TWO DETECTION RESULTS OF KHOVANOV HOMOLOGY ON LINKS

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Abstract. We prove that Khovanov homology with $\mathbb{Z}/2$–coefficients detects the link L7n1, and the union of a trefoil and its meridian.

1. Introduction

Given an oriented link L in S^3 and a commutative ring R, Khovanov homology [Kho00] assigns a bi-graded R–module $\mathrm{Kh}(L;R)$ to the link L. In 2011, Kronheimer and Mrowka [KM11] proved that Khovanov homology detects the unknot. Since then, many other detection results of Khovanov homology have been obtained. It is now known that Khovanov homology detects the unlink [BS15, HN13], the trefoil [BS18], the Hopf link [BSX19], the forest of unknots [XZ19], the splitting of links [LS19], and the torus link $T(2,6)$ [Mar20].

In [XZ20], a classification is given for all links L such that $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2)\leq 8$ and all 3-component links L such that $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2)\leq 12$. By [Shu14, Corollary 3.2.C], $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2) = 2\,\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Khr}(L;\mathbb{Z}/2)$, where Khr denotes the reduced Khovanov homology. Moreover, the parity of $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Khr}(L;\mathbb{Z}/2)$ is invariant under crossing changes and hence is always even for 2-component links (as is the case for the 2-component unlink). Therefore $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2)$ is always a multiple of 4. As a consequence, if a 2-component link L satisfies $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2)>8$, then $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2)\geq 12$.

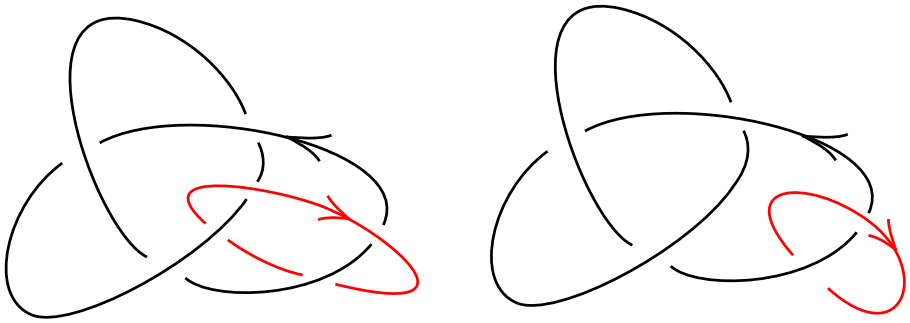
This paper studies 2-component links L such that $\mathrm{rank}_{\mathbb{Z}/2}(L;\mathbb{Z}/2)=12$. Among 2-component links with crossing numbers less than or equal to 7, there are four links (up to mirror images) satisfying $\mathrm{rank}_{\mathbb{Z}/2}(L;\mathbb{Z}/2)=12$. These links are:

(1) L7n1 in the Thistlethwaite Link Table,
(2) L6a3 in the Thistlethwaite Link Table,
(3) the disjoint union of a trefoil and an unknot,
(4) the union of a trefoil and its meridian.

Question 1.1. *Suppose L is a 2-component link with $\mathrm{rank}_{\mathbb{Z}/2}(L;\mathbb{Z}/2)=12$, is it true that L must be isotopic (up to mirror image) to one of the links listed above?*

Instead of giving a full answer to the question above, we show that Khovanov homology (with the bi-grading) detects the link L7n1, and the union of a trefoil with its meridian, from the list above. Since [XZ19] proved that Khovanov homology detects the disjoint union of a trefoil and an unknot, and Martin [Mar20] recently proved that Khovanov homology detects L6a3, we conclude that Khovanov homology detects all the links on the list.

In the following, we will call the link L7n1 as L_1, and the union of a trefoil with a meridian L_2. Moreover, we fix the chirality and orientation of these two links by



$L_1 = \text{L7n1}$ $L_2 = \text{trefoil} \cup \text{meridian}$

FIGURE 1. The two links L_1 and L_2

Figure 1. Notice that the link L_1 can also be described as the closure of the 2-braid σ_1^3 together with an axis unknot.

Recall that the internal grading of Khovanov homology is defined by $h - q$ in [BS15], where h is the homological grading and q is the quantum grading. The precise statement of our detection result is given as follows.

Theorem 1.2. *Let $L_1, L_2 \subset S^3$ be the oriented links as shown in Figure 1, and let $i \in \{1, 2\}$. Suppose $L \subset S^3$ is a 2-component oriented link, such that*

$$\mathrm{Kh}(L; \mathbb{Z}/2) \cong \mathrm{Kh}(L_i; \mathbb{Z}/2)$$

as abelian groups equipped with the internal gradings, then L is isotopic to L_i as oriented links.

The proof of Theorem 1.2 depends on a rank inequality between reduced Khovanov homology and knot Floer homology by Dowlin [Dow18], and a braid detection property of link Floer homology by Martin [Mar20]. The main ingredient of the proof of Theorem 1.2 is the following proposition, which is established in Section 3.

Proposition 1.3. *Let $L = K \cup U$ be a link such that U is an unknot and K is either an unknot or a trefoil. Let $l = |\operatorname{lk}(K, U)|$ be the linking number of K and U. Suppose $l > 0$, and*

$$\dim_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L; \mathbb{Q}) \leq 12, \tag{1.1}$$

where $\widehat{\mathrm{HFK}}$ is the knot Floer homology defined in [Ras03, OS04]. Then at least one of the following holds:

1. *K is the closure of an l-braid with axis U.*
2. *$l = 1$, K is an unknot.*
3. *$l = 1$, K is a trefoil, U is the meridian of K.*

Recall that a 2-component link $K_1 \cup K_2$ is said to be *exchangeably braided* (or *mutually braided*) if both K_1 and K_2 are unknots, K_1 is a braid closure with axis K_2, and K_2 is a braid closure with axis K_1. The concept of exchangeably braided links was introduced and studied by Morton [Mor85]. We will also need the following result from [XZ20].

Proposition 1.4 ([XZ20, Corollary 3.9]). *Suppose L is an exchangeably braided link with linking number $l \geq 3$, then we have* $\operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L;\mathbb{Q}) \geq 12$. *Moreover, if $l > 3$, then* $\operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L;\mathbb{Q}) > 12$.

We will prove Theorem 1.2 in Section 4 as a consequence of Proposition 1.3, Proposition 1.4, Dowlin's rank inequality [Dow18, Corollary 1.7], and Batson-Seed's spectral sequence [BS15].

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2. Link Floer homology

This section reviews the basic properties of link Floer homology and proves a result on the rank of link Floer homology that will play an important role in the proof of Proposition 1.3.

The link Floer homology was originally defined for $\mathbb{Z}/2$–coefficients by Ozsváth and Szabó in [OS04], and was generalized to $\mathbb{Z}$–coefficients in [Sar11]. We will work with $\mathbb{Q}$–coefficients in order to invoke Dowlin's spectral sequence [Dow18]. For the rest of this section, all Floer homology groups are with $\mathbb{Q}$–coefficients and it will be omitted from the notation.

Given an oriented n–component link $L \subset S^3$, its link Floer homology $\widehat{\mathrm{HFL}}(L)$ carries a homological grading over $\mathbb{Z}$ and n Alexander gradings associated to the n components of L. The Alexander grading associated to the i-th component K_i takes values in either $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, which depends on the parity of the linking number $\mathrm{lk}(K_i, L-K_i)$.

By [OS08a], when $n \geq 2$, the link Floer homology recovers the multi-variable Alexander polynomial in the following sense:

$$\sum_{a_1,\cdots,a_n} \chi\big(\widehat{\mathrm{HFL}}(L,a_1,\cdots,a_n)\big)\cdot T_1^{a_1}\cdots T_n^{a_n} \doteq (T_1^{1/2}-T_1^{-1/2})\cdots(T_n^{1/2}-T_n^{-1/2})\Delta_L(T_1,\cdots,T_n), \quad (2.1)$$

where $\widehat{\mathrm{HFL}}(L,a_1,\cdots,a_n)$ is the component of $\widehat{\mathrm{HFL}}(L)$ with multi-Alexander grading $(a_1,\cdots,a_n)$, and $\chi(\cdot)$ denotes the Euler characteristic with respect to the homological grading. The notation "$\doteq$" means that the two sides are equal up to a multiplication by $\pm T_1^{b_1}\cdots T_n^{b_n}$ for some $b_1,\cdots b_n \in \frac{1}{2}\mathbb{Z}$. There is also a symmetry

$$\widehat{\mathrm{HFL}}(L,a_1,\cdots,a_n) \cong \widehat{\mathrm{HFL}}(L,-a_1,\cdots,-a_n). \quad (2.2)$$

The following proposition is a special case of the Thurston norm detection property of link Floer homology.

Proposition 2.1 ([OS08b, Theorem 1.1]). *Suppose $L = K \cup U \subset S^3$ is a 2-component link with an unknotted component U and $l = |\mathrm{lk}(K,U)| > 0$. Then the top Alexander grading of $\widehat{\mathrm{HFL}}(L)$ associated to U is $\frac{l}{2}$ if and only if U has a Seifert disk that intersects K transversely at l points.*

Remark 2.2. The proof of [OS08b, Theorem 1.1] was originally given for $\mathbb{Z}/2$–coefficients, but the same argument applies to $\mathbb{Q}$–coefficients. Alternatively, a similar norm-detection property for instanton Floer homology was established by [GL19]

using sutured manifold decompositions and the formal properties of Floer homology, and the same argument can be carried over to Heegaard Floer homology with $\mathbb{Q}$–coefficients.

The following is a weaker version of a result from [Mar20].

Proposition 2.3 ([Mar20, Corollary 2])**.** *Let $L = K \cup U \subset S^3$ be a 2-component link such that U an unknot and $l = |\operatorname{lk}(K, U)| > 0$. Then K is the closure of a braid with axis U if and only if the dimension of $\widehat{\mathrm{HFL}}(L)$ is 2 at the top Alexander grading associated to U.*

The link Floer homology $\widehat{\mathrm{HFL}}$ can be interpreted by sutured Floer homology using the following proposition. Here we use SFH to denote the sutured Floer homology defined by Juhász in [Juh06].

Proposition 2.4 ([Juh06, Proposition 9.2])**.** *Suppose $L = K_1 \cup ... \cup K_n$ is an oriented link, and let $S^3 - N(L)$ be the link complement. Let γ be a suture on $\partial(S^3 - N(L))$ which consists of two meridians of each K_i. Then there is an isomorphism*

$$\widehat{\mathrm{HFL}}(L) \cong \mathrm{SFH}(S^3 - N(L), \gamma).$$

Moreover, the Alexander grading associated to K_i corresponds to the grading induced by a Seifert surface of K_i on $\mathrm{SFH}(S^3 - N(L), \gamma)$.

Remark 2.5. The original statement is for $\mathbb{Z}/2$–coefficients, but the proof is done by examining the Heegaard diagrams, which also works for $\mathbb{Q}$–coefficients.

We also need the following proposition from [Juh10].

Proposition 2.6 ([Juh10, Proposition 9.2])**.** *Suppose (M, γ) is a balanced sutured manifold. Suppose γ_0 is a component of γ that is homologically essential on ∂M. Let γ' be a suture on ∂M obtained by adding two parallel copies of γ_0 to γ. Then we have*

$$\mathrm{SFH}(M, \gamma') \cong \mathrm{SFH}(M, \gamma) \otimes_{\mathbb{Q}} \mathbb{Q}^2.$$

The main result of this section is the following proposition.

Proposition 2.7. *Suppose $L = K \cup U \subset S^3$ is a 2-component link with an unknotted component U and $l = |\operatorname{lk}(K, U)| > 0$, and suppose U has a Seifert disk D that intersects K transversely at l points. Then*

$$\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L, \frac{l}{2}) \equiv 2 \bmod 4,$$

where $\widehat{\mathrm{HFL}}(L, l/2)$ is the component of $\widehat{\mathrm{HFL}}(L)$ with degree $l/2$ on the Alexander grading associated to U.

In order to prove Proposition 2.7, we need to establish the following property of sutured Floer homology.

Proposition 2.8. *Let $l \in \mathbb{Z}^+$, let $T \subset [-1, 1] \times D^2$ be a tangle given by $T = \alpha_1 \cup \cdots \cup \alpha_l$, where α_i is an arc connecting $\{-1\} \times D^2$ and $\{1\} \times D^2$ for all i. Let $M_T = [-1, 1] \times D^2 - N(T)$, let $\gamma_T \subset \partial M_T$ be a suture on M_T with $(l + 1)$ components: one meridian component on each one of $\partial N(\alpha_1), \cdots \partial N(\alpha_l)$, and a component on $[-1, 1] \times \partial D^2$ given by $\{\mathrm{pt}\} \times \partial D^2$. Then $\dim_{\mathbb{Q}} \mathrm{SFH}(M_T, \gamma_T)$ is odd.*

We start the proof of Proposition 2.8 by verifying the trivial case.

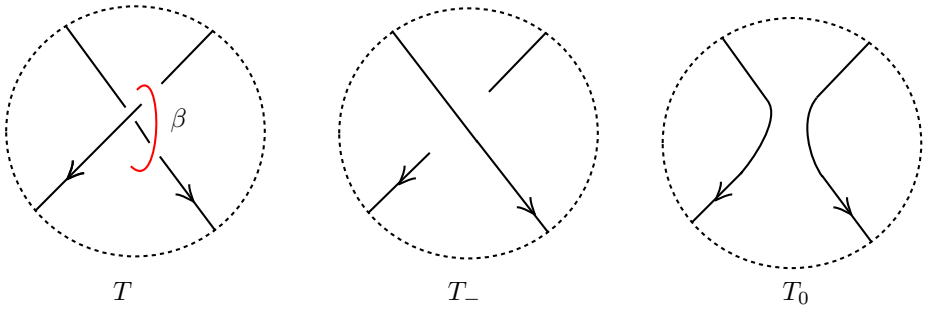

FIGURE 2. Surgery along β

Lemma 2.9. *If T is a product tangle, i.e., there are points $p_1, ..., p_n \subset \mathrm{int}(D^2)$ so that $\alpha_i = [-1,1] \times \{p_i\}$ for all i, then $\dim_{\mathbb{Q}} \mathrm{SFH}(M_T, \gamma_T)$ is odd.*

Proof. When T is a product tangle, (M_T, γ_T) is a product sutured manifold. Hence it follows from [Juh06] that the dimension of $\mathrm{SFH}(M_T, \gamma_T)$ is one. □

Let T be the tangle in Proposition 2.8. Orient T so that each α_i goes from $\{-1\} \times D^2$ to $\{1\} \times D^2$. Fix a diagram on $[-1,1] \times [-1,1]$ that represents the tangle T. We will also denote the diagram by T when there is no source of confusion. For a positive crossing of T, we can perform surgeries along the curve β as depicted in Figure 2. Let $M_{T,-1}$ be the manifold obtained by performing the (-1)–surgery along β, and let $M_{T,0}$ be the manifold obtained by performing the 0–surgery along β. Let T_- be the tangle that only differs from T at the crossing linked by β as depicted in Figure 2. It straightforward to show that $(M_{T,-1}, \gamma_T) \cong (M_{T_-}, \gamma_{T_-})$.

Definition 2.10. We call the operation of switching from T to T_- or from T_- to T a *crossing change*.

Lemma 2.11. *For any vertical tangle $T \subset [-1,1] \times D^2$, there is a finite sequence of crossing changes that takes T to the product tangle.* □

Now we study the sutured manifold $(M_{T,0}, \gamma_T)$. Inside $[-1,1] \times D^2$, the circle β bounds a disk D that intersects the tangle T twice. After performing the 0–surgery, the boundary ∂D can be capped by a meridian disk in the surgery solid torus, and hence we obtain a 2–sphere S that intersects the tangle T twice. The intersection of S and $M_{T,0}$ is a properly embedded annulus $A_\beta \subset M_{T,0}$. We can pick the suture γ_T so that one boundary component of A_β lies in $R_+(\gamma_T)$ and the other lies in $R_-(\gamma_T)$. Then there is a sutured manifold decomposition

$$(M_{T,0}, \gamma_T) \overset{A_\beta}{\rightsquigarrow} (M', \gamma').$$

From [KM10, Section 3.1], we know that $M' \cong M_{T_0} := [-1,1] \times D^2 - N(T_0)$, where T_0 is another tangle on $[-1,1] \times D^2$, possibly having closed components, such that T_0 only differs from T near the crossing linked by β as depicted in Figure 2.

Definition 2.12. We say that T_0 is obtained from T_+ by an oriented smoothing.

Lemma 2.13. *We have*

$$\mathrm{SFH}(M_{T,0}, \gamma_T) \cong \mathrm{SFH}(M_{T_0}, \gamma'),$$

and $\dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_0}, \gamma')$ is even.

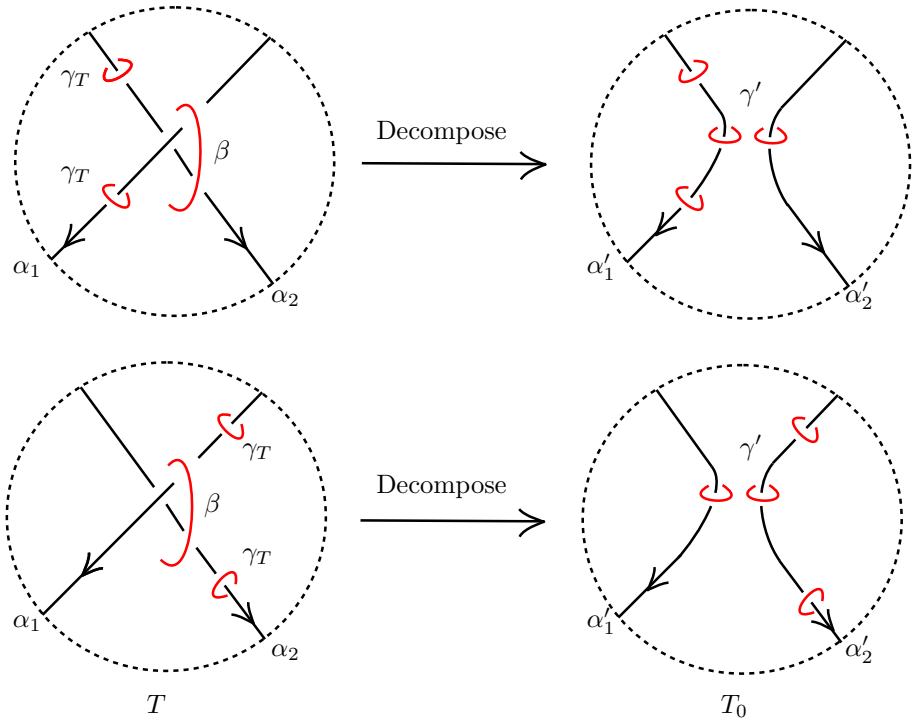

FIGURE 3. Oriented smoothing

Proof. The isomorphism

$$\mathrm{SFH}(M_{T,0}, \gamma_T) \cong \mathrm{SFH}(M_{T_0}, \gamma')$$

follows from [Juh08, Lemma 8.9]. For the parity statement, we argue in two cases.

Case 1. The crossing linked by β involves two different components of T. Without loss of generality we can assume that they are α_1 and α_2. See Figure 3. Recall we have an annulus $A_\beta \subset M_{T,0}$ after performing the 0-surgery along β. To make sure that the two boundary components of A_β lie in two different component of $R(\gamma_T)$, the suture γ_T must be arranged as in one of the two possibilities shown in Figure 3. After performing the sutured manifold decomposition along A_β, the new tangle T_0 has two new arcs α_1' and α_2'. For $i = 1, 2$, let $C_i' = \partial N(\alpha_i') - \{-1, 1\} \times D^2$. It is straightforward to check that, after the sutured manifold decomposition along A_β, one and exactly one of the following two possibilities happens, as shown in Figure 3:

- $\gamma' \cap C_1'$ consists of three parallel copies of meridians of α_1'
- $\gamma' \cap C_2'$ consists of three parallel copies of meridians of α_2'.

Without loss of generality, we assume that the first possibility happens, i.e., γ' contains three copies meridians of α_1'. Removing two such copies, we obtain a new sutured manifold (M_{T_0}, γ_{T_0}), and by Proposition 2.6 we have

$$\dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_0}, \gamma') = 2 \dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_0}, \gamma_{T_0}).$$

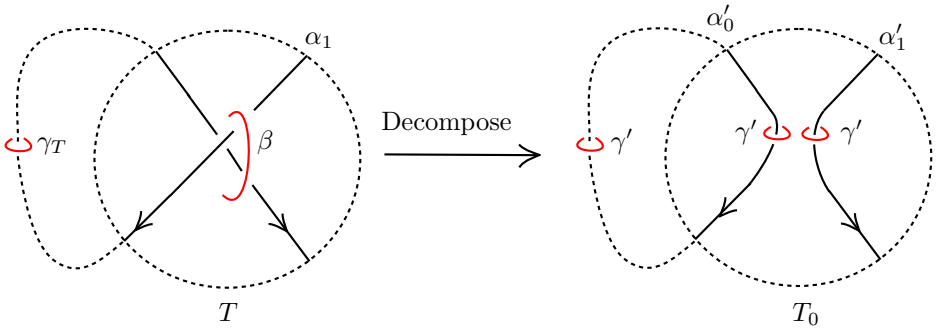

FIGURE 4. Oriented smoothing

As a result, $\dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_0}, \gamma')$ is even.

Case 2. The crossing linked by β involves only one component of T. Without loss of generality, we assume it is α_1, see Figure 4. Let $C_i = \partial N(\alpha_i) - \{-1, 1\} \times D^2$ for all i. To make sure that the two boundary components of A_β lie in two different components of $R(\gamma_T)$, we must have the suture on C_1 to be in the position as depicted in Figure 4. After the decomposition along A_β, the new tangle T_0 now has a closed component, which we call α_0', and an arc that we call α_1'. Let $C_0' = \partial N(\alpha_0')$ and $C_1' = \partial N(\alpha_1') - \{-1, 1\} \times D^2$. Note that C_0' is a torus while C_1' is an annulus. The suture γ' contains two meridians on C_0', and one meridian on C_1' (and one meridian on every other C_i).

Recall that our goal is to show that $\dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_0}, \gamma')$ is even. Write

$$T_0' = T_0 \backslash \alpha_0'.$$

Case 2.1. When α_0' is split from T_0', i.e., there is a 3-ball $B^3 \subset (-1, 1) \times D^2$ so that

$$B^3 \cap T_0 = \alpha_0'.$$

In this case, we know that (M_{T_0}, γ') is a connected sum:

$$(M_{T_0}, \gamma') \cong (M_{T_0'}, \gamma' - B^3) \# (S^3(\alpha_0'), \gamma' \cap B^3).$$

Here $S^3(\alpha_0')$ is the knot complement of the knot $\alpha_0' \subset B^3 \subset S^3$. It then follows from [Juh06, Proposition 9.15] that the dimension of $\mathrm{SFH}(M_{T_0}, \gamma')$ is even.

Case 2.2. When α_0' is not split from T_0'. Suppse there is a positive crossing of T_0 involving both α_0' and T_0'. Pick the circle θ as depicted in Figure 5. Suppose the component of T_0' involved in the crossing is α'. There is a surgery exact triangle associated to θ:

$$\begin{array}{ccc} \mathrm{SFH}(M_{T_0}, \gamma') & \longrightarrow & \mathrm{SFH}(M_{T_{0,-}}, \gamma_-') \\ & \searrow \quad \swarrow & \\ & \mathrm{SFH}(M_{T_{0,0}}, \gamma_0') & \end{array}$$

As above, $T_{0,-}$ is obtained from T_0 by a crossing change and $(M_{T_{0,-}}, \gamma_-')$ is the corresponding sutured manifold. The tangle $T_{0,0}$ is obtained from T_0 by an oriented smoothing. As in Figure 5, α_0' and α' merge into a single component $\alpha_0'' \subset T_{0,0}$. It is then straightforward to check that the new suture γ_0' consists of five meridians of α_0'': The two meridians of α_0' and one meridian of α' all survive, and there are

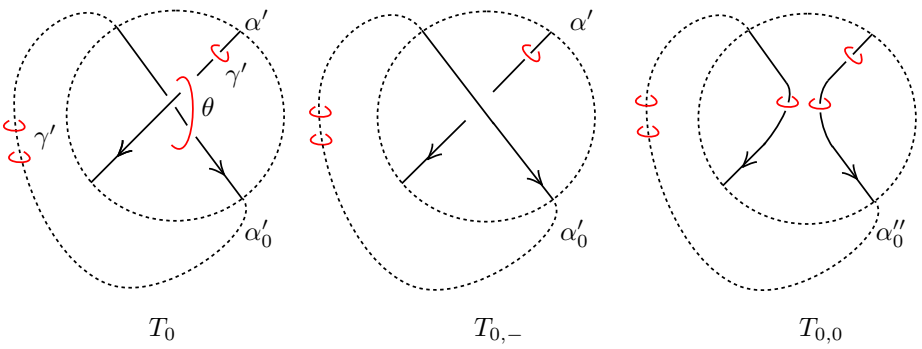

FIGURE 5. Surgery along θ

two more meridians coming from the decomposition along an annulus A_θ (similar to the annulus A_β above). By Proposition 2.6, we know that the dimension of $\mathrm{SFH}(M_{T_{0,0}}, \gamma'_0)$ is even, and hence

$$\dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_0}, \gamma') \equiv \dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_{0,-}}, \gamma'_-) \bmod 2. \tag{2.3}$$

However, T_0 and $T_{0,-}$ only differ by a crossing change, and following the same line of Lemma 2.11, there is a finite sequence of such crossing changes that makes α'_0 split from T'_0. Hence, it follows from Case 2.1 and (2.3) that the dimension of $\mathrm{SFH}(M_{T_0}, \gamma')$ must be even. This concludes the proof of Lemma 2.13. □

Proof of Proposition 2.8. There is a surgery exact triangle associated to β:

$$\begin{array}{ccc} \mathrm{SFH}(M_T, \gamma_T) & \longrightarrow & \mathrm{SFH}(M_{T_-}, \gamma_{T_-}) \\ & \nwarrow \quad \swarrow & \\ & \mathrm{SFH}(M_{T_0}, \gamma') & \end{array}$$

Therefore Lemma 2.13 implies $\dim_{\mathbb{Q}} \mathrm{SFH}(M_T, \gamma_T) \equiv \dim_{\mathbb{Q}} \mathrm{SFH}(M_{T_-}, \gamma_{T_-}) \bmod 2$. Proposition 2.8 then follows from Lemma 2.9 and Lemma 2.11. □

Remark 2.14. The statement and the proof of Proposition 2.8 can be applied to sutured monopole theory and sutured instanton theory as well (with suitable choices of coefficients).

Proof of Proposition 2.7. Take a pair of oppositely oriented meridional sutures to each boundary component of $S^3 - N(L)$, then $S^3 - N(L)$ becomes a balanced sutured manifold.

Decompose $S^3 - N(L)$ along the disk D, we obtain a sutured manifold (M, γ). The manifold M is given by $[-1,1] \times D^2 - N(T)$, where T is a tangle in $[-1,1] \times D^2$. Since the linking number of K and U is equal to $|K \cap U|$, we have $T = \alpha_1 \cup \cdots \cup \alpha_l$ where α_i is an arc from $\{1\} \times D^2$ to $\{-1\} \times D^2$ for each i. The suture γ consists of $(l+3)$ components: one meridian on each of $\partial N(\alpha_1), \cdots \partial N(\alpha_{l-1})$, three parallel meridians on $\partial N(\alpha_l)$, and one component on $[-1,1] \times \partial D^2$ given by $\{\mathrm{pt}\} \times \partial D^2$. We have

$$\widehat{\mathrm{HFL}}(L, \frac{l}{2}) \cong \mathrm{SFH}(M, \gamma).$$

Removing two sutures from $\partial N(\alpha_l)$, we obtain the sutured manifold (M_T, γ_T) as in Proposition 2.8. By Proposition 2.6, we have

$$\dim_{\mathbb{Q}} \mathrm{SFH}(M_T, \gamma) = 2 \dim_{\mathbb{Q}} \mathrm{SFH}(M_T, \gamma_T).$$

Therefore the desired result follows from Proposition 2.8. □

3. Proof of Proposition 1.3

The strategy of our proof of Proposition 1.3 is to exploit the properties of the multi-variable Alexander polynomial so that we can apply the braid detection property of link Floer homology by Martin [Mar20, Corollary 2]. The link Floer homology and the multi-variable Alexander polynomial are related by (2.1).

Suppose L is a 2-component link, let $\Delta_L(x, y) \in \mathbb{Z}[x, y, x^{-1}, y^{-1}]$ be the multi-variable Alexander polynomial of L. Then $\Delta_L(x, y)$ is *a priori* only well-defined up to a multiplication by $\pm x^a y^b$. It is possible to normalize the Alexander polynomial, for example, using Equation (2.1). However, the Alexander polynomial normalized by (2.1) can be a Laurent polynomial with half-integer exponents. For our purpose, it is more convenient to take $\Delta_L(x, y)$ as Laurent polynomial with integer exponents, and therefore we will not normalize $\Delta_L(x, y)$.

For $f_1, f_2 \in \mathbb{Z}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$, we write $f_1 \doteq f_2$ if and only if there exists a multiplicative unit ϵ such that $f_1 = \epsilon\, f_2$.

For $f \in \mathbb{Z}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$, we use $\|f\|$ to denote the sum of the absolute values of the coefficients of f. By (2.1), we have

$$\mathrm{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L; \mathbb{Q}) = \mathrm{rank}_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L; \mathbb{Q}) \geq \|(1-x)(1-y)\Delta_L(x, y)\|.$$

We need the following result.

Theorem 3.1 ([Tor53])**.** *Suppose $L = K_1 \cup K_2$ is a 2-component link with multi-variable Alexander polynomial $\Delta_L(x, y)$, where x, y are the variables associated to K_1, K_2 respectively. Then we have*

$$\Delta_L(x, 1) \doteq \frac{1 - x^l}{1 - x} \Delta_{K_1}(x),$$

where $\Delta_{K_1}(x)$ is the Alexander polynomial of K_1 and $l = \mathrm{lk}(K_1, K_2)$.

From now on, let $L = K \cup U$ be a 2-component link such that

(1) U is an unknot,
(2) K is either a trefoil or an unknot,
(3) the linking number $l = \mathrm{lk}(K, U)$ is positive.

Let $\Delta_L(x, y)$ be the multi-variable Alexander polynomial of L, where x, y are the variables corresponding to K and U respectively. Define

$$F(x, y) = (1 - x)(1 - y)\Delta_L(x, y).$$

By Theorem 3.1, we have

$$\Delta_L(1, y) \doteq (1 + y + \cdots + y^{l-1})\Delta_U(y) = 1 + y + \cdots + y^{l-1}. \tag{3.1}$$

Write

$$(1 - y)\Delta_L(x, y) =: \sum_{m=-\infty}^{+\infty} g_m(x) y^m, \tag{3.2}$$

then by definiton,

$$F(x,y)=\sum_m (1-x)g_m(x)y^m. \tag{3.3}$$

By (3.1), have

$$(1-y)\Delta_L(1,y)\doteq(1-y)(1+y+\cdots+y^{l-1})=1-y^l.$$

Therefore, after multiplying $\Delta_L(x,y)$ by $\pm y^a$, we may assume without loss of generality that

$$g_0(1)=-g_l(1)=1,\ g_m(1)=0 \text{ for all } m\neq 0,l. \tag{3.4}$$

We establish the following two technical lemmas, which allow us to deduce topological properties of L from the sequence of Laurent polynomials $\{g_m(x)\}_{m\in\mathbb{Z}}$.

Lemma 3.2. *Let L, $\{g_m(x)\}_{m\in\mathbb{Z}}$ be as above. If $g_m(x)=0$ for all $m\neq 0,l$, then we have $l=1$.*

Proof. By the assumption and (3.2),

$$(1-y)\,\Delta_L(x,y)=g_0(x)+g_l(x)y^l.$$

Plugging in $y=1$, we have $g_l(x)=-g_0(x)$, therefore

$$\Delta_L(x,y)=\frac{(1-y^l)\,g_0(x)}{1-y}=(1+y+\cdots+y^{l-1})\,g_0(x),$$

and hence

$$\Delta_L(x,1)\doteq l\,g_0(x).$$

On the other hand, by Theorem 3.1,

$$\Delta_L(x,1)\doteq(1+x+\cdots+x^{l-1})\Delta_K(x).$$

Recall that l is assumed to be positive. Comparing the two equations above, we have

$$\frac{\Delta_K(x)}{l}\in\mathbb{Z}[x,x^{-1}].$$

Since $\Delta_K(1)=\pm1$, this implies $l=1$. □

Recall that for a Laurent polynomial f, we use $\|f\|$ to denote the sum of the absolute values of the coefficients of f.

Lemma 3.3. *Let L, $\{g_m(x)\}_{m\in\mathbb{Z}}$ be as above. Suppose the following two conditions hold:*

1. *There exists $k\in\mathbb{Z}^+$, such that $g_m(x)=0$ for all $m\neq 0,l,-k,l+k$,*
2. *$\|(1-x)g_0(x)\|=\|(1-x)g_l(x)\|=2$,*

then $l=1$ and K is an unknot.

Proof. By Condition (1),

$$(1-y)\Delta_L(x,y)=g_{-k}(x)y^{-k}+g_0(x)+g_l(x)y^l+g_{l+k}(x)y^{l+k}. \tag{3.5}$$

By Condition (2), $\|(1-x)\,g_0(x)\|=2$. Hence there exists an integer $s>0$ such that $(1-x)\,g_0(x)\doteq 1-x^s$, thus

$$g_0(x)\doteq 1+\cdots+x^{s-1}.$$

By (3.4), this implies $s=1$, therefore $g_0(x)\doteq 1$. Similarly $g_l(x)\doteq 1$. By (3.4), there exist integers a,b, such that

$$g_0(x)=x^a, g_l(x)=-x^b.$$

Plugging in $y = 1$ to (3.5), we obtain

$$g_{l+k}(x) = x^b - x^a - g_{-k}(x), \tag{3.6}$$

and hence

$$\begin{aligned}(1-y)\Delta_L(x,y) &= g_{-k}(x)y^{-k} + x^a - x^b y^l + \big(x^b - x^a - g_{-k}(x)\big)y^{l+k} \\ &= g_{-k}(x)(y^{-k} - y^{l+k}) + x^a(1-y^{l+k}) - x^b(y^l - y^{l+k}).\end{aligned}$$

Therefore

$$\begin{aligned}\Delta_L(x,1) &= \lim_{y\to 1}\left(\frac{g_{-k}(x)(y^{-k} - y^{l+k}) + x^a(1-y^{l+k}) - x^b(y^l - y^{l+k})}{1-y}\right) \\ &= (l+2k)\, g_{-k}(x) + (l+k)x^a - kx^b. \end{aligned}\tag{3.7}$$

By (3.7), if $\Delta_L(x,1)$ has more than two terms, then at least one of its coefficients is a multiple of $(l+2k)$.

On the other hand, by Theorem 3.1,

$$\Delta_L(x,1) \doteq (1 + x + \cdots + x^{l-1})\Delta_K(x),$$

and hence (recall we have assumed that K is either a trefoil or an unknot)

$$\Delta_L(x,1) \doteq \begin{cases} 1 + x + \cdots + x^{l-1} & \text{if } K \text{ is an unknot,} \\ 1 - x + x^2 & \text{if } K \text{ is a trefoil and } l = 1, \\ 1 + \sum_{k=2}^{l-1} x^k + x^{l+1} & \text{if } K \text{ is a trefoil and } l \geq 2. \end{cases} \tag{3.8}$$

In particular, all the coefficients of $\Delta_L(x,1) \in \mathbb{Z}[x, x^{-1}]$ are ± 1. Since $l + 2k \geq 3$, the previous argument implies that $\Delta_L(x,1)$ has at most two terms, and hence there are three possibilities:

(1) K is an unknot, $l = 1$,
(2) K is an unknot, $l = 2$,
(3) K is a trefoil, $l = 2$.

To eliminate the second and third possibilities, notice that in these cases, (3.7) and (3.8) yield

$$(2k+2)g_{-k}(x) + (k+2)x^a - kx^b \doteq x + 1 \text{ or } x^3 + 1.$$

By the assumptions, we have $k \in \mathbb{Z}^+$, and hence

$$(k+2)x^a - kx^b \doteq x + 1 \text{ or } x^3 + 1 \mod (2k+2),$$

therefore we have

$$a \neq b,$$

and

$$k + 2 \equiv \pm 1, k \equiv \mp 1 \mod (2k+2),$$

which imply $k = 1$, thus $g_{-k}(x) = -x^a$. This yields a contradiction to (3.4). □

Proof of Proposition 1.3. Without loss of generality, we assume $l = \mathrm{lk}(K, U) > 0$. By (1.1), we have

$$\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L;\mathbb{Q}) = \dim_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L;\mathbb{Q}) \leq 12. \tag{3.9}$$

For $a \in \frac{1}{2}\mathbb{Z}$, we use $\widehat{\mathrm{HFL}}(L,a;\mathbb{Q})$ to denote the component of $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$ with degree a on the Alexander grading associated to U. Let $\Delta_L(x,y)$, $F(x,y)$, $g_m(x)$ be as above, and we choose $\Delta_L(x,y)$ such that (3.4) holds.

Recall that by (2.1), the coefficients of $F(x,y)$ are the bi-graded Euler characteristics of $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$. Since $F(1,y)=0$, we have $\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,a;\mathbb{Q})$ is even for all a. By (2.2), we have

$$\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,a;\mathbb{Q}) = \dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,-a;\mathbb{Q}). \tag{3.10}$$

Since $0 \neq \Delta_L(x,y) \doteq \Delta_L(x^{-1},y^{-1})$, there is a unique $(a,b) \in \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$, such that $\hat{F}(x,y) := x^a y^b F(x,y)$ satisfies $\hat{F}(x,y) = \pm \hat{F}(x^{-1},y^{-1})$. Write

$$\hat{F}(x,y) = \sum_{m \in \frac{1}{2}\mathbb{Z}} \hat{f}_m(x) y^m,$$

then by (3.4), we have $\hat{f}_{l/2}(x) = \pm \hat{f}_{-l/2}(x^{-1}) \neq 0$. Therefore by (2.1) and (2.2),

$$\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,l/2;\mathbb{Q}) = \dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,-l/2;\mathbb{Q}) \neq 0. \tag{3.11}$$

Let $s \in \frac{1}{2}\mathbb{Z}$ be the maximum degree such that $\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,s;\mathbb{Q}) \neq 0$, then $s \geq l/2 > 0$. Since $\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,s;\mathbb{Q})$ is even, by (3.9) and (3.10), we have

$$\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,s;\mathbb{Q}) = 2, 4, \text{ or } 6.$$

We discuss four cases.

Case 1. $\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,s;\mathbb{Q}) = 2$. By Proposition 2.3, K is a braid closure with axis U, therefore Case (1) of the proposition holds.

Case 2. $\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,s;\mathbb{Q}) = 4$, and $s = \frac{l}{2}$. By Proposition 2.1, U has a Seifert disk that intersects K transversely at l points, therefore this assumption contradicts Proposition 2.7.

Case 3. $\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,s;\mathbb{Q}) = 4$, and $s > \frac{l}{2}$. By (3.9) and (3.11),

$$\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,\pm s;\mathbb{Q}) = 4, \ \dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,\pm\frac{l}{2};\mathbb{Q}) = 2,$$

and $\widehat{\mathrm{HFL}}(L,a;\mathbb{Q})$ vanishes at all the other degrees. By (2.1), $\{g_m(x)\}_{m\in\mathbb{Z}}$ satisfies the assumption of Lemma 3.3, therefore $l=1$ and K is an unknot, and hence Case (2) holds.

Case 4. $\dim_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L,s;\mathbb{Q}) = 6$. By (3.9) and (3.11), we have $s = \frac{l}{2}$. By Proposition 2.1, there is a Seifert disk of U that intersects K transversely at l points. By (2.1), $F(x,y)$ is supported at only two degrees in y, and hence $\{g_m(x)\}_{m\in\mathbb{Z}}$ satisfies the assumption of Lemma 3.2. By Lemma 3.2, we have $l=1$, therefore U is a meridian of K. If K is an unknot, then L is the Hopf link, which satisfies Case (1). Otherwise, K is a trefoil, and hence Case (3) holds.

□

4. Proof of the main theorem

In this section, we prove that Khovanov homology detects the links L_1 and L_2 given by Figure 1.

Recall that the internal grading of the Khovanov homology of a link L is introduced in [BS15, Section 2] as $h-q$, where h is the homological grading, and q is the

quantum grading. Following [BS15], we use l to denote the internal grading. The next theorem is a special case of a more general result due to Batson and Seed.

Theorem 4.1 ([BS15, Corollary 4.4])**.** *Suppose $L = K_1 \cup K_2$ is a 2-component oriented link. Then we have*

$$\operatorname{rank}^l_{\mathbb{F}} \operatorname{Kh}(L;\mathbb{F}) \geq \operatorname{rank}^{l+2\operatorname{lk}(K_1,K_2)}_{\mathbb{F}}(\operatorname{Kh}(K_1;\mathbb{F}) \otimes \operatorname{Kh}(K_2;\mathbb{F}))$$

for all $l \in \mathbb{Z}$, where $\mathbb{F}$ is an arbitrary field and rank^k denotes the rank of the summand with internal grading k.

Let $\mathbb{F} = \mathbb{Z}/2$ from now on. We have

$$\operatorname{Kh}(U_2;\mathbb{F}) = \mathbb{F}_{(-2)} \oplus \mathbb{F}^2_{(0)} \oplus \mathbb{F}_{(2)}, \tag{4.1}$$

where U_2 is the 2-component unlink and the subscripts denote the internal gradings. Let T be the left-handed trefoil, and let U be the unknot, we have

$$\operatorname{Kh}(T;\mathbb{F}) \otimes \operatorname{Kh}(U;\mathbb{F}) = \mathbb{F}_{(0)} \oplus \mathbb{F}^3_{(2)} \oplus \mathbb{F}_{(3)} \oplus \mathbb{F}^3_{(4)} \oplus \mathbb{F}^2_{(5)} \oplus \mathbb{F}_{(6)} \oplus \mathbb{F}_{(7)} \tag{4.2}$$

Let $\bar{T}$ be the right-handed trefoil, we have

$$\operatorname{Kh}(\bar{T};\mathbb{F}) \otimes \operatorname{Kh}(U;\mathbb{F}) = \mathbb{F}_{(0)} \oplus \mathbb{F}^3_{(-2)} \oplus \mathbb{F}_{(-3)} \oplus \mathbb{F}^3_{(-4)} \oplus \mathbb{F}^2_{(-5)} \oplus \mathbb{F}_{(-6)} \oplus \mathbb{F}_{(-7)} \tag{4.3}$$

Recall that the link $L_1 = \text{L7n1}$ can be described as $\hat{\sigma}_1^3 \cup U$ where $\hat{\sigma}_1^3$ is the closure of the 2-braid σ_1^3 with axis unknot U, and we choose the orientation as given by Figure 1 and hence the linking number is 2. The link L_2 is given by $T \cup U$ where U is a meridian of T and the orientation is chosen so that the linking number is 1.

We have

$$\operatorname{Kh}(L_1;\mathbb{F}) = \mathbb{F}_{(4)} \oplus \mathbb{F}^3_{(6)} \oplus \mathbb{F}_{(7)} \oplus \mathbb{F}^3_{(8)} \oplus \mathbb{F}^2_{(9)} \oplus \mathbb{F}_{(10)} \oplus \mathbb{F}_{(11)}, \tag{4.4}$$

$$\operatorname{Kh}(L_2;\mathbb{F}) = \mathbb{F}_{(2)} \oplus \mathbb{F}^3_{(4)} \oplus \mathbb{F}_{(5)} \oplus \mathbb{F}^3_{(6)} \oplus \mathbb{F}^2_{(7)} \oplus \mathbb{F}_{(8)} \oplus \mathbb{F}_{(9)}. \tag{4.5}$$

Besides the above links, we define the link $L_3 = \widehat{\sigma_1\sigma_2} \cup U$, which is the union of the closure of the 3-braid $\sigma_1\sigma_2$ and its axis unknot U. This is the torus link $T(2,6)$, which is denoted by L6a3 in the Thistlethwaite Link Table. We pick the orientation properly so that the linking number is positive, then

$$\operatorname{Kh}(L_3;\mathbb{F}) = \mathbb{F}_{(4)} \oplus \mathbb{F}^2_{(6)} \oplus \mathbb{F}_{(7)} \oplus \mathbb{F}^2_{(8)} \oplus \mathbb{F}^2_{(9)} \oplus \mathbb{F}^2_{(10)} \oplus \mathbb{F}_{(11)} \oplus \mathbb{F}_{(12)}. \tag{4.6}$$

We now prove Theorem 1.2.

Theorem 1.2. *Suppose $L = K_1 \cup K_2$ is a 2-component oriented link and $i \in \{1,2\}$. If $\operatorname{Kh}(L;\mathbb{F}) \cong \operatorname{Kh}(L_i;\mathbb{F})$ $(i = 1,2)$ as l-graded abelian groups, then L is isotopic to L_i as oriented links.*

Proof. Recall that $\mathbb{F} = \mathbb{Z}/2$. By the assumptions, we have

$$\operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(L;\mathbb{F}) = 12. \tag{4.7}$$

By [Dow18, Corollary 1.7], we have

$$\operatorname{rank}_{\mathbb{Q}} \widehat{\operatorname{HFK}}(L;\mathbb{Q}) \leq 2 \operatorname{rank}_{\mathbb{Q}} \operatorname{Khr}(L;\mathbb{Q}) \leq 2 \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(L;\mathbb{Z}/2) = 12. \tag{4.8}$$

Theorem 4.1 yields

$$\operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_i;\mathbb{F}) = \frac{1}{2} \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_i;\mathbb{F}) \leq \frac{1}{2}\frac{12}{2} = 3.$$

Therefore K_i $(i = 1,2)$ is either the unknot or a trefoil according to [KM11, BS18]. By Theorem 4.1 again, we have at least one of K_1 and K_2 is the unknot. Without

loss of generality we assume K_2 is an unknot, and we discuss two cases.

Case 1. K_1 is also an unknot. We show that this is contradictory to the assumptions. In fact, Theorem 4.1 and (4.1), (4.4), (4.5) imply that $l = \mathrm{lk}(K_1, K_2)$ is no less than 2. Therefore K_1 is the closure of an l-braid with axis K_2 by Proposition 1.3. Switching the role of K_1 and K_2 we obtain that K_2 is the closure of an l-braid with axis K_1. By Proposition 1.4, we have $l \leq 3$. If $l = 2$, then $L = \hat{\sigma}_1^{\pm 1} \cup U$, which is the link L4a1 in the Thistlethwaite Link Table. This contradicts (4.7) because $\mathrm{rank}_{\mathbb{F}} \mathrm{Kh}(\mathrm{L4a1}; \mathbb{F}) = 8$. If $l = 3$, since the only 3-braid representations of the unknot are given by $\sigma_1^{\pm 1}\sigma_2^{\pm 1}$ and $\sigma_1^{\pm 1}\sigma_2^{\mp 1}$, we further divide into two cases:

Case 1.1. $L = \widehat{\sigma_1\sigma_2} \cup U = L_3$ or $L = \widehat{\sigma_1^{-1}\sigma_2^{-1}} \cup U = \bar{L}_3$. Recall that $\mathrm{Kh}(L_3; \mathbb{F})$ is given by (4.6). Changing the orientation or taking the mirror image will shift the l-grading or change the sign of the l-grading, respectively. In any case, the l-graded Khovanov homology of L cannot be isomorphic to $\mathrm{Kh}(L_i; \mathbb{F})$ $(i \in \{1, 2\})$, contradicting the assumptions.

Case 1.2. $L = \widehat{\sigma_1\sigma_2^{-1}} \cup U$ or $L = \widehat{\sigma_1^{-1}\sigma_2} \cup U$. In this case L is the link L6a2 (or its mirror image) in the Thistlethwaite Link Table. We have $\mathrm{rank}_{\mathbb{F}} \mathrm{Kh}(L; \mathbb{F}) = 20$, which is not the same as L_1 and L_2.

In conclusion, K_1 cannot be an unknot.

Case 2. K_1 is a trefoil. There are two cases.

Case 2.1. K_1 is the right-handed trefoil $\bar{T}$. Then Theorem 4.1 and (4.3), (4.4), (4.5) yield a contradiction.

Case 2.2. K_1 is the left-handed trefoil T.

If $\mathrm{Kh}(L; \mathbb{F}) \cong \mathrm{Kh}(L_1; \mathbb{F})$, then by Theorem 4.1, we have $\mathrm{lk}(K_1, K_2) = 2$. By Proposition 1.3, the knot K_1 is the closure of a 2-braid in $S^3 - N(K_2)$. A 2-braid representing the left-handed trefoil can only be σ_1^3. Therefore L is isotopic L_1.

If $\mathrm{Kh}(L; \mathbb{F}) \cong \mathrm{Kh}(L_2; \mathbb{F})$, then by Theorem 4.1, we have $\mathrm{lk}(K_1, K_2) = 1$. By Proposition 1.3, the knot K_2 is a meridian of the left-handed trefoil K_1. Therefore L is isotopic to L_2. □

Remark 4.2. The argument above gives an alternative proof of Martin's theorem that Khovanov homology detects the torus link $T(2, 6)$ [Mar20, Theorem 4]. In fact, the link $T(2, 6)$ is detected by Case 1.1 in the argument above.

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