary conditions can also be considered along the same lines. We assume the exact solution of (1.1) is smooth, this is true for all time t for the linear case with smooth coefficients, and when t is small for the nonlinear case, since we assume the initial condition $u_0(x,y)$ is smooth.

The first version of the DG method was introduced in 1973 by Reed and Hill [13] in the framework of neutron linear transport. It was later developed into the Runge-Kutta DG (RKDG) methods by Cockburn et al [4, 5, 6, 8]. For one-dimensional and some multidimensional cases, optimal a priori error estimates of order $k+1$ can be obtained for the DG schemes when upwind fluxes are used [3, 7, 14, 16]. In [11], Meng et al. obtained similar optimal a priori error estimates when upwind-biased fluxes are used. For higher order equations by utilizing and fully making use of the so called Gauss-Radau projections, Xu and Shu [15] introduced a general approach for proving optimal error estimates.

However, for multidimensional Cartesian meshes, the above optimal results are based on using Q^k, the space of tensor-product polynomials of degree at most k in each variable.

The numerical results show that the optimal accuracy $(k+1)$-th convergence order holds true using the P^k space. The number of degrees of freedom of the space P^k is $(k+1)(k+2)/2$, which is only about half of that for the space Q^k for large k. The most critical point to obtain the optimal error estimates is to construct a suitable projection. The projection can help us to deal with the troublesome terms in the analysis. However, since the number of degrees of freedom of P^k is only about half of that for Q_k, a suitable projection is elusive for this case. Recently, we have constructed a special projection to obtain the optimal error estimate for the central DG scheme by using a shifting technique [10]. We continue to use this technique to construct a special projection to obtain optimal error estimates for the DG methods based on the P^k space over uniform Cartesian meshes. We separately give the analysis of optimal error estimates in three cases, namely the case with linear constant coefficients, the case with linear variable coefficients, and the nonlinear case. First, the optimal $(k+1)$-th order is proved for smooth solutions of linear constant coefficient conservation laws when upwind numerical fluxes are used. This proof holds true for uniform meshes and for polynomials of arbitrary degree $k \geq 0$. For linear variable coefficient and nonlinear equations, we give the proof of optimal convergence results for $k = 0, 1, 2, 3$ and $k = 2, 3$, respectively, under the condition that $f'(u)$, $g'(u)$ do not change sign. Let us emphasize that this restriction appears to be artificial due to the limitation of our techniques in the proof; the optimal $(k+1)$-th order convergence appears to hold true for nonlinear conservation laws with general flux functions; see our numerical results in section 5. As far as we know, this is the first optimal error estimate proof for DG methods applied to time-dependent nonlinear hyperbolic equations using P^k elements on Cartesian meshes. To deal with the nonlinearity of the flux, Taylor expansion and an a priori assumption about the numerical solution are used.

The remainder of the paper is organized as follows. In section 2, we give the proof of the optimal error estimates for the semi-discrete DG scheme solving linear constant coefficient hyperbolic equations. In section 3, we provide the proof of the uniform boundedness and superconvergence properties of a special projection and the proof of the optimal error estimates for linear variable coefficients case. In section 4, we analyze nonlinear hyperbolic equations. Some numerical examples are provided in section 5. Finally, we conclude and give a few perspectives for future work in section 6. Some technical proof of the error estimates is provided in the Appendix.

2 Linear constant coefficients

In this section, we consider the two-dimensional scalar linear constant coefficient conservation law equation

$$\begin{cases} u_t + au_x + bu_y = 0,\ (x, y) \in \Omega,\ t \geq 0 \\ u(x, y, 0) = u_0(x, y),\ (x, y) \in \Omega, \end{cases} \tag{2.1}$$

with periodic boundary condition, where a and b are constants. Without loss of generality, we assume $\Omega = [0, 1]^2$ and $a, b > 0$.

We recall the two-dimensional formulation of the DG scheme in [4]. Let $\{K_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\}$, $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$ be a partition of Ω into rectangular cells. Let $V_h := \{v \in L^2(\Omega) : v|_{K_{i,j}} \in P^k(K_{i,j})\ \forall i, j\}$, where $P^k(K_{i,j})$ denotes the space of polynomials of degrees at most k defined on $K_{i,j}$; no continuity is assumed across cell boundaries. We denote $h_x^i = (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})$, $h_y^j = (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})$ and $h = \max(h_x^i, h_y^j)$. We also introduce some standard Sobolev spaces notations. For any integer $m > 0$, let $W^{m,p}(D)$ be the standard Sobolev spaces on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$.

The semidiscrete DG method with the upwind flux is described as follows: We seek an approximate solution $u_h \in V_h$ such that for all admissible test function $v \in V_h$ and $K_{i,j}$,

$$((u_h)_t, v)_{K_{i,j}} - (u_h, v_\beta)_{K_{i,j}}$$

$$
+\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\left(u_h(x^-_{i+\frac{1}{2}},y)v(x^-_{i+\frac{1}{2}},y)-u_h(x^-_{i-\frac{1}{2}},y)v(x^+_{i-\frac{1}{2}},y)\right)\,dy
$$

$$
+\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b\left(u_h(x,y^-_{j+\frac{1}{2}})v(x,y^-_{j+\frac{1}{2}})-u_h(x,y^-_{j-\frac{1}{2}})v(x,y^+_{j-\frac{1}{2}})\right)\,dx=0, \quad (2.2)
$$

where $(\cdot,\cdot)$ denotes the $L^2(K_{i,j})$-inner product. Here we have used the notation v_β for the (unnormalized) directional derivative of v with respect to $\beta=(a,b)$, namely $v_\beta = av_x + bv_y$, and,

$$
u_h(x,y^\pm_{j+\frac{1}{2}}) = \lim_{\varepsilon\to 0^+} u_h(x,y_{j+\frac{1}{2}}\pm\varepsilon),\ \forall x\in(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}), \quad (2.3)
$$

$$
u_h(x^\pm_{i+\frac{1}{2}},y) = \lim_{\varepsilon\to 0^+} u_h(x_{i+\frac{1}{2}}\pm\varepsilon,y),\ \forall y\in(y_{j-\frac{1}{2}},y_{j+\frac{1}{2}}). \quad (2.4)
$$

For the initial condition, we simply take $u_h(0)=\mathbb{P}_h u_0$, where $\mathbb{P}_h$ is the L^2 projection into V_h, and we have

$$
\|u_0-\mathbb{P}_h u_0\| \le Ch^{k+1}\|u_0\|_{k+1}, \quad (2.5)
$$

where the constant C depends on k. Here and below, an unmarked norm $\|\cdot\|$ denotes the L^2 norm.

The DG scheme using the upwind numerical fluxes for the two-dimensional linear conservation laws satisfies the following L^2-stability (e.g. [11]).

Proposition 2.1. *The solution of the semidiscrete DG method defined by (2.2) satisfies*

$$
\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + \frac{a}{2}\sum_{j=1}^{N_2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sum_{i=1}^{N_1}\left(u_h(x^+_{i+\frac{1}{2}},y)-u_h(x^-_{i+\frac{1}{2}},y)\right)^2\,dy
$$

$$
+\frac{b}{2}\sum_{i=1}^{N_1}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\sum_{j=1}^{N_2}\left(u_h(x,y^+_{j+\frac{1}{2}})-u_h(x,y^-_{j+\frac{1}{2}})\right)^2\,dx = 0. \quad (2.6)
$$

2.1 A priori error estimates

Now, we only consider uniform meshes, i.e. $h_x = h_x^i$ and $h_y = h_y^j$. Let us now state our main result as a theorem, whose proof will be given in the next subsection.

Theorem 2.1. *Suppose u_h is the approximate solution of the DG scheme(2.2) using uniform meshes for (2.1) with a smooth initial condition $u(\cdot,0) \in H^{k+2}$ and u is the exact solution of (2.1), then the scheme satisfies the following L^2 error estimate:*

$$\|u(\cdot,T) - u_h(\cdot,T)\| \leq Ch^{k+1}, \tag{2.7}$$

where k is the degree of the piecewise polynomials in the finite element spaces V_h, and the constant C depends on the $(k+2)$-th order Sobolev norm of the initial condition $\|u(\cdot,0)\|_{k+2}$ as well as on the final time T but is independent of the mesh size h.

Let us first introduce a few notations. We define

$$\begin{aligned} B_{i,j}(u_h, v; a, b) =& ((u_h)_t, v)_{K_{i,j}} - (u_h, v_\beta)_{K_{i,j}} \\ &+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b\left(u_h(x, y^-_{j+\frac{1}{2}})v(x, y^-_{j+\frac{1}{2}}) - u_h(x, y^-_{j-\frac{1}{2}})v(x, y^+_{j-\frac{1}{2}})\right)\, dx \\ &+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\left(u_h(x^-_{i+\frac{1}{2}}, y)v(x^-_{i+\frac{1}{2}}, y) - u_h(x^-_{i-\frac{1}{2}}, y)v(x^+_{i-\frac{1}{2}}, y)\right)\, dy. \end{aligned} \tag{2.8}$$

We also define

$$\begin{aligned} \tilde{B}_{i,j}(u_h, v; a, b) =& - (u_h, v_\beta)_{K_{i,j}} \\ &+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b\left(u_h(x, y^-_{j+\frac{1}{2}})v(x, y^-_{j+\frac{1}{2}}) - u_h(x, y^-_{j-\frac{1}{2}})v(x, y^+_{j-\frac{1}{2}})\right)\, dx \\ &+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\left(u_h(x^-_{i+\frac{1}{2}}, y)v(x^-_{i+\frac{1}{2}}, y) - u_h(x^-_{i-\frac{1}{2}}, y)v(x^+_{i-\frac{1}{2}}, y)\right)\, dy. \end{aligned} \tag{2.9}$$

Clearly, we have:

$$B_{i,j}(u_h, v; a, b) = 0, \tag{2.10}$$

for all i, j and all $v \in V_h$. It is also clear that the exact solution u of the PDE (2.1) satisfies

$$B_{i,j}(u, v; a, b) = 0, \tag{2.11}$$

for all i, j and all $v \in V_h$. Subtracting (2.10) from (2.11), we obtain the error equation

$$B_{i,j}(u - u_h, v; a, b) = 0, \tag{2.12}$$

for all i, j and all $v \in V_h$.

2.2 Proof of the error estimates

In this subsection, we divide it into several steps to prove Theorem 2.1. First, we construct the special local projection $\mathbb{P}^\star$ and prove the projection is well defined and has the optimal approximation properties. Next, we prove a few propositions and superconvergence properties of the special projection. Finally, the proof of Theorem 2.1 is completed in subsection 2.2.3.

2.2.1 The special projection $\mathbb{P}^\star$

We now define $\mathbb{P}^\star$ as the following projection into V_h. For each $K_{i,j}$,

$$\int_{K_{i,j}} \mathbb{P}^\star \omega(x)dx = \int_{K_{i,j}} \omega(x)dx, \tag{2.13a}$$

$$\widetilde{P_h}(\mathbb{P}^\star\omega, v; a, b)_{i,j} = \widetilde{P_h}(\omega, v; a, b)_{i,j} \qquad \forall v \in P^k(K_{i,j}), \tag{2.13b}$$

where $\widetilde{P_h}(\omega, v; a, b)_{i,j}$ is defined as follows

$$\begin{aligned}\widetilde{P_h}(\omega, v; a, b)_{i,j} = -(\omega, v_\beta)_{K_{i,j}} + &\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b\omega(x, y^-_{j+\frac{1}{2}})(v(x, y^-_{j+\frac{1}{2}}) - v(x, y^+_{j-\frac{1}{2}}))\, dx \\ &+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\omega(x^-_{i+\frac{1}{2}}, y)(v(x^-_{i+\frac{1}{2}}, y) - v(x^+_{i-\frac{1}{2}}, y))\, dy. \end{aligned}\tag{2.14}$$

Next, we prove the projection $\mathbb{P}^\star$ is well defined. Note that $\mathbb{P}^\star$ is a local projection, so we only consider the projection defined on the reference cell $[-1, 1] \times [-1, 1]$.

Remark 2.1. *We could also similarly define the projection $\mathbb{P}^\star$ for different signs of a, b, by simply changing the definition of $\widetilde{P_h}(\omega, v; a, b)_{i,j}$. We list the other cases below for completeness:*

$$\begin{aligned}\widetilde{P_h}(\omega, v; a, b)_{i,j} = &-(\omega, v_\beta)_{K_{i,j}} + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b\omega(x, y^-_{j+\frac{1}{2}})(v(x, y^-_{j+\frac{1}{2}}) - v(x, y^+_{j-\frac{1}{2}}))\, dx \\ &+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\omega(x^+_{i-\frac{1}{2}}, y)(v(x^-_{i+\frac{1}{2}}, y) - v(x^+_{i-\frac{1}{2}}, y))\, dy, \qquad \text{if } a < 0 \text{ and } b > 0; \end{aligned}\tag{2.15}$$

$$\widetilde{P_h}(\omega, v; a, b)_{i,j} = -(\omega, v_\beta)_{K_{i,j}} + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b\omega(x, y^+_{j-\frac{1}{2}})(v(x, y^-_{j+\frac{1}{2}}) - v(x, y^+_{j-\frac{1}{2}}))\, dx$$

$$+\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\omega(x^-_{i+\frac{1}{2}},y)(v(x^-_{i+\frac{1}{2}},y)-v(x^+_{i-\frac{1}{2}},y))\,dy, \qquad \text{if } a>0 \text{ and } b<0; \tag{2.16}$$

$$\begin{aligned}\widetilde{P_h}(\omega,v;a,b)_{i,j} = &-(\omega,v_\beta)_{K_{i,j}} + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b\omega(x,y^+_{j-\frac{1}{2}})(v(x,y^-_{j+\frac{1}{2}})-v(x,y^+_{j-\frac{1}{2}}))\,dx \\ &+\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\omega(x^+_{i-\frac{1}{2}},y)(v(x^-_{i+\frac{1}{2}},y)-v(x^+_{i-\frac{1}{2}},y))\,dy, \qquad \text{if } a<0 \text{ and } b<0.\end{aligned} \tag{2.17}$$

Lemma 2.1. *The projection $\mathbb{P}^\star$ defined by (2.13) on the cell $[-1,1]\times[-1,1]$ exists and is unique for any smooth function ω, and the projection is bounded in the L^∞ norm, i.e.*

$$\|\mathbb{P}^\star\omega\|_\infty \le C(k,a,b)\|\omega\|_\infty, \tag{2.18}$$

where C(k,a,b) is a constant that only depends on k,a,b but is independent of ω.

Proof. The proof of this lemma is provided in the Appendix; see section A.1.

Since the projection is a k-th degree polynomial preserving local projection, standard approximation theory [2] implies, for a smooth function ω,

$$\|\omega-\mathbb{P}^\star\omega\|_{L^2(K_{i,j})} \le Ch^{k+1}\|\omega\|_{k+1,K_{i,j}}, \tag{2.19}$$

where $C=C(k,a,b)$ is independent of the element $K_{i,j}$ and the mesh size h.

We also recall that [2], for any $\omega_h\in V_h$, there exists a positive constant C independent of ω_h and h, such that

$$\|\partial_x\omega_h\|\le Ch^{-1}\|\omega_h\|, \quad \|\omega_h\|_{L^2(\partial K_{i,j})}\le Ch^{-1/2}\|\omega_h\|, \quad \|\omega_h\|_\infty\le Ch^{-1}\|\omega_h\| \tag{2.20}$$

where $\partial K_{i,j}$ is the boundary of cell $K_{i,j}$.

Remark 2.2. *In fact, we can prove the bounding constant C only depends on k and is independent of a,b. Hence the projection $\mathbb{P}^\star$ is uniformly bounded for (a,b). We give the proof of this property in section 3 for $k=0,1,2,3$.*

2.2.2 Properties of the projection $\mathbb{P}^\star$

To obtain the optimal L^2 error estimate, we need the following lemmas.

Lemma 2.2. *Assume that* $u = x^{k+1-l}y^l, l = 0, 1, \ldots, k+1$. *Let* $u_{(i,j)} = \mathbb{P}^\star_{K_{i,j}}u$. *Then* $\forall (x, y) \in K_{i,j}$ *we have following relationship:*

$$\begin{aligned} x^{k+1-l}y^l - u_{(i,j)}(x,y) =&(x-h_x)^{k+1-l}y^l - u_{(i-1,j)}(x-h_x,y) \\ =&x^{k+1-l}(y-h_y)^l - u_{(i,j-1)}(x,y-h_y), \end{aligned} \tag{2.21}$$

where $\mathbb{P}^\star_{K_{i,j}}u$ *means that the projection of* u *is defined on the element* $K_{i,j}$ *and* $u_{(i-1,j)}(x-h_x,y)$, $u_{(i,j-1)}(x,y-h_y)$ *refer to the projection of* u *on the element* $K_{i-1,j}$ *and* $K_{i,j-1}$, *respectively, since* $(x,y) \in K_{i,j}$ *implies* $(x-h_x,y) \in K_{i-1,j}$ *and* $(x,y-h_y) \in K_{i,j-1}$.

Proof. The details of the proof for this lemma are provided in the Appendix; see section A.2.

Besides the standard approximation results (2.19), we also can prove the following superconvergence result of the special projection $\mathbb{P}^\star$.

Proposition 2.2. *Assume that* u *is a* $(k+1)$*-th degree polynomial function in* $P^{k+1}(\Omega)$. *For a uniform partition on the domain* Ω, *we have*

$$\tilde{B}_{i,j}(\mathbb{P}^\star u, v; a, b) = \tilde{B}_{i,j}(u, v; a, b) \quad \forall v \in P^k(K_{i,j}), \tag{2.22}$$

where $\tilde{B}$ *is defined by (2.9)*

Proof. The proof of this proposition is provided in the Appendix; see section A.3.

2.2.3 Proof of Theorem 2.1

We now take

$$\xi = \mathbb{P}^\star u - u_h; \quad \eta = \mathbb{P}^\star u - u. \tag{2.23}$$

From the error equation (2.12), we have

$$B_{i,j}(\xi, v; a, b) = B_{i,j}(\eta, v; a, b). \tag{2.24}$$

For the left-hand side of (2.24), we can use the stability result (2.6) to obtain

$$\sum_{i,j} B_{i,j}(\xi,\xi;a,b) = \frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \frac{a}{2}\sum_{j=1}^{N_2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sum_{i=1}^{N_1}\Big(\xi(x^+_{i+\frac{1}{2}},y) - \xi(x^-_{i+\frac{1}{2}},y)\Big)^2\ dy$$
$$+\frac{b}{2}\sum_{i=1}^{N_1}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\sum_{j=1}^{N_2}\Big(\xi(x,y^+_{j+\frac{1}{2}}) - \xi(x,y^-_{j+\frac{1}{2}})\Big)^2\ dx, \quad (2.25)$$

here we have already taken the test function $v = \xi \in V_h$. From Proposition 2.2, we know that on an arbitrary element $K_{i,j}$, we have the following results

$$\tilde{B}_{i,j}(\mathbb{P}^\star u, v; a,b) = \tilde{B}_{i,j}(u,v;a,b), \quad \forall u \in P^{k+1}(K_{i,j}\cup K_{i-1,j}\cup K_{i,j-1}). \quad (2.26)$$

Next, on each element $K_{i,j}$, we consider the Taylor expansion of u around (x_i,y_j):

$$u = Tu + Ru,$$

where

$$Tu = \sum_{l=0}^{k+1}\sum_{m=0}^{l}\frac{1}{(l-m)!m!}\frac{\partial^l u(x_i,y_j)}{\partial x^{l-m}\partial y^m}(x-x_i)^{l-m}(y-y_j)^m,$$

$$Ru = (k+2)\sum_{m=0}^{k+2}\frac{(x-x_i)^{k+2-m}(y-y_j)^m}{(k+2-m)!m!}\int_0^1(1-s)\frac{\partial^{k+2}u(x_i^s,y_j^s)}{\partial x^{k+2-m}\partial y^m}\ ds.$$

with $x_i^s = x_i + s(x-x_i)$, $y_j^s = y_j + s(y-y_j)$. Clearly, $Tu \in P^{k+1}(K_{i,j}\cup K_{i-1,j}\cup K_{i,j-1})$, Note that the operator $\mathbb{P}^\star$ is linear, and thus $\mathbb{P}^\star u = \mathbb{P}^\star Tu + \mathbb{P}^\star Ru$. From (2.26), we then get

$$\tilde{B}_{i,j}(\eta,v;a,b) = \tilde{B}_{i,j}(\mathbb{P}^\star Tu - Tu, v;a,b) + \tilde{B}_{i,j}(\mathbb{P}^\star Ru - Ru, v;a,b)$$
$$= \tilde{B}_{i,j}(\mathbb{P}^\star Ru - Ru, v;a,b). \quad (2.27)$$

Again recalling the Bramble-Hilbert lemma in [2], we have

$$\|Ru\|_{L^\infty(K_{i,j})} \le Ch^{k+1}|u|_{H^{k+2}(K_{i,j})}. \quad (2.28)$$

Next, using the simple inequality

$$\mu\nu \le \frac{1}{2}(\mu^2+\nu^2), \quad (2.29)$$

and standard approximate proposition of the projection (2.19), the property (2.28) for Ru, and the inequality in (2.20) for ξ, we have

$$\sum_{i,j} B_{i,j}(\eta, \xi; a, b) \leq Ch^{2k+2}|u|^2_{H^{k+2}} + C\|\xi\|^2. \tag{2.30}$$

Combining (2.24), (2.25), and (2.30), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi\| \leq C\|\xi\| + Ch^{2k+2}|u|^2_{H^{k+2}}. \tag{2.31}$$

An application Gronwall's inequality together with the approximation result (2.19) give us the desired error estimate (2.7).

3 Linear variable coefficients

In this section, we consider the two-dimensional scalar variable coefficient conservation law equation

$$\begin{cases} u_t + (a(x,y)u)_x + (b(x,y)u)_y = 0, \ (x,y) \in \Omega, \ t \geq 0 \\ u(x,y,0) = u_0(x,y), \ (x,y) \in \Omega \end{cases} \tag{3.1}$$

with periodic boundary condition. The functions $a(x,y), b(x,y)$ are smooth periodic functions in Ω. The semidiscrete DG method with upwind flux is described as follows: We seek an approximate solution $u_h \in V_h$ such that for all admissible test functions $v \in V_h$ and $K_{i,j}$,

$$\begin{aligned} &\int_{K_{i,j}} (u_h)_t v \ dxdy = \int_{K_{i,j}} a(x,y)u_h v_x + b(x,y)u_h v_y \ dxdy \\ &- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a(x_{i+\frac{1}{2}}, y)\hat{u_h}(x_{i+\frac{1}{2}}, y)v(x^-_{i+\frac{1}{2}}, y) - a(x_{i-\frac{1}{2}}, y)\hat{u_h}(x_{i-\frac{1}{2}}, y)v(x^+_{i-\frac{1}{2}}, y) \ dy \\ &- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b(x, y_{j+\frac{1}{2}})\tilde{u_h}(x, y_{j+\frac{1}{2}})v(x, y^-_{j+\frac{1}{2}}) - b(x, y_{j-\frac{1}{2}})\tilde{u_h}(x, y_{j-\frac{1}{2}})v(x, y^+_{j-\frac{1}{2}}) \ dx, \end{aligned} \tag{3.2}$$

where the upwind fluxes $\hat{u_h}, \tilde{u_h}$ are defined as follows

$$\hat{u_h}(x_{i+\frac{1}{2}}, y) = \begin{cases} u_h(x^-_{i+\frac{1}{2}}, y), \ \text{if } a(x_{i+\frac{1}{2}}, y) \geq 0 \\ u_h(x^+_{i+\frac{1}{2}}, y), \ \text{if } a(x_{i+\frac{1}{2}}, y) < 0 \end{cases} \tag{3.3}$$

$$\tilde{u_h}(x, y_{j+\frac{1}{2}}) = \begin{cases} u_h(x, y^-_{j+\frac{1}{2}}), \text{ if } b(x, y_{j+\frac{1}{2}}) \geq 0 \\ u_h(x, y^+_{j+\frac{1}{2}}), \text{ if } b(x, y_{j+\frac{1}{2}}) < 0. \end{cases} \tag{3.4}$$

For the initial condition, we simply take the L^2 projection into V_h, $u_h(0) = \mathbb{P}_h u_0$, and we have

$$\|u_0 - \mathbb{P}_h u_0\| \leq C h^{k+1} \|u_0\|_{k+1}. \tag{3.5}$$

The DG scheme satisfies the following L^2-stability,

Proposition 3.1. *The solution of the semidiscrete DG method defined by (3.2) satisfies*

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 \leq C\|u_h\|^2, \tag{3.6}$$

where the constant $C = \max\{\|a_x\|_\infty, \|b_y\|_\infty\}$.

Proof. The proof is similar to that for the linear constant coefficient case (e.g. in [11]), by taking the test function $v = u_h$ and applying integration by parts. □

3.1 A priori error estimates

Firstly, we state the a priori error estimates as a theorem whose proof will given in the next subsection. Here we assume that $a(x, y)$ and $b(x, y)$ do not change sign. Without loss of generality, we assumed $a(x, y) \geq 0$ and $b(x, y) \geq 0$.

Theorem 3.1. *The numerical solution* u_h *of the DG scheme (3.2)-(3.4) using uniform meshes for (3.1) with a smooth exact condition* $u(\cdot, t) \in H^{k+2}$ *satisfies the following* L^2 *error estimate:*

$$\|u(\cdot, T) - u_h(\cdot, T)\| \leq C h^{k+1}, \tag{3.7}$$

where u *is the exact solution of (3.1),* $k = 0, 1, 2, 3$ *is the degree of the piecewise polynomials in the finite element spaces* V_h*, and the constant* C *depends on the* $(k + 2)$*-th order Sobolev norm of the solution* $\|u(\cdot, t)\|_{k+2}$*, the* H^1 *norm of the coefficients* a, b *and the final time* T*, but is in dependent of the mesh size* h*.*

3.2 Proof of the error estimates

To prove Theorem 3.1 for the $k = 0, 1, 2, 3$ cases stated in the previous subsection, we proceed as follows. First, in subsection 3.2.1, we prove the uniform boundedness properties with respect to the coefficients a, b of the projection $\mathbb{P}^\star$ which is defined in (2.13) and a superconvergence result of the special projection. Then, we complete the proof of Theorem 3.1 in subsection 3.2.2.

3.2.1 The special projection

Notice that the definition of the special projection $\mathbb{P}^\star$ in (2.13) depends on the constants a, b. Thus, we use the new notation $\mathbb{P}_h^{a,b}$ to denote this projection. To obtain the optimal L^2 error estimate, we need the following results.

Lemma 3.1. *For $k = 0, 1, 2, 3$, the projection $\mathbb{P}_h^{a,b}$ defined by (2.13) on the reference cell $[-1, 1] \times [-1, 1]$ is uniformly bounded in the L^∞ norm with respect to the coefficients a, b, i.e.*

$$\|\mathbb{P}_h^{a,b}\omega\|_\infty \leq C(k)\|\omega\|_\infty, \tag{3.8}$$

where $C(k)$ is constant that only depends on k and not on a, b.

Proof. The proof of this lemma is provided in the Appendix; see section A.4.

From Lemma 3.1, we have the straightforward corollary as following.

Corollary 3.1. *For $k = 0, 1, 2, 3$, the projection $\mathbb{P}_h^{a,b}$ has the optimal approximation, for a smooth function ω,*

$$\|\omega - \mathbb{P}_h^{a,b}\omega\|_{L^2(K_{i,j})} \leq C(k)h^{k+1}\|\omega\|_{k+1,K_{i,j}}. \tag{3.9}$$

Besides the standard approximation results (3.9), the special projection $\mathbb{P}_h^{a,b}$ also has the following superconvergence result.

Proposition 3.2. *For $k = 0, 1, 2, 3$, if $a_1 a_2 > 0$, $b_1 b_2 > 0$ and $|a_1 - a_2| + |b_1 - b_2| \le Ch$, then the projections $\mathbb{P}_h^{a_l,b_l}$, $l = 1, 2$ defined by (2.13) on the rectangular cell $K_{i,j}$ have*

$$\max_l(|a_l|, |b_l|)\|\mathbb{P}_h^{a_1,b_1}\omega - \mathbb{P}_h^{a_2,b_2}\omega\|_{L^\infty(K_{i,j})} \le Ch^{k+2}|\omega|_{H^{k+1}(K_{i,j})}. \tag{3.10}$$

Proof. The proof of this proposition is provided in the Appendix; see section A.5.

Remark 3.1. *Lemma 3.1 shows that the projection is uniformly bounded with respect to a, b. Since $\mathbb{P}_h^{a,b}\omega$ is a polynomial of degree at most k, we only need to check that the coefficients for a particular set of basis functions, such as the Legendre polynomials, are uniformly bounded by ω. Also, the coefficients should be homogeneous rational functions in a, b. Thus Proposition 3.2 can be viewed as a corollary of Lemma 3.1. We will give more details in section A.4.*

Remark 3.2. *Here, we only provide the proof of the uniform boundedness of the special projection for $k = 0, 1, 2, 3$. In fact, it is straightforward to verify this property for any finite k. We have verified this until $k = 7$, without giving details here to save space. For a general proof for arbitrary k, it is challenging to find a unified general formula of the coefficients for a particular set of basis functions.*

3.2.2 Proof of Theorem 3.1

Now, we begin the proof of Theorem 3.1. Let us first introduce a few notations. We define

$$\begin{aligned}
A_{i,j}(u_h, v) = &\int_{K_{i,j}} (u_h)_t v \, dxdy - \int_{K_{i,j}} a(x,y)u_h v_x + b(x,y)u_h v_y \, dxdy \\
&+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a(x_{i+\frac{1}{2}}, y)u_h(x_{i+\frac{1}{2}}^-, y)v(x_{i+\frac{1}{2}}^-, y) - a(x_{i-\frac{1}{2}}, y)u_h(x_{i-\frac{1}{2}}^-, y)v(x_{i-\frac{1}{2}}^+, y) \, dy \\
&+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b(x, y_{j+\frac{1}{2}})u_h(x, y_{j+\frac{1}{2}}^-)v(x, y_{j+\frac{1}{2}}^-) - b(x, y_{j-\frac{1}{2}})u_h(x, y_{j-\frac{1}{2}}^-)v(x, y_{j-\frac{1}{2}}^+) \, dx.
\end{aligned}$$

We also define

$$\tilde{A}_{i,j}(u_h, v) = -\int_{K_{i,j}} a(x,y)u_h v_x + b(x,y)u_h v_y \, dxdy$$

$$+\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a(x_{i+\frac{1}{2}},y)u_h(x^-_{i+\frac{1}{2}},y)v(x^-_{i+\frac{1}{2}},y) - a(x_{i-\frac{1}{2}},y)u_h(x^-_{i-\frac{1}{2}},y)v(x^+_{i-\frac{1}{2}},y)\ dy$$
$$+\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b(x,y_{j+\frac{1}{2}})u_h(x,y^-_{j+\frac{1}{2}})v(x,y^-_{j+\frac{1}{2}}) - b(x,y_{j-\frac{1}{2}})u_h(x,y^-_{j-\frac{1}{2}})v(x,y^+_{j-\frac{1}{2}})\ dx.$$

Thus,

$$A_{i,j}(u_h,v) = \int_{K_{i,j}} (u_h)_t v\ dxdy + \tilde{A}_{i,j}(u_h,v). \tag{3.11}$$

Clearly, we have

$$A_{i,j}(u_h,v) = 0, \tag{3.12}$$

for all i,j and $v \in V_h$. It is also clear that the exact solution u of the PDE (3.1) satisfies

$$A_{i,j}(u,v) = 0, \tag{3.13}$$

for all i,j and $v \in V_h$. Subtracting (3.12) from (3.13), we obtain the error equation

$$A_{i,j}(u-u_h,v) = 0, \tag{3.14}$$

for all i,j and $v \in V_h$.

We now define $\mathbb{P}$ as the projection into V_h. We denote $a_{ij} = a(x_i,y_j)$, $b_{ij} = b(x_i,y_j)$, then, for cell $K_{i,j}$

$$\mathbb{P}u|_{K_{i,j}} = \mathbb{P}_h^{a_{ij},b_{ij}} u. \tag{3.15}$$

Remark 3.3. *Under our assumption, we have $a_{ij} \geq 0$, $b_{ij} \geq 0$. If a_{ij} or $b_{ij} = 0$, then $\|a\|_{L^\infty(K_{i,j})} = O(h)$ or $\|b\|_{L^\infty(K_{i,j})} = O(h)$. We can just set $a_{ij} = h$ or $b_{ij} = h$ to make sure that all a_{ij} and $b_{ij} > 0$ and then apply the projection $\mathbb{P}_h^{a_{ij},b_{ij}}$.*

We now take the test function $v = \mathbb{P}u - u_h$ in the error equation (3.14) and define

$$\eta = \mathbb{P}u - u \tag{3.16}$$

to obtain

$$A_{i,j}(v,v) = A_{i,j}(\eta,v). \tag{3.17}$$

For the left-hand side of (3.17), we use Proposition 3.1 to conclude

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le C\|v\|^2 + |\sum_{i,j} A_{i,j}(\eta, v)|. \tag{3.18}$$

We then write the right-hand side of (3.17) as a sum of two terms

$$A_{i,j}(\eta, v) = A^1_{i,j} + A^2_{i,j}, \tag{3.19}$$

where

$$\begin{aligned} A^1_{i,j} &= \int_{K_{i,j}} (\eta)_t v \; dxdy, \\ A^2_{i,j} &= \tilde{A}_{i,j}(\eta, v), \end{aligned}$$

and we will estimate each term separately.

By using the simple inequality (2.29) and the special projection properties (3.9) for $\partial_t \eta$, we have

$$\sum_{i,j} |A^1_{i,j}| \le \frac{1}{2}\|v\|^2 + Ch^{2k+2}. \tag{3.20}$$

For $A^2_{i,j}$, from Taylor expansion,

$$\|a(x,y) - a_{ij}\|_{L^\infty(K_{i,j})} = O(h), \quad \|b(x,y) - b_{ij}\|_{L^\infty(K_{i,j})} = O(h), \tag{3.21}$$

then

$$\begin{aligned} A^2_{i,j} &= -\int_{K_{i,j}} (a(x,y) - a_{ij})\eta v_x + (b(x,y) - b_{ij})\eta v_y \; dxdy \\ &+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (a(x_{i+\frac{1}{2}}, y) - a_{ij})\eta(x^-_{i+\frac{1}{2}}, y)v(x^-_{i+\frac{1}{2}}, y) - (a(x_{i-\frac{1}{2}}, y) - a_{ij})\eta(x^-_{i-\frac{1}{2}}, y)v(x^+_{i-\frac{1}{2}}, y) \; dy \\ &+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (b(x, y_{j+\frac{1}{2}}) - b_{ij})\eta(x, y^-_{j+\frac{1}{2}})v(x, y^-_{j+\frac{1}{2}}) - (b(x, y_{j-\frac{1}{2}}) - b_{ij})\eta(x, y^-_{j-\frac{1}{2}})v(x, y^+_{j-\frac{1}{2}}) \; dx \\ &+ \tilde{B}_{i,j}(u - \mathbb{P}_h^{a_{ij}, b_{ij}} u, v; a_{ij}, b_{ij}) \\ &+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a_{ij} \left(\mathbb{P}_h^{a_{i-1j}, b_{i-1j}} u - \mathbb{P}_h^{a_{ij}, b_{ij}} u \right) (x^-_{i-\frac{1}{2}}, y) v(x^+_{i-\frac{1}{2}}, y) \; dy \\ &+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b_{ij} \left(\mathbb{P}_h^{a_{ij-1}, b_{ij-1}} u - \mathbb{P}_h^{a_{ij}, b_{ij}} u \right) (x, y^-_{j-\frac{1}{2}}) v(x, y^+_{j-\frac{1}{2}}) \; dx \end{aligned}$$

$$\leq Ch^{2k+2}\|u\|^2_{k+2,K_{i,j}\cup K_{i-1,j}\cup K_{i,j-1}} + \|v\|^2_{0,K_{i,j}}, \tag{3.22}$$

where we have used the inequalities (2.29), (2.20) and (3.21) for the first three terms. For $\tilde{B}_{i,j}$, we have used the same argument as for the linear constant coefficient case. Finally, for the last two terms, we have used the superconvergence result of the special projection (3.10). We now sum over all i, j to obtain

$$\sum_{i,j} |A^2_{i,j}| \leq Ch^{2k+2} + \|v\|^2. \tag{3.23}$$

Combining (3.20), (3.23) with (3.18), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq C\|v\|^2 + Ch^{2k+2}\|u\|^2_{k+2,\Omega}. \tag{3.24}$$

This together with the approximation results (3.5), implies the desired error estimate (3.7).

4 The nonlinear case

In this section, we consider the two-dimensional scalar nonlinear conservation law equation

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \ (x, y) \in \Omega, \ t \geq 0 \\ u(x, y, 0) = u_0(x, y), \ (x, y) \in \Omega \end{cases} \tag{4.1}$$

The semidiscrete DG method with upwind fluxes is described as follows: We seek an approximate solution $u_h \in V_h$ such that for all admissible test functions $v \in V_h$ and $K_{i,j}$:

$$\int_{K_{i,j}} (u_h)_t v \, dxdy = \mathcal{H}_{i,j}(u_h, v), \tag{4.2}$$

where

$$\begin{aligned} \mathcal{H}_{i,j}(u_h, v) = &\int_{K_{i,j}} f(u_h)v_x + g(u_h)v_y \, dxdy \\ &- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{f}(u_h)(x_{i+\frac{1}{2}}, y)v(x^-_{i+\frac{1}{2}}, y) - \hat{f}(u_h)(x_{i-\frac{1}{2}}, y)v(x^+_{i-\frac{1}{2}}, y) \, dy \\ &- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{g}(u_h)(x, y_{j+\frac{1}{2}})v(x, y^-_{j+\frac{1}{2}}) - \tilde{g}(u_h)(x, y_{j-\frac{1}{2}})v(x, y^+_{j-\frac{1}{2}}) \, dx, \end{aligned} \tag{4.3}$$

where

$$\hat{f}(u_h)(x_{i+\frac{1}{2}}, y) \equiv \hat{f}(u_h(x^-_{i+\frac{1}{2}}, y), u_h(x^+_{i+\frac{1}{2}}, y)),\ \tilde{g}(u_h)(x, y_{j+\frac{1}{2}}) \equiv \tilde{g}(u_h(x, y^-_{j+\frac{1}{2}}), u_h(x, y^+_{j+\frac{1}{2}}))$$

are upwind monotone numerical fluxes that depend on the two values of the function u_h at the element interface point. For more details, see, for example, [12].

4.1 A priori error estimates

Let us state the a priori error estimate for the two-dimensional nonlinear equations. Here we assume that $f'(u)$ and $g'(u)$ do not change sign. Without loss of generality, we assume $f'(u) \geq 0$ and $g'(u) \geq 0$. In this case, all upwind monotone fluxes become $\hat{f}(u^-, u^+) = f(u^-)$ and $\hat{g}(u^-, u^+) = g(u^-)$. To deal with the nonlinearity of the flux $f(u)$ and $g(u)$, we could adopt the a priori assumption that, for $e = u - u_h$,

$$\|e\|_\infty \leq h, \tag{4.4}$$

which holds for h small enough. We will justify this assumption for piecewise polynomials of degree $k > 1$. This assumption is frequently used in the DG error analysis for nonlinear problems; see, e.g., [11, 1].

Theorem 4.1. *Let $u(\cdot, t) \in H^{k+2}$ be the solution of (4.1) with the flux function $f(u)$ and $g(u)$ sufficiently smooth such that $|f^{(m)}(u)| \lesssim 1$, $|g^{(m)}(u)| \lesssim 1$ $(m = 1, 2)$ and $|(\ln(f'(u)))_t| \lesssim 1$, $|(\ln(g'(u)))_t| \lesssim 1$. Suppose u_h is the numerical solution of the DG scheme (4.2) using uniform meshes satisfying the error assumption (4.4). If the initial discretization satisfies (2.5) then*

$$\|(u - u_h)(\cdot, T)\| \leq Ch^{k+1}, \tag{4.5}$$

where $k = 2, 3$, and the constant C depends on the exact solution u, the polynomial degree k, the final time T, and the maximum of $|f^{(m)}|$, $|g^{(m)}|$ $(m = 1, 2)$ and $|(\ln(f'(u)))_t|$, $|(\ln(g'(u)))_t|$ but is independent of h and the approximate solution u_h.

Remark 4.1. *We remark that the bounds we take for $|f^{(m)}|$ and $|g^{(m)}|$ are over $[m-h, M+h]$, m and M are the minimum and maximum of the initial condition $u_0(x,y)$ respectively.*

4.2 Proof of the error estimates

As before, we have the error equation,

$$\int_{K_{i,j}} (u-u_h)_t v \, dxdy = \mathcal{H}_{i,j}(u,v) - \mathcal{H}_{i,j}(u_h,v). \tag{4.6}$$

We now define the projection $\mathbb{P}u$ into V_h. We denote $u_{i,j} = u(x_i, y_j, t)$, then, for the element $K_{i,j}$

$$\mathbb{P}u|_{K_{i,j}} = \mathbb{P}_h^{f'(u_{i,j}),g'(u_{i,j})} u. \tag{4.7}$$

Now, we take the test function $v = u_h - \mathbb{P}u$ in the error equation (4.6) and denote $\eta = u - \mathbb{P}u$. Then the error $e = \eta - v$. To deal with the nonlinearity of the flux functions f, g, we used the Taylor expansion for $f(u)$ and $g(u)$,

$$\begin{aligned} f(u) &= f(u_h) + f'(u)(u-u_h) - \frac{\bar{f}''_u}{2}(u-u_h)^2, \\ g(u) &= g(u_h) + g'(u)(u-u_h) - \frac{\bar{g}''_u}{2}(u-u_h)^2, \end{aligned} \tag{4.8}$$

where $\bar{f}''_u = f''(\theta_1 u + (1-\theta_1)u_h)$ and $\bar{g}''_u = g''(\theta_2 u + (1-\theta_2)u_h)$ with $0 \le \theta_1, \theta_2 \le 1$. Then, we have

$$\mathcal{H}_{i,j}(u,v) - \mathcal{H}_{i,j}(u_h,v) = \mathcal{B}_{i,j}(e,\eta;v) - \mathcal{B}_{i,j}(e,v;v), \tag{4.9}$$

where

$$\begin{aligned} &\mathcal{B}_{i,j}(e,\eta;v) = \int_{K_{i,j}} \left(f'(u)\eta - \frac{\bar{f}''_u}{2}e\eta\right) v_x + \left(g'(u)\eta - \frac{\bar{g}''_u}{2}e\eta\right) v_y \, dxdy \\ &- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(f'(u)\eta - \frac{\bar{f}''_u}{2}e\eta\right)(x^-_{i+\frac{1}{2}}, y) v(x^-_{i+\frac{1}{2}}, y) - \left(f'(u)\eta - \frac{\bar{f}''_u}{2}e\eta\right)(x^-_{i-\frac{1}{2}}, y) v(x^+_{i-\frac{1}{2}}, y) \, dy \\ &- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(g'(u)\eta - \frac{\bar{g}''_u}{2}e\eta\right)(x, y^-_{j+\frac{1}{2}}) v(x, y^-_{j+\frac{1}{2}}) - \left(g'(u)\eta - \frac{\bar{g}''_u}{2}e\eta\right)(x, y^-_{j-\frac{1}{2}}) v(x, y^+_{j-\frac{1}{2}}) \, dx. \end{aligned} \tag{4.10}$$

By the assumption (4.4) and $\|f'(u) - f'(u_{i,j})\| = O(h)$, $\|g'(u) - g'(u_{i,j})\| = O(h)$, we have

$$\begin{aligned}\mathcal{B}_{i,j}(e,\eta;v) &\le Ch^{2k+2}\|u\|^2_{k+1,K_{i,j}} + \|v\|^2_{0,K_{i,j}} - \tilde{B}_{i,j}(\eta, v; f'(u_{i,j}), g'(u_{i,j}))\\ &\le Ch^{2k+2}\|u\|^2_{k+2,K_{i,j}} + \|v\|^2_{0,K_{i,j}},\end{aligned}\tag{4.11}$$

where for the last inequality we have used the same argument as that for the variable coefficient case (3.22). For $\mathcal{B}_{i,j}(e,v;v)$, we have the following estimate

$$\begin{aligned}-\mathcal{B}_{i,j}(e,v;v) \le& C\|v\|^2_{0,K_{i,j}} - f'(u_{i,j})\left(\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \frac{1}{2}v(x^-_{i+\frac{1}{2}},y)^2 - v(x^-_{i-\frac{1}{2}},y)v(x^+_{i-\frac{1}{2}},y) + \frac{1}{2}v(x^+_{i-\frac{1}{2}},y)^2\,dy\right)\\ &- g'(u_{i,j})\left(\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{1}{2}v(x,y^-_{j+\frac{1}{2}})^2 - v(x,y^-_{j-\frac{1}{2}})v(x,y^+_{j-\frac{1}{2}}) + \frac{1}{2}v(x,y^-_{j-\frac{1}{2}})^2\,dx\right).\end{aligned}$$

Summing over i,j and using the periodic boundary condition,

$$\begin{aligned}\sum_{i,j} -\mathcal{B}_{i,j}(e,v;v) \le& C\|v\|^2 - \sum_{i,j}\frac{f'(u_{i,j})}{2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left(v(x^-_{i-\frac{1}{2}},y) - v(x^+_{i-\frac{1}{2}},y)\right)^2 dy\\ &- \sum_{i,j}\frac{g'(u_{i,j})}{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(v(x,y^-_{j-\frac{1}{2}}) - v(x,y^+_{j-\frac{1}{2}})\right)^2 dx\\ \le& C\|v\|^2,\end{aligned}\tag{4.12}$$

where we have used the inverse inequality (2.20) and the fact $|f'(u_{i,j}) - f'(u_{i-1,j})| = O(h)$ and $|g'(u_{i,j}) - g'(u_{i,j-1})| = O(h)$. For the estimation of $\|\eta_t\|$, we denote $a = f'(u_{i,j})$ and $b = g'(u_{i,j})$ and take time derivative on the both sides of (2.13) to obtain

$$\begin{aligned}\widetilde{P_h}((\mathbb{P}^{a,b}_h u)_t, v; a, b)_{i,j} + \widetilde{P_h}(\mathbb{P}^{a,b}_h u, v; a_t, b_t)_{i,j} &= \widetilde{P_h}(u_t, v; a, b)_{i,j} + \widetilde{P_h}(u, v; a_t, b_t)_{i,j}\\ &= \widetilde{P_h}(\mathbb{P}^{a,b}_h u_t, v; a, b)_{i,j} + \widetilde{P_h}(u, v; a_t, b_t)_{i,j}\end{aligned}\tag{4.13}$$

then

$$\widetilde{P_h}(\mathbb{P}^{a,b}_h u_t - (\mathbb{P}^{a,b}_h u)_t, v; a, b)_{i,j} = \widetilde{P_h}(\mathbb{P}^{a,b}_h u - u, v; a_t, b_t)_{i,j}. \tag{4.14}$$

From the proof of lemma 3.1, there holds

$$\|\mathbb{P}_h^{a,b}u_t - (\mathbb{P}_h^{a,b}u)_t\| \lesssim \|\mathbb{P}_h^{a,b}u - u\| \lesssim h^{k+1} \tag{4.15}$$

if $|\frac{a_t}{a}| \lesssim 1$ and $|\frac{b_t}{b}| \lesssim 1$. Thus

$$\|\eta_t\| \leq \|u_t - \mathbb{P}_h^{a,b}u_t\| + \|\mathbb{P}_h^{a,b}u_t - (\mathbb{P}_h^{a,b}u)_t\| \lesssim h^{k+1}. \tag{4.16}$$

Combining (4.6), (4.11), (4.12) and (4.16), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,dxdy \leq C\|v\|^2 + Ch^{2k+2}\|u\|_{k+2,\Omega}^2. \tag{4.17}$$

An application of Gronwall's inequality together with the fact that $\|v(\cdot,0)\| \leq Ch^{k+1}$ give us,

$$\|v(\cdot,t)\| \leq Ch^{k+1}. \tag{4.18}$$

This, together with the approximation results (3.9), implies the desired error estimate (4.5).

Remark 4.2. *Let us justify the a priori assumption (4.4) for $k > 1$. Actually, we note that we only need to justify that*

$$\|\mathbb{P}u(t) - u_h(t)\| \leq h^{\frac{5}{2}} \tag{4.19}$$

holds for $t \in [0,T]$. If (4.19) holds for $t \in [0,T]$, then

$$\begin{aligned} \|u(t) - u_h(t)\|_\infty &\leq \|u(t) - \mathbb{P}u(t)\|_\infty + \|\mathbb{P}u(t) - u_h(t)\|_\infty \\ &\leq \|u(t) - \mathbb{P}u(t)\|_\infty + C_1h^{-1}\|\mathbb{P}u(t) - u_h(t)\| \\ &\leq \|u(t) - \mathbb{P}u(t)\|_\infty + C_1h^{\frac{3}{2}} \\ &\leq C_2h^k + C_1h^{\frac{3}{2}} \\ &\leq h, \end{aligned} \tag{4.20}$$

when h is small enough and $k > 1$. Therefore, the assumption (4.4) holds for $t \in [0, T]$. In the derivation above, C_1 is the constant in the third inequality in (2.20) and C_2 is the constant for the estimate of $\|u(t) - \mathbb{P}u(t)\|_\infty$, which is obtained as follows

$$\begin{aligned} \|u(t) - \mathbb{P}u(t)\|_\infty &\le \|u(t) - I_h u(t)\|_\infty + \|\mathbb{P}(u(t) - I_h u(t))\|_\infty \\ &\le \|u(t) - I_h u(t)\|_\infty + C\|u(t) - I_h u(t)\|_\infty \quad \textit{from (3.8)} \\ &\le (1 + C)C_3 h^k \end{aligned} \tag{4.21}$$

where $I_h u$ is the interpolation of u and C_3 is the constant for the interpolation error. Next we justify (4.19). First, (4.19) is satisfied at $t = 0$ since $u_h(0) = \mathbb{P}_h u_0$,

$$\|\mathbb{P}u(0) - u_h(0)\| = \|\mathbb{P}u_0 - \mathbb{P}_h u_0\| \le Ch^{k+1} \le h^{\frac{5}{2}} \tag{4.22}$$

when $k > 1$ and h is small enough. Define $t^\star = \sup\{s \le T : \|\mathbb{P}u(t) - u_h(t)\| \le h^{\frac{5}{2}}$ for all $t \in [0, s]\}$, then we have $\|\mathbb{P}u(t^\star) - u_h(t^\star)\| = h^{\frac{5}{2}}$ by continuity if $t^\star < T$. Clearly, (4.18) holds for $t = t^\star$. Since $k > 1$, when h is small enough we have $Ch^{k+1} \le \frac{1}{2}h^{\frac{5}{2}}$, where C is the constant in (4.18) determined by the time $t^\star$. Therefore, $\|\mathbb{P}u(t^\star) - u_h(t^\star)\| \le Ch^{k+1} \le \frac{1}{2}h^{\frac{5}{2}}$ which is a contraction. Thus we have $t^\star = T$, and the a priori assumption (4.4) is justified.

Remark 4.3. *We should remark that the restriction, $k > 1$, for the nonlinear case is artificial due to the technique in the proof. In our numerical example 5.3, we observe optimal convergence also for $k = 0$ and 1.*

5 Numerical examples

In this section, we present some numerical examples to verify our theoretical findings. In our numerical experiments, we presents the E_1, E_2, and E_∞ errors, respectively. They are defined by

$$E_1 = \int_\Omega |(u - u_h)(x, y, T)| \, dxdy, \tag{5.1}$$

$$E_2 = \left(\int_\Omega |(u-u_h)(x,y,T)|^2 \, dxdy \right)^{\frac{1}{2}}, \tag{5.2}$$

$$E_\infty = \max_\Omega |(u-u_h)(x,y,T)|. \tag{5.3}$$

In our all experiments, we used the DG scheme (2.2) using P^k polynomials with $k = 0, 1, 2, 3$ respectively. The computational domain, $[0, 2\pi] \times [0, 2\pi]$, is equally divided into $N \times N$ rectangles with $N = 10, 20, 40, 80, 160$ in our experiments. To reduce the time discretization error, the seventh-order strong stability-preserving Runge-Kutta method [9] with the time step $\Delta t = 0.05h$, $h = \frac{2\pi}{N}$ is used.

Example 5.1. *We firstly consider a linear constant coefficient equation with periodic boundary condition:*

$$\begin{cases} u_t + u_x + u_y = 0, \quad (x,y,t) \in [0,2\pi] \times [0,2\pi] \times (0,2\pi) \\ u(x,y,0) = \sin(x+y), \\ u(0,y,t) = u(2\pi,y,t), \quad u(x,0,t) = u(x,2\pi,t). \end{cases} \tag{5.4}$$

The exact solution to this problem is

$$u(x,y,t) = \sin(x+y-2t). \tag{5.5}$$

Table 5.1 shows that the order of convergence of the error achieves the expected $(k+1)$-th order of accuracy.

Example 5.2. *Next, we consider the linear variable coefficients equation with periodic boundary condition:*

$$\begin{cases} u_t + (a(x,y)u)_x + (b(x,y)u)_y = f, \quad (x,y,t) \in [0,2\pi] \times [0,2\pi] \times (0,2\pi) \\ u(x,y,0) = \sin(x+y), \\ u(0,y,t) = u(2\pi,y,t), \quad u(x,0,t) = u(x,2\pi,t). \end{cases} \tag{5.6}$$

where $a(x,y) = \sin(x+y)$, $b(x,y) = \cos(x+y)$ *and* $f = \cos(x+y-t) - 2\cos(x+y-2t) + \sin(2(x+y-t))$.

The exact solution to this problem is

$$u(x,y,t) = \sin(x+y-2t). \tag{5.7}$$

Table 5.1. The errors and corresponding convergence rates for the cases $k = 0, 1, 2, 3$. $T = 2\pi$ for Example 5.1

	$N \times N$	E_1	Rate	E_2	Rate	E_∞	Rate
$k=0$	10×10	5.63E+00	–	2.09E+00	–	9.88E-01	–
	20×20	4.95E+00	0.19	1.83E+00	0.19	8.76E-01	0.17
	40×40	3.61E+00	0.46	1.34E+00	0.46	6.35E-01	0.46
	80×80	2.24E+00	0.69	8.29E-01	0.69	3.94E-01	0.69
	160×160	1.26E+00	0.83	4.65E-01	0.83	2.21E-01	0.84
	320×320	6.67E-01	0.91	2.47E-01	0.91	1.17E-01	0.92
$k=1$	10×10	8.28E-01	–	3.14E-01	–	1.67E-01	–
	20×20	1.28E-01	2.70	4.94E-02	2.67	2.87E-02	2.54
	40×40	2.07E-02	2.63	8.51E-03	2.54	5.80E-03	2.30
	80×80	3.94E-03	2.39	1.83E-03	2.22	1.79E-03	1.69
	160×160	9.51E-04	2.05	4.41E-04	2.05	4.91E-04	1.87
	320×320	2.34E-04	2.02	1.10E-04	2.00	1.28E-04	1.94
$k=2$	10×10	3.18E-02	–	1.31E-02	–	1.33E-02	–
	20×20	3.48E-03	3.19	1.57E-03	3.06	1.77E-03	2.91
	40×40	4.28E-04	3.02	1.96E-04	3.00	2.22E-04	2.99
	80×80	5.34E-05	3.00	2.45E-05	3.00	2.78E-05	3.00
	160×160	6.67E-06	3.00	3.07E-06	3.00	3.48E-06	3.00
	320×320	8.34E-07	3.00	3.83E-07	3.00	4.35E-07	3.00
$k=3$	10×10	1.52E-02	–	3.67E-03	–	3.38E-03	–
	20×20	9.66E-04	3.98	2.33E-04	3.98	2.23E-04	3.92
	40×40	6.06E-05	3.99	1.46E-05	3.99	1.40E-05	3.99
	80×80	3.79E-06	4.00	9.15E-07	4.00	8.78E-07	4.00
	160×160	2.37E-07	4.00	5.72E-08	4.00	5.49E-08	4.00
	320×320	1.48E-08	4.00	3.58E-09	4.00	3.43E-09	4.00

The results in Table 5.2 show that the order of convergence of the error, $\|u-u_h\|_{L^2(\Omega)}$, achieves the expected $(k+1)$-th order of accuracy. We note that the coefficients $a(x, y)$ and $b(x, y)$ do change signs in this example, thus this example is not covered by our analysis. This indicates that probably the restriction in our analysis is artificial and due to the technique in our proof.

Example 5.3. *Finally, we consider the following nonlinear equation with periodic boundary condition:*

$$\begin{cases} u_t + (u^3)_x + (\exp(u))_y = f, \quad (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \times (0, 2\pi) \\ u(x, y, 0) = \sin(x + y), \\ u(0, y, t) = u(2\pi, y, t), \quad u(x, 0, t) = u(x, 2\pi, t). \end{cases} \tag{5.8}$$

where $f = \cos(2t - x - y)\,(-2 + \exp(-\sin(2t - x - y)) + 3\sin(2t - x - y)^2)$.

Table 5.2. The errors and corresponding convergence rates in cases $k = 0, 1, 2, 3$. $T = 2\pi$ for Example 5.2

	$N \times N$	E_1	Rate	E_2	Rate	E_∞	Rate
$k=0$	10×10	1.43E+01	–	2.25E+00	–	8.22E-01	–
	20×20	7.67E+00	0.90	1.23E+00	0.87	5.13E-01	0.68
	40×40	3.98E+00	0.95	6.44E-01	0.93	2.84E-01	0.85
	80×80	2.03E+00	0.97	3.31E-01	0.96	1.51E-01	0.92
	160×160	1.03E+00	0.98	1.68E-01	0.98	7.93E-02	0.93
	320×320	5.16E-01	0.99	8.46E-02	0.99	4.14E-02	0.94
$k=1$	10×10	1.97E+00	–	3.41E-01	–	2.00E-01	–
	20×20	4.92E-01	2.00	8.69E-02	1.97	5.04E-02	1.99
	40×40	1.20E-01	2.04	2.18E-02	2.00	1.34E-02	1.91
	80×80	2.95E-02	2.02	5.44E-03	2.00	3.37E-03	1.99
	160×160	7.34E-03	2.01	1.36E-03	2.00	8.36E-04	2.01
	320×320	1.84E-03	2.00	3.42E-04	2.00	2.08E-04	2.01
$k=2$	10×10	2.34E-01	–	4.13E-02	–	2.60E-02	–
	20×20	2.60E-02	3.17	4.93E-03	3.07	3.59E-03	2.86
	40×40	3.12E-03	3.06	6.05E-04	3.03	4.43E-04	3.02
	80×80	3.81E-04	3.03	7.51E-05	3.01	5.52E-05	3.00
	160×160	4.73E-05	3.01	9.37E-06	3.00	7.06E-06	2.97
	320×320	5.90E-06	3.00	1.17E-06	3.00	9.07E-07	2.96
$k=3$	10×10	2.08E-02	–	3.91E-03	–	3.39E-03	–
	20×20	1.12E-03	4.22	2.34E-04	4.06	2.32E-04	3.87
	40×40	6.57E-05	4.09	1.44E-05	4.02	1.44E-05	4.00
	80×80	4.06E-06	4.02	8.99E-07	4.00	9.02E-07	4.00
	160×160	2.53E-07	4.01	5.62E-08	4.00	5.57E-08	4.02
	320×320	1.58E-08	4.00	3.51E-09	4.00	3.45E-09	4.01

The exact solution to this problem is

$$u(x, y, t) = \sin(x + y - 2t). \tag{5.9}$$

The results in Table 5.3 also show the expected optimal order of convergence.

6 Concluding remarks

In this paper, optimal L^2 error estimates to DG methods applied to 2D hyperbolic equations are proved. Our analysis is carried out for both linear and nonlinear cases for uniform Cartesian meshes and piecewise P^k polynomial spaces. The result is valid for arbitrary polynomial degree $k \geq 0$ for linear constant coefficient equations. For variable

Table 5.3. The errors and corresponding convergence rates in cases $k = 0, 1, 2, 3$. $T = 1$ for Example 5.3

	$N \times N$	E_1	Rate	E_2	Rate	E_∞	Rate
$k = 0$	10×10	1.30E+01	–	2.41E+00	–	8.59E-01	–
	20×20	7.22E+00	0.85	1.42E+00	0.77	6.18E-01	0.48
	40×40	3.79E+00	0.93	7.99E-01	0.83	4.30E-01	0.52
	80×80	1.96E+00	0.95	4.43E-01	0.85	2.78E-01	0.63
	160×160	1.02E+00	0.95	2.45E-01	0.86	1.76E-01	0.66
	320×320	5.33E-01	0.93	1.34E-01	0.86	1.10E-01	0.68
$k = 1$	10×10	2.84E+00	–	4.54E-01	–	2.16E-01	–
	20×20	6.14E-01	2.21	1.02E-01	2.16	5.28E-02	2.03
	40×40	1.35E-01	2.18	2.34E-02	2.12	1.32E-02	2.00
	80×80	3.14E-02	2.11	5.62E-03	2.06	3.30E-03	2.00
	160×160	7.52E-03	2.06	1.38E-03	2.02	8.27E-04	2.00
	320×320	1.84E-03	2.03	3.44E-04	2.01	2.07E-04	2.00
$k = 2$	10×10	2.34E-01	–	4.56E-02	–	3.70E-02	–
	20×20	2.78E-02	3.08	5.72E-03	3.00	5.18E-03	2.84
	40×40	3.39E-03	3.03	7.04E-04	3.02	6.24E-04	3.05
	80×80	4.19E-04	3.02	8.69E-05	3.02	7.58E-05	3.04
	160×160	5.21E-05	3.01	1.08E-05	3.01	9.17E-06	3.05
	320×320	6.51E-06	3.00	1.34E-06	3.01	1.12E-06	3.04
$k = 3$	10×10	1.75E-02	–	4.15E-03	–	3.91E-03	–
	20×20	1.06E-03	4.04	2.43E-04	4.09	2.29E-04	4.10
	40×40	6.48E-05	4.04	1.49E-05	4.03	1.41E-05	4.02
	80×80	4.01E-06	4.01	9.23E-07	4.01	8.79E-07	4.01
	160×160	2.50E-07	4.01	5.76E-08	4.00	5.49E-08	4.00
	320×320	1.56E-08	4.00	3.60E-09	4.00	3.43E-09	4.00

coefficients and nonlinear equations, it holds true for polynomial degree $k = 0, 1, 2, 3$ and $k = 2, 3$, respectively, under the condition that $f'(u)$, $g'(u)$ do not change sign. The main ingredients in the proof are the construction and analysis of a special projection. The numerical examples also verify the results of our theoretical analysis. Extension of this work to nonuniform meshes and to arbitrary polynomial degree k for the variable coefficient and nonlinear equations is interesting and challenging, and constitutes our future work.

A Appendix: Proof of a few technical lemmas and propositions

In this appendix, we collect the proof of some of the technical lemmas and propositions in the error estimates.

A.1 Proof of Lemma 2.1

Proof. Note that the procedure to find $\mathbb{P}^{\star}\omega \in P^k([-1,1]^2)$ is to solve a linear system, so the existence and uniqueness are equivalent. Thus, we only prove the uniqueness of the projection $\mathbb{P}^{\star}$. We set $\omega_I(x) = \mathbb{P}^{\star}\omega(x)$ with $\omega(x) = 0$, and would like to prove $\omega_I(x) = 0$. By the definition of the projection $\mathbb{P}^{\star}$, we have

$$\begin{aligned}\widetilde{P_h}(\omega_I, v; a, b) = &- \int_{-1}^{1}\int_{-1}^{1} \omega_I v_\beta \, dxdy + \int_{-1}^{1} b\omega_I(x,1)(v(x,1) - v(x,-1)) \, dx \\ &+ \int_{-1}^{1} a\omega_I(1,y)(v(1,y) - v(-1,y)) \, dy = 0, \quad \forall v \in P^k([-1,1]^2), \end{aligned} \tag{A.1}$$

and

$$\int_{-1}^{1}\int_{-1}^{1} \omega_I(x,y) \, dxdy = 0. \tag{A.2}$$

Specially, we set $v = \omega_I \in P^k([-1,1]^2)$ to get

$$\widetilde{P_h}(\omega_I, \omega_I; a, b) = \frac{b}{2}\int_{-1}^{1} (\omega_I(x,1) - \omega_I(x,-1))^2 \, dx + \frac{a}{2}\int_{-1}^{1} (\omega_I(1,y) - \omega_I(-1,y))^2 \, dy = 0. \tag{A.3}$$

Thus

$$\omega_I(x,1) = \omega_I(x,-1), \forall x \in [-1,1]; \tag{A.4}$$

$$\omega_I(1,y) = \omega_I(-1,y), \forall y \in [-1,1]. \tag{A.5}$$

Then, we set $v = (\omega_I)_\beta \in P^k([-1,1]^2)$ and use (A.4) and (A.5) to obtain

$$\int_{-1}^{1}\int_{-1}^{1} (\omega_I)_\beta^2 \, dxdy = 0. \tag{A.6}$$

Therefore, we have:

$$(\omega_I)_\beta = a(\omega_I)_x + b(\omega_I)_y = 0. \tag{A.7}$$

This, together with (A.4), (A.5) and $\beta \neq (0,1)$ or $(1,0)$, implies $\omega_I(x,y) \equiv C$. Finally, (2.13a) implies $\omega_I \equiv 0$. We have now finished the proof of uniqueness.

We now move to the proof of the second part (2.18). We denote

$$\mathbb{P}^\star \omega(x,y) = \omega_I(x,y) = \sum_{i=1}^{M} a_i v_i(x,y). \tag{A.8}$$

where $M = \frac{(k+1)(k+2)}{2}$ is the number of the basis functions of $P^k([-1,1]^2)$, $\{v_1, v_2, \ldots, v_M\} = \{1, x, y, \ldots, x^m y^{l-m}, \ldots, y^k\}$, then we set the test function $v = v_i, 2 \le i \le M$. Thus:

$$\widetilde{P_h}(\omega_I, v_i; a, b) = \sum_{l=1}^{M} \alpha_{il} a_l, \quad 2 \le i \le M, \tag{A.9}$$

$$\int_{-1}^{1}\int_{-1}^{1} \omega_I(x,y)\ dxdy = \sum_{l=1}^{M} \alpha_{1l} a_l. \tag{A.10}$$

It is easy to prove $|\widetilde{P_h}(\omega, v_i; a, b)| \le C\|\omega\|_\infty$, and the coefficients $\alpha_{il}, 1 \le i \le M, 1 \le l \le M$ are independent of ω. We denote $\zeta = (a_1, a_2, \ldots, a_M)^T$, $A_{il} = \alpha_{il}$, and $b_1 = \int_{-1}^{1}\int_{-1}^{1} \omega(x,y)\ dxdy$, $b_l = \widetilde{P_h}(\omega; v_l), l = 2, \ldots, M$, $\gamma = (b_1, b_2, \ldots, b_M)^T$. We can solve the following linear system:

$$A\zeta = \gamma \tag{A.11}$$

to get $\zeta = A^{-1}\gamma$. Since each component of γ is bounded by $\|\omega\|_\infty$ and each component of A is dependent on the constants a, b, k, each component of ζ is bounded by $\|\omega\|_\infty$, i.e. $|a_i| \lesssim \|\omega\|_\infty$, $i = 1, 2, \ldots, M$. Thus $\|\mathbb{P}_h^\star \omega\|_\infty \le C\|\omega\|_\infty$, where C is dependent on a, b, k. □

A.2 Proof of Lemma 2.2

Proof. We just need to prove $x^{k+1-l}y^l - u_{(i,j)}(x,y) = (x-h_x)^{k+1-l}y^l - u_{(i-1,j)}(x-h_x, y), \forall (x,y) \in K_{i,j}$. We set $\tilde{v}(x,y) = x^{k+1-l}y^l - (x-h_x)^{k+1-l}y^l + u_{(i-1,j)}(x-h_x, y)$, then we just need to prove $u_{(i,j)}(x,y) = \tilde{v}(x,y)$. By the uniqueness of the projection $\mathbb{P}^\star$, we just need to check the following equations:

$$\int_{K_{i,j}} \tilde{v}(x,y)\ dxdy = \int_{K_{i,j}} u(x,y)\ dxdy, \tag{A.12}$$

$$\widetilde{P_h}(\tilde{v}, v; a, b)_{i,j} = \widetilde{P_h}(u, v; a, b)_{i,j} \quad \forall v \in P^k(K_{i,j}). \tag{A.13}$$

The first equation can be checked as follows

$$\begin{aligned}
&\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{v}(x,y)\ dxdy \\
&= \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_{(i-1,j)}(x-h_x,y) - (x-h_x)^{k+1-l} y^l + x^{k+1-l} y^l\ dxdy \\
&= \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{3}{2}}}^{x_{i-\frac{1}{2}}} u_{(i-1,j)}(x,y) - x^{k+1-l} y^l + (x+h_x)^{k+1-l} y^l\ dxdy \\
&= \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{3}{2}}}^{x_{i-\frac{1}{2}}} u_{(i-1,j)}(x,y) - x^{k+1-l} y^l\ dxdy + \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} x^{k+1-l} y^l\ dxdy \\
&= \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} x^{k+1-l} y^l\ dxdy,
\end{aligned}$$

where we have used the definition of projection $\mathbb{P}^\star$ in (2.13a). The second equation can be checked as follows

$$\begin{aligned}
\widetilde{P_h}(\tilde{v}, v; a, b)_{i,j} &= \widetilde{P_h}(u_{(i-1,j)}(x,y) - u(x,y), v(x+h_x, y); a, b)_{i-1,j} + \widetilde{P_h}(x^{k+1-l} y^l, v; a, b)_{i,j} \\
&= \widetilde{P_h}(x^{k+1-l} y^l, v; a, b)_{i,j} \quad \forall v \in P^k(K_{i,j}),
\end{aligned}$$

where we have used the fact $v(x+h_x, y) \in P^k(K_{i-1,j})$. Therefore the uniqueness of the projection $\mathbb{P}^\star$ implies that $u_{i,j}(x,y) = \tilde{v}(x,y)$. □

A.3 Proof of Proposition 2.2

Proof. We just prove one case $\tilde{B}_{i,j}(\mathbb{P}^\star u, v; a, b) = \tilde{B}_{i,j}(u, v; a, b)$, where $u = x^{k+1-l} y^l$, as the other cases follow the same lines. We use Lemma 2.2 to $\tilde{B}_{i,j}(\mathbb{P}^\star u, v; a, b)$:

$$\begin{aligned}
\tilde{B}_{i,j}(\mathbb{P}^\star u, v; a, b) = &-(\mathbb{P}^\star u, v_\beta)_{K_{i,j}} \\
&+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b \left(\mathbb{P}^\star u(x, y^-_{j+\frac{1}{2}}) v(x, y^-_{j+\frac{1}{2}}) - \mathbb{P}^\star u(x, y^-_{j-\frac{1}{2}}) v(x, y^+_{j-\frac{1}{2}}) \right)\ dx
\end{aligned}$$

$$+\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a\left(\mathbb{P}^\star u(x^-_{i+\frac{1}{2}},y)v(x^-_{i+\frac{1}{2}},y)-\mathbb{P}^\star u(x^-_{i-\frac{1}{2}},y)v(x^+_{i-\frac{1}{2}},y)\right)\,dy$$
$$=\widetilde{P_h}(\mathbb{P}^\star u-u,v;a,b)_{i,j}+\tilde{B}_{i,j}(u,v;a,b)$$
$$=\tilde{B}_{i,j}(u,v;a,b)\quad \forall v\in P^k(K_{i,j}).$$

□

A.4 Proof of Lemma 3.1

Proof. Without loss of generality, we assume $a,b>0$. Firstly, we denote $\{\varphi_l(x,y)\}_{l=1}^{10}$ as the standard orthogonal basis functions on $[-1,1]^2$, which are defined as follows for P^3,

$$\varphi_1(x,y)=\frac{1}{2};\quad \varphi_2(x,y)=\frac{\sqrt{3}}{2}x;\quad \varphi_3(x,y)=\frac{\sqrt{3}}{2}y;\quad \varphi_4(x,y)=\frac{3}{2}xy;$$
$$\varphi_5(x,y)=\frac{\sqrt{5}}{4}(3x^2-1);\quad \varphi_6(x,y)=\frac{\sqrt{5}}{4}(3y^2-1);\quad \varphi_7(x,y)=\frac{\sqrt{7}}{4}(-3x+5x^3);$$
$$\varphi_8(x,y)=\frac{\sqrt{15}}{4}(-1+3x^2)y;\quad \varphi_9(x,y)=\frac{\sqrt{15}}{4}(-1+3y^2)x;\quad \varphi_{10}(x,y)=\frac{\sqrt{7}}{4}(-3y+5y^3).$$

Since $\mathbb{P}_h^{a,b}\omega\in P^k$, we have the following representation,

$$\mathbb{P}_h^{a,b}\omega=\sum_{i=1}^{\frac{(k+1)(k+2)}{2}}\alpha_i\varphi_i(x,y). \tag{A.14}$$

It is easy to see that we just need to verify the coefficients are uniformly bounded by $\|\omega\|_\infty$ with a constant which does not depend on a,b. Next, we give the coefficients for $k=0,1,2,3$.

For $k=0$,

$$\alpha_1=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\omega(x,y)\,dxdy. \tag{A.15}$$

For $k=1$,

$$\alpha_1=\frac{\int_{-1}^1\int_{-1}^1\omega(x,y)\,dxdy}{2};\quad \alpha_2=\frac{\int_{-1}^1\sqrt{3}\omega(1,y)\,dy-\frac{1}{2}\sqrt{3}\int_{-1}^1\int_{-1}^1\omega(x,y)dydx}{3};$$
$$\alpha_3=\frac{\int_{-1}^1\sqrt{3}\omega(x,1)\,dx-\frac{1}{2}\sqrt{3}\int_{-1}^1\int_{-1}^1\omega(x,y)dydx}{3}. \tag{A.16}$$

For $k = 2$,

$$\alpha_1 = \frac{\int_{-1}^{1}\int_{-1}^{1}\omega(x,y)\ dxdy}{2}; \quad \alpha_2 = \frac{\sqrt{3}\int_{-1}^{1}\int_{-1}^{1}x\omega(x,y)dydx}{2};$$

$$\alpha_3 = \frac{\sqrt{3}\int_{-1}^{1}\int_{-1}^{1}y\omega(x,y)dydx}{2};$$

$$\alpha_4 = \frac{\int_{-1}^{1}3ay\omega(1,y)\,dy + \int_{-1}^{1}3bx\omega(x,1)\,dx - \frac{3}{2}\int_{-1}^{1}\int_{-1}^{1}(bx+ay)\omega(x,y)dydx}{3(a+b)};$$

$$\alpha_5 = \frac{2\sqrt{3}\int_{-1}^{1}\sqrt{3}\omega(1,y)\,dy - 3\left(\int_{-1}^{1}\int_{-1}^{1}\omega(x,y)dydx + 3\int_{-1}^{1}\int_{-1}^{1}x\omega(x,y)dydx\right)}{6\sqrt{5}};$$

$$\alpha_6 = \frac{2\sqrt{3}\int_{-1}^{1}\sqrt{3}\omega(x,1)\,dx - 3\left(\int_{-1}^{1}\int_{-1}^{1}\omega(x,y)dydx + 3\int_{-1}^{1}\int_{-1}^{1}y\omega(x,y)dydx\right)}{6\sqrt{5}}. \quad \text{(A.17)}$$

Since $\frac{a}{a+b} \leq 1$ and $\frac{b}{a+b} \leq 1$, α_4 is uniformly bounded by $\|\omega\|_\infty$.

For $k = 3$,

$$\alpha_1 = \frac{\int_{-1}^{1}\int_{-1}^{1}\omega(x,y)\ dxdy}{2}; \quad \alpha_2 = \frac{\sqrt{3}\int_{-1}^{1}\int_{-1}^{1}x\omega(x,y)dydx}{2};$$

$$\alpha_3 = \frac{\sqrt{3}\int_{-1}^{1}\int_{-1}^{1}y\omega(x,y)dydx}{2};$$

$$\alpha_4 = \frac{1}{12\left(5a^3+3a^2b+3ab^2+5b^3\right)}\left(36a^2b\int_{-1}^{1}y\omega(1,y)\,dy - 30ab^2\int_{-1}^{1}\left(-1+3y^2\right)\omega(1,y)\,dy\right.$$
$$+36ab^2\int_{-1}^{1}x\omega(x,1)\,dx - 30a^2b\int_{-1}^{1}\left(-1+3x^2\right)\omega(x,1)\,dx$$
$$-18ab\int_{-1}^{1}\int_{-1}^{1}(bx+ay)\omega(x,y)dydx + 15a^2\int_{-1}^{1}\int_{-1}^{1}\left(b\left(-1+3x^2\right)+6axy\right)\omega(x,y)dydx$$
$$\left.+15b^2\int_{-1}^{1}\int_{-1}^{1}\left(6bxy + a\left(-1+3y^2\right)\right)\omega(x,y)dydx\right);$$

$$\alpha_5 = \frac{1}{84\sqrt{5}}\left(-42\int_{-1}^{1}\int_{-1}^{1}\omega(x,y)dydx + 63\int_{-1}^{1}\int_{-1}^{1}\left(-1+5x^2\right)\omega(x,y)dydx\right);$$

$$\alpha_6 = \frac{1}{84\sqrt{5}}\left(-42\int_{-1}^{1}\int_{-1}^{1}\omega(x,y)dydx + 63\int_{-1}^{1}\int_{-1}^{1}\left(-1+5y^2\right)\omega(x,y)dydx\right);$$

$$\alpha_7 = \frac{\int_{-1}^{1}\sqrt{7}\omega(1,y)\,dy}{7} - \frac{3}{4\sqrt{7}}\left(2\int_{-1}^{1}\int_{-1}^{1}x\omega(x,y)dydx + \int_{-1}^{1}\int_{-1}^{1}\left(-1+5x^2\right)\omega(x,y)dydx\right);$$

$$\alpha_8 = \frac{1}{12\left(5a^3+3a^2b+3ab^2+5b^3\right)}\left(12\sqrt{15}a^3\int_{-1}^{1}y\omega(1,y)\,dy - 10\sqrt{15}a^2b\int_{-1}^{1}\left(-1+3y^2\right)\omega(1,y)\,dy\right.$$
$$+2\sqrt{15}b\left(3a^2+3ab+5b^2\right)\int_{-1}^{1}\left(-1+3x^2\right)\omega(x,1)\,dx + \sqrt{15}\left(12a^2b\int_{-1}^{1}x\omega(x,1)\,dx\right.$$
$$-6a^2\int_{-1}^{1}\int_{-1}^{1}(bx+ay)\omega(x,y)dydx - \left(3a^2+3ab+5b^2\right)\int_{-1}^{1}\int_{-1}^{1}\left(b\left(-1+3x^2\right)+6axy\right)\omega(x,y)dydx$$

$$
+5ab\int_{-1}^{1}\int_{-1}^{1}\left(6bxy+a\left(-1+3y^2\right)\right)\omega(x,y)dydx\Big)\Big);
$$

$$
\alpha_9=\frac{1}{12\left(5a^3+3a^2b+3ab^2+5b^3\right)}\Big(12\sqrt{15}ab^2\int_{-1}^{1}y\omega(1,y)\,dy-10ab^2\int_{-1}^{1}\left(-1+3x^2\right)\omega(x,1)\,dx
$$

$$
+2\sqrt{15}a\left(5a^2+3ab+3b^2\right)\int_{-1}^{1}\left(-1+3y^2\right)\omega(1,y)\,dy+\sqrt{15}\Big(12b^3\int_{-1}^{1}x\omega(x,1)\,dx
$$

$$
-6b^2\int_{-1}^{1}\int_{-1}^{1}(bx+ay)\omega(x,y)dydx+5ab\int_{-1}^{1}\int_{-1}^{1}\left(b\left(-1+3x^2\right)+6axy\right)\omega(x,y)dydx
$$

$$
-\left(5a^2+3ab+3b^2\right)\int_{-1}^{1}\int_{-1}^{1}\left(6bxy+a\left(-1+3y^2\right)\right)\omega(x,y)dydx\Big)\Big);
$$

$$
\alpha_{10}=\frac{\int_{-1}^{1}\sqrt{7}\omega(x,1)\,dx}{7}-\frac{3}{4\sqrt{7}}\left(2\int_{-1}^{1}\int_{-1}^{1}y\omega(x,y)dydx+\int_{-1}^{1}\int_{-1}^{1}\left(-1+5y^2\right)\omega(x,y)dydx\right). \tag{A.18}
$$

We just need to check $\alpha_4,\alpha_8,\alpha_9$ which are homogeneous rational functions of $a,b>0$. Thus by the Young inequality,

$$
a^lb^{k-l}\le\frac{l}{k}a^k+(1-\frac{l}{k})b^k,\quad 0\le l\le k, \tag{A.19}
$$

these coefficients are uniformly bounded by $\|\omega\|_\infty$. □

A.5 Proof of Proposition 3.2

Proof. We consider the projection on the reference cell $[-1,1]^2$. From the proof of Lemma 3.1, we can see that the coefficients are the homogeneous rational functions of $a,b>0$ and the denominators of the rational functions are positive. By the Young inequality (A.19), we can prove

$$
|\max(a,b)\frac{\partial\alpha_i}{\partial a}|\le C,\quad |\max(a,b)\frac{\partial\alpha_i}{\partial b}|\le C, \tag{A.20}
$$

where C is constant which depends on $\|\omega\|_\infty$. Thus, we have

$$
\begin{aligned}
\|\max_{l=1,2}(a_l,b_l)(\mathbb{P}_h^{a_1,b_1}\omega-\mathbb{P}_h^{a_2,b_2}\omega)\|_\infty &= \|\max_{l=1,2}(a_l,b_l)(\mathbb{P}_h^{a_1,b_1}(\omega-I\omega)-\mathbb{P}_h^{a_2,b_2}(\omega-I\omega))\|_\infty\\
&\le Ch\|\omega-I\omega\|_\infty,
\end{aligned} \tag{A.21}
$$

where $I\omega\in P^k$ is the interpolation approximation of ω. We finished the proof by the scaling argument. □

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