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Polynomially bounded error estimates for Trapezoidal Rule Convolution Quadrature[☆]

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ABSTRACT

We clarify the dependence with respect to the time variable of some estimates about the convergence of the Trapezoidal Rule based Convolution Quadrature method applied to hyperbolic problems. This requires a careful investigation of the article of Lehel Banjai where the first convergence estimates were introduced, and of some technical results from a classical paper of Christian Lubich.

1. Introduction

We present a careful analysis of Convolution Quadrature (CQ) based on the Trapezoidal Rule (TR). The reason for this is multiple. The original work of Lubich [1] extends his results [2] on multistep CQ for parabolic problems (parabolic character is reflected in having the Laplace transform of the operator extended to a sector around the negative axis) to hyperbolic problems (where the Laplace transform is defined only on a half plane). Because of Dahlquist barrier, only second order multistep CQ methods are available for hyperbolic problems, and the analysis in [1] excludes the TRCQ method for technical reasons. However, it is well known (and it has been tested repeatedly in the area of Time Domain Boundary Integral Equations — TDBIE) that the TR based method outperforms the first order backward Euler method and BDF2 which is much too dispersive. For a much detailed comparison of BDF2 and TR based CQ methods, including the former one's computational cost advantage in certain cases, we refer to [3]. Note that Runge–Kutta CQ schemes [4] with higher order and less dispersion are also available, and that a detailed time domain analysis is also missing from [5,6].

As a warning to the reader, let us say that this paper is quite technical, but it closes an important question (left open in the monograph [7]) as to how error estimates for TRCQ behave polynomially in time and there is no hidden Gronwall Lemma argument that would lead to exponential in time upper bounds. In the appendix of Banjai's paper [3], which we are polishing up, the estimates are written for finite time intervals and the behavior with respect to the final time is not specified.

Let us now briefly introduce the mathematical aspects of TRCQ. For algorithmic and practical introductions to the CQ methods, we recommend [8,9]. For a detailed introduction to the distributional language required for a deep understanding of CQ applied to TDBIE, see [7]. Our starting point is a couple of Banach spaces X and Y and the space

☆ In the first version of this paper, the title proposed by the second author was "Brushing up a theorem by Lehel Banjai on the convergence of Trapezoidal Rule Convolution Quadrature". During revision, the current title was adopted, however, the arXiv version of the paper retains its original title.

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$\mathcal{B}(X,Y)$ of bounded linear operators $X \to Y$, whose norm will be denoted $\|\cdot\|_{X\to Y}$. The second ingredient is the symbol of a momentarily hidden convolutional operator: we assume that we have an analytic function

$$\mathrm{F}:\mathbb{C}_+ \to \mathcal{B}(X,Y), \qquad \mathbb{C}_+ := \{s \in \mathbb{C} \,:\, \mathrm{Re}\, s > 0\} \tag{1.1}$$

satisfying

$$\|\mathrm{F}(s)\|_{X\to Y} \le C_F(\mathrm{Re}\, s)\,|s|^{\mu} \qquad \forall s \in \mathbb{C}_+ \tag{1.2}$$

where $C_F : (0,\infty) \to (0,\infty)$ is non-increasing and $C_F(x) \le c_0\, x^{-m}$ for some $m \ge 0$ when x is close to zero. We will be interested in symbols F where the parameter $\mu \ge 0$ in (1.2), but we will show some results (based on [1]), for negative values of μ as well. The TRCQ approximation of this symbol consists of defining

$$\mathrm{F}_\kappa(s) := \mathrm{F}(s_\kappa), \qquad s_\kappa := \frac{\delta(e^{-\kappa s})}{\kappa} = \frac{2}{\kappa}\tanh\left(\frac{\kappa s}{2}\right), \qquad \delta(\zeta) := 2\,\frac{1-\zeta}{1+\zeta}. \tag{1.3}$$

Here $\kappa > 0$ is the constant time-step (see more explanations later) of the underlying TR scheme (recall that δ in (1.3) is the characteristic function of the TR scheme). We will show that $s_\kappa \in \mathbb{C}_+$ for every $s \in \mathbb{C}_+$, so that the definition of F_κ makes sense, and we will also show that F_κ is a symbol with properties (1.1)–(1.2), although with different parameters to those of F.

Properties (1.1)–(1.2) ensure that F is the Laplace transform of a causal tempered $\mathcal{B}(X,Y)$-valued distribution, which we will name f. Moreover, F is the Laplace transform of the distributional time derivative of a certain order (depending on μ) applied to a function $h:\mathbb{R}\to\mathcal{B}(X,Y)$ which is causal ($h \equiv 0$ in $(-\infty,0)$), continuous, and polynomially bounded. See full details in [7, Chapters 2 & 3]. Under these conditions, we can define a convolutional product $f * g$ of the operator valued f acting on a causal X-valued distribution g, outputting a causal Y-valued distribution. Informally, we are dealing with

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau.$$

Similarly, F_κ is the Laplace transform of a causal tempered $\mathcal{B}(X,Y)$-valued distribution f_κ, and the TRCQ approximation consists of substituting $f*g$ by $f_\kappa * g$. In practice, what is computed are the values

$$(f_\kappa * g)(t_n) \qquad t_n := n\kappa \qquad n \in \mathbb{Z}, \quad n \ge 0, \tag{1.4}$$

although the theory is developed for the full real distribution $f_\kappa * g$. The time-step values of $f_\kappa * g$ are given by the discrete convolution

$$(f_\kappa * g)(t_n) := \sum_{m=0}^{n} \omega^F_{n-m}(\kappa) g(t_m), \qquad \mathrm{F}(\delta(\zeta)/\kappa) = \sum_{m=0}^{\infty} \omega^F_m(\kappa)\zeta^m. \tag{1.5}$$

In practice, the discrete convolutions (1.5) are computed using a parallel process, FFTs, and some kind of contour integration [8–10]. If applied to a linear system of ODEs with vanishing initial conditions, TRCQ is reduced to the TR scheme applied to the original system. One of the main field of applications of CQ for hyperbolic problems is in the area of TDBIE, using the language and ideas of the seminal papers of Bamberger and Ha-Duong [11,12]. More examples, including coupled systems of wave equations in bounded domains with TDBIE in their exterior, can be found in [13].

Even before its convergence analysis was entirely completed, TRCQ already found applications in the literature. Studying time-dependent scattering problem involving various types of obstacles, such as, anisotropic, thermoelastic, or piezoelectric, the authors of [14–17] presented numerical evidence for the convergence of TRCQ (see [17, Table 3–4 & Section 7], [16, Table 4-6]). The second order convergence properties of TRCQ were also observed studying the two-dimensional Schrödinger equation [18], Volterra equation with a convex kernel [19], and PDEs with memory [20]. More recently, a rich domain of applications of TRCQ has been opened in the numerical approximation of propagation of viscoelastic waves [21], as well as the HDG discretization of transient elastic waves [22].

2. The main theorem

To state the main theorem, we will use the Sobolev–Bochner space

$$W^n_+(\mathbb{R};X) := \{g \in \mathcal{C}^{n-1}(\mathbb{R};X) \,:\, g \equiv 0 \text{ in } (-\infty,0), \quad g^{(n)} \in L^1(\mathbb{R};X)\}.$$

Note that if $g \in W^n_+(\mathbb{R};X)$, then $g^{(m)}(0) = 0$ for $m \le n-1$. Moreover, since $g^{(n)} \in L^1(\mathbb{R};X)$, the functions $g^{(m)}:\mathbb{R}\to X$, for $m \le n-1$, are polynomially bounded, hence have well-defined Laplace transforms. We will also need the mth order linear differential operator

$$(\mathcal{P}_m g)(t) := e^{-t}(e^{\cdot} g)^{(m)}(t) = \sum_{\ell=0}^{m} \binom{m}{\ell} g^{(\ell)}(t).$$

The remainder of this paper will consist of a proof of the next theorem. Some easy (but somewhat cumbersome details) are avoided. The reader is welcome to look for the arXiv version of this document to find a much detailed proof of each single step. In the appendix, for a convenient comparison, we also included [1, Theorem 3.1] and [3, Theorem A.2] which are improved by the following main result of this paper.

Theorem 2.1. *Let* F *satisfy (1.1)–(1.2) for* $\mu \geq 0$ *and with* C_F *fulfilling the conditions given after (1.2). Let* f *be the distributional inverse Laplace transform of* F *and* f_κ *be its TRCQ approximation, i.e., the inverse Laplace transform of* F_κ *given in (1.3), for any given time-step* $\kappa \in (0, 1]$. *Consider the parameters*

$$m := \lceil \mu \rceil, \qquad \alpha := \lfloor \mu - m \rfloor + 5, \qquad \beta := \max\{2m+4, m+\alpha\}. \tag{2.1}$$

For any $g \in W_+^\beta(\mathbb{R}; X)$ *and* $t \geq 0$, *we have*

$$\|(f_\kappa - f) * g(t)\|_Y \leq \kappa^2 C(t^{-1}) \left(\int_0^t \|g^{(m+\alpha)}(\tau)\|_X \mathrm{d}\tau + \int_0^t \|\mathcal{P}_m g^{(m+4)}(\tau)\|_X \mathrm{d}\tau \right), \tag{2.2}$$

where

$$C(x) := C_F\,(\min\{x, 1\}/4)\,\frac{C_\mu}{\min\{x^\varepsilon, 1\}}, \qquad \varepsilon := \max\{2m - \mu + 1, \lfloor \mu \rfloor - \mu + 3\},$$

and C_μ *is a positive constant depending only on* μ.

Note that

$$\alpha = \begin{cases} 5, & \mu = m, \\ 4, & \mu \neq m, \end{cases} \qquad \beta = \begin{cases} 5, & \mu = 0, \\ 2m+4, & \mu > 0, \end{cases} \qquad 1 + \max\{m, 1\} \leq \varepsilon \leq 2 + \max\{m, 1\}.$$

3. The TRCQ discrete derivative

We now introduce some key functions for the estimates that follow. First of all, note that the function

$$\omega \longmapsto 2\,\frac{\tanh(\omega/2) - \omega/2}{\omega^3} = \frac{\delta(e^{-\omega}) - \omega}{\omega^3} = \sum_{\ell=0}^{\infty} b_\ell\,\omega^{2\ell}$$

is even and analytic in $B(0; \pi)$. We then define

$$D(\omega) := \sum_{\ell=0}^{\infty} \alpha_\ell\,\omega^{2\ell}, \qquad \alpha_\ell := |b_\ell|,$$

and note that D is also analytic in the same disk and that the function $[0, \pi) \ni x \mapsto x^2 D(x)$ is strictly increasing, non-negative and diverges as $x \to \pi$. Therefore, there exists a unique

$$c_0 \in (0, \pi), \qquad \text{such that} \qquad c_0^2 D(c_0) = 1.$$

Next, we define

$$E_m(\omega) := \max\{D^j(\omega) : j = 1, \ldots, m\}\,\frac{(1+\omega^2)^m - 1}{\omega^2},$$

and notice that $E_1 \equiv D$. Using these, we are going to present some properties of the characteristic function of the TR rule. At the end of this section we will give a technical result which will be a key tool in the proof of Theorem 2.1.

Lemma 3.1. *The following inequalities hold:*

(a) $\mathrm{Re}\,\delta(e^{-z}) \geq \dfrac{1}{2}\min\{\mathrm{Re}\,z, 1\}$ *for all* $z \in \mathbb{C}_+$.

(b) $|\delta(e^{-z})| \leq \dfrac{8}{\min\{\mathrm{Re}\,z, 1\}}$ *for all* $z \in \mathbb{C}_+$.

(c) $|\delta^m(e^{-z}) - z^m| \leq E_m(|z|)|z|^{m+2}$ *for all* $m \geq 1$ *and all* $z \in \mathbb{C}_+$ *with* $|z| < \pi$.

(d) $\mathrm{Re}\,\dfrac{\delta(e^{-z})}{z} \geq 1 - |z|^2 D(|z|)$ *for all* $z \in \mathbb{C}_+$ *with* $0 < |z| < c_0 < \pi$.

Proof. For $|\zeta| < 1$, it is easy to verify that $\mathrm{Re}\,\frac{1-\zeta}{1+\zeta} \geq \frac{1-|\zeta|}{1+|\zeta|}$. Using this with $\zeta = e^{-z}$ for $\mathrm{Re}\,z > 0$ and noting that $|e^{-z}| = e^{-\mathrm{Re}\,z}$, we write

$$\frac{1}{2}\mathrm{Re}\,\delta(e^{-z}) \geq \frac{1 - e^{-\mathrm{Re}\,z}}{1 + e^{-\mathrm{Re}\,z}} = \tanh\left(\frac{\mathrm{Re}\,z}{2}\right).$$

The proof of (a) follows then from

$$\tanh\frac{x}{2} \geq \frac{1}{4}\min\{x, 1\}. \tag{3.1}$$

We prove (b) by using (3.1) together with the triangle and reverse triangle inequalities in the following way

$$\frac{1}{2}|\delta(e^{-z})| \leq \frac{1+e^{-\mathrm{Re}\,z}}{1-e^{-\mathrm{Re}\,z}} = \coth\left(\frac{\mathrm{Re}\,z}{2}\right) \leq \frac{4}{\min\{\mathrm{Re}\,z, 1\}}.$$

To show (c) and (d), we define

$$\mathbb{C}_+ \cap B(0;\pi) \ni z \longmapsto q(z) := \frac{\delta(e^{-z}) - z}{z^3},$$

and observe that $|q(z)| \leq D(|z|)$ which holds because of the definition of D. Using this, it is not hard to see that

$$\begin{aligned}\left|\delta^m(e^{-z}) - z^m\right| &= \left|(z + z^3 q(z))^m - z^m\right| \\ &\leq \sum_{j=0}^{m-1}\binom{m}{j}|z|^j D^{m-j}(|z|)|z|^{3m-3j} \;\leq\; E_m(|z|)|z|^{m+2},\end{aligned}$$

which proves (c). For (d), we write

$$\mathrm{Re}\,\frac{\delta(e^{-z})}{z} \geq 1 - |z|^2|q(z)| \geq 1 - |z|^2 D(|z|) \qquad \forall z \in \mathbb{C}_+ \cap B(0;\pi).$$

Note that the result is stated (and later used) only for $|z| < c_0 < \pi$, which ensures that the right-hand side of the above inequality is positive. □

The discrete version of this lemma will be a building block for the rest of this paper.

Proposition 3.2. *For $\kappa \in (0, 1]$, the following inequalities hold:*

(a) $\mathrm{Re}\,s_\kappa \geq \dfrac{1}{2}\min\{\mathrm{Re}\,s, 1\}$ *for all* $s \in \mathbb{C}_+$.
(b) $|s_\kappa| \leq \dfrac{8}{\kappa^2\min\{\mathrm{Re}\,s, 1\}}$ *for all* $s \in \mathbb{C}_+$.
(c) $|s_\kappa^m - s^m| \leq E_m(|\kappa s|)\kappa^2|s|^{m+2}$ *for all* $m \geq 1$ *and all* $s \in \mathbb{C}_+$ *with* $|\kappa s| < \pi$.
(d) $\mathrm{Re}\,\dfrac{s_\kappa}{s} \geq 1 - |\kappa s|^2 D(|\kappa s|)$ *for all* $s \in \mathbb{C}_+$ *with* $0 < |\kappa s| < c_0 < \pi$.

Proof. The result follows from Lemma 3.1 by simply inserting $z = \kappa s$ and noting that $\kappa \in (0, 1]$. □

Lemma 3.3. *For $g \in W^2_+(\mathbb{R}; X)$, $\mathrm{G} := \mathcal{L}\{g\}$, and $\sigma > 0$ we have*

$$\int_{-\infty}^{\infty}\|\mathrm{G}(\sigma + i\omega)\|_X \mathrm{d}\omega \leq \frac{\pi}{\sigma}\int_0^{\infty}\|\ddot{g}(\tau)\|_X \mathrm{d}\tau.$$

Proof. We can easily estimate

$$\begin{aligned}\int_{-\infty}^{\infty}\|\mathrm{G}(\sigma + i\omega)\|_X \mathrm{d}\omega &= \int_{-\infty}^{\infty}\frac{1}{\sigma^2+\omega^2}\|(\sigma + i\omega)^2\mathrm{G}(\sigma + i\omega)\|_X \mathrm{d}\omega \\ &\leq \sup_{\mathrm{Re}\,s=\sigma}\|s^2\mathrm{G}(s)\|_X \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\sigma^2+\omega^2} \;=\; \sup_{\mathrm{Re}\,s=\sigma}\|\mathcal{L}\{\ddot{g}\}(s)\|_X\,\frac{\pi}{\sigma}.\end{aligned}$$

This finishes the proof. □

Proposition 3.4. *If $g \in W^{2m+4}_+(\mathbb{R}; X)$ with $m \geq 0$, and*

$$\mathrm{G}(s) := \mathcal{L}\{g\}(s), \qquad \mathrm{H}(s) := (s_\kappa^m - s^m)\mathrm{G}(s),$$

then for all $\sigma > 0$ we have

$$\int_{-\infty}^{\infty}\|\mathrm{H}(\sigma + i\omega)\|_X \mathrm{d}\omega \leq \kappa^2\frac{C_m^1}{\sigma\min\{\sigma^m, 1\}}\int_0^{\infty}\|(\mathcal{P}_m g^{(m+4)})(\tau)\|_X \mathrm{d}\tau,$$

where C_m^1 is a positive constant depending only on m.

Proof. For the sake of convenience, we will abuse notation by eliminating the explicit dependence with respect to ω in

$$s = s(\omega) := \sigma + i\omega, \qquad s_\kappa = s_\kappa(\omega) := \frac{1}{\kappa}\delta(e^{-(\sigma+i\omega)\kappa}),$$

where $\sigma > 0$ is fixed. We now take an arbitrary but fixed value of $c \in (0,\pi)$, and define the integration regions $I^1 := \{\omega \in \mathbb{R} : |\sigma + i\omega| \le c/\kappa\}$ and $I^2 := \{\omega \in \mathbb{R} : |\sigma + i\omega| \ge c/\kappa\}$, covering the entire real line. We split our target integral into three pieces, and work on them one by one

$$\int_{-\infty}^{\infty} \|\mathrm{H}(s)\|_X \mathrm{d}\omega \le \int_{I^1} |s_\kappa^m - s^m|\, \|\mathrm{G}(s)\|_X \mathrm{d}\omega + \int_{I^2} |s_\kappa|^m\, \|\mathrm{G}(s)\|_X \mathrm{d}\omega + \int_{I^2} |s|^m\, \|\mathrm{G}(s)\|_X \mathrm{d}\omega.$$

Since $|\kappa s| \le c$ on I^1 and E_m is increasing, Proposition 3.2(c) yields

$$\int_{I^1} |s_\kappa^m - s^m|\, \|\mathrm{G}(s)\|_X \mathrm{d}\omega \le \kappa^2 E_m(c) \int_{-\infty}^{\infty} |s|^{m+2}\, \|\mathrm{G}(s)\|_X \mathrm{d}\omega.$$

For the second integral, using Proposition 3.2(b) and the fact that $c \le |\kappa s|$ on I^2, we have

$$\int_{I^2} |s_\kappa|^m\, \|\mathrm{G}(s)\|_X \mathrm{d}\omega \le \frac{\kappa^2}{c^{2m+2}} \frac{8^m}{\min\{\sigma^m, 1\}} \int_{-\infty}^{\infty} \|s^{2m+2}\mathrm{G}(s)\|_X \mathrm{d}\omega.$$

Lastly, the definition of I^2 implies that

$$\int_{I^2} |s|^m\, \|\mathrm{G}(s)\|_X \mathrm{d}\omega \le \frac{\kappa^2}{c^2} \int_{-\infty}^{\infty} \|s^{m+2}\mathrm{G}(s)\|_X \mathrm{d}\omega.$$

Combining these three estimates we can write

$$\int_{-\infty}^{\infty} \|\mathrm{H}(s)\|_X \mathrm{d}\omega \le \kappa^2 \frac{e_{m,c}}{\min\{\sigma^m, 1\}} \int_{-\infty}^{\infty} (1 + |s|^m) \|s^{m+2}\mathrm{G}(s)\|_X \mathrm{d}\omega, \tag{3.2}$$

where

$$e_{m,c} := \max\left\{E_m(c) + \frac{1}{c^2}, \frac{8^m}{c^{2m+2}}\right\}.$$

The definition of $\mathcal{P}_m$ implies that

$$(1+s)^m s^{m+2}\mathrm{G}(s) = \mathcal{L}\{\mathcal{P}_m g^{m+2}\}(s).$$

Therefore, using that $1 + |s|^m \le 2^{m/2}|1+s|^m$ for $s \in \mathbb{C}_+$ and Lemma 3.3, we can write

$$\begin{aligned}\int_{-\infty}^{\infty} \|(1+|s|)^m s^{m+2}\mathrm{G}(s)\|_X \mathrm{d}\omega &\le 2^{m/2} \int_{-\infty}^{\infty} \|(1+s)^m s^{m+2}\mathrm{G}(s)\|_X \mathrm{d}\omega \\ &\le 2^{m/2}\frac{\pi}{\sigma} \int_0^{\infty} \|(\mathcal{P}_m g^{(m+4)})(\tau)\|_X \mathrm{d}\tau.\end{aligned}$$

This inequality and (3.2) prove the result with $C_m = 2^{m/2}\pi\, e_{m,c}$. However, the dependence on $c \in (0,\pi)$ is limited to $e_{m,c}$, so we can eliminate c by taking $e_m := \min_{c\in(0,\pi)} e_{m,c}$ in the bounds. □

4. Revisiting a result of Christian Lubich

In this section, we work on some key results when F satisfies (1.1)–(1.2) with $\mu \le 0$. We start with showing that F_κ, like F, is the Laplace transform of a causal tempered $\mathcal{B}(X,Y)$-valued distribution. In Proposition 4.4 we revisit Lubich's [1, Theorem 3.1], and prove it for $-1 < \mu \le 0$ by including the case $\mu = 0$ which was missing in that manuscript, and add the explicit dependence with respect to the time variable in the bounds.

Proposition 4.1. *If* F *satisfies* (1.1)–(1.2) *with* $\mu \le 0$, *then*

(a) $\|\mathrm{F}_\kappa(s)\|_{X\to Y} \le \Theta_1(\mathrm{Re}\, s)$ *for all* $s \in \mathbb{C}_+$.
(b) $\|\mathrm{F}'(s)\|_{X\to Y} \le \Theta_2(\mathrm{Re}\, s)|s|^\mu$ *for all* $s \in \mathbb{C}_+$.
(c) $\|\mathrm{F}_\kappa(s) - \mathrm{F}(s)\|_{X\to Y} \le \kappa^2\Theta_2\left(\tfrac{1}{2}\min\{\mathrm{Re}\, s, 1\}\right)\Theta_3(|\kappa s|)|s|^{\mu+3}$ *for all* $s \in \mathbb{C}_+ \cap B(0; c_0/\kappa)$.

In the above bounds

$$\Theta_1(x) := (\tfrac{1}{2}\min\{x,1\})^{\mu}\, C_F(\tfrac{1}{2}\min\{x,1\}),$$
$$\Theta_2(x) := \frac{2^{1-\mu}}{x} C_F(\tfrac{1}{2}x),$$
$$\Theta_3(x) := D(x)(1-x^2D(x))^{\mu}.$$

The functions Θ_1 and Θ_2 are defined on $(0,\infty)$ and they can be bounded by a negative power of x as $x \to 0$. The function Θ_3 is defined on $(0, c_0)$, is increasing, and when $\mu \neq 0$ it diverges as $x \to c_0$.

Proof. To prove (a), we first observe that Proposition 3.2(a) implies $s_\kappa \in \mathbb{C}_+$, therefore (1.2) gives

$$\|\mathrm{F}_\kappa(s)\|_{X\to Y} \le C_F(\mathrm{Re}\, s_\kappa)|s_\kappa|^{\mu}.$$

The rest of the proof follows from Proposition 3.2(a) and the fact that C_F and $(\cdot)^{\mu}$ are non-decreasing functions on $(0,\infty)$.

For (b), we use same ideas given in the proof of [7, Proposition 4.5.3]. Defining the curve $\Xi(s) := \{z \in \mathbb{C} : |z-s| = \frac{1}{2}\mathrm{Re}\, s\}$ with positive orientation, we write

$$\mathrm{F}'(s) = \frac{1}{2\pi i}\int_{\Xi(s)} \frac{\mathrm{F}(z)}{(z-s)^2}\mathrm{d}z.$$

We finish the proof of (b) using the fact that $\frac{1}{2}|s| \le |z|$ and $\frac{1}{2}\mathrm{Re}\, s \le \mathrm{Re}\, z$ for $z \in \Xi(s)$.

To show (c), we write the following by using the Mean Value Theorem

$$\|\mathrm{F}_\kappa(s) - \mathrm{F}(s)\|_{X\to Y} \le \|\mathrm{F}'(\lambda s_\kappa + (1-\lambda)s)\|_{X\to Y}\, |s_\kappa - s|,$$

for some $\lambda \in (0,1)$. Now, we define $z(s) := \lambda s_\kappa + (1-\lambda)s$, and use (a) to write

$$\|\mathrm{F}'(z)\|_{X\to Y} \le \Theta_2(\mathrm{Re}\, z)|z|^{\mu}.$$

This can be bounded by observing

$$\mathrm{Re}\, z \ge \min\{\mathrm{Re}\, s_\kappa, \mathrm{Re}\, s\}, \qquad |z| \ge |s|\,\mathrm{Re}\,\frac{z}{s} \ge |s|\min\left\{\mathrm{Re}\,\frac{s_\kappa}{s}, 1\right\},$$

and then using Proposition 3.2(a) and (d). We finish the proof by using Proposition 3.2(c) with $m = 1$ and noting that $E_1 \equiv D$. □

Lemma 4.2. *The following holds for all $\sigma > 0$, $\alpha > 1$, $\kappa \in (0,1]$ and $c > 0$:*

(a) $\displaystyle\int_{|\sigma+i\omega|\ge c/\kappa} |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega \le \frac{2\alpha}{\alpha-1}\left(\frac{\kappa}{c}\right)^{\alpha-1}.$

(b) $\displaystyle\int_{-\infty}^{\infty} |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega \le \frac{2}{\sigma^{\alpha}} + \frac{2}{\alpha-1}.$

Proof. In order to prove (a), for fixed σ, c and κ, we define the domains of integration $I^1 := \{\omega \in \mathbb{R} : |\sigma+i\omega| \ge c/\kappa, |\omega| \le c/\kappa\}$ and $I^2 := \{\omega \in \mathbb{R} : |\omega| \ge c/\kappa\}$, which give

$$\int_{|\sigma+i\omega|\ge c/\kappa} |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega = \int_{I^1} |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega + \int_{I^2} |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega.$$

We bound the first integral using the fact that $|\sigma+i\omega|^{-\alpha} \le (c/\kappa)^{-\alpha}$ on I^1. We rewrite the second integral using a change of variables and bound it in the following way

$$\kappa^{\alpha-1}\int_{|\omega|\ge c} |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega \le \kappa^{\alpha-1}\int_{|\omega|\ge c} |\omega|^{-\alpha}\,\mathrm{d}\omega = \frac{2}{\alpha-1}\left(\frac{\kappa}{c}\right)^{\alpha-1},$$

which finishes the proof of (a). We prove (b) by simply writing

$$\int_{-\infty}^{\infty} |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega \le 2\int_0^1 \sigma^{-\alpha}\,\mathrm{d}\omega + 2\int_1^{\infty}\omega^{-\alpha}\,\mathrm{d}\omega = \frac{2}{\sigma^{\alpha}} + \frac{2}{\alpha-1}. \quad \square$$

Proposition 4.3. *If F satisfies (1.1)–(1.2) with $-1 < \mu \le 0$ and $\alpha = \lfloor \mu + 5 \rfloor$, then for all $\sigma > 0$ we have*

$$\int_{-\infty}^{\infty} \|\mathrm{F}_\kappa(\sigma+i\omega) - \mathrm{F}(\sigma+i\omega)\|_{X\to Y}\, |\sigma+i\omega|^{-\alpha}\,\mathrm{d}\omega \le \kappa^2\, C_\mu^1\, C_2(\sigma),$$

where

$$C_2(x) := \frac{C_F(\frac{1}{4}\min\{x,1\})}{\min\{x^{2+\delta},1\}}, \qquad \delta := \lfloor\mu\rfloor - \mu + 1 \in (0,1],$$

and C_μ^1 is a positive constant depending only on μ.

Proof. For fixed σ, κ and $c \in (0, c_0)$ we will make use of the following domains of integration

$$I^1 := \{\omega \in \mathbb{R} : |\sigma + i\omega| \le c/\kappa\}, \qquad I^2 := \{\omega \in \mathbb{R} : |\sigma + i\omega| \ge c/\kappa\},$$

and the notation $s = s(\omega) := \sigma + i\omega$. Using these we can write

$$\int_{-\infty}^{\infty} \|\mathrm{F}_\kappa(s) - \mathrm{F}(s)\|_{X\to Y}\,|s|^{-\alpha}\,\mathrm{d}\omega \le \int_{I^1} \|\mathrm{F}_\kappa(s) - \mathrm{F}(s)\|_{X\to Y}\,|s|^{-\alpha}\,\mathrm{d}\omega + \int_{I^2} \left(\|\mathrm{F}_\kappa(s)\|_{X\to Y} + \|\mathrm{F}(s)\|_{X\to Y}\right)\,|s|^{-\alpha}\,\mathrm{d}\omega.$$

We bound the first integral on the right-hand side using Proposition 4.1(c) and Lemma 4.2(b)

$$\begin{aligned}\int_{I^1} \|\mathrm{F}_\kappa(s) - \mathrm{F}(s)\|_{X\to Y}\,|s|^{-\alpha}\,\mathrm{d}\omega &\le \kappa^2 \Theta_2(\min\{\sigma,1\}/2)\Theta_3(c)\left(\frac{2}{\sigma^{\alpha-\mu-3}} + \frac{2}{\alpha-\mu-4}\right)\\ &\le \kappa^2 C_F(\min\{\sigma,1\}/4)\frac{e_\mu^1\,\Theta_3(c)}{\min\{\sigma^{2+\delta},1\}},\end{aligned}$$

where e_μ^1 is a positive constant depending only on μ. Next, with the help of Proposition 4.1(a) and Lemma 4.2(a), the second integral is bounded in the following way

$$\begin{aligned}\int_{I^2} \left(\|\mathrm{F}_\kappa(s)\|_{X\to Y} + \|\mathrm{F}(s)\|_{X\to Y}\right)\,|s|^{-\alpha}\,\mathrm{d}\omega &\le \left(\Theta_1(\sigma) + C_F(\sigma)\sigma^\mu\right)\frac{2\alpha}{\alpha-1}\left(\frac{\kappa}{c}\right)^{\alpha-1}\\ &\le \kappa^2 C_F(\min\{\sigma,1\}/2)\frac{e_\mu^2\,c^{1-\alpha}}{\min\{\sigma,1\}}.\end{aligned}$$

Here e_μ^2 is a positive constant depending only on μ. Combining these estimates we write

$$\int_{-\infty}^{\infty} \|\mathrm{F}_\kappa(s) - \mathrm{F}(s)\|_{X\to Y}\,|s|^{-\alpha}\,\mathrm{d}\omega \le \kappa^2 C_F(\min\{\sigma,1\}/4)\frac{C_{\mu,c}}{\min\{\sigma^{2+\delta},1\}},$$

with $C_{\mu,c} := e_\mu^1\,\Theta_3(c) + e_\mu^2\,c^{1-\alpha}$. Note that this estimate holds for all $c \in (0, c_0)$, therefore we finish the proof by replacing the constant $C_{\mu,c}$ with $C_\mu^1 := \min_{c\in(0,c_0)} C_{\mu,c}$. $\square$

Proposition 4.4. *If* F *satisfies* (1.1)–(1.2) *with* $-1 < \mu \le 0$, *and* $g \in W_+^\alpha(\mathbb{R}; X)$ *with* $\alpha := \lfloor\mu + 5\rfloor$, *then*

$$\|(f_\kappa - f) * g(t)\|_Y \le \kappa^2\, C_\mu^2\, C_2(t^{-1}) \int_0^t \|g^{(\alpha)}(\tau)\|_X \mathrm{d}\tau$$

holds for all $t \ge 0$, where C_μ^2 is a positive constant depending only on μ.

Proof. For any $\sigma > 0$ and $t > 0$, using the inverse Laplace transformation, we write

$$\|(f_\kappa - f) * g(t)\|_Y \le \frac{e^{\sigma t}}{2\pi} \sup_{\mathrm{Re}\,s=\sigma} \|s^\alpha \mathcal{L}\{g\}(s)\|_X \int_{-\infty}^{\infty} \|(\mathrm{F}_\kappa - \mathrm{F})(\sigma + i\omega)\|_{X\to Y}\,|\sigma + i\omega|^{-\alpha}\,\mathrm{d}\omega.$$

Next, the definition of Laplace transformation together with Proposition 4.3 and setting $\sigma = t^{-1}$ give

$$\|(f_\kappa - f) * g(t)\|_Y \le \kappa^2\, C_\mu^2\, C_2(t^{-1}) \int_0^\infty \|g^{(\alpha)}(\tau)\|_X \mathrm{d}\tau,$$

for a positive constant C_μ^2. Now, we are going to obtain an integral bound over the interval $(0, t)$. For a fixed $t > 0$, we define the following function

$$p(\tau) := \begin{cases} g(\tau), & \tau \le t,\\ \displaystyle\sum_{\ell=0}^{\alpha-1} \frac{(\tau - t)^\ell}{\ell!} g^{(\ell)}(t), & \tau \ge t.\end{cases}$$

It is not hard to see that $p \in W^{\alpha}_{+}(\mathbb{R}; X)$, in other words, p satisfies the conditions of the proposition. Using the fact that $p \equiv g$ on $(-\infty, t)$ and $p^{(\alpha)} \equiv 0$ on (t, ∞), we write

$$\begin{aligned}\|(f_\kappa - f) * g(t)\|_Y = \|(f_\kappa - f) * p(t)\|_Y &\le \kappa^2\, C_\mu^2\, C_2(t^{-1}) \int_0^\infty \|p^{(\alpha)}(\tau)\|_X \,\mathrm{d}\tau \\ &= \kappa^2\, C_\mu^2\, C_2(t^{-1}) \int_0^t \|g^{(\alpha)}(\tau)\|_X \,\mathrm{d}\tau,\end{aligned}$$

which finishes the proof. □

5. Proof of Theorem 2.1

In this section we prove the main theorem. We start with presenting a lemma to obtain upper-bounds integrated over the interval $(0, t)$ rather than $(0, \infty)$, which will be used in the proof of the main theorem.

Lemma 5.1. *Let $m \ge 0$, $g \in W^{2m+4}_{+}(\mathbb{R}; X)$, $h := e^{\cdot} g^{(m+4)}$, and $t > 0$ be a fixed real number. We define the function*

$$\mathbb{R} \ni \omega \longmapsto j(\omega) := e^{-\omega} \sum_{\ell=0}^{m-1} \frac{(\omega - t)^\ell}{\ell!} h^{(\ell)}(t),$$

and, for $n \ge 1$, the integration operator

$$(\partial^{-n} f)(\tau) := \int_t^\tau \int_t^{\omega_1} \cdots \int_t^{\omega_{n-1}} f(\omega_n)\,\mathrm{d}\omega_n \,\ldots\, \mathrm{d}\omega_2\,\mathrm{d}\omega_1.$$

The function

$$p(\tau) := \left\{ \begin{array}{ll} g(\tau), & \tau \le t, \\ \displaystyle\sum_{\ell=0}^{m+3} \frac{(\tau - t)^\ell}{\ell!} g^{(\ell)}(t) + (\partial^{-m-4} j)(\tau), & \tau \ge t, \end{array} \right.$$

satisfies that $p \in W^{2m+4}_{+}(\mathbb{R}; X)$ and $\mathcal{P}_m p^{(m+4)} \equiv 0$ on (t, ∞).

Proof. We observe that, for $0 \le k \le m + 3$, the functions

$$\mathbb{R} \ni \tau \longmapsto (\partial^k\, \partial^{-m-4} j)(\tau) = (\partial^{-m-4+k} j)(\tau)$$

vanish when $\tau = t$. From there, it is not hard to see that $p \in \mathcal{C}^{m+3}(\mathbb{R}; X)$. Next, since $h \in \mathcal{C}^{m-1}(\mathbb{R}; X)$, we know that

$$q(\tau) := \left\{ \begin{array}{ll} h(\tau), & \tau \le t, \\ \displaystyle\sum_{\ell=0}^{m-1} \frac{(\tau - t)^\ell}{\ell!} h^{(\ell)}(t), & \tau \ge t, \end{array} \right.$$

is also in $\mathcal{C}^{m-1}(\mathbb{R}; X)$, and so is $e^{-\cdot} q = p^{(m+4)}$. This shows that $p \in \mathcal{C}^{2m+3}(\mathbb{R}; X)$. The function $j^{(m)} \in L^1(t, \infty; X)$ and therefore $p^{(2m+4)} \in L^1(\mathbb{R}; X)$. The rest of the proof follows from the fact that

$$(\mathcal{P}_m p^{(m+4)})(\tau) = e^{-\tau} \frac{\mathrm{d}^m}{\mathrm{d}\tau^m} \left(\sum_{\ell=0}^{m-1} \frac{(\tau - t)^\ell}{\ell!} h^{(\ell)}(t) \right)(\tau) = 0 \qquad \forall \tau \in (t, \infty). \quad \square$$

Proposition 5.2. *Let $g \in W^{2m+4}_{+}(\mathbb{R}; X)$ with $m \ge 0$, and F satisfy (1.1)–(1.2). We define $(\partial_t^\kappa)^m g$ such that*

$$\mathcal{L}\{(\partial_t^\kappa)^m g\}(s) := s_\kappa^m \mathrm{G}(s), \qquad \mathrm{G}(s) := \mathcal{L}\{g\}(s).$$

The following estimate holds for all $t > 0$

$$\|f * ((\partial_t^\kappa)^m g - g^{(m)})(t)\|_Y \le \kappa^2 C_1(t^{-1}) \sup_{\mathrm{Re}\, s = t^{-1}} \|\mathrm{F}(s)\|_{X \to Y} \int_0^t \|\mathcal{P}_m g^{(m+4)}(\tau)\|_X \mathrm{d}\tau,$$

where

$$C_1(x) := \frac{C_m}{x \min\{x^m, 1\}},$$

and C_m is a positive constant depending only on m.

Proof. For any $\sigma > 0$, using the inverse Laplace transformation, we write

$$\|f * ((\partial_t^\kappa)^m g - g^{(m)})(t)\|_Y \le \frac{e^{\sigma t}}{2\pi} \sup_{\mathrm{Re}\, s=\sigma} \|\mathrm{F}(s)\|_{X\to Y} \int_{-\infty}^{\infty} \|\mathrm{H}(\sigma + i\omega)\|_X \mathrm{d}\omega,$$

where $\mathrm{H}(s) := s_\kappa^m \mathrm{G}(s) - s^m \mathrm{G}(s)$. Here, with the help of Proposition 3.4 and inserting $\sigma = t^{-1}$, we obtain

$$\|f * ((\partial_t^\kappa)^m g - g^{(m)})(t)\|_Y \le \kappa^2 C_1(t^{-1}) \sup_{\mathrm{Re}\, s=t^{-1}} \|\mathrm{F}(s)\|_{X\to Y} \int_0^{\infty} \|\mathcal{P}_m g^{(m+4)}(\tau)\|_X \mathrm{d}\tau. \tag{5.1}$$

Now, our goal is to have an integral bound over the interval $(0, t)$. To do that, for fixed $t > 0$, we consider the function p introduced in Lemma 5.1. Since $p \in W_+^{2m+4}(\mathbb{R}; X)$, in other words, it satisfies the conditions of this proposition, we can have the estimate (5.1) for p as well. Therefore, using the properties of this function, we write

$$\begin{aligned}\|f * ((\partial_t^\kappa)^m g - g^{(m)})(t)\|_Y &= \|f * ((\partial_t^\kappa)^m p - p^{(m)})(t)\|_Y \\ &\le \kappa^2 C_1(t^{-1}) \sup_{\mathrm{Re}\, s=t^{-1}} \|\mathrm{F}(s)\|_{X\to Y} \int_0^{\infty} \|\mathcal{P}_m p^{(m+4)}(\tau)\|_X \mathrm{d}\tau \\ &= \kappa^2 C_1(t^{-1}) \sup_{\mathrm{Re}\, s=t^{-1}} \|\mathrm{F}(s)\|_{X\to Y} \int_0^{t} \|\mathcal{P}_m g^{(m+4)}(\tau)\|_X \mathrm{d}\tau.\end{aligned}$$

This finishes the proof. □

Proof of Theorem 2.1. For $s \in \mathbb{C}_+$, we define

$$\mathrm{F}^m(s) := s^{-m}\mathrm{F}(s) = \mathcal{L}\{f^m\}(s), \qquad \mathrm{F}_\kappa^m(s) := s_\kappa^{-m}\mathrm{F}(s_\kappa) = \mathcal{L}\{f_\kappa^m\}(s),$$

and

$$\mathrm{G}(s) := \mathcal{L}\{g\}(s), \qquad \mathrm{H}(s) := s_\kappa^m \mathrm{G}(s) - s^m \mathrm{G}(s) = \mathcal{L}\{h\}(s).$$

Using these definitions it is not hard to see that

$$(f_\kappa - f) * g = (f_\kappa^m - f^m) * g^{(m)} + f_\kappa^m * h.$$

Now, we will obtain bounds for the terms on the right-hand side. For the first term, since $g^{(m)} \in W_+^\alpha(\mathbb{R}; X)$ and

$$\|\mathrm{F}^m\|_{X\to Y} \le C_F(\mathrm{Re}\, s)|s|^{\mu-m} \qquad \forall s \in \mathbb{C}_+,$$

where $\mu - m \in (-1, 0]$, we can use Proposition 4.4 to write

$$\|(f_\kappa^m - f^m) * g^{(m)}(t)\|_Y \le \kappa^2 C_{\mu-m}^2 C_2(t^{-1}) \int_0^t \|g^{(m+\alpha)}(\tau)\|_X \,\mathrm{d}\tau. \tag{5.2}$$

Next, we bound the second term using Proposition 5.2 in the following way

$$\|(f_\kappa^m * h)(t)\|_Y \le \kappa^2 C_1(t^{-1}) \sup_{\mathrm{Re}\, s=t^{-1}} \|\mathrm{F}_\kappa^m(s)\|_{X\to Y} \int_0^t \|\mathcal{P}_m g^{(m+4)}(\tau)\|_X \,\mathrm{d}\tau.$$

Here, using the definition of C_1 and Proposition 4.1(a) we have

$$C_1(x) \sup_{\mathrm{Re}\, s=x} \|\mathrm{F}_\kappa^m(s)\|_{X\to Y} \le C_F(\min\{x, 1\}/2) \frac{C_m\, 2^{-\mu+m}}{x \min\{x^{2m-\mu}, 1\}},$$

for all $x > 0$. Combining this with (5.2) and defining $C_\mu = \max\{C_{\mu-m}^2, C_m 2^{-\mu+m}\}$ finish the proof. □

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Appendix

Here, for the reader's convenience, we summarize existing results related to the convergence of TRCQ that have been improved by this paper. In this section, $\rho(\zeta)$ denotes the characteristic function of a general multistep CQ method, e.g., for BDF2 we have $\rho(\zeta) = \frac{3}{2} - 2\zeta + \frac{1}{2}\zeta^2$, and for the TR rule $\rho(\zeta) = 2\frac{1-\zeta}{1+\zeta}$. We start with the following hypotheses:

(H1) The method is *A-stable*, i.e., Re $\rho(\zeta) > 0$ for $|\zeta| < 1$.

(H2) The method is of *order* p, i.e., $\frac{1}{h}\rho(e^{-h}) = 1 + \mathcal{O}(h^p)$ as $h \to 0$.
(H3) $\rho(\zeta)$ has no poles on the unit circle.
(H4) $\rho(\zeta)$ has no zeros on the unit circle, with the exception of $\zeta = 1$.

Note that, all these hypotheses except (H3) are satisfied by the TR rule.

Theorem A.1 (*[1, Theorem 3.1]*)**.** *Let $\sigma_0 > 0$ and F satisfy (1.1) with $\|\mathrm{F}(s)\|_{X\to Y} \le C_F(\sigma_0)|s|^\mu$ for $\mathrm{Re}\, s > \sigma_0$. Let f be the distributional inverse Laplace transform of F and f_κ be its CQ approximation for any given time-step $\kappa \in (0, \kappa_0]$. Here, κ_0 is a constant that depends on σ_0 and $\rho(\zeta)$. For the discretization method, assumptions (H1) and (H2) are to be satisfied. Furthermore, (H3) for $\mu > 0$, or (H4) for $\mu < 0$ should hold.*

Let $m \ge \max\{p + 2 + \mu, p\}$ and $g \in \mathcal{C}^m([0, T]; X)$ with $g(0) = \cdots = g^{(m-1)}(0) = 0$. For all $t \in [0, T]$, the following estimate holds

$$\|(f_\kappa - f) * g(t)\|_Y \le \kappa^p C \int_0^t \|g^{(m)}(\tau)\|_X \mathrm{d}\tau,$$

where C is proportional to $C_F(\sigma_0)$, and it depends on μ, T, σ_0 as well as the discretization method.

Theorem A.2 (*[3, Theorem A.2]*)**.** *Let F and f be as in Theorem A.1 with $\mu > 0$, and f_κ be TRCQ approximation of f. Let $m > 2\lceil\mu\rceil + 3$, $g \in \mathcal{C}^m([0, T]; X)$ with $g(0) = \cdots = g^{(m-1)}(0) = 0$, and the interval $[0, T]$ be discretized with $t_j = j\kappa$ for $j = 0, 1, \ldots, N$. The following estimate holds for all $\kappa \in (0, 1]$*

$$\left(\kappa \sum_{j=0}^{N} \|(f_\kappa - f) * g(t_j)\|_Y^2\right)^{1/2} = \mathcal{O}(\kappa^2).$$

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