Analyticity up to the boundary for the Stokes and the Navier-Stokes systems

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1. Introduction

In this paper we consider the incompressible Navier-Stokes equations in a bounded domain Ω in $\mathbb{R}^d$ with analytic boundary $\partial\Omega$, and homogeneous Dirichlet boundary conditions

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, &\qquad \text{in } \Omega, \\ \nabla \cdot u = 0, &\qquad \text{in } \Omega, \\ u = 0, &\qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The forcing f is assumed to be analytic in space and time, and the system (1.1) is supplemented with a Sobolev smooth initial condition

$$u(x,0) = u_0(x), \qquad \text{in } \Omega. \tag{1.2}$$

For simplicity we assume $d \in \{2,3\}$; higher dimensions can be treated in the same way.

The main goal of this paper is to establish the immediate gain of space-time analyticity for solutions to (1.1)–(1.2), using a direct energy-type method, in the case of a domain with curved boundary. Our main result is Theorem 2.8 below, which shows that from a Sobolev smooth initial datum the solution instantaneously becomes space-time analytic, with analyticity radius which is uniform up to the curved analytic boundary. The direct energy-type approach utilized in this paper was presented in [33] for the Stokes system and in [12] for the Navier-Stokes equations on the half space. This method is robust and easily expendable to the case of non-analytic Gevrey-classes, jointly in space-time, provided the boundary belongs to the same Gevrey class.

Analyticity and Gevrey-class regularity have proven to be important for studying the vanishing viscosity problem for the Navier-Stokes equations in bounded domains [45, 46, 37, 30, 14, 50, 41, 19], and for establishing nonlinear inviscid damping near the Couette flow [3, 4, 5]. Moreover, the analyticity radius provides a measure of the minimal scale in a turbulent flow [24].

Analyticity and Gevrey-class regularity for the Navier-Stokes equation is a classical subject [13, 49]. Initially, interior analyticity for the Navier-Stokes system in $d \geq 2$ space dimensions was proven by Kahane [25], using an iteration of high order Sobolev norms. The problem of interior space-time analyticity was then addressed by Masuda [39], and then by Kato-Masuda [26], assuming that the external force is analytic. Analyticity up to the boundary of the domain was established by Komatsu in [28, 29], based on earlier work by Kinderlehrer and Nirenberg [27] for parabolic type equations. Subsequently, Giga [20] developed a semigroup approach for analyticity up to the boundary for the Navier-Stokes system.

On the other hand, in the absence of boundaries, Foias and Temam [18] introduced an alternative approach to analyticity and Gevrey-class regularity which is based on L^2 energy estimates and Fourier analysis (cf. [17, 13] for an earlier energy approach for the time analyticity). This method has proven to be a powerful tool to establish analyticity as well as to estimate the analyticity or the Gevrey radius. The

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method of Foias and Temam [18], which was based on L^2 energy estimates was extended to L^p by the second author and Grujić [22, 23], taking advantage of the mild formulation (cf. also [10] for the local variant). An elegant Fourier variant of the approach using the mild formulation and Fourier analysis was then introduced by Lemarié-Rieusset in [34]. For other results based on the Foias-Temam method, see [6, 7, 8, 9, 16, 15, 21, 31, 34, 36, 42, 43] and references therein.

The main motivation for the present work is to provide a direct energy approach to analyticity for the Navier-Stokes and related systems for domains with boundary. As it is well-known the main difficulty is the presence of normal derivatives in the diffusion term and its non-vanishing due to Dirichlet boundary conditions. For a finite Sobolev regularity, this is typically overcome by the Agmon-Douglis-Nirenberg approach to parabolic/elliptic regularity. For the analyticity, this requires a carefully designed iteration based on this parabolic regularity and binomial inequalities (cf. [28] for instance).

Recently, there have been two works where a variant of this has been employed in the case of the half space. In [50, 14], the authors provided an interior analyticity approach for the half space, using conormal, rather than normal, derivatives (cf. [38]). Recently, in [12] the authors of the present paper have found an alternative method based on a derivative reduction estimates and ellipticity; the main idea is to use the elliptic regularity to find a Grönwall type inequality for a simple series consisting of Taylor coefficients.

The main difficulty for curved domains is the non-commutativity and possible vanishing of tangential and normal derivatives (e.g. the singularity of the polar coordinates used for a disc). We overcome this by Komatsu's system of tangential vector fields, which was in turn inspired by an earlier work by Nelson [40].

As in the case of our previous paper which treated the half-space [12], the main idea is to use derivative reductions by means of the global elliptic regularity. While the method is technical, it is also robust and we believe it is going to be applicable in other settings. In particular, we hope that it will be useful for the vanishing viscosity problem in a curved domain.

There are several difficulties when trying to extend the results from the half-space setting [12]. Since there is no analytic partition of unity, the energy approach requires working with global tangential and normal vector fields. However, there is a possibility of vanishing of the tangential or normal derivatives in the interior. To overcome this problem, we use Komatsu's system of tangential vector fields, which in turn is based on earlier work by Nelson [40], with the main idea of allowing the number of tangential derivatives to be higher than the space dimension d. The analytic vector field setting is recalled in Section 2 below. An important aspect of this analytic theory is that the iterated tangential derivatives form high order tensors. To deal with this, we use summation of all the norms (rather then, say, the Euclidean or the sup convention). Regarding the Stokes problem in the case of half-space, it was not necessary to include the pressure in the energy as we were able to recover the pressure from ellipticity. However, in the case of the curved domain, three pressure commutator terms arise which thus require inclusion of the pressure space-time analyticity norm. For the case of the Navier-Stokes equations, the main problem is the product estimate for the term $u \cdot \nabla u$. While the leading terms have been treated in [12], the Leibniz rule is more complicated here and the high order commutators require special care.

In the case of Euler equations, one cannot expect instantaneous gain of analyticity; however, it is possible to obtain a lower bound on the uniform radius. On the other hand, the difficulties arising from the Laplacian are absent. Thus the above mentioned energy methods have already been employed to obtain rather precise bounds on the analyticity decay (cf. [31]). In the case of Euler equation, it is actually possible to use the partition of unity due to finite speed of Lagrangian trajectories [32]. Analyticity results for the Euler equations are classical; see the work by Bardos-Benachour [2] and Bardos [1]. For other applications of analyticity for the Euler equations, cf. [48, 11, 3].

The paper is structured as follows. In Section 2, we recall the analytic vector field setting for bounded domains with an analytic boundary, and the necessary commutator estimates. The three main theorems address separately the non-homogeneous heat equation, the non-homogeneous Stokes equation, and the Navier-Stokes equations. The next two sections contain the derivative reduction estimates and the proof of the analyticity for the heat operator. Section 5 contains the derivative reduction estimates for the Stokes operator. Finally, Section 6 contains the proof of the space-time analyticity for the Navier-Stokes system.

2. Analytic vector fields in a bounded domain and the main results

Assume throughout the paper that Ω is a bounded domain in $\mathbb{R}^d$ with analytic boundary $\partial\Omega$.

2.1. Analytic vector fields. Denote by $\delta = \delta(x)$ the distance function to the boundary $\partial\Omega$, taking positive values inside Ω and negative outside Ω. For $\delta_0 > 0$, we set

$$\Omega_{\delta_0} = \{x \in \Omega : \delta(x) < \delta_0\}, \qquad \Omega^{\delta_0} = \Omega \setminus \bar{\Omega}_{\delta_0}.$$

A vector field X is said to be *tangential* to $\partial\Omega$ if $X\delta = 0$ on $\partial\Omega$. Such X may be restricted to $\partial\Omega$ by $Xf = (X\tilde{f})\big|_{\partial\Omega}$ for $f \in C^\infty(\partial\Omega)$, where $\tilde{f} \in C^\infty(\bar{\Omega})$ is an arbitrary extension of f.

Existence of global analytic vector fields in the following proposition is due to Komatsu [**29**]. For the convenience of the reader, we state it next.

PROPOSITION 2.1 ([**29**, Section 2]). *For any sufficiently small $\delta_0 > 0$ there exists a finite number of analytic vector fields $X_0, T_1, \ldots, T_{N'}, T_{N'+1}, \ldots, T_N$ defined globally on $\bar{\Omega}$ having the following properties:*

1. *$T_1, \ldots, T_N$ are tangential to $\partial\Omega$.*
2. *On $\bar{\Omega}$ there are global expressions*

$$\frac{\partial}{\partial x_k} = \xi_k(x) X_0 + \sum_{j=1}^{N} \eta_{jk}(x) T_j, \qquad k = 1, \ldots, d, \tag{2.1}$$

with analytic coefficients $\xi_k(x)$ and $\eta_{jk}(x)$.

3. *On $\bar{\Omega}^{\delta_0}$, we have*

$$\frac{\partial}{\partial x_k} = \sum_{j=1}^{N'} \zeta_{jk}(x) T_j, \qquad k = 1, \ldots, d,$$

where $\zeta_{jk}(x)$ are analytic functions on $\bar{\Omega}^{\delta_0}$.

REMARK 2.2. The vector field X_0 is a non-tangential vector field to $\partial\Omega$ in the sense that $X_0\delta \neq 0$ on the boundary.

EXAMPLE 2.3. For $\Omega = B_1(0) \subset \mathbb{R}^2$, an example of a system of the vector fields postulated in Proposition 2.1 is as follows: $X_0 = x_1\partial_{x_1} + x_2\partial_{x_2} = x \cdot \nabla$, $T_1 = x_1\partial_{x_2} - x_2\partial_{x_1} = x^\perp \cdot \nabla$, $T_2 = (1 - x_1^2 - x_2^2)\partial_{x_1} = (1 - |x|^2)\partial_{x_1}$, and $T_3 = (1 - x_1^2 - x_2^2)\partial_2 = (1 - |x|^2)\partial_{x_2}$. Indeed, $\partial_{x_1} = x_1 X_0 - x_2 T_1 + T_2$, and $\partial_{x_2} = x_1 X_0 - x_2 T_1 + T_3$.

REMARK 2.4. *We use the letter T to denote the endpoint of the time interval $[0, T]$, and we use the notation $\boldsymbol{T} = (T_1, \ldots, T_N)$ to denote the vector of tangential differential operators T_j from Proposition 2.1.*

REMARK 2.5. Denote by $I = \{1, \ldots, N\}$ the index set for the tangential vector fields $T_1, \ldots, T_N$ given by Proposition 2.1. We adopt the following agreement for the iterated derivatives ∂_x^j and $\mathbf{T}^k$. The symbol $\mathbf{T}^k$, where $k \in \mathbb{N}$, is understood in tensorial sense, i.e., it denotes the list of all the possible operators $T_{\beta_1} \cdots T_{\beta_k}$, where $\beta = (\beta_1, \ldots, \beta_k) \in I^k$, with an analogous agreement for ∂_x^j. On the other hand, when the symbol occurs inside a norm, it has the following meaning. For $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we define

$$\|\partial_x^j \mathbf{T}^k u\|_{L^2_{x,t}} = \sum_{\substack{\alpha \in \mathbb{N}_0^d, |\alpha| = j \\ \beta \in I^k}} \|\partial_x^\alpha \mathbf{T}^\beta u\|_{L^2_{x,t}} \tag{2.2}$$

and

$$\|\partial_x^j \mathbf{T}^k u\|_{L^\infty_{x,t}} = \sum_{\substack{\alpha \in \mathbb{N}_0^d, |\alpha| = j \\ \beta \in I^k}} \|\partial_x^\alpha \mathbf{T}^\beta u\|_{L^\infty_{x,t}},$$

where $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ and $\mathbf{T}^\beta = T_{\beta_1} \cdots T_{\beta_k}$ with $\beta = (\beta_1, \ldots, \beta_k) \in I^k$. In the same way, we define

$$\|\partial_x^j \mathbf{T}^k u\|_{L_t^2 \dot{H}_x^1} = \|\partial_x^{j+1} \mathbf{T}^k u\|_{L^2_{x,t}} \tag{2.3}$$

and

$$\|\partial_x^j \mathbf{T}^k u\|_{L_t^2 \dot{H}_x^2} = \|\partial_x^{j+2} \mathbf{T}^k u\|_{L^2_{x,t}}. \tag{2.4}$$

2.2. Main results. Fix $T > 0$ and let

$$0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1. \tag{2.5}$$

In order to explain the main ideas of the proof, it is convenient to first consider the inhomogeneous heat equation

$$\partial_t u - \Delta u = f \quad \text{in } \Omega \tag{2.6}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{2.7}$$

with the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{2.8}$$

For $r \geq 3$, define the index sets

$$B = \{(i,j,k) : i,j,k \in \mathbb{N}_0, i+j+k \geq r\}, \qquad B^c = \mathbb{N}_0^3 \backslash B. \tag{2.9}$$

For the system (2.6)–(2.8), we define

$$\begin{aligned}\phi(u) &= \sum_B \frac{(i+j+k)^r}{(i+j+k)!} \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k \|t^{i+j+k-r} \partial_x^j \mathbf{T}^k \partial_t^i u\|_{L^2_{x,t}(\Omega\times[0,T])} + \sum_{B^c} \|\partial_x^j \mathbf{T}^k \partial_t^i u\|_{L^2_{x,t}(\Omega\times[0,T])} \\ &= \bar{\phi}(u) + \phi_0(u).\end{aligned} \tag{2.10}$$

We refer the reader to [33] for the same problem posed on the half space. Both on the half space and in a curved domain, the sum $\phi(u)$ is based on Taylor-like coefficients. We note that in general domains it is more convenient to use the full gradient ∂_x instead of the normal derivative X_0 in the analyticity norm (2.10) as the former commutes with the Laplacian.

THEOREM 2.6 (Heat equation). *Let $0 < T < 1$ and $r \geq 3$. Then there exist $0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1$, which depend only on r, d, and the analyticity radius of the tangential vector field $\boldsymbol{T}$ such that for any $u_0 \in H_0^1(\Omega) \cap H^{2r-1}(\Omega)$ which satisfies the compatibility conditions, and f sufficiently smooth, the solution u of (2.6)–(2.8) satisfies the estimate*

$$\phi(u) \lesssim \phi_0(u) + M_T(f) + \|u_0\|_{H^{2r-1}}$$

where

$$\begin{aligned} M_T(f) = &\sum_{i+j+k \geq (r-2)_+} \frac{(i+j+k+2)^r \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^k}{(i+j+k+2)!} \|t^{i+j+k+2-r} \partial_x^j \boldsymbol{T}^k \partial_t^i f\|_{L^2_{x,t}(\Omega\times(0,T))} \\ &+ \sum_{i+k \geq (r-2)_+} \frac{(i+k+2)^r \epsilon^i \bar{\epsilon}^{k+2}}{(i+k+2)!} \|t^{i+k+2-r} \boldsymbol{T}^k \partial_t^i f\|_{L^2_{x,t}(\Omega\times(0,T))} \\ &+ \sum_{i \geq r-1} \frac{(i+1)^r \epsilon^{i+1}}{(i+1)!} \|t^{i+1-r} \partial_t^i f\|_{L^2_{x,t}(\Omega\times(0,T))}. \end{aligned} \tag{2.11}$$

In an analogous way, we adapt our result to the case of the Stokes system with the Dirichlet boundary condition

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.12}$$

Firstly, we need to adjust our analyticity norm for the Stokes system (2.12). For $r \geq 3$, define the index set

$$\tilde{B} = \{(i,j,k) \in \mathbb{N}_0^3 : i+j+k \geq r-1, j+k \geq 1\}$$

and

$$\begin{aligned} \psi(u,p) &= \sum_{B} \frac{(i+j+k)^r}{(i+j+k)!} \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k \| t^{i+j+k-r} \partial_x^j \mathbf{T}^k \partial_t^i u \|_{L^2_{x,t}(\Omega \times [0,T])} \\ &\quad + \sum_{\tilde{B}} \frac{(i+j+k+1)^{r-1}}{(i+j+k)!} \epsilon^i \tilde{\epsilon}^{j+1} \bar{\epsilon}^k \| t^{i+j+k+1-r} \partial_x^j \mathbf{T}^k \partial_t^i p \|_{L^2_{x,t}(\Omega \times [0,T])} \\ &\quad + \sum_{B^c} \| \partial_x^j \mathbf{T}^k \partial_t^i u \|_{L^2_{x,t}(\Omega \times [0,T])} + \sum_{\tilde{B}^c \cap \{j+k \geq 1\}} \| \partial_x^j \mathbf{T}^k \partial_t^i p \|_{L^2_{x,t}(\Omega \times [0,T])} \\ &= \bar{\psi}(u,p) + \psi_0(u,p), \end{aligned} \tag{2.13}$$

where $\bar{\psi}(u,p)$ and $\psi_0(u,p)$ consist of the first and the last two terms, respectively.

THEOREM 2.7 (Stokes equations). *Let $0 < T < 1$ and $r \geq 3$. Then there exist $0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1$, which depend on r, d, and the analyticity radius of the tangential vector field $\boldsymbol{T}$ such that for any divergence-free $u_0 \in H_0^1(\Omega) \cap H^{2r-1}(\Omega)$ which satisfies the compatibility conditions, and f sufficiently smooth, the solution u of the Cauchy problem (2.12) satisfies the estimate*

$$\psi(u,p) \lesssim \psi_0(u,p) + M_T(f) + \|u_0\|_{H^{2r-1}}, \tag{2.14}$$

where $M_T(f)$ is defined in (2.11).

The essential ingredients in the proof of Theorem 2.6 are the derivative reduction estimates in the normal, tangential, and time components, which are provided in the next section. The proof of Theorem 2.7 follows by the same method and we outline the modifications in Section 5.

Using the result on the Stokes system with the force $-u \cdot \nabla u$, we may address the Navier-Stokes system with the Dirichlet boundary conditions (1.1).

THEOREM 2.8 (Navier-Stokes equations). *Let $d \in \{2,3\}$ and $r = 3$. Then there exist $0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1$, which depend on the analyticity radius of the tangential vector field $\boldsymbol{T}$, such that the following statement holds: For any divergence-free $u_0 \in H_0^1(\Omega) \cap H^5(\Omega)$ which satisfies suitable compatibility conditions, and a space-time real-analytic $f \in L^\infty(0,1;H^3(\Omega)) \cap \dot{W}^{1,\infty}(0,1;H^1(\Omega)) \cap \dot{W}^{2,\infty}(0,1;L^2(\Omega))$, for which $M_1(f) < \infty$, there exists $T_* \in (0,1]$ such that the solution u of the Cauchy problem for (1.1) satisfies the estimate*

$$\psi_T(u,p) \lesssim 1 + M_T(f) + \|u_0\|_{H^5},$$

for any $T \in (0, T_*]$, where $M_T(f)$ is given in (2.11). The implicit constant depends only on Ω.

The dimension is restricted to $d \in \{2,3\}$ for simplicity of computations. With this choice, we also fix the index $r = 3$ in the definition of $\phi(u)$ and $\psi(u,p)$. In order to obtain boundedness for the lower part of the energy norm $\psi_0(u,p)$, we appeal to the local existence theory for the Navier-Stokes system, cf. [**47, 13, 49, 44**] for instance. Since we need to set $r = 3$, we require the data to belong to H^5 (note that then $2r-1=5$). However, due to the regularizing effect, we might also assume that the data lie in H_0^1.

3. Derivative reductions for the heat operator

Here, we state the normal, tangential, and time derivative reduction estimates for a smooth solution u of (2.6)–(2.8) in terms of the vector fields introduced in Section 2. The same discussion on the half space, splitting the gradient operator into tangential and normal components, was provided in detail in [**33**, Section 3]. Here, we outline these ideas and deal with the additional commutator terms. Throughout this section we require $i+j+k \geq r$.

3.1. Normal derivative reduction. Here we consider (2.6)–(2.8).

LEMMA 3.1. *For $j \geq 2$, we have*

$$\begin{aligned} \|t^{i+j+k-r}\partial_x^j \boldsymbol{T}^k \partial_t^i u\|_{L^2_{x,t}} &\lesssim \|t^{i+j+k-r}\partial_x^{j-2}\boldsymbol{T}^k\partial_t^i f\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_x^{j-2}\boldsymbol{T}^k\partial_t^{i+1}u\|_{L^2_{x,t}} \\ &\quad + \|t^{i+j+k-r}\partial_x^{j-2}\boldsymbol{T}^{k+1}\partial_t^i u\|_{L^2_t H^1_x} + \|t^{i+j+k-r}\partial_x^{j-2}\boldsymbol{T}^k\partial_t^i u\|_{L^2_{x,t}} \\ &\quad + \|t^{i+j+k-r}\partial_x^{j-2}[\boldsymbol{T}^k,\Delta]\partial_t^i u\|_{L^2_{x,t}}. \end{aligned} \tag{3.1}$$

Similarly, for $j = 1$ and $k \geq 1$ we have

$$\begin{aligned} \|t^{i+1+k-r}\partial_x\boldsymbol{T}^k\partial_t^i u\|_{L^2_{x,t}} &\lesssim \|t^{i+1+k-r}\boldsymbol{T}^{k-1}\partial_t^{i+1}u\|_{L^2_{x,t}} + \|t^{i+1+k-r}\boldsymbol{T}^{k-1}\partial_t^i f\|_{L^2_{x,t}} \\ &\quad + \|t^{i+1+k-r}[\boldsymbol{T}^{k-1},\Delta]\partial_t^i u\|_{L^2_{x,t}}, \end{aligned} \tag{3.2}$$

while for $j = 1$ and $k = 0$, we get

$$\|t^{i+1-r}\partial_x\partial_t^i u\|_{L^2_{x,t}} \lesssim \|t^{i+1-r}\partial_t^i u\|^{1/2}_{L^2_{x,t}}\|t^{i+1-r}\partial_t^{i+1}u\|^{1/2}_{L^2_{x,t}} + \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}. \tag{3.3}$$

Before the proof, we recall the H^2 regularity for the Laplace equation which, combined with the trace theorem (cf. [35]), yields the estimate

$$\|u\|_{H^2(\Omega)} \lesssim \|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^{3/2}(\partial\Omega)} \lesssim \|\Delta u\|_{L^2(\Omega)} + \|\mathbf{T}u\|_{H^{1/2}(\partial\Omega)} + \|u\|_{H^{1/2}(\partial\Omega)} \tag{3.4}$$

from where

$$\|u\|_{H^2(\Omega)} \lesssim \|\Delta u\|_{L^2(\Omega)} + \|\mathbf{T}u\|_{H^1} + \|u\|_{L^2}. \tag{3.5}$$

If, in addition, $u\big|_{\partial\Omega} = 0$, then we have

$$\|u\|_{H^2(\Omega)} \lesssim \|\Delta u\|_{L^2(\Omega)}. \tag{3.6}$$

Besides (3.5)–(3.6), we recall

$$\|\nabla u\|_{L^2(\Omega)} \lesssim \|u\|^{1/2}_{L^2(\Omega)}\|\partial_x^2 u\|^{1/2}_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}. \tag{3.7}$$

PROOF OF LEMMA 3.1. Using (2.6), we compute

$$\begin{aligned} &\Delta(t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i u) \\ &\quad = t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^{i+1}u - t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i f - t^{i+j+k-r}\partial_x^{j-2}[\mathbf{T}^k,\Delta]\partial_t^i u \end{aligned} \tag{3.8}$$

for $j \geq 2$. By the H^2 regularity estimate (3.5), we get

$$\begin{aligned} \|t^{i+j+k-r}\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2} &\lesssim \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i u\|_{H^2} \\ &\lesssim \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^{i+1}u\|_{L^2} + \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i f\|_{L^2} \\ &\quad + \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^{k+1}\partial_t^i u\|_{H^1} + \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i u\|_{L^2} \\ &\quad + \|t^{i+j+k-r}\partial_x^{j-2}[\mathbf{T}^k,\Delta]\partial_t^i u\|_{L^2}, \end{aligned}$$

and (3.1) follows.

For (3.2), let $k \geq 1$. We have

$$\Delta(t^{i+1+k-r}\mathbf{T}^{k-1}\partial_t^i u) = t^{i+1+k-r}\mathbf{T}^{k-1}\partial_t^{i+1}u - t^{i+1+k-r}\mathbf{T}^{k-1}\partial_t^i f - t^{i+1+k-r}[\mathbf{T}^{k-1},\Delta]\partial_t^i u.$$

Since $\mathbf{T}^{k-1}\partial_t^i u\big|_{\partial\Omega} = 0$, the H^2 regularity estimate (3.6) leads to (3.2). In order to prove (3.3), we use the equation

$$\Delta(t^{i+1-r}\partial_t^i u) = t^{i+1-r}\partial_t^{i+1}u - t^{i+1-r}\partial_t^i f.$$

Using the interpolation inequality (3.7) and the H^2 regularity estimate (3.6) we obtain (3.3) (cf. [33]). □

3.2. Tangential derivative reduction. The following lemma allows us to reduce the number of tangential derivatives.

LEMMA 3.2. *For $k \geq 2$ we have*

$$\begin{aligned}\|t^{i+k-r}\boldsymbol{T}^k\partial_t^i u\|_{L^2_{x,t}} \lesssim \|t^{i+k-r}\boldsymbol{T}^{k-2}\partial_t^{i+1}u\|_{L^2_{x,t}} + \|t^{i+k-r}\boldsymbol{T}^{k-2}\partial_t^i f\|_{L^2_{x,t}} \\ + \|t^{i+k-r}[\boldsymbol{T}^{k-2},\Delta]\partial_t^i u\|_{L^2_{x,t}},\end{aligned} \tag{3.9}$$

while for $k = 1$, we have

$$\|t^{i+1-r}\boldsymbol{T}\partial_t^i u\|_{L^2_{x,t}} \lesssim \|t^{i+1-r}\partial_t^i u\|^{1/2}_{L^2_{x,t}}\|t^{i+1-r}\partial_t^{i+1}u\|^{1/2}_{L^2_{x,t}} + \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} \tag{3.10}$$

for all $i \geq r-1$.

PROOF OF LEMMA 3.2. Setting $j = 2$ and replacing k with $k-2$ in (3.8), we have

$$\Delta(t^{i+k-r}\mathbf{T}^{k-2}\partial_t^i u) = t^{i+k-r}\mathbf{T}^{k-2}\partial_t^{i+1}u - t^{i+k-r}\mathbf{T}^{k-2}\partial_t^i f - t^{i+k-r}[\mathbf{T}^{k-2},\Delta]\partial_t^i u$$

for $k \geq 2$. As $\mathbf{T}^{k-2}\partial_t^i u\big|_{\partial\Omega} = 0$, the rest of the proof is obtained following the arguments in [33]. □

3.3. Time derivative reduction. In this part, we consider the expressions of the form $\|t^{i-r}\partial_t^i u\|_{L^2_{x,t}}$, which do not involve spatial derivative operators. Therefore, the time derivative reduction estimate on the half space [33] is still applicable here. For completeness, we recall the statement.

LEMMA 3.3. *For $i \geq r$, we have*

$$\|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} \lesssim (i-r)\|t^{i-1-r}\partial_t^{i-1}u\|_{L^2_{x,t}} + \|t^{i-r}\partial_t^{i-1}f\|_{L^2_{x,t}} + \mathbb{1}_{i=r}\|\nabla\partial_t^{r-1}u(0)\|_{L^2}. \tag{3.11}$$

The proof follows from the energy inequality for the system (2.6); cf. [33]. In particular, for $i = r$, we apply ∂_t^{r-1} to (2.6) and test the resulting equation with $\partial_t^r u$.

3.4. A Leibniz type formula. Having a nice analytic expansion as in (2.1) comes with a cost of losing the equality of mixed derivatives. Below, we recall the Leibniz formula for k-folded commutator terms (cf. [40]). Given two linear operators Y, Z, the adjoint operator ad $Y(Z)$ is defined as

$$\text{ad}\, Y(Z) = [Y,Z] = YZ - ZY.$$

LEMMA 3.4 ([29, 40]). *Let $k \geq 1$. Given a differential operator Z, we have*

$$[\boldsymbol{T}^k, Z] = \sum_{m=1}^{k}\binom{k}{m}((ad\,\boldsymbol{T})^m(Z))\boldsymbol{T}^{k-m}. \tag{3.12}$$

In addition, when $Z = Z_1Z_2$, a similar formula is given by

$$[\boldsymbol{T}^k, Z] = \sum_{m=1}^{k}\sum_{\alpha\in\mathbb{N}_0^2,|\alpha|=m}\frac{k!}{\alpha!(k-m)!}\prod_{j=1}^{2}((ad\,\boldsymbol{T})^{\alpha_j}(Z_j))\boldsymbol{T}^{k-m}. \tag{3.13}$$

PROOF OF LEMMA 3.4. Formulas (3.12) and (3.13) follow from an induction based commutator expansion fact. If $Y_1,\ldots,Y_m$ and Z are linear differential operators, then

$$[Y_1\cdots Y_k, Z] = \sum_{m=1}^{k}\sum_{\tau\in\pi(k,m)}(\text{ad}\,Y_{\tau(1)}\cdots\text{ad}\,Y_{\tau(m)}(Z))Y_{\tau(m+1)}\cdots Y_{\tau(k)},$$

where $\pi(k,m)$ denotes the set of all $\binom{k}{m}$ permutations τ of $1,\ldots,k$ such that $\tau(1) < \cdots < \tau(m)$ and $\tau(m+1) < \cdots < \tau(k)$. Noting our notational convention on $\mathbf{T}^k$ given in Remark 2.5, we treat the permutations of the same order equal and rewrite the above formula in the tensor form to deduce (3.12).

Similarly, when $Z = Z_1 Z_2$, the expansion given above becomes

$$[Y_1 \cdots Y_k, Z_1 Z_2] = \sum_{|\alpha|=1, \alpha\in\mathbb{N}_0^2}^{k} \sum_{\tau\in\pi(k,\alpha)} a_{1,\tau,\alpha} a_{2,\tau,\alpha} Y_{\tau(|\alpha|+1)} \cdots Y_{\tau(k)}$$

where τ is a permutation of $\{1,\ldots,k\}$ with $\tau(1) < \cdots < \tau(\alpha_1)$, $\tau(\alpha_1+1) < \cdots < \tau(\alpha_1+\alpha_2)$, and $\tau(\alpha_1+\alpha_2+1) < \tau(k)$. We drop the respective monotonicity restrictions on τ in case $\alpha_1 = 0$, $\alpha_2 = 0$, or $\alpha_1 + \alpha_2 = k$. The coefficients above read $a_{1,\tau,\alpha} = \text{ad } Y_{\tau(1)} \cdots \text{ad } Y_{\tau(\alpha_1)}(Z_1)$ and $a_{2,\tau,\alpha} = \text{ad } Y_{\tau(\alpha_1+1} \cdots \text{ad } Y_{\tau(\alpha_1+\alpha_2)}(Z_2)$. When $\alpha_1 = 0$ or $\alpha_2 = 0$, the coefficients become $a_{1,\tau,\alpha} = Z_1$, $a_{2,\tau,\alpha} = Z_2$, respectively. □

If Y_j and Z are vector fields, then so is $(\text{ad } Y_k \cdots \text{ad } Y_1)(Z)$. Regarding the analyticity properties of the latter vector field, we recall the following result from [29].

LEMMA 3.5 ([29]). *Let $Y_1 \ldots, Y_m$ and Z be analytic vector fields defined on a domain in $\mathbb{R}^d$ such that*

$$Y_j = \sum_{i=1}^{d} a_j^i \partial_i,$$

where

$$\max_{|\alpha|=k} |\partial_x^\alpha a_j^i| \lesssim k!\, K_1^k, \qquad i = 1,\ldots,d, \qquad j = 1,\ldots,m$$

for some $K_1 \geq 1$. Then there exist $\bar{K}_1, \bar{K}_2 \geq 1$ such that

$$(ad\, Y_m \cdots ad\, Y_1)(Z) = \sum_{i=1}^{n} b_m^i \partial_i,$$

where

$$\max_{|\alpha|=k} |\partial_x^\alpha b_m^i| \lesssim (k+m)!\, \bar{K}_1^k \bar{K}_2^m$$

for $i = 1,\ldots,d$.

Note that in the above formula the constants $\bar{K}_1$ and $\bar{K}_2$ give the radius of analyticity for the vector field $\text{ad } Y_m \cdots \text{ad } Y_1$.

In the following two lemmas, we derive upper bounds for the commutators with the Laplacian. Formulas, which are close to these but different, were stated in [29] for the case of the double gradients.

LEMMA 3.6. *For $i + k \geq r - 2$, we have*

$$\begin{aligned} \|t^{i+k+2-r}[\boldsymbol{T}^k, \Delta]\partial_t^i u\|_{L^2_{x,t}} &\lesssim \sum_{k'=1}^{k} \frac{k!}{(k-k')!} K^{k'} \|t^{i+k+2-r}\partial_x^2 \boldsymbol{T}^{k-k'} \partial_t^i u\|_{L^2_{x,t}} \\ &\quad + \sum_{k'=1}^{k} \frac{k!}{(k-k')!} K^{k'} k' \|t^{i+k+2-r}\partial_x \boldsymbol{T}^{k-k'} \partial_t^i u\|_{L^2_{x,t}} \end{aligned} \tag{3.14}$$

for some $K \geq \max(K_2, \bar{K}_2)$, where K_2 and $\bar{K}_2$ are given by Lemma 3.5.

PROOF OF LEMMA 3.6. By Lemma 3.4, we have the expansion formula

$$[\mathbf{T}^k, \Delta] = [\mathbf{T}^k, \partial_{jj}] = \sum_{k'=1}^{k} \sum_{\substack{\alpha\in\mathbb{N}_0^2 \\ |\alpha|=k'}} \frac{k!}{\alpha!\,(k-k')!} (\text{ad } \mathbf{T})^{\alpha_1}(\partial_j)(\text{ad } \mathbf{T})^{\alpha_2}(\partial_j)\mathbf{T}^{k-k'}. \tag{3.15}$$

Using Lemma 3.5 with $m \in \mathbb{N}_0$, we obtain

$$(\mathrm{ad}\,\mathbf{T})^m(\partial_j) = \sum_{i=1}^{n} b^i_{m,j}\partial_i \tag{3.16}$$

where

$$\max_{|\beta|=k} |\partial_x^\beta b^i_{m,j}| \le (k+m)!\, K_2^k \bar{K_2}^m \tag{3.17}$$

for some $K_2, \bar{K}_2 \ge 1$ depending on the vector field $\mathbf{T}$.

The vector field expression in (3.15) gives

$$[\mathbf{T}^k,\Delta] = \sum_{k'=1}^{k} \sum_{\substack{\alpha\in\mathbb{N}_0^2\\ |\alpha|=k'}} \frac{k!}{\alpha!\,(k-k')!} b^l_{\alpha_1,j} b^m_{\alpha_2,j}\, \partial_l\partial_m \mathbf{T}^{k-k'} + \sum_{k'=1}^{k} \sum_{\substack{\alpha\in\mathbb{N}_0^2\\ |\alpha|=k'}} \frac{k!}{\alpha!\,(k-k')!} b^l_{\alpha_1,j}\, \partial_l b^m_{\alpha_2,j}\, \partial_m \mathbf{T}^{k-k'}.$$

Using the analyticity bounds in (3.17), we then obtain

$$\begin{aligned}
&\|t^{i+k+2-r}[\mathbf{T}^k,\Delta]\partial_t^i u\|_{L^2_{x,t}} \\
&\quad\le \sum_{k'=1}^{k} \sum_{\substack{\alpha\in\mathbb{N}_0^2\\ |\alpha|=k'}} \frac{k!}{\alpha!\,(k-k')!} K_2^{\alpha_1}\bar{K_2}^{\alpha_2}\alpha_1!\,\alpha_2!\, \|t^{i+k+2-r}\partial_x^2\mathbf{T}^{k-k'}\partial_t^i u\|_{L^2_{x,t}} \\
&\qquad + \sum_{k'=1}^{k} \sum_{\substack{\alpha\in\mathbb{N}_0^2\\ |\alpha|=k'}} \frac{k!}{\alpha!\,(k-k')!} K_2^{\alpha_1}\bar{K_2}^{\alpha_2}\alpha_1!\,\alpha_2!\,(\alpha_2+1)\|t^{i+k+2-r}\partial_x\mathbf{T}^{k-k'}\partial_t^i u\|_{L^2_{x,t}}.
\end{aligned}$$

Bounding the terms involving α and setting $K \ge \max(K_2, \bar{K}_2)$ we get (3.14). ☐

Next, we examine the operator $\partial_x^j[\mathbf{T}^k,\Delta]$ for $j,k \ge 1$. We use the binomial formula for multi-indices in order to sum up the coefficients that results from the Leibniz rule. Recall that for multi-indices β, $\beta' \in \mathbb{N}_0^n$ with $m = |\beta|$, we have

$$\sum_{\beta'\le\beta, |\beta'|=l} \binom{\beta}{\beta'} = \binom{m}{l}. \tag{3.18}$$

LEMMA 3.7. *For $(i,j,k) \in \mathbb{N}_0^3$ with $j,k \ge 1$, we have*

$$\begin{aligned}
&\|t^{i+j+k+2-r}\partial_x^j[\boldsymbol{T}^k,\Delta]\partial_t^i u\|_{L^2_{x,t}} \\
&\quad\lesssim \sum_{k'=0}^{k-1} \sum_{j'+j_3=j} \binom{j'+k-k'}{j'} \frac{j!\,k!}{j_3!\,k'!} K^{j'+k-k'} \|t^{i+j+k+2-r}\partial_x^{j_3+2}\boldsymbol{T}^{k'}\partial_t^i u\|_{L^2_{x,t}}
\end{aligned}$$

for some $K > 0$.

PROOF OF LEMMA 3.7. Differentiating both sides of the equation (3.15) and using the Leibniz rule, we get

$$\begin{aligned}
\partial_x^j[\mathbf{T}^k,\Delta] &= \partial_x^j[\mathbf{T}^k,\partial_{\ell\ell}] \\
&= \sum_{k'=1}^{k} \sum_{\substack{\alpha\in\mathbb{N}_0^2\\ |\alpha|=k'}} \frac{k!}{\alpha!\,(k-k')!} \sum_{\substack{j_1,j_2,j_3\ge0\\ j_1+j_2+j_3=j}} \binom{j}{j_1\ j_2} \partial_x^{j_1}(\mathrm{ad}\,\mathbf{T})^{\alpha_1}(\partial_\ell)\partial_x^{j_2}(\mathrm{ad}\,\mathbf{T})^{\alpha_2}(\partial_\ell)\partial_x^{j_3}\mathbf{T}^{k-k'}.
\end{aligned}$$

Analogously, the analyticity of the vector fields $(\mathrm{ad}\,\mathbf{T})^{\alpha_1}(\partial_\ell)$ and $(\mathrm{ad}\,\mathbf{T})^{\alpha_2}(\partial_\ell)$ gives the upper bound

$$\begin{aligned}
&\|t^{i+j+k+2-r}\partial_x^j[\mathbf{T}^k,\Delta]\partial_t^i u\|_{L^2_{x,t}}\\
&\lesssim \sum_{k'=1}^{k}\sum_{\substack{\alpha\in\mathbb{N}_0^2\\|\alpha|=k'}}\frac{k!}{\alpha!\,(k-k')!}\sum_{\substack{j_1,j_2,j_3\geq 0\\ j_1+j_2+j_3=j}}\Bigg(\frac{j!}{j_1!\,j_2!\,j_3!}(j_1+\alpha_1)!(j_2+\alpha_2)!\\
&\qquad\times K_2^{j_1+\alpha_1}\bar{K}_2^{j_2+\alpha_2}\|t^{i+j+k+2-r}\partial_x^{j_3+2}\mathbf{T}^{k-k'}\partial_t^i u\|_{L^2_{x,t}}\Bigg)
\end{aligned}$$

for $K_2,\bar{K}_2>0$.

Applying the binomial formula (3.18) for multi-indices $\beta=(j_1+\alpha_1,j_2+\alpha_2)\in\mathbb{N}^2$ and $\beta'=(j_1,j_2)\in\mathbb{N}^2$ we find that

$$\sum_{\substack{\alpha\in\mathbb{N}_0^2\\|\alpha|=k'}}\sum_{\substack{j_1+j_2=j'\\ j'=j-j_3}}\frac{(j_1+\alpha_1)!(j_2+\alpha_2)!}{j_1!\alpha_1!\,j_2!\,\alpha_2!}=\binom{j_1+j_2+\alpha_1+\alpha_2}{j_1+j_2}=\binom{j'+k'}{j'}$$

for fixed integers k' and $0\leq j_3\leq j$. This yields

$$\begin{aligned}
&\|t^{i+j+k+2-r}\partial_x^j[\mathbf{T}^k,\Delta]\partial_t^i u\|_{L^2_{x,t}}\\
&\lesssim\sum_{k'=1}^{k}\sum_{j'+j_3=j}\binom{j'+k'}{j'}\frac{j!\,k!}{j_3!\,(k-k')!}K^{j'+k'}\|t^{i+j+k+2-r}\partial_x^{j_3+2}\mathbf{T}^{k-k'}\partial_t^i u\|_{L^2_{x,t}}\\
&\lesssim\sum_{k'=0}^{k-1}\sum_{j'+j_3=j}\binom{j'+k-k'}{j'}\frac{j!\,k!}{j_3!\,k'!}K^{j'+k-k'}\|t^{i+j+k+2-r}\partial_x^{j_3+2}\mathbf{T}^{k'}\partial_t^i u\|_{L^2_{x,t}}
\end{aligned}\tag{3.19}$$

for $K\geq\max(K_2,\bar{K}_2)$. □

We use Lemmas 3.6 and 3.7 when estimating the commutator terms appearing in the derivative reduction estimates (3.1), (3.2), and (3.9).

4. Proof of the analyticity result for the heat operator

We recall the analyticity norm as given in (2.10):

$$\begin{aligned}
\phi(u)&=\sum_{B}\frac{(i+j+k)^r}{(i+j+k)!}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{i+j+k-r}\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}(\Omega\times[0,T])}+\sum_{B^c}\|\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}(\Omega\times[0,T])}\\
&=\bar{\phi}(u)+\phi_0(u).
\end{aligned}$$

PROOF OF THEOREM 2.6. In order to apply the derivative reduction estimates, we split $\bar{\phi}(u)$ as

$$\bar{\phi}(u)=\sum_{\ell=1}^{6}S_\ell$$

where

$$S_\ell=\sum_{B_\ell}\frac{(i+j+k)^r}{(i+j+k)!}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{i+j+k-r}\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}},\qquad \ell=1,\ldots,6$$

and

$$\begin{aligned}
&B_1=\{(i,j,k)\in B: j\geq 2\}, && B_2=\{(i,j,k)\in B: j=1,k\geq 1\}\\
&B_3=\{(i,j,k)\in B: j=1,k=0\}, && B_4=\{(i,j,k)\in B: j=0,k\geq 2\}\\
&B_5=\{(i,j,k)\in B: j=0,k=1\}, && B_6=\{(i,j,k)\in B: j=0,k=0\}.
\end{aligned}\tag{4.1}$$

We start with the two main sums S_1 and S_4, which are estimated using (3.1) and (3.9), respectively.

4.1. The S_4 term. First, we check the sum S_4 which consists of terms with only tangential and time derivatives with $j=0$ and $k\geq 2$. By (3.9),

$$\begin{aligned}
S_4 \lesssim& \sum_{(i,j,k)\in B_4} \frac{(i+j+k)^r \epsilon^i \tilde\epsilon^j \bar\epsilon^k}{(i+j+k)!} \|t^{i+k-r}\mathbf{T}^{k-2}\partial_t^{i+1}u\|_{L^2_{x,t}}\\
&+ \sum_{(i,j,k)\in B_4} \frac{(i+j+k)^r \epsilon^i \tilde\epsilon^j \bar\epsilon^k}{(i+j+k)!} \|t^{i+k-r}\mathbf{T}^{k-2}\partial_t^{i}f\|_{L^2_{x,t}}\\
&+ \sum_{(i,j,k)\in B_4} \frac{(i+j+k)^r \epsilon^i \tilde\epsilon^j \bar\epsilon^k}{(i+j+k)!} \|t^{i+k-r}[\mathbf{T}^{k-2},\Delta]\partial_t^{i}u\|_{L^2_{x,t}}.
\end{aligned}$$

By relabeling, we get

$$\begin{aligned}
S_4 \lesssim& \sum_{(i-1,j,k+2)\in B_4} \frac{(i+j+k+1)^r \epsilon^{i-1} \bar\epsilon^{k+2}}{(i+j+k+1)!} \|t^{i+k+1-r}\mathbf{T}^{k}\partial_t^{i}u\|_{L^2_{x,t}}\\
&+ \sum_{(i,j,k+2)\in B_4} \frac{(i+j+k+2)^r \epsilon^i \bar\epsilon^{k+2}}{(i+j+k+2)!} \|t^{i+k+2-r}\mathbf{T}^{k}\partial_t^{i}f\|_{L^2_{x,t}}\\
&+ \sum_{(i,j,k+2)\in B_4} \frac{(i+j+k+2)^r \epsilon^i \bar\epsilon^{k+2}}{(i+j+k+2)!} \|t^{i+k+2-r}[\mathbf{T}^{k},\Delta]\partial_t^{i}u\|_{L^2_{x,t}}\\
=& I_{41}+I_{42}+I_{43}.
\end{aligned}$$

We utilize the shift in the indices to estimate I_{41} and I_{42}, leading to

$$\begin{aligned}
I_{41}+I_{42} \lesssim& \frac{\bar\epsilon^2 T}{\epsilon} \sum_{(i-1,j,k+2)\in B_4} \frac{(i+j+k+1)^r \epsilon^{i} \bar\epsilon^{k}}{(i+j+k+1)!} \|t^{i+k-r}\mathbf{T}^{k}\partial_t^{i}u\|_{L^2_{x,t}}\\
&+ \sum_{(i,j,k+2)\in B_4} \frac{(i+j+k+2)^r \epsilon^i \bar\epsilon^{k+2}}{(i+j+k+2)!} \|t^{i+k+2-r}\mathbf{T}^{k}\partial_t^{i}f\|_{L^2_{x,t}}\\
\lesssim& \frac{\bar\epsilon^2 T}{\epsilon}\phi(u) + \sum_{(i,j,k+2)\in B_4} \frac{(i+j+k+2)^r \epsilon^i \bar\epsilon^{k+2}}{(i+j+k+2)!} \|t^{i+k+2-r}\mathbf{T}^{k}\partial_t^{i}f\|_{L^2_{x,t}}.
\end{aligned}$$

For I_{43}, we use the bound (3.14) on $\|t^{i+k+2-r}[\mathbf{T}^k,\Delta]\partial_t^i u\|_{L^2_{x,t}}$ and we obtain

$$\begin{aligned}
I_{43} \lesssim& \sum_{i\in\mathbb{N}_0, k\in\mathbb{N}} \frac{(i+k+2)^r \epsilon^i \bar\epsilon^{k+2}}{(i+k+2)!} \|t^{i+k+2-r}[\mathbf{T}^{k},\Delta]\partial_t^{i}u\|_{L^2_{x,t}}\\
\lesssim& \sum_{\substack{(i,0,k+2)\in B_4\\ k\geq 1}} \frac{(i+k+2)^r \epsilon^i \bar\epsilon^{k+2}}{(i+k+2)!} \sum_{k'=0}^{k-1} \frac{k!}{k'!} K^{k-k'} \|t^{i+k+2-r}\partial_x^2\mathbf{T}^{k'}\partial_t^{i}u\|_{L^2_{x,t}}.
\end{aligned} \tag{4.2}$$

Note that we only kept the first term from the equation (3.14) as the coefficient k in the second term is compensated by the decrease in the total number of derivatives. Also, the terms in the sum with $k=0$ are excluded above as there is no commutator term appearing. Changing the order of summation, the sum I_{43}

can be further estimated as

$$
\begin{aligned}
I_{43} \lesssim & \sum_{\substack{i,k'=0\\(i,k')\neq(0,0)}}^{\infty} \frac{(i+k'+2)^r\epsilon^i\tilde{\epsilon}^2\bar{\epsilon}^{k'}}{(i+k'+2)!}\|t^{i+k'+2-r}\partial_x^2\mathbf{T}^{k'}\partial_t^i u\|_{L^2_{x,t}} \\
& \times\left(\sum_{k=k'+1}^{\infty}\frac{\bar{\epsilon}^2}{\tilde{\epsilon}^2}\frac{(i+k+2)^r}{(i+k'+2)^r}\frac{(i+k'+2)!\,k!}{(i+k+2)!\,k'!}(K\bar{\epsilon}t)^{k-k'}\right).
\end{aligned}
\tag{4.3}
$$

Note that the sum in k in (4.3) is dominated by

$$
\frac{\bar{\epsilon}^2}{\tilde{\epsilon}^2}\sum_{k=k'+1}^{\infty}\frac{(i+k+2)^{r-2}}{(i+k'+2)^{r-2}}\frac{k!}{(i+k)!}\frac{(i+k')!}{k'!}(K\bar{\epsilon}T)^{k-k'}\lesssim 1 \tag{4.4}
$$

uniformly for all $i\geq 0$, whenever $T\leq\min(1/K\bar{\epsilon},1)$. From here on, we assume $0<T\leq 1$ and reduce the range for $\bar{\epsilon}$ to $0<\bar{\epsilon}<1/K$ to assure convergence.

Therefore,

$$
I_{43}\lesssim KT\frac{\bar{\epsilon}^3}{\tilde{\epsilon}^2}\bar{\phi}(u)+KT\bar{\epsilon}^3\phi_0(u)
$$

for $0<\bar{\epsilon}<1/K$.

Adding the estimates for I_{41}, I_{42}, and I_{43}, we obtain

$$
\begin{aligned}
S_4 \lesssim & KT\frac{\bar{\epsilon}^3}{\tilde{\epsilon}^2}\bar{\phi}(u)+KT\bar{\epsilon}^3\phi_0(u)+\frac{\bar{\epsilon}^2T}{\epsilon}\phi(u) \\
& +\sum_{(i,j,k+2)\in B_4}\frac{(i+j+k+2)^r\epsilon^i\bar{\epsilon}^{k+2}}{(i+j+k+2)!}\|t^{i+k+2-r}\mathbf{T}^k\partial_t^i f\|_{L^2_{x,t}}.
\end{aligned}
\tag{4.5}
$$

4.2. The S_1 term. Next, we consider S_1. By (3.1),

$$
\begin{aligned}
S_1 \lesssim & \sum_{B_1}\frac{(i+j+k)^r\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k}{(i+j+k)!}\|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i f\|_{L^2_{x,t}} \\
& +\sum_{B_1}\frac{(i+j+k)^r\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k}{(i+j+k)!}\|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^{i+1} u\|_{L^2_{x,t}} \\
& +\sum_{B_1}\frac{(i+j+k)^r\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k}{(i+j+k)!}\|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^{k+1}\partial_t^{i} u\|_{L^2_t H^1_x} \\
& +\sum_{B_1}\frac{(i+j+k)^r\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k}{(i+j+k)!}\|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^{i} u\|_{L^2_{x,t}} \\
& +\sum_{B_1}\frac{(i+j+k)^r\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k}{(i+j+k)!}\|t^{i+j+k-r}\partial_x^{j-2}[\mathbf{T}^k,\Delta]\partial_t^{i} u\|_{L^2_{x,t}}.
\end{aligned}
$$

We then relabel to obtain

$$
\begin{aligned}
S_1 \lesssim & \sum_{(i,j+2,k)\in B_1}\frac{(i+j+k+2)^r\epsilon^i\tilde{\epsilon}^{j+2}\bar{\epsilon}^k}{(i+j+k+2)!}\|t^{i+j+k+2-r}\partial_x^{j}\mathbf{T}^k\partial_t^i f\|_{L^2_{x,t}} \\
& +\sum_{(i-1,j+2,k)\in B_1}\frac{(i+j+k+1)^r\epsilon^{i-1}\tilde{\epsilon}^{j+2}\bar{\epsilon}^k}{(i+j+k+1)!}\|t^{i+j+k+1-r}\partial_x^{j}\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}} \\
& +\sum_{(i,j+2,k-1)\in B_1}\frac{(i+j+k+1)^r\epsilon^{i}\tilde{\epsilon}^{j+2}\bar{\epsilon}^{k-1}}{(i+j+k+1)!}\|t^{i+j+k+1-r}\partial_x^{j}\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{(i,j+1,k-1)\in B_1} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^{j+1} \bar{\epsilon}^{k-1}}{(i+j+k)!} \|t^{i+j+k-r} \partial_x^j \mathbf{T}^k \partial_t^i u\|_{L^2_{x,t}} \\
&+ \sum_{(i,j+2,k-2)\in B_1} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k-2}}{(i+j+k)!} \|t^{i+j+k-r} \partial_x^j \mathbf{T}^k \partial_t^i u\|_{L^2_{x,t}} \\
&+ \sum_{(i,j+2,k)\in B_1} \frac{(i+j+k+2)^r \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!} \|t^{i+j+k+2-r} \partial_x^j \mathbf{T}^k \partial_t^i u\|_{L^2_{x,t}} \\
&+ \sum_{(i,j+2,k)\in B_1} \frac{(i+j+k+2)^r \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!} \|t^{i+j+k+2-r} \partial_x^j [\mathbf{T}^k, \Delta] \partial_t^i u\|_{L^2_{x,t}} \\
&= I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16} + I_{17}.
\end{aligned}
$$

Firstly, using the definition of the norm $\phi(u)$ the non-commutator terms can be bounded from above

$$
\begin{aligned}
&I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16} \\
&\leq \left(T\frac{\tilde{\epsilon}^2}{\epsilon} + T\frac{\tilde{\epsilon}^2}{\bar{\epsilon}} + \frac{\tilde{\epsilon}}{\bar{\epsilon}} + \frac{\tilde{\epsilon}^2}{\bar{\epsilon}^2} + T^2\tilde{\epsilon}^2 \right) \phi(u) \\
&\quad + \sum_{(i,j+2,k)\in B_1} \frac{(i+j+k+2)^r \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!} \|t^{i+j+k+2-r} \partial_x^j \mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}}.
\end{aligned}
\tag{4.6}
$$

Next, we estimate the sum I_{17}. In order to do this, we use the estimate (3.19), and then change the order of summation. Note that $k \geq 1$ as I_{17} is a commutator term. That results in

$$
\begin{aligned}
I_{17} &\lesssim \sum_{(i,j+2,k)\in B_1} \sum_{k'=0}^{k-1} \sum_{j'+j_3=j} \left(\frac{(i+j+k+2)^r}{(i+j+k+2)!} \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^k \binom{j'+k-k'}{j'} \frac{j!\,k!}{j_3!\,k'!} \right. \\
&\qquad \left. \times K^{j'+k-k'} \|t^{i+j+k+2-r} \partial_x^{j_3+2} \mathbf{T}^{k'} \partial_t^i u\|_{L^2_{x,t}} \right) \\
&\lesssim \sum_{(i,j_3+2,k')\in B_1\cup B^c} \sum_{k=k'+1}^{\infty} \sum_{j=j_3}^{\infty} \left(\frac{(i+j+k+2)^r}{(i+j+k+2)!} \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^k \binom{j-j_3+k-k'}{j-j_3} \frac{j!\,k!}{j_3!\,k'!} \right. \\
&\qquad \left. \times K^{j-j_3+k-k'} \|t^{i+j+k+2-r} \partial_x^{j_3+2} \mathbf{T}^{k'} \partial_t^i u\|_{L^2_{x,t}} \right).
\end{aligned}
\tag{4.7}
$$

Next, we split up the first sum in the equation (4.7) into two parts–over B_1 and B^c. Then, rearranging the coefficients according to our analyticity norm we obtain

$$
\begin{aligned}
I_{17} &\lesssim \sum_{(i,j_3+2,k')\in B_1} \frac{(i+j_3+k'+2)^r}{(i+j_3+k'+2)!} \epsilon^i \tilde{\epsilon}^{j_3+2} \bar{\epsilon}^{k'} A_1 \|t^{i+j_3+k'+2-r} \partial_x^{j_3+2} \mathbf{T}^{k'} \partial_t^i u\|_{L^2_{x,t}} \\
&\quad + \sum_{(i,j_3+2,k')\in B^c} A_0 \|\partial_x^{j_3+2} \mathbf{T}^{k'} \partial_t^i u\|_{L^2_{x,t}},
\end{aligned}
$$

where the coefficients A_1 and A_0 enclose the second sums due to the Fubini Theorem. Explicitly,

$$
\begin{aligned}
A_1 &= \sum_{k=k'+1}^{\infty} \sum_{j=j_3}^{\infty} \frac{(i+j_3+k'+2)!}{(i+j+k+2)!} \frac{(i+j+k+2)^r}{(i+j_3+k'+2)^r} \frac{j!k!}{j_3!\,k'!} \binom{j-j_3+k-k'}{j-j_3} \\
&\qquad \times K^{j-j_3+k-k'} (T\tilde{\epsilon})^{j-j_3} (T\bar{\epsilon})^{k-k'} \\
&\lesssim \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{(i+j_3+k'+2-r)!}{(i+j+j_3+k+k'+2-r)!} \frac{(j+j_3)!\,(k+k')!\,(j+k)!}{j_3!\,k'!\,j!\,k!} (KT\tilde{\epsilon})^j (KT\bar{\epsilon})^k
\end{aligned}
\tag{4.8}
$$

for $(i, j_3, k') \in \mathbb{N}_0^3$ with $i + j_3 + k \geq r - 2$, and

$$A_0 = \sum_{\substack{j,k=0 \\ i+j+k \geq r-2}}^{\infty} \frac{(i+j+k+2)^r}{(i+j+k+2)!} \binom{j - j_3 + k - k'}{j - j_3} \frac{j!\, k!}{j_3!\, k'!} K^{j-j_3+k-k'} T^{i+j+k+2-r} \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^k$$

for $(i, j_3, k') \in \mathbb{N}_0^3$ with $i + j_3 + k' < r - 2$. We omit the dependence on i, j_3, k', and T. The coefficient in front of $(KT\tilde{\epsilon})^j (KT\bar{\epsilon})^k$ in (4.8) is bounded above by

$$\begin{aligned} &\frac{(i+j_3+k'+2-r)!}{(i+j+j_3+k+k'+2-r)!} \frac{(j+j_3)!\,(k+k')!(j+k)!}{j_3!\,k'!\,j!\,k!} = \frac{\binom{j+j_3}{j}\binom{k+k'}{k}}{\binom{i+j+j_3+k+k'+2-r}{j+k}} \\ &\leq \frac{\binom{j+j_3}{j}\binom{k+k'}{k}}{\binom{j+j_3+k+k'}{j+k}} \frac{(i+j_3+k'+2-r)!}{(j_3+k')!} \frac{(j+j_3+k+k')!}{(i+j+j_3+k+k'+2-r)!} \\ &\leq \frac{(i+j_3+k'+2-r)!}{(j_3+k')!} \frac{(j+j_3+k+k')!}{(i+j+j_3+k+k'+2-r)!} \end{aligned} \tag{4.9}$$

for $i + j_3 + k' + 2 \geq r$. Note that when $i + 2 \geq r$, the far right side of (4.9) is bounded above by 1. On the other hand, when $0 \leq i < r - 2$, it is bounded by a constant multiple of $(j + k)^{r-i-2}$. Combining this with our assumption $T \leq 1$, we arrive at

$$\sup_{(i,j_3+2,k') \in B_1} A_1 \lesssim \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} (j+k)^{r-2} (KT\tilde{\epsilon})^j (KT\bar{\epsilon})^k \lesssim KT\bar{\epsilon},$$

when $\max(\tilde{\epsilon}, \bar{\epsilon}) < 1/K$.

Likewise, the coefficients appearing in the sum A_0 obey

$$\begin{aligned} &\frac{(i+j+k+2)^r}{(i+j+k+2)!} \frac{j!\,k!}{j_3!\,k'!} \binom{j - j_3 + k - k'}{j - j_3} \lesssim \frac{(j+k)!}{(i+j+k+2-r)!} \frac{j!\,k!}{(j+k)!} \frac{(j-j_3+k-k')!}{j_3!\,k'!\,(j-j_3)!\,(k-k')!} \\ &= \frac{(j+k)!}{(i+j+k+2-r)!} \frac{\binom{j}{j_3}\binom{k}{k'}}{\binom{j+k}{j_3+k'}} \frac{1}{(j_3+k')!} \leq \frac{(j+k)!}{(i+j+k+2-r)!}. \end{aligned} \tag{4.10}$$

Similarly to (4.9), the right side of (4.10) is bounded by 1 when $i+2 \geq r$. Otherwise, the bound is a constant multiple of $(j + k)^{r-i-2}$. Thus, A_0 is also summable whenever $\max(\tilde{\epsilon}, \bar{\epsilon}) < 1/K$, and is bounded by

$$\sup_{(i,j_3+2,k') \in B^c} A_0 \leq \sum_{\substack{j,k=0 \\ i+j+k \geq r-2}}^{\infty} (j+k)^{r-2}\, \tilde{\epsilon}^2 (K\tilde{\epsilon})^j (K\bar{\epsilon})^k \lesssim \tilde{\epsilon}^2.$$

We then conclude that

$$I_{17} \lesssim \tilde{\epsilon}^2 \phi_0(u) + KT\bar{\epsilon}\bar{\phi}(u). \tag{4.11}$$

Combining the estimates (4.6) and (4.11), we obtain

$$\begin{aligned} S_1 &\lesssim \tilde{\epsilon}^2 \phi_0(u) + KT\bar{\epsilon}\bar{\phi}(u) + \left(T\frac{\tilde{\epsilon}^2}{\epsilon} + T\frac{\tilde{\epsilon}^2}{\bar{\epsilon}} + \frac{\tilde{\epsilon}}{\bar{\epsilon}} + \frac{\tilde{\epsilon}^2}{\bar{\epsilon}^2} + T^2\tilde{\epsilon}^2 \right) \phi(u) \\ &\quad + \sum_{(i,j+2,k) \in B_1} \frac{(i+j+k+2)^r \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^k}{(i+j+k+2)!} \| t^{i+j+k+2-r} \partial_x^j \mathbf{T}^k \partial_t^i f \|_{L^2_{x,t}}. \end{aligned} \tag{4.12}$$

4.3. The S_2 term. Next, we estimate S_2. Using the tangential derivative reductions, given in (3.2), we write

$$S_2 \lesssim \sum_{B_2} \frac{(i+1+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^k}{(i+1+k)!} \|t^{i+1+k-r} \mathbf{T}^{k-1} \partial_t^{i+1} u\|_{L^2_{x,t}} + \sum_{B_2} \frac{(i+1+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^k}{(i+1+k)!} \|t^{i+1+k-r} \mathbf{T}^{k-1} \partial_t^i f\|_{L^2_{x,t}}$$
$$+ \sum_{B_2} \frac{(i+1+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^k}{(i+1+k)!} \|t^{i+1+k-r} [\mathbf{T}^{k-1}, \Delta] \partial_t^i u\|_{L^2_{x,t}}$$

and then, by relabeling

$$\begin{aligned} S_2 \lesssim & \sum_{(i-1,1,k+1) \in B_2} \frac{(i+1+k)^r \epsilon^{i-1} \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+1+k)!} \|t^{i+1+k-r} \mathbf{T}^k \partial_t^i u\|_{L^2_{x,t}} \\ & + \sum_{(i,1,k+1) \in B_2} \frac{(i+2+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+2+k)!} \|t^{i+2+k-r} \mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}} \\ & + \sum_{(i,1,k+1) \in B_2} \frac{(i+2+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+2+k)!} \|t^{i+2+k-r} [\mathbf{T}^k, \Delta] \partial_t^i u\|_{L^2_{x,t}} \\ = & \, I_{21} + I_{22} + I_{23}. \end{aligned}$$

Once again, we start with the first two terms I_{21} and I_{22}. For $t \leq T$, we get

$$I_{21} + I_{22} \lesssim \frac{T \bar{\epsilon}}{\epsilon} \phi(u) + \sum_{(i,1,k+1) \in B_2} \frac{(i+2+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+2+k)!} \|t^{i+2+k-r} \mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}}.$$

Next, we treat the commutator term I_{23}. Note that multiplying I_{23} with an $\bar{\epsilon}/\tilde{\epsilon}$ prefactor yields a summation over B_2 whose terms are identical to those of I_{43}. Therefore,

$$\begin{aligned} \frac{\bar{\epsilon}}{\tilde{\epsilon}} I_{23} & \lesssim \sum_{\substack{(i,1,k+1) \in B_2 \\ k \geq 1}} \frac{(i+k+2)^r \epsilon^i \bar{\epsilon}^{k+2}}{(i+k+2)!} \|t^{i+k+2-r} [\mathbf{T}^k, \Delta] \partial_t^i u\|_{L^2_{x,t}} \\ & \lesssim \sum_{\substack{(i,1,k+1) \in B_2 \\ k \geq 1}} \frac{(i+k+2)^r \epsilon^i \bar{\epsilon}^{k+2}}{(i+k+2)!} \sum_{k'=0}^{k-1} \frac{k!}{k'!} K^{k-k'} \|t^{i+k+2-r} \partial_x^2 \mathbf{T}^{k'} \partial_t^i u\|_{L^2_{x,t}} \end{aligned} \tag{4.13}$$

for $K > 0$ as determined in Lemma 3.6. Changing the order of summation as in (4.3)–(4.4), we arrive at

$$I_{23} \lesssim KT \frac{\bar{\epsilon}^2}{\tilde{\epsilon}} \bar{\phi}(u) + KT \tilde{\epsilon} \bar{\epsilon}^2 \phi_0(u)$$

for $T \leq 1$ and $0 < \bar{\epsilon} < 1/K$. Therefore,

$$S_2 \lesssim KT \tilde{\epsilon} \bar{\epsilon}^2 \phi_0(u) + KT \left(\frac{\bar{\epsilon}^2}{\tilde{\epsilon}} + \frac{\bar{\epsilon}}{\epsilon} \right) \phi(u) + \sum_{(i,1,k+1) \in B_2} \frac{(i+2+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+2+k)!} \|t^{i+2+k-r} \mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}}. \tag{4.14}$$

4.4. The S_3 term. Using (3.3) for S_3, we have

$$\begin{aligned} S_3 \lesssim & \sum_{(i,j,k) \in B_3} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i+1-r} \partial_t^i u\|_{L^2_{x,t}}^{1/2} \|t^{i+1-r} \partial_t^{i+1} u\|_{L^2_{x,t}}^{1/2} \\ & + \sum_{(i,j,k) \in B_3} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i+1-r} \partial_t^i u\|_{L^2_{x,t}} + \sum_{(i,j,k) \in B_3} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i+1-r} \partial_t^i f\|_{L^2_{x,t}}. \end{aligned} \tag{4.15}$$

We note that $j = 1$ and $k = 0$ for $(i, j, k) \in B_3$. Using the Cauchy-Schwarz inequality in (4.15) and relabeling in one of the sums, we immediately get

$$S_3 \lesssim \left(T\tilde{\epsilon} \sum_{(i,j,k)\in B_3} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} \right)^{1/2} \left(\frac{\tilde{\epsilon}}{\epsilon} \sum_{(i-1,j,k)\in B_3} \frac{i^r \epsilon^i}{i!} \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} \right)^{1/2}$$
$$+ T\tilde{\epsilon} \sum_{(i,j,k)\in B_3} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} + \tilde{\epsilon} \sum_{(i,j,k)\in B_3} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}.$$

Therefore, we obtain

$$S_3 \lesssim \left(\frac{T^{1/2}\tilde{\epsilon}}{\epsilon^{1/2}} + T\tilde{\epsilon} \right) \phi(u) + \tilde{\epsilon} \sum_{(i,j,k)\in B_3} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}. \tag{4.16}$$

4.5. The S_5 term. For S_5, we use (3.10) and write

$$S_5 \lesssim \sum_{(i,j,k)\in B_5} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}}^{1/2} \|t^{i+1-r}\partial_t^{i+1} u\|_{L^2_{x,t}}^{1/2}$$
$$+ \sum_{(i,j,k)\in B_5} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}} + \sum_{(i,j,k)\in B_5} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}.$$

Note that $j = 0$ and $k = 1$ for $(i, j, k) \in B_5$. By using Cauchy-Schwarz inequality on the first term and then relabeling the indices on the right side, we get

$$S_5 \lesssim \left(T\bar{\epsilon} \sum_{(i,j,k)\in B_5} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} \right)^{1/2} \left(\frac{\bar{\epsilon}}{\epsilon} \sum_{(i-1,j,k)\in B_5} \frac{i^r \epsilon^i}{i!} \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} \right)^{1/2}$$
$$+ T\bar{\epsilon} \sum_{(i,j,k)\in B_5} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} + \bar{\epsilon} \sum_{(i,j,k)\in B_5} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}.$$

We deduce that

$$S_5 \lesssim \left(\frac{T\bar{\epsilon}}{\epsilon^{1/2}} + T\bar{\epsilon} \right) \phi(u) + \bar{\epsilon} \sum_{(i,j,k)\in B_5} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}. \tag{4.17}$$

4.6. The S_6 term. Finally, for S_6 we use (3.11) and obtain

$$S_6 \lesssim \sum_{(i,j,k)\in B_6} \frac{(i-r)(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i-1-r}\partial_t^{i-1} u\|_{L^2_{x,t}}$$
$$+ \sum_{(i,j,k)\in B_6} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} \|t^{i-r}\partial_t^{i-1} f\|_{L^2_{x,t}} + \frac{r^r}{r!} \epsilon^r \|\nabla \partial_t^{r-1} u(0)\|_{L^2}$$
$$= \sum_{(i+1,j,k)\in B_6} \frac{(i+1)^r (i-r+1) \epsilon^{i+1}}{(i+1)!} \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}}$$
$$+ \sum_{(i+1,j,k)\in B_6} \frac{(i+1)^r \epsilon^{i+1}}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} + \frac{r^r}{r!} \epsilon^r \|\nabla \partial_t^{r-1} u(0)\|_{L^2}$$

since for any triple $(i, j, k) \in B_6$ we have $j = k = 0$. Therefore,

$$S_6 \lesssim \epsilon\phi(u) + \epsilon \sum_{(i+1,j,k)\in B_6} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} + \|u_0\|_{H^{2r-1}}. \tag{4.18}$$

4.7. Conclusion of the proof. Combining all the estimates (4.5), (4.12), (4.14), (4.16), (4.17), and (4.18), we write

$$
\begin{aligned}
\bar{\phi}(u) \lesssim \Bigg(& KT\frac{\bar{\epsilon}^3}{\tilde{\epsilon}^2} + \frac{\bar{\epsilon}^2 T}{\epsilon} + KT\bar{\epsilon} + T\frac{\tilde{\epsilon}^2}{\epsilon} + T\frac{\tilde{\epsilon}^2}{\bar{\epsilon}} + \frac{\tilde{\epsilon}}{\bar{\epsilon}} + \frac{\tilde{\epsilon}^2}{\bar{\epsilon}^2} + T^2\tilde{\epsilon}^2 \\
& + KT\left(\frac{\bar{\epsilon}^2}{\tilde{\epsilon}} + \frac{\bar{\epsilon}}{\epsilon}\right) + \frac{T^{1/2}(\tilde{\epsilon}+\bar{\epsilon})}{\epsilon^{1/2}} + T(\tilde{\epsilon}+\bar{\epsilon}) + \epsilon \Bigg)\phi(u) \\
& + (KT\bar{\epsilon}^3 + 1 + KT\tilde{\epsilon}\bar{\epsilon}^2)\phi_0(u) \\
& + \sum_{(i,j,k+2)\in B_4} \frac{(i+j+k+2)^r \epsilon^i \bar{\epsilon}^{k+2}}{(i+j+k+2)!} \|t^{i+k+2-r}\mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}} \\
& + \sum_{(i,j+2,k)\in B_1} \frac{(i+j+k+2)^r \epsilon^i \tilde{\epsilon}^{j+2}\bar{\epsilon}^k}{(i+j+k+2)!} \|t^{i+j+k+2-r}\partial_x^j \mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}} \\
& + \sum_{(i,1,k+1)\in B_2} \frac{(i+2+k)^r \epsilon^i \tilde{\epsilon}\bar{\epsilon}^{k+1}}{(i+2+k)!} \|t^{i+2+k-r}\mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}} \\
& + \tilde{\epsilon} \sum_{(i,j,k)\in B_3} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} + \bar{\epsilon} \sum_{(i,j,k)\in B_5} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} \\
& + \epsilon \sum_{(i+1,j,k)\in B_6} \frac{(i+1)^r \epsilon^i}{(i+1)!} \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} + \|u_0\|_{H^{2r-1}}.
\end{aligned} \tag{4.19}
$$

Denote by C the implicit constant associated with the symbol $\lesssim$ in (4.19). Recall that we already fixed $0 < T \leq 1$. Next, we set our radii of analyticity $\epsilon, \tilde{\epsilon}, \bar{\epsilon}$ in order to keep the coefficient of $\phi(u)$ on the right side of the equation (4.19) sufficiently small. Starting with $\epsilon = \epsilon(C) > 0$, we enforce

$$\epsilon \leq \frac{1}{8C}. \tag{4.20}$$

With this choice of ϵ, we choose $0 < \bar{\epsilon} = \bar{\epsilon}(K, C) \leq \min(\epsilon, \frac{1}{K})$ such that

$$T\frac{\bar{\epsilon}^2}{\epsilon} + KT\bar{\epsilon} + KT\frac{\bar{\epsilon}}{\epsilon} + T^{1/2}\frac{\bar{\epsilon}}{\epsilon^{1/2}} + T\bar{\epsilon} \leq \frac{1}{8C}. \tag{4.21}$$

Since $T \leq 1$, it suffices to take $0 < \bar{\epsilon} \leq \epsilon/8^2KC$. Next, we pick $0 < \tilde{\epsilon} = \tilde{\epsilon}(\bar{\epsilon}, \epsilon, K, C) \leq \bar{\epsilon}$ such that

$$T\frac{\tilde{\epsilon}^2}{\epsilon} + T\frac{\tilde{\epsilon}^2}{\bar{\epsilon}} + \frac{\tilde{\epsilon}}{\bar{\epsilon}} + \frac{\tilde{\epsilon}^2}{\bar{\epsilon}^2} + T^2\tilde{\epsilon}^2 + T^{1/2}\frac{\tilde{\epsilon}}{\epsilon^{1/2}} + T\tilde{\epsilon} \leq \frac{1}{8C}.$$

Once again, setting $0 < \tilde{\epsilon} < \bar{\epsilon}/8^2C$ is enough. Finally, we require that

$$KT\left(\frac{\bar{\epsilon}^3}{\tilde{\epsilon}^2} + \frac{\bar{\epsilon}^2}{\tilde{\epsilon}}\right) \leq \frac{1}{8C}. \tag{4.22}$$

Note that the conditions (4.20)–(4.22) hold with the choice $\tilde{\epsilon} = \epsilon/8^4C^2K$, $\bar{\epsilon} = \epsilon/8^2CK$, and $\epsilon = 1/8^3C^2$ without any extra requirement on the size of T.

Using the selection (4.20)–(4.22) we made for $0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1$, we rewrite (4.19) as

$$
\begin{aligned}
\bar{\phi}(u) \leq \frac{1}{2}\phi(u) + C\phi_0(u) + C & \sum_{(i,j+2,k)\in B_1} \frac{(i+j+k+2)^r \epsilon^i \tilde{\epsilon}^{j+2}\bar{\epsilon}^k}{(i+j+k+2)!} \|t^{i+j+k+2-r}\partial_x^j \mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}} \\
& + C\left(\tilde{\epsilon}\bar{\epsilon} \sum_{(i,j,k+1)\in B_2} + \bar{\epsilon}^2 \sum_{(i,j,k+2)\in B_4}\right)\left(\frac{(i+k+2)^r \epsilon^i \bar{\epsilon}^k}{(i+k+2)!} \|t^{i+2+k-r}\mathbf{T}^k \partial_t^i f\|_{L^2_{x,t}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ C\left(\tilde{\epsilon}\sum_{(i,j,k)\in B_3} + \bar{\epsilon}\sum_{(i,j,k)\in B_5} + \epsilon\sum_{(i+1,j,k)\in B_6}\right)\left(\frac{(i+1)^r\epsilon^i}{(i+1)!}\|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}\right)\\
&\le \frac{1}{2}\phi(u) + C\phi_0(u) + 2C\sum_{i+j+k\ge(r-2)_+}\frac{(i+j+k+2)^r\epsilon^i\tilde{\epsilon}^{j+2}\bar{\epsilon}^k}{(i+j+k+2)!}\|t^{i+j+k+2-r}\partial_x^j\mathbf{T}^k\partial_t^i f\|_{L^2_{x,t}}\\
&\quad + 2C\sum_{i+k\ge(r-2)_+}\frac{(i+k+2)^r\epsilon^i\bar{\epsilon}^{k+2}}{(i+k+2)!}\|t^{i+k+2-r}\mathbf{T}^k\partial_t^i f\|_{L^2_{x,t}}\\
&\quad + 3C\sum_{i\ge(r-1)_+}\frac{(i+1)^r\epsilon^{i+1}}{(i+1)!}\|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} + C\|u_0\|_{H^{2r-1}},
\end{aligned}
$$

and the proof is concluded ☐

5. Derivative reductions for the Stokes system

In this section, we adapt the derivative reduction estimates in Section 3 to the time-dependent Stokes equations. Analogously to the heat equation, the method is based on the H^2 inequalities for the stationary Stokes system

$$
\begin{aligned}
-\Delta u + \nabla p &= f \quad \text{in } \Omega\\
\nabla\cdot u &= 0 \quad \text{in } \Omega,
\end{aligned}
$$

which read

$$
\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|\mathbf{T}u\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)}. \tag{5.1}
$$

If also $u\big|_{\partial\Omega} = 0$ holds, then the above estimate becomes

$$
\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \tag{5.2}
$$

Next, we state the normal, tangential, and time reduction estimates for the Stokes operator. Assume $i+j+k\ge r$.

5.1. Normal derivative reductions for the Stokes operator.

For $j\ge 2$, we claim

$$
\begin{aligned}
&\|t^{i+j+k-r}\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_x^{j-1}\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}}\\
&\quad\lesssim \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i f\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^{i+1} u\|_{L^2_{x,t}}\\
&\qquad + \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^{k+1}\partial_t^i u\|_{L^2_t H^1_x} + \|t^{i+j+k-r}\partial_x^{j-2}\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}}\\
&\qquad + \|t^{i+j+k-r}\partial_x^{j-2}[\mathbf{T}^k,\Delta]\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_x^{j-2}[\mathbf{T}^k,\nabla]\partial_t^i p\|_{L^2_{x,t}}.
\end{aligned}
\tag{5.3}
$$

Similarly, for $j=1$, we have

$$
\begin{aligned}
&\|t^{i+1+k-r}\partial_x\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+1+k-r}\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}}\\
&\quad\lesssim \|t^{i+1+k-r}\mathbf{T}^{k-1}\partial_t^{i+1} u\|_{L^2_{x,t}} + \|t^{i+1+k-r}\mathbf{T}^{k-1}\partial_t^i f\|_{L^2_{x,t}}\\
&\qquad + \|t^{i+1+k-r}[\mathbf{T}^{k-1},\Delta]\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+1+k-r}[\mathbf{T}^{k-1},\nabla]\partial_t^i p\|_{L^2_{x,t}}.
\end{aligned}
\tag{5.4}
$$

For $j=1$ and $k=0$, we obtain

$$
\|t^{i+1-r}\partial_x\partial_t^i u\|_{L^2_{x,t}} \lesssim \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}}^{1/2}\|t^{i+1-r}\partial_t^{i+1} u\|_{L^2_{x,t}}^{1/2} + \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}. \tag{5.5}
$$

The equations (5.3)–(5.5) constitute the analogues of (3.1)–(3.3).

5.2. Tangential derivative reductions for the Stokes operator. We claim that, for $k \geq 2$, the tangential derivatives may be reduced using

$$\begin{aligned}
&\|t^{i+k-r}\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+k-r}\mathbf{T}^{k-1}\partial_t^i p\|_{L^2_{x,t}} \\
&\quad\lesssim \|t^{i+k-r}\mathbf{T}^{k-2}\partial_t^{i+1} u\|_{L^2_{x,t}} + \|t^{i+k-r}\mathbf{T}^{k-2}\partial_t^i f\|_{L^2_{x,t}} \\
&\qquad + \|t^{i+k-r}[\mathbf{T}^{k-2},\Delta]\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+k-r}[\mathbf{T}^{k-2},\nabla]\partial_t^i p\|_{L^2_{x,t}},
\end{aligned} \tag{5.6}$$

while for $k = 1$, we have

$$\|t^{i+1-r}\mathbf{T}\partial_t^i u\|_{L^2_{x,t}} \lesssim \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}}^{1/2}\|t^{i+1-r}\partial_t^{i+1} u\|_{L^2_{x,t}}^{1/2} + \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}} \tag{5.7}$$

for all $i \geq r-1$. The equations (5.6)–(5.7) represent the analogues of (3.9)–(3.10).

5.3. Time derivative reductions for the Stokes operator. For $i \geq r$, we have

$$\|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} \lesssim (i-r)\|t^{i-1-r}\partial_t^{i-1} u\|_{L^2_{x,t}} + \mathbb{1}_{i=r}\|\nabla\partial_t^{r-1}u(0)\|_{L^2} + \|t^{i-r}\partial_t^{i-1} f\|_{L^2_{x,t}}. \tag{5.8}$$

In order to prove the inequalities (5.3)–(5.7), we follow the ideas in Section 3 by appealing to the H^2 inequalities (5.1)–(5.2). Since the proofs are completely analogous, we omit further details. For the equation (5.8), note that the energy inequality for Stokes equation is the same with (3.11), as the integral with the pressure term vanishes due to $\nabla \cdot u = 0$.

PROOF OF THEOREM 2.7. Instead of considering $\psi(u,p)$ in (2.13) directly, it is more convenient to introduce an alternate norm $\psi_1(u,p)$, which is larger than $\psi(u,p)$, modulo a multiplicative constant. The function ψ_1 is defined as follows. Fix $T > 0$ and let $0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1$. Using the index sets B and B^c as in (2.9), we define

$$\psi_1(u,p) = \bar{\psi}_1(u,p) + \psi_0(u,p)$$

where $\bar{\psi}_1(u,p) = \sum_{\ell=1}^{6} S_\ell$ with

$$S_\ell = \sum_{B_\ell} \frac{(i+j+k)^r \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k}{(i+j+k)!} R_\ell, \qquad \ell = 1, \ldots, 6$$

and

$$\begin{aligned}
R_1 &= \|t^{i+j+k-r}\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_x^{j-1}\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}}, \qquad j \geq 2 \\
R_2 &= \|t^{i+1+k-r}\partial_x\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}}, \qquad k \geq 1 \\
R_3 &= \|t^{i+1-r}\partial_x\partial_t^i u\|_{L^2_{x,t}} \\
R_4 &= \|t^{i+k-r}\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}} + \|t^{i+k-r}\mathbf{T}^{k-1}\partial_t^i p\|_{L^2_{x,t}}, \qquad k \geq 2 \\
R_5 &= \|t^{i+1-r}\mathbf{T}\partial_t^i u\|_{L^2_{x,t}} \\
R_6 &= \|t^{i-r}\partial_t^i u\|_{L^2_{x,t}}.
\end{aligned}$$

Recall that the sets B_1–B_6 were defined in (4.1). First, we sketch an argument showing that

$$\bar{\psi}(u,p) \leq \bar{\psi}_1(u,p).$$

It is easy to verify that the terms containing u are the same, so we only need to check that

$$\begin{aligned}\sum_{\tilde B} &\frac{(i+j+k+1)^{r-1}}{(i+j+k)!}\epsilon^i\tilde\epsilon^{j+1}\bar\epsilon^k\|t^{i+j+k+1-r}\partial_x^j\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}(\Omega\times[0,T])}\\ &\lesssim \sum_{B_1}\frac{(i+j+k)^r}{(i+j+k)!}\epsilon^i\tilde\epsilon^j\bar\epsilon^k\|t^{i+j+k-r}\partial_x^{j-1}\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}(\Omega\times[0,T])}\\ &\quad+\sum_{B_4}\frac{(i+j+k)^r}{(i+j+k)!}\epsilon^i\tilde\epsilon^j\bar\epsilon^k\|t^{i+j+k-r}\mathbf{T}^{k-1}\partial_t^i p\|_{L^2_{x,t}(\Omega\times[0,T])}.\end{aligned}\tag{5.9}$$

The sum on the left side equals

$$\begin{aligned}&\sum_{\tilde B\cap\{j\ge1\}}\frac{(i+j+k+1)^{r-1}}{(i+j+k)!}\epsilon^i\tilde\epsilon^{j+1}\bar\epsilon^k\|t^{i+j+k+1-r}\partial_x^j\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}(\Omega\times[0,T])}\\ &\quad+\sum_{\tilde B\cap\{j=0\}\cap\{k\ge1\}}\frac{(i+j+k+1)^{r-1}}{(i+j+k)!}\epsilon^i\tilde\epsilon^{j+1}\bar\epsilon^k\|t^{i+j+k+1-r}\partial_x^j\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}(\Omega\times[0,T])}\\ &=\sum_{\{(i,j,k):i+j+k\ge r,j\ge2\}}\frac{(i+j+k)^{r-1}}{(i+j+k-1)!}\epsilon^i\tilde\epsilon^j\bar\epsilon^k\|t^{i+j+k-r}\partial_x^{j-1}\mathbf{T}^k\partial_t^i p\|_{L^2_{x,t}(\Omega\times[0,T])}\\ &\quad+\sum_{\{(i,j,k):j=0,i+k\ge r,k\ge2\}}\frac{(i+j+k)^{r-1}}{(i+j+k-1)!}\epsilon^i\tilde\epsilon^{j+1}\bar\epsilon^{k-1}\|t^{i+j+k-r}\mathbf{T}^{k-1}\partial_t^i p\|_{L^2_{x,t}(\Omega\times[0,T])},\end{aligned}\tag{5.10}$$

where we changed j to $j-1$ in the first sum and k to $k-1$ in the second; we also used that $j+k\ge1$ on $\tilde B$. It is now easy to check that the right hand side in (5.10) is dominated by a constant multiple the right hand side of (5.9) by using $\tilde\epsilon\le\bar\epsilon$.

The rest of the proof is identical to that of Theorem 2.6. Namely, we follow the arguments in Subsections 4.1–4.6 and appeal to the derivative reduction estimates (5.3)–(5.8). The only difference is due to the pressure terms appearing as the last terms in (5.3), (5.4), and (5.6).

First, we estimate the term $\|t^{i+k+2-r}[\mathbf{T}^k,\nabla]\partial_t^i p\|_{L^2_{x,t}}$. By Lemma 3.4, we have

$$[\mathbf{T}^k,\partial_\ell]\partial_t^i p=\sum_{m=1}^k\binom{k}{m}(\mathrm{ad}\ \mathbf{T})^m(\partial_\ell)\mathbf{T}^{k-m}\partial_t^i p\tag{5.11}$$

for $k\ge1$. Using (3.16)–(3.17), we express the term $(\mathrm{ad}\ \mathbf{T})^m(\partial_\ell)$ as

$$[\mathbf{T}^k,\partial_\ell]\partial_t^i p=\sum_{m=1}^k\binom{k}{m}b^i_{m,\ell}\partial_i\mathbf{T}^{k-m}\partial_t^i p$$

with coefficients that obey

$$\max|b^i_{m,\ell}|\lesssim m!\,\bar K_2^m.$$

This yields

$$\|t^{i+k+2-r}[\mathbf{T}^k,\nabla]\partial_t^i p\|_{L^2_{x,t}}\lesssim\sum_{m=1}^k\frac{k!}{(k-m)!}\bar K_2^m\|t^{i+k+2-r}\partial_x^1\mathbf{T}^{k-m}\partial_t^i p\|_{L^2_{x,t}}.\tag{5.12}$$

Therefore, the last term in (5.4) is bounded as

$$\begin{aligned}\|t^{i+k+1-r}[\mathbf{T}^{k-1},\nabla]\partial_t^i p\|_{L^2_{x,t}} &\lesssim \sum_{m=1}^{k-1}\frac{(k-1)!}{(k-1-m)!}\bar{K}_2^m\|t^{i+k+1-r}\partial_x\mathbf{T}^{k-1-m}\partial_t^i p\|_{L^2_{x,t}}\\ &\lesssim \sum_{k'=0}^{k-2}\frac{(k-1)!}{k'!}\bar{K}_2^{k-k'-1}\|t^{i+k+1-r}\partial_x\mathbf{T}^{k'}\partial_t^i p\|_{L^2_{x,t}},\end{aligned} \tag{5.13}$$

changing the variable $k'=k-1-m$. Similarly, the last term in (5.6) is estimated as

$$\|t^{i+k-r}[\mathbf{T}^{k-2},\nabla]\partial_t^i p\|_{L^2_{x,t}} \lesssim \sum_{k'=0}^{k-3}\frac{(k-2)!}{k'!}\bar{K}_2^{k-k'-2}\|t^{i+k-r}\partial_x\mathbf{T}^{k'}\partial_t^i p\|_{L^2_{x,t}}. \tag{5.14}$$

In order to bound $\|t^{i+j+k-r}\partial_x^{j-2}[\mathbf{T}^k,\nabla]\partial_t^i p\|_{L^2_{x,t}}$ appearing in (5.3), we differentiate the formula (5.11) and apply the Leibniz rule, which gives

$$\partial_x^{j-2}[\mathbf{T}^k,\partial_\ell]\partial_t^i p = \sum_{m=1}^{k}\binom{k}{m}\sum_{j_1=0}^{j-2}\binom{j-2}{j_1}\partial_x^{j_1}(\text{ad }\mathbf{T})^m(\partial_\ell)\,\partial_x^{j-2-j_1}\mathbf{T}^{k-m}\partial_t^i p.$$

Taking the L^2 norm and using (3.16) and (3.17), we get

$$\begin{aligned}&\|t^{i+j+k-r}\partial_x^{j-2}[\mathbf{T}^k,\partial_\ell]\partial_t^i p\|_{L^2_{x,t}}\\ &\lesssim \sum_{k'=1}^{k}\sum_{j'=0}^{j-2}\binom{k}{k'}\binom{j-2}{j'}(k'+j')!K_2^{j'}\bar{K}_2^{k'}\|t^{i+j+k-r}\partial_x^{j-2-j'+1}T^{k-k'}\partial_t^i p\|_{L^2_{x,t}}\\ &\lesssim \sum_{k'=0}^{k-1}\sum_{j'=0}^{j-2}\binom{k}{k-k'}\binom{j-2}{j'}(j'+k-k')!K^{j'+k-k'}\|t^{i+j+k-r}\partial_x^{j-1-j'}T^{k'}\partial_t^i p\|_{L^2_{x,t}}\\ &\lesssim \sum_{k'=0}^{k-1}\sum_{j'=0}^{j-2}\binom{j'+k-k'}{j'}\frac{(j-2)!\,k!}{(j-2-j')!\,k'!}K^{j'+k-k'}\|t^{i+j+k-r}\partial_x^{j-1-j'}T^{k'}\partial_t^i p\|_{L^2_{x,t}}\end{aligned} \tag{5.15}$$

for $K\geq\max(K_2,\bar{K}_2)$.

Now we show that the pressure terms in (5.13), (5.14), and (5.15) may be absorbed in ψ, starting with (5.15). For this purpose, we need to bound

$$\begin{aligned}&\sum_{(i,j,k)\in B_1}\frac{(i+j+k)^r\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k}{(i+j+k)!}\\ &\qquad\times\sum_{k'=0}^{k-1}\sum_{j'=0}^{j-2}\binom{j'+k-k'}{j'}\frac{(j-2)!\,k!}{(j-2-j')!\,k'!}K^{j'+k-k'}\|t^{i+j+k-r}\partial_x^{j-1-j'}T^{k'}\partial_t^i p\|_{L^2_{x,t}}\\ &=\sum_{(i,j+2,k)\in B_1}\frac{(i+j+2+k)^r\epsilon^i\tilde{\epsilon}^{j+2}\bar{\epsilon}^k}{(i+j+2+k)!}\\ &\qquad\times\sum_{k'=0}^{k-1}\sum_{j'=0}^{j}\binom{j'+k-k'}{j'}\frac{j!\,k!}{(j-j')!\,k'!}K^{j'+k-k'}\|t^{i+j+2+k-r}\partial_x^{j+1-j'}T^{k'}\partial_t^i p\|_{L^2_{x,t}}.\end{aligned} \tag{5.16}$$

Observe that the right side of (5.16) is the same as (4.7) except that $\partial_x^{j_3+2}$ needs to be replaced by $\partial_x^{j_3+1}$, and u with p. Therefore, the right side of (5.16) is bounded, up to a constant, by

$$\begin{aligned}
&\sum_{(i,j_3+2,k')\in B_1} \frac{(i+j_3+k'+2)^r \epsilon^i \tilde{\epsilon}^{j_3+2} \bar{\epsilon}^{k'}}{(i+j_3+k'+2)!} (KT\bar{\epsilon}) \\
&\qquad \times \|t^{i+j_3+k'+2-r} \partial_x^{j_3+1} T^{k'} \partial_t^i p\|_{L^2_{x,t}} + \sum_{(i,j_3+2,k')\in B^c} \tilde{\epsilon}^2 \|\partial_x^{j_3+1} T^{k'} \partial_t^i p\|_{L^2_{x,t}} \\
&\qquad \lesssim KT\bar{\epsilon}(\bar{\psi}(u,p) - \bar{\phi}(u)) + \tilde{\epsilon}^2(\psi_0(u,p) - \phi_0(u)).
\end{aligned} \tag{5.17}$$

Note that the upper bound in (5.17) contains only the pressure terms in the norm $\psi(u,p)$.

Similarly, we point out a comparison between (5.14) and (4.2). We have

$$\begin{aligned}
&\sum_{\substack{(i,0,k)\in B_4 \\ k\geq 3}} \frac{(i+k)^r \epsilon^i \bar{\epsilon}^k}{(i+k)!} \sum_{k'=0}^{k-3} \frac{k-2!}{k'!} K^{k-k'-2} \|t^{i+k-r} \partial_x \mathbf{T}^{k'} \partial_t^i p\|_{L^2_{x,t}} \\
&= \sum_{\substack{(i,0,k+2)\in B_4 \\ k\geq 1}} \frac{(i+k+2)^r \epsilon^i \bar{\epsilon}^{k+2}}{(i+k+2)!} \sum_{k'=0}^{k-1} \frac{k!}{k'!} K^{k-k'} \|t^{i+k+2-r} \partial_x \mathbf{T}^{k'} \partial_t^i p\|_{L^2_{x,t}}.
\end{aligned} \tag{5.18}$$

Changing the order of summation as done in (4.3), the right side of (5.18) may be bounded above by

$$\begin{aligned}
&\sum_{\substack{i,k'=0 \\ (i,k')\neq(0,0)}}^{\infty} \frac{(i+k'+2)^r \epsilon^i \tilde{\epsilon}^2 \bar{\epsilon}^{k'}}{(i+k'+2)!} \|t^{i+k'+2-r} \partial_x \mathbf{T}^{k'} \partial_t^i p\|_{L^2_{x,t}} \\
&\qquad \times \left(\sum_{k=k'+1}^{\infty} \frac{\bar{\epsilon}^2}{\tilde{\epsilon}^2} \frac{(i+k+2)^r}{(i+k'+2)^r} \frac{(i+k'+2)!\, k!}{(i+k+2)!\, k'!} (K\bar{\epsilon}T)^{k-k'} \right) \\
&\qquad \lesssim KT \frac{\bar{\epsilon}^3}{\tilde{\epsilon}^2} (\bar{\psi}(u,p) - \bar{\phi}(u)) + KT\bar{\epsilon}^3(\psi_0(u,p) - \phi_0(u)),
\end{aligned} \tag{5.19}$$

where at the last line we recall the definition of the norm ψ in (2.13) and note the bound in (4.4).

For the pressure term in (5.13), we check the sum

$$\begin{aligned}
&\sum_{\substack{(i,1,k)\in B_2 \\ k\geq 2}} \frac{(i+1+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^k}{(i+1+k)!} \sum_{k'=0}^{k-2} \frac{(k-1)!}{k'!} K^{k-k'-1} \|t^{i+k+1-r} \partial_x \mathbf{T}^{k'} \partial_t^i p\|_{L^2_{x,t}} \\
&= \sum_{\substack{(i,1,k+1)\in B_2 \\ k\geq 1}} \frac{(i+2+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+2+k)!} \sum_{k'=0}^{k-1} \frac{k!}{k'!} K^{k-k'} \|t^{i+k+2-r} \partial_x \mathbf{T}^{k'} \partial_t^i p\|_{L^2_{x,t}}.
\end{aligned} \tag{5.20}$$

Once again, noting the similarity between (5.20) and (4.13), we obtain

$$\begin{aligned}
&\sum_{\substack{(i,1,k)\in B_2 \\ k\geq 2}} \frac{(i+1+k)^r \epsilon^i \tilde{\epsilon} \bar{\epsilon}^k}{(i+1+k)!} \|t^{i+k+1-r} [\mathbf{T}^{k-1}, \nabla] \partial_t^i p\|_{L^2_{x,t}} \\
&\qquad \lesssim KT \frac{\bar{\epsilon}^2}{\tilde{\epsilon}} (\bar{\psi}(u,p) - \bar{\phi}(u)) + KT\tilde{\epsilon}\bar{\epsilon}^2(\psi_0(u,p) - \phi_0(u)).
\end{aligned} \tag{5.21}$$

Finally, we combine the estimates (5.17), (5.19), and (5.21) and add them to those obtained in Section 4 for $\phi(u)$. Selecting $\epsilon, \tilde{\epsilon}, \bar{\epsilon} \in (0,1]$ according to the conditions (4.20)–(4.22), we arrive at the estimate

$$\begin{aligned}\bar{\psi}(u,p) \le \bar{\psi}_1(u,p) &\le \frac{1}{2}\phi(u,p) + C\phi_0(u,p) + C_1(KT\bar{\epsilon} + KT\frac{\bar{\epsilon}^3}{\tilde{\epsilon}^2} + KT\frac{\tilde{\epsilon}^2}{\epsilon})(\bar{\psi}(u,p) - \bar{\phi}(u)) \\ &\quad + C_1(\psi_0(u,p) - \phi_0(u)) + CM_T(f) + C\|u_0\|_{H^{2r-1}},\end{aligned}$$

where $C_1 > 0$ denotes the implicit constant associated with the symbol $\lesssim$ throughout this section, and $M_T(f)$ is given in (2.11). By (4.21) and (4.22), we have

$$KT\bar{\epsilon} + KT\frac{\bar{\epsilon}^3}{\tilde{\epsilon}^2} + KT\frac{\tilde{\epsilon}^2}{\epsilon} \le \frac{1}{4C}.$$

Dividing our choice of radii ϵ, $\tilde{\epsilon}$, and $\bar{\epsilon}$ fixed in Subsection 4.7 by C_1 if necessary, we conclude that

$$\bar{\psi}(u,p) \le \frac{1}{2}\psi(u,p) + C\psi_0(u,p) + CM_T(f) + C\|u_0\|_{H^{2r-1}}$$

which concludes the proof of Theorem 2.7. □

6. Analyticity for the Navier-Stokes equations

In this section, we apply Theorem 2.7 to the Navier-Stokes equations. Writing the equations (1.1) as a forced Stokes system, we have

$$\begin{aligned}&\partial_t u - \Delta u + \nabla p = -u\cdot\nabla u + f, \qquad \text{in } \Omega \\ &\nabla\cdot u = 0, \qquad \text{in } \Omega,\end{aligned} \tag{6.1}$$

where Ω is a bounded domain $\Omega \in \mathbb{R}^3$ with analytic boundary $\partial\Omega$.

6.1. The Stokes estimate. Applying the estimate (2.14) in Theorem 2.7 to the Stokes system (6.1), we obtain

$$\begin{aligned}\psi(u,p) &\lesssim \psi_0(u,p) + M_T(f) + \|u_0\|_{H^{2r-1}} \\ &\quad + \sum_{i+j+k\ge(r-2)_+} \frac{(i+j+k+2)^r\epsilon^i\tilde{\epsilon}^{j+2}\bar{\epsilon}^k}{(i+j+k+2)!}\|t^{i+j+k+2-r}\partial_x^j\mathbf{T}^k\partial_t^i(u\cdot\nabla u)\|_{L^2_{x,t}(\Omega\times(0,T))} \\ &\quad + \sum_{i+k\ge(r-2)_+} \frac{(i+k+2)^r\epsilon^i\bar{\epsilon}^{k+2}}{(i+k+2)!}\|t^{i+k+2-r}\mathbf{T}^k\partial_t^i(u\cdot\nabla u)\|_{L^2_{x,t}(\Omega\times(0,T))} \\ &\quad + \sum_{i\ge r-1} \frac{(i+1)^r\epsilon^{i+1}}{(i+1)!}\|t^{i+1-r}\partial_t^i(u\cdot\nabla u)\|_{L^2_{x,t}(\Omega\times(0,T))} \\ &= \psi_0(u,p) + M_T(f) + \|u_0\|_{H^{2r-1}} + \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3,\end{aligned} \tag{6.2}$$

where $M_T(f)$ is given explicitly in (2.11). The parameters $\epsilon, \tilde{\epsilon}, \bar{\epsilon}$ are determined according to (2.5) and the equations (4.20)–(4.22). As stated in Theorem 2.7, these parameters depend on the choices of r and d.

Before we start with the proof of Theorem 2.8 we recall that any smooth vector field X on a differential manifold M satisfies

$$X(fg) = X(f)g + fX(g), \qquad f,g\in C^\infty(M). \tag{6.3}$$

For the vector fields X_0 and $\mathbf{T}$ on $\bar{\Omega}$ introduced in Section 2, the Leibniz rule (6.3) holds, i.e., for any $\beta \in I$, we have

$$\mathbf{T}_\beta(u\cdot v) = \mathbf{T}_\beta u\cdot v + u\cdot\mathbf{T}_\beta v.$$

We then check the product rule for $\|T^k(u\cdot v)\|_{L^2_{x,t}}$ with the notation given in (2.2)–(2.4).

LEMMA 6.1. *For $u \in H^1(0,T;H^{k+2}(\Omega))$ and $v \in H^1(0,T;H^{k+1}(\Omega))$, where $k \in \mathbb{N}$, we have*

$$\|\boldsymbol{T}^k(u\cdot v)\|_{L^2_{x,t}} \lesssim \sum_{m=0}^{k}\binom{k}{m}\|\boldsymbol{T}^m u\|_{L^\infty_{x,t}}\|\boldsymbol{T}^{k-m}v\|_{L^2_{x,t}}.$$

Similarly, for $u,v \in H^1(0,T;H^{k+1}(\Omega))$, where $k \in \mathbb{N}$, we have

$$\|\boldsymbol{T}^k(u\cdot v)\|_{L^2_{x,t}} \lesssim \sum_{m=0}^{k}\binom{k}{m}\|\boldsymbol{T}^m u\|_{L^\infty_t L^4_x}\|\boldsymbol{T}^{k-m}v\|_{L^2_t L^4_x}. \tag{6.4}$$

The issue with this Leibniz rule, even in two space dimensions, is that the above norms are represented by sums and thus the inequalities need to be checked directly.

PROOF OF LEMMA 6.1. First, we write

$$\begin{aligned}\|\mathbf{T}^k(u\cdot v)\|_{L^2_{x,t}} &= \sum_{\beta\in I^k}\|\mathbf{T}^\beta(u\cdot v)\|_{L^2_{x,t}} = \sum_{(\beta_1,\ldots,\beta_k)\in I^k}\|\mathbf{T}_{\beta_1}\cdots\mathbf{T}_{\beta_k}(u\cdot v)\|_{L^2_{x,t}}\\ &= \sum_{\beta\in I^k}\sum_{m=0}^{k}\sum_{\tau\in\pi(k,m)}\|\mathbf{T}_{\beta_{\tau_1}}\cdots\mathbf{T}_{\beta_{\tau_m}}u\cdot\mathbf{T}_{\beta_{\tau_{m+1}}}\cdots\mathbf{T}_{\beta_{\tau_k}}v\|_{L^2_{x,t}}.\end{aligned} \tag{6.5}$$

Using the Cauchy-Schwarz inequality, we arrive at the upper bound

$$\sum_{\beta\in I^k}\sum_{m=0}^{k}\sum_{\tau\in\pi(k,m)}\|\mathbf{T}_{\beta_{\tau_1}}\cdots\mathbf{T}_{\beta_{\tau_m}}u\|_{L^\infty_{x,t}}\|\mathbf{T}_{\beta_{\tau_{m+1}}}\cdots\mathbf{T}_{\beta_{\tau_k}}v\|_{L^2_{x,t}} \lesssim \sum_{m=0}^{k}\binom{k}{m}\|\mathbf{T}^m u\|_{L^\infty_{x,t}}\|\mathbf{T}^{k-m}\|_{L^2_{x,t}}.$$

Similarly, the inequality (6.4) follows from (6.5) and the Cauchy-Schwarz inequality. ☐

Lemma 6.1 is applied below on the sums $\tilde{M}_1$, $\tilde{M}_2$, and $\tilde{M}_3$ with $v = \nabla u$.

The proof of Theorem 2.8 parallels the proof of Theorem 2.3 in [**12**] in which we considered the same problem in the half space $\Omega = \{x = (x_1,\ldots,x_d) \in \mathbb{R}^d : x_d > 0\}$. In both cases, the bulk of the proof comprises of bounds on the sums $\tilde{M}_1$, $\tilde{M}_2$, and $\tilde{M}_3$ appearing in (6.2) in terms of the analyticity norm. In order to avoid repetition, we shall frequently refer to the proof of Theorem 2.3 in [**12**].

6.2. Space-time analytic estimates for the nonlinear term. We use the notation $|(i,j,k)| = i+j+k$, which indicates the length of the multi-index, and denote

$$U_{i,j,k} := \begin{cases} N_{i+j+k}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{i+j+k-r}\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}}, & |(i,j,k)| \geq r,\\ \|\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^2_{x,t}}, & 0 \leq |(i,j,k)| \leq r-1,\end{cases} \tag{6.6}$$

with $N_{i+j+k} = |(i,j,k)|^r/|(i,j,k)|! = (i+j+k)^r/(i+j+k)!$. From here on, we set $r = 3$, which is a suitable choice for the space dimensions 2 and 3. Using the definition of $\psi(u,p)$ in (2.13), we obtain

$$\bar{\psi}(u,p) \geq \sum_{i+j+k\geq r} U_{i,j,k} \quad\text{and}\quad \psi_0(u,p) \geq \sum_{0\leq i+j+k\leq r-1} U_{i,j,k}.$$

6.3. Gagliardo-Nirenberg inequalities. We recall the space-time Gagliardo-Nirenberg inequalities from [**12**] that are frequently used below in order to bound the nonlinear term appearing on the right side of (6.1). For $u \in H^2(\Omega)$, we utilize the following estimates:

$$\|u\|_{L^\infty(\Omega)} \lesssim \|u\|_{\dot{H}^2(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4} + \|u\|_{L^2(\Omega)}, \qquad u \in H^2(\Omega), \tag{6.7}$$

$$\|u\|_{L^\infty(\Omega)} \lesssim \|u\|_{\dot{H}^2(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4}, \qquad u \in H^2(\Omega),\ \text{with } u\big|_{\partial\Omega} = 0,$$

$$\|u\|_{L^4(\Omega)} \lesssim \|u\|_{\dot{H}^1(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4} + \|u\|_{L^2(\Omega)}, \qquad u \in H^1(\Omega), \tag{6.8}$$

$$\|u\|_{L^4(\Omega)} \lesssim \|u\|_{\dot H^1(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4}, \qquad u\in H^1(\Omega),\ \text{with } u\big|_{\partial\Omega}=0. \tag{6.9}$$

For $v\in H^1(0,T)$ such that $v|_{t=0}=0$, we use Agmon's inequality

$$\|v\|_{L^\infty(0,T)} \lesssim \|v\|_{L^2(0,T)}^{1/2}\|\partial_t v\|_{L^2(0,T)}^{1/2}, \tag{6.10}$$

while in the case $v|_{t=0}\neq 0$, a lower order term is needed in the above estimate, namely,

$$\|v\|_{L^\infty(0,T)} \lesssim \|v\|_{L^2(0,T)}^{1/2}\|\partial_t v\|_{L^2(0,T)}^{1/2} + \|v\|_{L^2(0,T)}. \tag{6.11}$$

Together, the estimates (6.7)–(6.11) imply that for $u\in H^1(0,T;H^2(\Omega))$, we have

$$\|u\|_{L^\infty_{x,t}} \lesssim \|\partial_t u\|_{L^2_t\dot H^2_x}^{1/2}\|u\|_{L^2_t\dot H^2_x}^{1/2} + \|u\|_{L^2_t\dot H^2_x} + \|\partial_t u\|_{L^2_{x,t}} + \|u\|_{L^2_{x,t}}. \tag{6.12}$$

Similarly, for $u\in H^1(0,T;H^1(\Omega))$, we may bound

$$\|u\|_{L^\infty_t L^4_x} \lesssim \|\partial_t u\|_{L^2_t\dot H^1_x}^{1/2}\|u\|_{L^2_t\dot H^1_x}^{1/2} + \|u\|_{L^2_t\dot H^1_x} + \|\partial_t u\|_{L^2_{x,t}} + \|u\|_{L^2_{x,t}}. \tag{6.13}$$

Next, we rewrite (6.10) and (6.11) for a function of the form $t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell u$. The inequality (6.11) becomes

$$\begin{aligned}
\|t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell u\|_{L^\infty_{x,t}} &= \sum_{\alpha\in\mathbb{N}_0^d,|\alpha|=n,\beta\in I^m} \|t^{\ell+n+m}\partial_x^\alpha\mathbf{T}^\beta\partial_t^\ell u\|_{L^\infty_{x,t}}\\
&\lesssim \sum_{\alpha\in\mathbb{N}_0^d,|\alpha|=n,\beta\in I^m}\Big(\|\partial_t(t^{\ell+n+m}\partial_x^\alpha\mathbf{T}^\beta\partial_t^\ell u)\|_{L^2_t\dot H^2_x}^{1/2}\|t^{\ell+n+m}\partial_x^\alpha\mathbf{T}^\beta\partial_t^\ell u\|_{L^2_t\dot H^2_x}^{1/2}\Big)\\
&\quad+\sum_{\alpha\in\mathbb{N}_0^d,|\alpha|=n,\beta\in I^m}\Big(\|t^{\ell+n+m}\partial_x^\alpha\mathbf{T}^\beta\partial_t^\ell u\|_{L^2_t\dot H^2_x} + \|\partial_t(t^{\ell+n+m}\partial_x^\alpha\mathbf{T}^\beta\partial_t^\ell u)\|_{L^2_{x,t}} + \|t^{\ell+n+m}\partial_x^\alpha\mathbf{T}^\beta\partial_t^\ell u\|_{L^2_{x,t}}\Big)\\
&\lesssim \Big\|\partial_t\left(t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell u\right)\Big\|_{L^2_t\dot H^2_x}^{1/2}\|t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell u\|_{L^2_t\dot H^2_x}^{1/2}\\
&\quad+\|t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell u\|_{L^2_t\dot H^2_x} + \|\partial_t(t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell u)\|_{L^2_{x,t}} + \|t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell u\|_{L^2_{x,t}}.
\end{aligned}$$

With the notation (2.4) and (6.6), we simply write

$$U_{\ell,n+2,m} = N_{\ell+n+m+2}\epsilon^\ell\tilde\epsilon^{n+2}\bar\epsilon^m\|t^{\ell+n+m-1}\partial_x^n\mathbf{T}^m\partial_t^\ell u\|_{L^2_t\dot H^2_x}$$

for $|(\ell,n,m)|\geq 1$. This allows us to express [**12**, Lemma 4.2] as follows.

LEMMA 6.2. *For $u\in H^1(0,T;H^2(\Omega))$ and all multi-indices $|(\ell,n,m)|\geq 1$, we have*

$$\begin{aligned}
&N_{\ell+n+m}\epsilon^\ell\tilde\epsilon^n\bar\epsilon^m\|t^{\ell+n+m}\partial_x^n\boldsymbol{T}^m\partial_t^\ell u\|_{L^\infty_{x,t}}\\
&\quad\lesssim U_{\ell+1,n+2,m}^{1/2}U_{\ell,n+2,m}^{1/2}T^{1/2}|(\ell,n,m)|^{5/2} + U_{\ell,n+2,m}T^{1/2}|(\ell,n,m)|^{5/2}\\
&\qquad+U_{\ell+1,n,m}\left(T\mathbb{1}_{\ell+n+m=1}+T^2\mathbb{1}_{\ell+n+m\geq 2}\right)|(\ell,n,m)|\\
&\qquad+U_{\ell,n,m}\left(T^{\ell+n+m-1}\mathbb{1}_{\ell+n+m\leq 2}+T^2\mathbb{1}_{\ell+n+m\geq 3}\right)|(\ell,n,m)|.
\end{aligned}$$

Analogously, the equation (4.13) in [**12**] is preserved when using the notation (2.2)–(2.4). We then rewrite [**12**, Lemma 4.3] as follows.

LEMMA 6.3. *For $u\in H^1(0,T;H^1(\Omega))$ and all multi-indices $|(\ell,n,m)|\geq 2$, we have*

$$\begin{aligned}
N_{\ell+n+m}\epsilon^\ell\tilde\epsilon^n\bar\epsilon^m\|t^{\ell+n+m-1}\partial_x^n\boldsymbol{T}^m\partial_t^\ell u\|_{L^\infty_tL^4_x} &\lesssim U_{\ell+1,n+1,m}^{1/2}U_{\ell,n+1,m}^{1/2}T^{1/2}|(\ell,n,m)|^{3/2}\\
&\quad+U_{\ell,n+1,m}T^{1/2}|(\ell,n,m)|^{3/2}+U_{\ell+1,n,m}T|(\ell,n,m)|\\
&\quad+U_{\ell,n,m}\left(\mathbb{1}_{\ell+n+m=2}+T\mathbb{1}_{\ell+n+m\geq 3}\right)|(\ell,n,m)|.
\end{aligned}$$

6.4. Terms with only time derivatives. In this section, we estimate $\tilde{M}_3$ in (6.2). Since we do not have any tangential or normal derivatives in this case, the arguments in [**12**, Subsection 4.2] apply without much change. The only difference arises from the treatment of the term ∇u. Instead of splitting the gradient $\nabla = (\bar{\partial}, \partial_d)$ into normal and tangential components as in [**12**], we keep it as a gradient, i.e., ∂_x^1, and switch the order with ∂_t. Then, applying the Leibniz rule, we have

$$\tilde{M}_3 \leq \sum_{i\geq 2}\sum_{\ell=0}^{i}\binom{i}{\ell} N_{i+1}\epsilon^i \|t^{i-2}\partial_t^\ell u \cdot \partial_x^1 \partial_t^{i-\ell} u\|_{L^2_{x,t}}.$$

Next, we show that the estimate for M_3 in [**12**, (4.15)] stays valid for $\tilde{M}_3$ as well.

LEMMA 6.4. *For solution u of the Cauchy problem* (6.1)*, we have*

$$\tilde{M}_3 \lesssim \phi_0(u)^{1/2}\phi(u)^{3/2} + T^{1/2}\phi(u)^2 \tag{6.14}$$

for $0 < T \leq 1$.

PROOF OF LEMMA 6.4. Analogously to the proof of Lemma 4.4 in [**12**], we split $\tilde{M}_3$ into sums $\tilde{M}_{31}$ and $\tilde{M}_{32}$ corresponding to $i = 2$ or $i \geq 3$, respectively. By applying the product rule and switching the order of ∇ and ∂_t, we have

$$\tilde{M}_{31} \lesssim \|u\|_{L^\infty_{x,t}}\|\partial_x^1\partial_t^2 u\|_{L^2_{x,t}} + \|\partial_t u\|_{L^\infty_t L^4_x}\|\partial_x^1 \partial_t u\|_{L^2_t L^4_x} + \|\partial_t^2 u\|_{L^2_t L^4_x}\|\partial_x^1 u\|_{L^\infty_t L^4_x}.$$

Noting the similarity of the estimates above with those in [**12**], we point out that the upper bounds on the expression $\partial_t^\ell \partial_d u$ in [**12**] may be used on $\partial_x^1 \partial_t^\ell u$ exactly with the same norms. Therefore, following the same strategy in [**12**] we apply (6.12) and (6.13) on $\tilde{M}_{31}$ to conclude that

$$\tilde{M}_{31} \lesssim \phi_0(u)^2 + \phi_0(u)^{1/2}\phi(u)^{3/2}.$$

Using the notation $\lfloor x \rfloor = [x]$ and $\lceil x \rceil = [x] + 1$, we get

$$\begin{aligned}
\tilde{M}_{32} &\leq \sum_{i\geq 3}\sum_{\ell=1}^{\lfloor i/2 \rfloor}\binom{i}{\ell} N_{i+1}\epsilon^i \|t^\ell \partial_t^\ell u\|_{L^\infty_{x,t}}\|t^{i-\ell-2}\partial_x^1\partial_t^{i-\ell}u\|_{L^2_{x,t}} + \sum_{i\geq 3} N_{i+1}\epsilon^i\|u\|_{L^\infty_{x,t}}\|t^{i-2}\partial_x^1\partial_t^i u\|_{L^2_{x,t}}\\
&\quad + \sum_{i\geq 3}\sum_{\ell=\lceil i/2 \rceil}^{i-1}\binom{i}{\ell} N_{i+1}\epsilon^i\|t^{\ell-2}\partial_t^\ell u\|_{L^2_t L^4_x}\|t^{i-\ell}\partial_x^1\partial_t^{i-\ell}u\|_{L^\infty_t L^4_x}\\
&\quad + \sum_{i\geq 3} N_{i+1}\epsilon^i \|t^{i-2}\partial_t^i u\|_{L^2_t L^4_x}\|\partial_x^1 u\|_{L^\infty_t L^4_x}\\
&= \tilde{M}_{321} + \tilde{M}_{322} + \tilde{M}_{323} + \tilde{M}_{324}.
\end{aligned} \tag{6.15}$$

First, we check the boundary terms $\tilde{M}_{322}$ and $\tilde{M}_{324}$. Using (6.12), we see that we get the same upper bound as on the term M_{322} in [**12**]. Therefore,

$$\begin{aligned}
\tilde{M}_{322} &\lesssim \sum_{i\geq 3} N_{i+1}\epsilon^i \left(\|\partial_t u\|^{1/2}_{L^2_t \dot{H}^2_x}\|u\|^{1/2}_{L^2_t\dot{H}^2_x} + \|u\|_{L^2_t\dot{H}^2_x} + \|\partial_t u\|_{L^2_{x,t}} + \|u\|_{L^2_{x,t}}\right)\|t^{i-2}\partial_x^1\partial_t^i u\|_{L^2_{x,t}}\\
&\lesssim (\phi_0(u) + \phi_0(u)^{1/2}\bar{\phi}(u)^{1/2})\sum_{i\geq 3} N_{i+1}\epsilon^i\|t^{i-2}\partial_x^1\partial_t^i u\|_{L^2_{x,t}}\\
&\lesssim \phi_0(u)\phi(u) + \phi_0(u)^{1/2}\phi(u)^{3/2}.
\end{aligned}$$

Likewise, we proceed with the Gagliardo-Nirenberg inequalities (6.8)–(6.13) for $\tilde{M}_{324}$ and write

$$\tilde{M}_{324} \lesssim \sum_{i\geq 3} N_{i+1}\epsilon^i \left(\|t^{i-2}\partial_t^i u\|^{d/4}_{L^2_t \dot{H}^1_x}\|t^{i-2}\partial_t^i u\|^{1-d/4}_{L^2_{x,t}}\right)$$

$$\times \left(\|\partial_x^1 \partial_t u\|_{L_t^2 \dot{H}_x^1}^{1/2} \|\partial_x^1 u\|_{L_t^2 \dot{H}_x^1}^{1/2} + \|\partial_x^1 u\|_{L_t^2 \dot{H}_x^1} + \|\partial_x^1 \partial_t u\|_{L_{x,t}^2} + \|\partial_x^1 u\|_{L_{x,t}^2} \right).$$

Expressing the estimates in terms of $U_{i,j,k}$ and recalling the notation (2.3)–(2.4), we note the slight change $\|t^{i-2}\partial_t^i u\|_{L_t^2 \dot{H}_x^1} = U_{i,1,0}$ and write

$$\begin{aligned} \tilde{M}_{324} &\lesssim \sum_{i\geq 3} N_{i+1}\epsilon^i U_{i,1,0}^{d/4} U_{i,0,0}^{1-d/4} (\bar{\phi}(u)^{1/2}\phi_0(u)^{1/2} + \phi_0(u)) \left(\frac{(i+1)!}{(i+1)^3\epsilon^i}\right)^{d/4} \left(\frac{i!T}{i^3\epsilon^i}\right)^{1-d/4} \\ &\lesssim T^{1-d/4}(\bar{\phi}(u)^{1/2}\phi_0(u)^{1/2} + \phi_0(u))\phi(u) \\ &\lesssim T^{1-d/4}\phi_0(u)\phi(u) + T^{1-d/4}\phi_0(u)^{1/2}\phi(u)^{3/2}, \end{aligned}$$

where we have used $0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1$.

For $\tilde{M}_{321}$, we express $\|t^{i-\ell-2}\partial_x^1\partial_t^{i-\ell} u\|_{L_{x,t}^2} = U_{i-\ell,1,0}$ and write

$$\tilde{M}_{321} \lesssim \sum_{i\geq 3} \sum_{\ell=1}^{\lfloor i/2 \rfloor} \|t^\ell \partial_t^\ell u\|_{L_{x,t}^\infty} U_{i-\ell,1,0} \frac{(i+1)^2\epsilon^\ell}{\ell!(i-\ell+1)^2}.$$

Following the steps used in [12], we utilize Lemma 6.2 to bound $\tilde{M}_{321}$, where we express $\|t^{\ell-1}\partial_t^\ell u\|_{L_t^2 \dot{H}_x^2} = U_{\ell,2,0}$ by the notational agreement (2.4). Despite this notational difference, the essence of the inequality stays the same and we still obtain the same estimate with the term M_{321} in [12, Equation (4.21)]:

$$\tilde{M}_{321} \lesssim \phi_0(u)\phi(u) + T^{1/2}\phi(u)^2.$$

Lastly, we estimate $\tilde{M}_{323}$. By appealing to (6.9), we obtain

$$\tilde{M}_{323} \lesssim \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} \binom{i}{\ell} N_{i+1}\epsilon^i \left(\|t^{\ell-2}\partial_t^\ell u\|_{L_t^2 \dot{H}_x^1}^{d/4} \|t^{\ell-2}\partial_t^\ell u\|_{L_{x,t}^2}^{1-d/4} \right) \|t^{i-\ell}\partial_x^1 \partial_t^{i-\ell} u\|_{L_t^\infty L_x^4}.$$

Applying Lemma 6.3 on $\|t^{i-\ell}\partial_x^1\partial_t^{i-\ell} u\|_{L_t^\infty L_x^4}$, we have

$$\begin{aligned} \tilde{M}_{323} &\lesssim \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} U_{\ell,1,0}^{d/4} U_{\ell,0,0}^{1-d/4} U_{i-\ell+1,2,0}^{1/2} U_{i-\ell,2,0}^{1/2} T^{1/2} + \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} U_{\ell,1,0}^{d/4} U_{\ell,0,0}^{1-d/4} U_{i-\ell,2,0} T^{1/2} \\ &\quad + \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} U_{\ell,1,0}^{d/4} U_{\ell,0,0}^{1-d/4} U_{i-\ell,1,0} (\mathbb{1}_{i-\ell=1} + T\mathbb{1}_{i-\ell\geq 2}). \end{aligned}$$

Once again, applying the discrete Young's inequality and selecting the maximal prefactors in T, we get

$$\tilde{M}_{323} \lesssim \phi_0(u)\phi(u) + T^{1/2}\phi(u)^2.$$

Combining all the estimates on the terms given in (6.15) and selecting the maximal prefactors in T and $\phi_0(u)$, we get the desired bound (6.14). ☐

6.5. Terms with no normal derivatives.

In this section we estimate $\tilde{M}_2$.

LEMMA 6.5. *For solutions u of the Cauchy problem (6.2), we get*

$$\tilde{M}_2 \lesssim \phi_0(u)^{3/2}\phi(u)^{1/2} + T\phi(u)^2 + T^{3/2-d/4}\phi(u)^2 \tag{6.16}$$

for $0 < T \leq 1$.

For $\tilde{M}_2$, we use the estimates obtained in [12, Subsection 4.3].

PROOF OF LEMMA 6.5. Separating the terms with $i + k = 1$, we split the estimate of $\tilde{M}_2$ sum into three parts as

$$\begin{aligned}\tilde{M}_2 &\le \sum_{i+k\ge 2}\sum_{|(\ell,m)|=0}^{i+k}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^i\bar{\epsilon}^k\|t^{i+k-1}\mathbf{T}^m\partial_t^\ell u\cdot\mathbf{T}^{k-m}\partial_t^{i-\ell}\nabla u\|_{L^2_{x,t}}\\ &\quad+\|\partial_t(u\cdot\nabla u)\|_{L^2_{x,t}}+\|\mathbf{T}(u\cdot\nabla u)\|_{L^2_{x,t}}\\ &=\tilde{M}_{21}+\tilde{M}_{22}+\tilde{M}_{23}.\end{aligned}\tag{6.17}$$

We use the Hölder's inequality on the lower order terms and write

$$\begin{aligned}\tilde{M}_{22}+\tilde{M}_{23}\lesssim\|\partial_t u\|_{L^\infty_t L^4_x}\|\nabla u\|_{L^2_t L^4_x}+\|u\|_{L^\infty_{x,t}}\|\partial_t\partial_x^1 u\|_{L^2_{x,t}}\\ +\|\mathbf{T}u\|_{L^\infty_t L^4_x}\|\nabla u\|_{L^2_t L^4_x}+\|u\|_{L^\infty_{x,t}}\|\mathbf{T}\nabla u\|_{L^2_{x,t}}.\end{aligned}$$

Recalling the definition of $\bar{\phi}(u)$ and $\phi_0(u)$, we obtain

$$\tilde{M}_{22}+\tilde{M}_{23}\lesssim\bar{\phi}(u)^{1/2}\phi_0(u)^{3/2}+\phi_0(u)^2.\tag{6.18}$$

Now, we split $\tilde{M}_{21}$ into two parts as

$$\begin{aligned}\tilde{M}_{21}&\lesssim\sum_{i+k\ge 2}\sum_{|(\ell,m)|=0}^{\lfloor(i+k)/2\rfloor}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^i\bar{\epsilon}^k\\ &\qquad\times\|t^{(i+k)-(\ell+m)-1}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_t L^4_x}\|t^{\ell+m}\mathbf{T}^m\partial_t^\ell\nabla u\|_{L^2_t L^4_x}\\ &\quad+\sum_{i+k\ge 2}\sum_{|(\ell,m)|\ge\lceil(i+k)/2\rceil}^{i+k}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^i\bar{\epsilon}^k\\ &\qquad\times\|t^{(i+k)-(\ell+m)}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\|t^{\ell+m-1}\mathbf{T}^m\partial_t^\ell\nabla u\|_{L^2_{x,t}}\\ &=\tilde{M}_{211}+\tilde{M}_{212}.\end{aligned}\tag{6.19}$$

Note that the inequality (6.19) is similar to the estimate (4.27) in [**12**]. In order to keep the commutator terms simpler, we took advantage of the symmetry in the binomial coefficients and switched the indices $(\ell, m, 0)$ and $(i-\ell, k-m, 0)$ of the operators on u and ∇u (cf. compare (6.17) and (6.19)). Therefore, the term $\tilde{M}_{211}$ corresponds to the term M_{212} and $\tilde{M}_{212}$ here corresponds to M_{211}.

First, we start with the second term $\tilde{M}_{212}$. Switching the order of ∇ with the tangential vector field $\mathbf{T}^m$, we obtain an upper bound with a commutator sum

$$\begin{aligned}\tilde{M}_{212}&\lesssim\sum_{i+k\ge 2}\sum_{|(\ell,m)|\ge\lceil(i+k)/2\rceil}^{i+k}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^{i-\ell}\bar{\epsilon}^{k-m}\|t^{(i+k)-(\ell+m)}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\\ &\qquad\times\epsilon^\ell\bar{\epsilon}^m\|t^{\ell+m-1}\partial_x^1\mathbf{T}^m\partial_t^\ell u\|_{L^2_{x,t}}\\ &\quad+\sum_{i+k\ge 2}\sum_{\substack{|(\ell,m)|\ge\lceil(i+k)/2\rceil\\ m\ge 1}}^{i+k}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^{i-\ell}\bar{\epsilon}^{k-m}\|t^{(i+k)-(\ell+m)}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\\ &\qquad\times\epsilon^\ell\bar{\epsilon}^m\|t^{\ell+m-1}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}\\ &=\tilde{M}_{2121}+\tilde{M}_{2122}.\end{aligned}$$

In order to have a nontrivial commutator, we take $m \geq 1$ for the second term. For $\tilde{M}_{2121}$, we follow the same steps in [12] for the term M_{211}. By applying Lemma 6.2 on $\|t^{(i+k)-(\ell+m)}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}$, we get

$$\begin{aligned}
&\epsilon^{i-\ell}\bar{\epsilon}^{k-m}\|t^{i+k-(\ell+m)}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\\
&\quad\lesssim U^{1/2}_{i-\ell+1,2,k-m}U^{1/2}_{i-\ell,2,k-m}T^{1/2}\frac{|(i-\ell,0,k-m)|^{5/2}}{N_{i-\ell+k-m}}\\
&\qquad+U_{i-\ell,2,k-m}T^{1/2}\frac{|(i-\ell,0,k-m)|^{5/2}}{N_{i-\ell+k-m}}\\
&\qquad+U_{i-\ell+1,0,k-m}(T\mathbb{1}_{i-\ell+k-m=1}+T^2\mathbb{1}_{i-\ell+k-m\geq 2})\frac{|(i-\ell,0,k-m)|}{N_{i-\ell+k-m}}\\
&\qquad+U_{i-\ell,0,k-m}(T^{i-\ell+k-m-1}\mathbb{1}_{i-\ell+k-m\leq 2}+T^2\mathbb{1}_{i-\ell+k-m\geq 3})\frac{|(i-\ell,0,k-m)|}{N_{i-\ell+k-m}}.
\end{aligned} \tag{6.20}$$

Also, expressing $N_{\ell+m+1}\tilde{\epsilon}\epsilon^\ell\bar{\epsilon}^m\|t^{\ell+m-1}\partial_x^1\mathbf{T}^m\partial_t^\ell u\|_{L^2_{x,t}} = TU_{\ell,1,m}$ for $|(\ell,0,m)| \geq 2$, we see that the binomial coefficients coming from the two norms in $\tilde{M}_{2121}$ obey the bound

$$\frac{N_{i+k+2}(i-\ell+k-m)!}{N_{\ell+m+1}(i-\ell+k-m)^{1/2}}\binom{i}{\ell}\binom{k}{m} \lesssim \frac{\binom{i}{\ell}\binom{k}{m}}{\binom{i+k}{\ell+m}} \lesssim 1 \tag{6.21}$$

for $|(\ell,m)| \geq \lceil (i+k)/2\rceil$. Using the bound (6.21) on the binomial coefficients and applying the discrete Young's inequality on the upper bounds for $\tilde{M}_{2121}$ as performed in [12] for M_{211}, we obtain that the estimate for M_{211} dominates the term $\tilde{M}_{2121}$. As a result, we get

$$\tilde{M}_{2121} \lesssim T\phi_0(u)\phi(u) + T\phi_0(u)^{1/2}\phi(u)^{3/2} + T^{3/2}\phi(u)^2. \tag{6.22}$$

We utilize the same strategy for $\tilde{M}_{2122}$. Denoting by $A_{i-\ell,k-m}$ the right side of (6.20), we get

$$\tilde{M}_{2121} \lesssim \sum_{i+k\geq 2}\sum_{|(\ell,m)|\geq\lceil (i+k)/2\rceil}^{i+k}\binom{i}{\ell}\binom{k}{m}\frac{N_{i+k+2}}{N_{\ell+m+1}}A_{i-\ell,k-m}N_{\ell+m+1}\epsilon^\ell\bar{\epsilon}^m\|t^{\ell+m-1}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}. \tag{6.23}$$

In the same way, we first bound the binomial coefficients using (6.21) and then apply the discrete Young's inequality on (6.23). Using the definition of $A_{i-\ell,k-m}$ and the norm $\phi(u)$, we obtain

$$\tilde{M}_{2121} \lesssim (\phi_0(u) + T^{1/2}\bar{\phi}(u))\sum_{|(\ell,m)|=2,m\geq 1}N_{\ell+m+1}\epsilon^\ell\bar{\epsilon}^m\|t^{\ell+m-1}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}. \tag{6.24}$$

We are now reduced to estimating the commutator sum on the right side of (6.24).

Using the estimate (5.12) derived for the commutator term in Section 5 with $r = 3$, we get

$$\|t^{\ell+m-1}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}} \lesssim \sum_{m'=0}^{m-1}\frac{m!}{m'!}K^{m-m'}\|t^{\ell+m-1}\partial_x^1\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}} \tag{6.25}$$

for $K > 1$. The estimate (6.25) yields a bound on the commutator term on the right side of (6.24) . Applying the Fubini Theorem on the double sums, we obtain

$$\begin{aligned}
&\sum_{\substack{|(\ell,m)|=2\\ m\geq 1}} N_{\ell+m+1}\epsilon^{\ell}\bar{\epsilon}^{m}\sum_{m'=0}^{m-1}\frac{m!}{m'!}K^{m-m'}\|t^{\ell+m-1}\partial_x^1\mathbf{T}^{m'}\partial_t^{\ell}u\|_{L^2_{x,t}}\\
&\quad\lesssim \sum_{|(\ell,m')|\leq 1}\|\partial_x^1\mathbf{T}^{m'}\partial_t^{\ell}u\|_{L^2_{x,t}}\sum_{m=m'+1}\frac{(\ell+m+1)^3}{(\ell+m+1)!}\frac{m!}{m'!}\bar{\epsilon}^{m}T^{m-1}K^{m-m'}\\
&\quad\quad+T\sum_{|(\ell,1,m')|=3}^{\infty} N_{\ell+m'+1}\tilde{\epsilon}\epsilon^{\ell}\bar{\epsilon}^{m'}\|t^{\ell+m-2}\partial_x^1\mathbf{T}^{m'}\partial_t^{\ell}u\|_{L^2_{x,t}}\\
&\quad\quad\quad\times\sum_{m=m'+1}\frac{(\ell+m+1)^3}{(\ell+m'+1)^3}\frac{(\ell+m'+1)!}{(\ell+m+1)!}\frac{m!}{m'!}(\bar{\epsilon}TK)^{m-m'}.
\end{aligned}\tag{6.26}$$

For the two geometric sums in m, we note that

$$\sum_{m=m'+1}\frac{(\ell+m+1)^3}{(\ell+m+1)!}\frac{m!}{m'!}\bar{\epsilon}^{m}T^{m-1}K^{m-m'}\lesssim\sum_{m=1}^{\infty}m^2(\bar{\epsilon}K)^m\lesssim 1$$

for $|(\ell,m')|\leq 1$, and similarly for $|(\ell,m')|\geq 2$,

$$\sum_{m=m'+1}\frac{(\ell+m+1)^3}{(\ell+m'+1)^3}\frac{(\ell+m'+1)!}{(\ell+m+1)!}\frac{m!}{m'!}(\bar{\epsilon}TK)^{m-m'}\lesssim\bar{\epsilon}K\lesssim 1$$

provided $0<\bar{\epsilon}\leq 1/2K$, as $0<T\leq 1$. Therefore, the right side of (6.26) is bounded above by $\phi_0(u)+T\bar{\phi}(u)$. Using this bound in (6.24), we get

$$\tilde{M}_{2121}\lesssim(\phi_0(u)+T^{1/2}\bar{\phi}(u))(\phi_0(u)+T\bar{\phi}(u))\lesssim\phi_0(u)^2+T\phi(u)^2.$$

Next, we treat the term $\tilde{M}_{211}$ in (6.19). Starting with a commutator argument, we have

$$\|t^{\ell+m}\mathbf{T}^{m}\partial_t^{\ell}\nabla u\|_{L_t^2L_x^4}\lesssim\|t^{\ell+m}\partial_x^1\mathbf{T}^{m}\partial_t^{\ell}u\|_{L_t^2L_x^4}+\|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^{\ell}u\|_{L_t^2L_x^4}.\tag{6.27}$$

Using (6.27), we write $\tilde{M}_{211}$ as

$$\begin{aligned}
\tilde{M}_{211}&\lesssim\sum_{i+k\geq 2}\sum_{|(\ell,m)|=0}^{\lfloor (i+k)/2\rfloor}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^{i}\bar{\epsilon}^{k}\|t^{(i+k)-(\ell+m)-1}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}\|t^{\ell+m}\partial_x^1\mathbf{T}^{m}\partial_t^{\ell}u\|_{L_t^2L_x^4}\\
&\quad+\sum_{i+k\geq 2}\sum_{\substack{|(\ell,m)|=1\\ m\geq 1}}^{\lfloor (i+k)/2\rfloor}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^{i}\bar{\epsilon}^{k}\|t^{(i+k)-(\ell+m)-1}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}\|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^{\ell}u\|_{L_t^2L_x^4}\\
&=\tilde{M}_{2111}+\tilde{M}_{2112}.
\end{aligned}\tag{6.28}$$

The first term $\tilde{M}_{2111}$ may be estimated in the same way with the term M_{2122} in [**12**]. Following the same arguments as in [**12**], we conclude that

$$\begin{aligned}
\tilde{M}_{2111}&\lesssim\phi_0(u)^2+T^{1/2}\phi_0(u)\phi(u)+T^{3/2}\phi_0(u)^{1-d/4}\phi(u)^{1+d/4}\\
&\quad+T\phi_0(u)^{2-d/4}\phi(u)^{d/4}+T^{3/2-d/4}\phi(u)^2.
\end{aligned}\tag{6.29}$$

For $\tilde{M}_{2112}$, we first use the Gagliardo-Nirenberg estimates and write

$$\begin{aligned}\|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2L_x^4} &\lesssim \|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2\dot{H}_x^1}^{d/4}\|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_{x,t}^2}^{1-d/4}\\ &\quad + \|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_{x,t}^2}\\ &\lesssim \|t^{\ell+m}\partial_x^1[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_{x,t}^2}^{d/4}\|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_{x,t}^2}^{1-d/4}\\ &\quad + \|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_{x,t}^2}.\end{aligned} \tag{6.30}$$

Recall that by (5.15), we have

$$\begin{aligned}\|t^{\ell+m}\partial_x^1[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_{x,t}^2} &\lesssim \sum_{m'=1}^{m}\binom{m}{m'}(m'+1)!K^{m'+1}\|t^{\ell+m}\partial_x^1\mathbf{T}^{m-m'}\partial_t^\ell u\|_{L_{x,t}^2}\\ &\quad + \sum_{m'=1}^{m}\binom{m}{m'}m'!K^{m'}\|t^{\ell+m}\partial_x^2\mathbf{T}^{m-m'}\partial_t^\ell u\|_{L_{x,t}^2}.\end{aligned} \tag{6.31}$$

By the discrete Young's inequality, we note that the bound given above controls both of the terms on the right side of (6.30). By changing the summation index to $\bar{m}=m-m'$, we get

$$\begin{aligned}\|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2L_x^4} &\lesssim \sum_{\bar{m}=0}^{m-1}\frac{m!}{\bar{m}!}(m-\bar{m})K^{m-\bar{m}}\|t^{\ell+m}\partial_x^1\mathbf{T}^{\bar{m}}\partial_t^\ell u\|_{L_{x,t}^2}\\ &\quad + \sum_{\bar{m}=0}^{m-1}\frac{m!}{\bar{m}!}K^{m-\bar{m}}\|t^{\ell+m}\partial_x^2\mathbf{T}^{\bar{m}}\partial_t^\ell u\|_{L_{x,t}^2}.\end{aligned} \tag{6.32}$$

Going back to (6.28), we apply Lemma 6.3 on the first factor. For $i+k-\ell-m\geq 2$, we get

$$\begin{aligned}&\epsilon^{i-\ell}\bar{\epsilon}^{k-m}\|t^{(i+k)-(\ell+m)-1}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}\\ &\lesssim U_{i-\ell+1,1,k-m}^{1/2}U_{i-\ell,1,k-m}^{1/2}\frac{(i+k-\ell-m)!}{(i+k-\ell-m)^{3/2}}T^{1/2}\\ &\quad + U_{i-\ell,1,k-m}\frac{(i+k-\ell-m)!}{(i+k-\ell-m)^{3/2}}T^{1/2} + U_{i-\ell+1,0,k-m}\frac{(i+k-\ell-m)!}{(i+k-\ell-m)^{2}}T\\ &\quad + U_{i-\ell,0,k-m}\frac{(i+k-\ell-m)!}{(i+k-\ell-m)^{2}}(\mathbb{1}_{i+k-\ell-m=2}+T\mathbb{1}_{i+k-\ell-m\geq 3}).\end{aligned} \tag{6.33}$$

Denote

$$B_{\ell,m}=\frac{\epsilon^\ell\bar{\epsilon}^m}{(\ell+m)!}\|t^{\ell+m}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2L_x^4}.$$

By separating the term $i+k=2$ from the rest of the sum and using the notation $B_{\ell,m}$, we rewrite $\tilde{M}_{2112}$ as

$$\begin{aligned}\tilde{M}_{2112} &\lesssim \Big(\|\mathbf{T}u\|_{L_t^\infty L_x^4}+\|\partial_t u\|_{L_t^\infty L_x^4}\Big)\|t[\mathbf{T},\nabla]u\|_{L_t^2L_x^4}\\ &\quad + \sum_{i+k\geq 3}\sum_{|(\ell,m)|=1}^{\lfloor (i+k)/2\rfloor}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}(\ell+m)!\\ &\qquad \times \epsilon^{i-\ell}\bar{\epsilon}^{k-m}\|t^{(i+k)-(\ell+m)-1}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}B_{\ell,m},\end{aligned} \tag{6.34}$$

where the binomial coefficients are bounded above by

$$\binom{i}{\ell}\binom{k}{m}N_{i+k+2}(\ell+m)!\frac{(i+k-\ell-m)!}{(i+k-\ell-m)^{3/2}}\lesssim \frac{\binom{i}{\ell}\binom{k}{m}}{\binom{i+k}{\ell+m}}\lesssim 1.$$

Bounding the binomial coefficients from above by 1, we then apply Young's inequality on (6.34). Noting (6.33), we deduce

$$\tilde{M}_{2112} \lesssim \left(\phi_0(u) + \phi_0(u)^{1/2}\bar{\phi}(u)^{1/2}\right) T\phi_0(u) + (\phi_0(u) + T^{1/2}\bar{\phi}(u)) \sum_{|(\ell,m)|=1}^{\infty} B_{\ell,m}. \tag{6.35}$$

Next, we deal with bounding the sum on the right side of (6.35). We recall the definition of $B_{\ell,m}$ and appeal to the formulas (6.31)–(6.32). We have

$$\begin{aligned}\sum_{|(\ell,m)|=1}^{\infty} B_{\ell,m} \lesssim &\sum_{|(\ell,m)|=1}^{\infty} \sum_{\bar{m}=0}^{m-1} \frac{m!}{\bar{m}!} \frac{\epsilon^\ell \bar{\epsilon}^m}{(\ell+m)!}(m-\bar{m}) K^{m-\bar{m}} \|t^{\ell+m}\partial_x^1 \mathbf{T}^{\bar{m}} \partial_t^\ell u\|_{L^2_{x,t}} \\ &+ \sum_{|(\ell,m)|=1}^{\infty} \sum_{\bar{m}=0}^{m-1} \frac{m!}{\bar{m}!} \frac{\epsilon^\ell \bar{\epsilon}^m}{(\ell+m)!} K^{m-\bar{m}} \|t^{\ell+m}\partial_x^2 \mathbf{T}^{\bar{m}} \partial_t^\ell u\|_{L^2_{x,t}}.\end{aligned}$$

Next, we separate the case $m = 1$, and then apply the Fubini Theorem to change the order of the summation. We get,

$$\sum_{|(\ell,m)|=1}^{\infty} B_{\ell,m} \lesssim \phi_0(u) \left(\sum_{m=1}^{\infty} m(T\bar{\epsilon}K)^m\right) + T^2 \sum_{|(\ell,\bar{m})|\geq 2} B_0\, U_{\ell,1,\bar{m}} + T \sum_{|(\ell,\bar{m})|\geq 2} B_1\, U_{\ell,2,\bar{m}},$$

where the coefficients B_0 and B_1 are given respectively as

$$B_0 = \sum_{m=\bar{m}+1}^{\infty} \frac{(\ell+\bar{m})!}{(\ell+\bar{m})^2} \frac{(m-\bar{m})}{(\ell+m)!} \frac{m!}{\bar{m}!} (T\bar{\epsilon}K)^{m-\bar{m}}$$

and

$$B_1 = \sum_{m=\bar{m}+1}^{\infty} \frac{(\ell+\bar{m})!}{(\ell+\bar{m})(\ell+m)!} \frac{m!}{\bar{m}!} (T\bar{\epsilon}K)^{m-\bar{m}}.$$

Note that the factorial terms for B_0 are bounded from above by

$$\frac{m!\,(\ell+\bar{m})!}{\bar{m}!\,(\ell+\bar{m})!}(m-\bar{m}) \leq (m-\bar{m})$$

and likewise for B_1 we have

$$\frac{m!\,(\ell+\bar{m})!}{\bar{m}!\,(\ell+m)!} \leq 1.$$

Using the bounds above we obtain that the two geometric sums in m above are finite for $0 < T \leq 1$ and $0 < \bar{\epsilon} \leq 1/2K$. Therefore,

$$\sum_{|(\ell,m)|=1}^{\infty} B_{\ell,m} \lesssim T\phi_0(u) + T^3\bar{\phi}(u) + T^2\bar{\phi}(u),$$

which implies

$$\tilde{M}_{2112} \lesssim T\phi_0(u)^2 + T\phi(u)^2. \tag{6.36}$$

Finally adding the estimates (6.18), (6.22), (6.29), (6.36), and selecting the maximal coefficients in T and $\phi_0(u)$, we obtain (6.16). □

6.6. Terms with mixed derivatives. In this section we estimate $\tilde{M}_1$.

LEMMA 6.6. *For solutions u of the Cauchy problem (6.2), we have*

$$\tilde{M}_1 \lesssim \phi_0(u)^{3/2}\phi(u)^{1/2} + T^{1/2}\phi_0(u)\phi(u) + T^{3/2}\phi(u)^2$$

for all $0 < T \leq 1$.

PROOF OF LEMMA 6.6. By the Leibniz rule we have

$$\tilde{M}_1 \leq \sum_{i+j+k\geq 1}\sum_{\ell=0}^{i}\sum_{n=0}^{j}\sum_{m=0}^{k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{i+j+k-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\cdot\partial_x^n\mathbf{T}^m\partial_t^\ell\nabla u\|_{L^2_{x,t}}.$$

Following the program in Subsection 4.4, [12], we separate the case $|(i+j+k)| = 1$ from the sum and split the rest into two parts

$$\begin{aligned}\tilde{M}_1 &\lesssim \sum_{i+j+k=1}\Big(\|u\|_{L^\infty_{x,t}}\|\partial_x^j\mathbf{T}^k\partial_t^i\nabla u\|_{L^2_{x,t}} + \|\partial_x^j\mathbf{T}^k\partial_t^i u\|_{L^\infty_t L^4_x}\|\nabla u\|_{L^2_t L^4_x}\Big)\\
&\quad + \sum_{i+j+k\geq 2}\sum_{|(\ell,n,m)|=\lceil(i+j+k)/2\rceil}^{i+j+k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\\
&\qquad \times N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{(i+j+k)-(\ell+n+m)}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\|t^{\ell+n+m-1}\partial_x^n\mathbf{T}^m\partial_t^\ell\nabla u\|_{L^2_{x,t}}\\
&\quad + \sum_{i+j+k\geq 2}\sum_{|(\ell,n,m)|=0}^{\lfloor(i+j+k)/2\rfloor}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\\
&\qquad \times N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{(i+j+k)-(\ell+n+m)-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_t L^4_x}\|t^{\ell+n+m}\partial_x^n\mathbf{T}^m\partial_t^\ell\nabla u\|_{L^2_t L^4_x}\\
&= \tilde{M}_{11} + \tilde{M}_{12} + \tilde{M}_{13}.\end{aligned}$$

The estimate above on $\tilde{M}_1$ is similar to the one on M_1 in Subsection 4.4, [12]. Note that we switched the indices (ℓ, m, n) and $(i-\ell, k-m, n-j)$ of the operators acting on u and ∇u to in order to keep the commutator terms simpler. Due to the symmetry in the binomial coefficients, we may still follow the corresponding estimates given in [12].

Analogously to [12], the contribution from $|(i, j, k)| = 1$ stays bounded by

$$\tilde{M}_{11} \lesssim \phi_0(u)^2 + \phi_0(u)^{3/2}\phi(u)^{1/2} \tag{6.37}$$

since the number of derivatives on u and on ∇u does not exceed three.

For $\tilde{M}_{12}$, we start with switching the order of ∇ with the tangential vector field $\mathbf{T}^m$ and obtain an upper bound with a commutator sum

$$\begin{aligned}\tilde{M}_{12} &\lesssim \sum_{i+j+k\geq 2}\sum_{|(\ell,n,m)|=\lceil(i+j+k)/2\rceil}^{i+j+k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\\
&\qquad \times N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{(i+j+k)-(\ell+n+m)}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\|t^{\ell+n+m-1}\partial_x^{n+1}\mathbf{T}^m\partial_t^\ell u\|_{L^2_{x,t}}\\
&\quad + \sum_{i+j+k\geq 2}\sum_{|(\ell,n,m)|=\lceil(i+j+k)/2\rceil}^{i+j+k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\\
&\qquad \times N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{(i+j+k)-(\ell+n+m)}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\|t^{\ell+n+m-1}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}\\
&= \tilde{M}_{121} + \tilde{M}_{122}.\end{aligned} \tag{6.38}$$

Noting the symmetry of the coefficients once again, we see that the sum $\tilde{M}_{121}$ above corresponds to the term M_{12} in [12]. Therefore, following the same steps in [12] to bound M_{12} we arrive at the conclusion that

$$\tilde{M}_{121} \lesssim \phi_0(u)^2 + T^{1/2}\phi_0(u)\phi(u) + T^{3/2}\phi(u)^2. \tag{6.39}$$

Next, we treat the term $\tilde{M}_{122}$ in (6.38). By the equation (5.15), we get

$$\begin{aligned}
&\|t^{\ell+n+m-1}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}} \\
&\quad\lesssim \sum_{m'=0}^{m-1}\sum_{n'+n_3=n}\binom{n'+m-m'}{n'}\frac{n!\,m!}{n_3!\,m'!}K^{n'+m-m'}\|t^{\ell+n+m-1}\partial_x^{n_3+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}
\end{aligned}$$

for $K > 1$. Applying Lemma 6.2 on the factor $\|t^{(i+j+k)-(\ell+n+m)}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}$, we obtain, with $a = (i+j+k)-(\ell+m+n)$, that

$$\begin{aligned}
&\epsilon^{i-\ell}\tilde{\epsilon}^{j-n}\bar{\epsilon}^{k-m}\|t^a\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}} \\
&\quad\lesssim U_{i-\ell+1,j-n+2,k-m}^{1/2}U_{i-\ell,j-n+2,k-m}^{1/2}T^{1/2}\frac{a^{5/2}}{N_a} + U_{i-\ell,j-n+2,k-m}T^{1/2}\frac{a^{5/2}}{N_a} \\
&\qquad + U_{i-\ell+1,j-n,k-m}(T\mathbb{1}_{a=1}+T^2\mathbb{1}_{a\ge2})\frac{a}{N_a} + U_{i-\ell,j-n,k-m}(T^{a-1}\mathbb{1}_{a\le2}+T^2\mathbb{1}_{a\ge3})\frac{a}{N_a}.
\end{aligned} \tag{6.40}$$

We write $\tilde{M}_{122}$ as

$$\begin{aligned}
\tilde{M}_{122} \lesssim &\sum_{i+j+k\ge2}\ \sum_{|(\ell,n,m)|=\lceil(i+j+k)/2\rceil}^{i+j+k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\frac{N_{i+j+k+2}}{N_{\ell+n+m+1}}\epsilon^{i-\ell}\tilde{\epsilon}^{j-n}\bar{\epsilon}^{k-m} \\
&\times\|t^{(i+j+k)-(\ell+n+m)}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_{x,t}}\frac{\epsilon^\ell\tilde{\epsilon}^n\bar{\epsilon}^m}{N_{\ell+n+m+1}}\|t^{\ell+n+m-1}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}
\end{aligned}$$

and note that the factorial terms obey

$$\begin{aligned}
&\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\frac{N_{i+j+k+2}}{N_{\ell+n+m+1}}\frac{|(i-\ell,j-n,k-m)|^{5/2}}{N_{i-\ell+j-n+k-m}} \\
&\quad\lesssim\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\frac{i+j+k}{(i+j+k)!}\frac{(\ell+n+m)!}{(\ell+n+m)^2}\frac{(i-\ell+j-n+k-m)!}{(i-\ell+j-n+k-m)^{1/2}} \le 1
\end{aligned}$$

as $|(\ell,n,m)| \ge \lceil(i+j+k)/2\rceil$. This gives

$$\tilde{M}_{122} \lesssim \sum_{i+j+k\ge2}\ \sum_{|(\ell,n,m)|=\lceil(i+j+k)/2\rceil}^{i+j+k} A_{\ell,n,m}^{i,j,k}\epsilon^\ell\tilde{\epsilon}^n\bar{\epsilon}^m N_{\ell+n+m+1}\|t^{\ell+n+m-1}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}},$$

where $A_{\ell,n,m}^{i,j,k}$ denotes the right side of (6.40) multiplied by $a^{5/2}/N_a$. Applying Young's inequality we bound the sum on $A_{\ell,n,m}^{i,j,k}$ by

$$\tilde{M}_{122} \lesssim \left(\phi_0(u) + T^{1/2}\bar{\phi}(u)\right)\left(\sum_{|(\ell,n,m)|=2}^{\infty}\epsilon^\ell\tilde{\epsilon}^n\bar{\epsilon}^m N_{\ell+n+m+1}\|t^{\ell+n+m-1}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}\right).$$

Next, we estimate the second factor above. Using the bounds on the commutator sum, we have

$$\begin{aligned}
&\sum_{\substack{|(\ell,n,m)|=2\\ n,m\geq 1}}^{\infty} \epsilon^{\ell}\tilde{\epsilon}^{n}\bar{\epsilon}^{m} N_{\ell+n+m+1}\|t^{\ell+n+m-1}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}\\
&\lesssim \sum_{\substack{|(\ell,n,m)|=2\\ n,m\geq 1}}^{\infty} \epsilon^{\ell}\tilde{\epsilon}^{n}\bar{\epsilon}^{m} N_{\ell+n+m+1}\sum_{m'=0}^{m-1}\sum_{n'+n_3=n}\binom{n'+m-m'}{n'}\frac{n!\,m!}{n_3!\,m'!}K^{n'+m-m'}\\
&\quad\times\|t^{\ell+n+m-1}\partial_x^{n_3+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}.
\end{aligned}\tag{6.41}$$

Changing the order of summation by the Fubini Theorem, the right side of (6.41) becomes

$$\begin{aligned}
&\sum_{(\ell,n_3+1,m')\in B^c\cup B_1}\sum_{m=m'+1}^{\infty}\sum_{n=n_3}^{\infty}\frac{(n-n_3+m-m')!}{(n-n_3)!\,(m-m')!}\frac{n!\,m!}{n_3!\,m'!}\frac{(\ell+n+m+1)^3}{(\ell+n+m+1)!}\\
&\qquad\times K^{n-n_3+m-m'}\|t^{\ell+n+m-1}\partial_x^{n_3+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}\\
&\lesssim \sum_{(\ell,n_3+1,m')\in B_1}\frac{(\ell+m'+n_3+1)^3}{(\ell+m'+n_3+1)!}\epsilon^{\ell}\tilde{\epsilon}^{n_3+1}\bar{\epsilon}^{m'}A_1\|t^{\ell+n_3+m'-2}\partial_x^{n_3+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}\\
&\quad+\sum_{(\ell,n_3+1,m')\in B^c}A_0\|\partial_x^{n_3+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}
\end{aligned}$$

where

$$\begin{aligned}
A_1 = &\sum_{m=m'+1}^{\infty}\sum_{n=n_3}^{\infty}\frac{(n-n_3+m-m')!}{(n-n_3)!\,(m-m')!}\frac{n!\,m!}{n_3!\,m'!}\frac{(\ell+n+m+1)^3}{(\ell+n_3+m'+1)^3}\frac{(\ell+n_3+m'+1)!}{(\ell+n+m+1)!}\\
&\times\frac{T}{\tilde{\epsilon}}(T\tilde{\epsilon}K)^{n-n_3}(T\bar{\epsilon}K)^{m-m'}
\end{aligned}$$

and

$$A_0=\sum_{n,m=1}^{\infty}\frac{(n-n_3+m-m')!}{(n-n_3)!\,(m-m')!}\frac{n!\,m!}{n_3!\,m'!}\frac{(\ell+n+m+1)^3}{(\ell+n+m+1)!}T^{\ell+n+m-1}K^{n-n_3+m-m'}\epsilon^{\ell}\tilde{\epsilon}^{n}\bar{\epsilon}^{m}.$$

We recall the computations in Section 4.2 and note that

$$\sup_{(\ell,n_3+2,m')\in B^c}A_0\lesssim T\qquad\text{and}\qquad\sup_{(\ell,n_3+2,m')\in B_1}A_1\lesssim T^2.$$

We recall the definition of ϕ_0 and $\bar{\phi}$ and using the bounds above for A_0 and A_1 we conclude that

$$\tilde{M}_{122}\lesssim(\phi_0(u)+T^{1/2}\bar{\phi}(u))(T\phi_0(u)+T^2\bar{\phi}(u)).\tag{6.42}$$

Next, we estimate $\tilde{M}_{13}$. Recalling the change of indices in the operators acting on u and ∇u, we note that $\tilde{M}_{13}$ corresponds to M_{13}. We follow the same program we applied in the previous section to bound $\tilde{M}_{211}$, and write

$$\begin{aligned}
\tilde{M}_{13}\lesssim&\sum_{i+j+k\geq 2}\sum_{|(\ell,n,m)|=0}^{\lfloor(i+j+k)/2\rfloor}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^{i}\tilde{\epsilon}^{j}\bar{\epsilon}^{k}\|t^{(i+j+k)-(\ell+n+m)-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_t L^4_x}\\
&\quad\times\|t^{\ell+n+m}\partial_x^{n+1}\mathbf{T}^m\partial_t^\ell u\|_{L^2_t L^4_x}\\
&+\sum_{i+j+k\geq 4}\sum_{\substack{|(\ell,n,m)|=2\\ n,m\geq 1}}^{\lfloor(i+j+k)/2\rfloor}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^{i}\tilde{\epsilon}^{j}\bar{\epsilon}^{k}\|t^{(i+j+k)-(\ell+n+m)-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L^\infty_t L^4_x}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \|t^{\ell+n+m}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2L_x^4}\\
&= \tilde{M}_{131}+\tilde{M}_{132}.
\end{aligned}
$$

Analogously to $\tilde{M}_{2111}$, the first term $\tilde{M}_{131}$ is dominated by the estimates on M_{13} in [12]. Therefore,

$$
\begin{aligned}
\tilde{M}_{131} &\lesssim \phi_0(u)^2+T\phi_0(u)^{2-d/4}\phi(u)^{d/4}+T^{1/2}\phi_0(u)\phi(u)\\
&\quad +T^{3/2}\phi_0(u)^{1-d/4}\phi(u)^{1+d/4}+T^{5/2-d/4}\phi(u)^2.
\end{aligned}
\tag{6.43}
$$

For $\tilde{M}_{132}$, using the Gagliardo-Nirenberg estimates we first write the commutator term in $L^2_{x,t}$ norm

$$
\begin{aligned}
\|t^{\ell+n+m}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2L_x^4} &\lesssim \|t^{\ell+n+m}\partial_x^{n+1}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}^{d/4}\|t^{\ell+n+m}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}^{1-d/4}\\
&\quad +\|t^{\ell+n+m}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}\\
&\lesssim \|t^{\ell+n+m}\partial_x^{n+1}[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}+\|t^{\ell+n+m}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L^2_{x,t}}.
\end{aligned}
\tag{6.44}
$$

Also, we estimate $\|t^{(i+j+k)-(\ell+n+m)-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}$ by Lemma 6.3. Putting $a=i+j+k-\ell-n-m\geq 2$, we get

$$
\begin{aligned}
&\epsilon^{i-\ell}\tilde{\epsilon}^{j-n}\bar{\epsilon}^{k-m}\|t^{a-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}\\
&\quad\lesssim U^{1/2}_{i-\ell+1,j-n+1,k-m}U^{1/2}_{i-\ell,j-n+1,k-m}\frac{a!}{a^{3/2}}T^{1/2}+U_{i-\ell,j-n+1,k-m}\frac{a!}{a^{3/2}}T^{1/2}\\
&\qquad +U_{i-\ell+1,j-n,k-m}\frac{a!}{a^2}T+U_{i-\ell,j-n,k-m}\frac{a!}{a^2}(\mathbb{1}_{a=2}+T\mathbb{1}_{a\geq 3}).
\end{aligned}
\tag{6.45}
$$

Denote

$$
B_{\ell,n,m}=\frac{\epsilon^\ell\tilde{\epsilon}^n\bar{\epsilon}^m}{(\ell+n+m)!}\|t^{\ell+n+m}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2L_x^4}.
\tag{6.46}
$$

Then, we rewrite $\tilde{M}_{132}$ in this notation as

$$
\begin{aligned}
\tilde{M}_{132} &\lesssim \sum_{i+j+k\geq 4}\sum_{|(\ell,n,m)|=1}^{\lfloor (i+j+k)/2\rfloor}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\,(\ell+n+m)!\,B_{\ell,m}\\
&\quad \times \epsilon^{i-\ell}\tilde{\epsilon}^{j-n}\bar{\epsilon}^{k-m}\|t^{(i+j+k)-(\ell+n+m)-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}.
\end{aligned}
\tag{6.47}
$$

Similarly, the factorial and the binomial terms appearing on (6.47) and on the right side of (6.45) are bounded from above by

$$
\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\,(\ell+n+m)!\frac{(i+j+k-\ell-n-m)!}{(i+j+k-\ell-n-m)^{3/2}}\lesssim 1.
$$

With the estimate for $\epsilon^{i-\ell}\tilde{\epsilon}^{j-n}\bar{\epsilon}^{k-m}\|t^{(i+j+k)-(\ell+n+m)-1}\partial_x^{j-n}\mathbf{T}^{k-m}\partial_t^{i-\ell}u\|_{L_t^\infty L_x^4}$ given in the equation (6.45), we bound the factorial terms in (6.47) from above by 1 , and then apply Young's inequality on the resulting double sum. This yields

$$
\tilde{M}_{132}\lesssim \Big(\phi_0(u)+T^{1/2}\bar{\phi}(u)\Big)\sum_{|(\ell,n,m)|=2,n,m\geq 1}B_{\ell,n,m}.
\tag{6.48}
$$

Before we bound the sum in (6.48), we recall the expansion (5.15) for the commutator term and combine it with the estimate (6.44)

$$
\begin{aligned}
&\|t^{\ell+n+m}\partial_x^n[\mathbf{T}^m,\nabla]\partial_t^\ell u\|_{L_t^2L_x^4}\\
&\lesssim \sum_{m'=0}^{m-1}\sum_{n'=0}^{n+1}\binom{n'+m-m'}{n'}\frac{(n+1)!\,m!}{(n+1-n')!\,m'!}K^{n'+m-m'}\|t^{\ell+n+m}\partial_x^{n+2-n'}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}\\
&\quad+\sum_{m'=0}^{m-1}\sum_{n'=0}^{n}\binom{n'+m-m'}{n'}\frac{n!\,m!}{(n-n')!\,m'!}K^{n'+m-m'}\|t^{\ell+n+m}\partial_x^{n+1-n'}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}.
\end{aligned}
$$

Recalling the definition of $B_{\ell,n,m}$ in (6.46) and using the inequality above, and changing the indices to $\tilde{n}=n+1$, $n_1=n+1-n'$ and $n_2=n-n'$, we obtain

$$
\begin{aligned}
\sum_{\substack{|(\ell,n,m)|=2\\ n,m\ge1}}B_{\ell,n,m}\lesssim&\sum_{\substack{|(\ell,\tilde{n},m)|=3\\ \tilde{n}\ge2,m\ge1}}\frac{\epsilon^\ell\tilde{\epsilon}^{\tilde{n}-1}\bar{\epsilon}^m}{(\ell+\tilde{n}+m-1)!}\sum_{m'=0}^{m-1}\sum_{n_1=0}^{\tilde{n}}\binom{\tilde{n}-n_1+m-m'}{\tilde{n}-n_1}\frac{\tilde{n}!\,m!}{n_1!\,m'!}K^{\tilde{n}-n_1+m-m'}\\
&\qquad\times\|t^{\ell+\tilde{n}+m-1}\partial_x^{n_1+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}\\
&+\sum_{\substack{|(\ell,n,m)|=2\\ n,m\ge1}}\frac{\epsilon^\ell\tilde{\epsilon}^n\bar{\epsilon}^m}{(\ell+n+m)!}\sum_{m'=0}^{m-1}\sum_{n_2=0}^{n}\binom{n-n_2+m-m'}{n-n_2}\frac{n!\,m!}{(n_2)!\,m'!}K^{n-n_2+m-m'}\\
&\qquad\times\|t^{\ell+n+m}\partial_x^{n_2+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}.
\end{aligned}
\tag{6.49}
$$

We then change the order of the sums on the right side of (6.49) by the Fubini Theorem, and obtain

$$
\begin{aligned}
&\sum_{(\ell,n_1,m')\in B_1\cup B^c}\ \sum_{m=m'+1}^{\infty}\ \sum_{\tilde{n}=n_1\wedge2}^{\infty}\binom{\tilde{n}-n_1+m-m'}{\tilde{n}-n_1}\frac{\tilde{n}!\,m!}{(\ell+\tilde{n}+m-1)!\,n_1!\,m'!}K^{\tilde{n}-n_1+m-m'}\\
&\qquad\times\epsilon^\ell\tilde{\epsilon}^{\tilde{n}-1}\bar{\epsilon}^m\|t^{\ell+\tilde{n}+m-1}\partial_x^{n_1+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}\\
&+\sum_{(\ell,n_2,m')\in B_1\cup B^c}\ \sum_{m=m'+1}^{\infty}\ \sum_{n=n_2}^{\infty}\binom{n-n_2+m-m'}{n-n_2}\frac{n!\,m!}{(\ell+n+m)!\,n_2!\,m'!}K^{n-n_2+m-m'}\\
&\qquad\times\epsilon^\ell\tilde{\epsilon}^n\bar{\epsilon}^m\|t^{\ell+n+m}\partial_x^{n_2+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}.
\end{aligned}
\tag{6.50}
$$

Using the notation $U_{\ell,n,m}$, we further bound the first sum in (6.50) by

$$
\sum_{(\ell,n_1,m')\in B^c}\|\partial_x^{n_1+1}\mathbf{T}^{m'}\partial_t^\ell u\|_{L^2_{x,t}}A_0+T\sum_{(\ell,n_1,m')\in B_1}U_{\ell,n_1+1,m'}A_1
\tag{6.51}
$$

where

$$
A_0=\frac{1}{T}\sum_{m=1}^{\infty}\sum_{\tilde{n}=2}^{\infty}\binom{\tilde{n}+m}{\tilde{n}}\frac{\tilde{n}!\,m!}{(\tilde{n}+m-1)!}(T\tilde{\epsilon}K)^{\tilde{n}}(T\bar{\epsilon}K)^m
$$

and

$$
A_1=\sum_{\substack{m=m'+1\\ \tilde{n}=n_1}}^{\infty}\binom{\tilde{n}-n_1+m-m'}{\tilde{n}-n_1}\frac{\tilde{n}!\,m!}{n_1!\,m'!}\frac{(\ell+n_1+m')!}{(\ell+\tilde{n}+m-1)!}\frac{1}{(\ell+n_1+m')^2}(T\tilde{\epsilon}K)^{\tilde{n}-n_1}(T\bar{\epsilon}K)^{m-m'}.
$$

After the cancellation of factorial terms, we see that A_0 is summable and it is bounded above by

$$A_0 \lesssim T\bar{\epsilon}K \sum_{m,\tilde{n}=0}^{\infty} (n+m)(T\tilde{\epsilon}K)^{\tilde{n}}(T\bar{\epsilon}K)^m \lesssim T \tag{6.52}$$

for $0 < \tilde{\epsilon} \lesssim \bar{\epsilon} < 1/K$.

Similarly, we first simplify the factorial terms for A_1. Recalling the hypergeometric inequality (cf. (4.9)), the first factor below is bounded above by 1, and we have

$$\frac{\binom{\tilde{n}}{n_1}\binom{m}{m'}}{\binom{\ell+\tilde{n}+m}{\tilde{n}-n_1+m-m'}} \frac{\ell+\tilde{n}+m}{(\ell+n_1+m')^2} \lesssim \frac{\ell+\tilde{n}+m}{(\ell+n_1+m')^2},$$

which in turn yields

$$\begin{aligned} \sup_{(\ell,n_1,m')\in B_1} A_1 &\lesssim \sup_{(\ell,n_1,m')\in B_1} \sum_{\substack{m=m'+1\\ \tilde{n}=n_1}}^{\infty} \frac{\ell+\tilde{n}+m}{(\ell+n_1+m')^2}(T\tilde{\epsilon}K)^{n-n_1}(T\bar{\epsilon}K)^{m-m'} \\ &\lesssim T\bar{\epsilon}K \lesssim T \end{aligned} \tag{6.53}$$

for $0 < \bar{\epsilon} \leq 1/2K$.

In the same fashion, the second sum in (6.50) is bounded above by

$$\sum_{(\ell,n_2,m')\in B^c} \|\partial_x^{n_2+1}\mathbf{T}^{m'}\partial_t^{\ell}u\|_{L^2_{x,t}}\tilde{A}_0 + T^2 \sum_{(\ell,n_2,m')\in B_1} U_{\ell,n_2+1,m'}\tilde{A}_1 \tag{6.54}$$

with the coefficients $\tilde{A}_0$ and $\tilde{A}_1$ obeying the same estimates with A_0 and A_1.

Back to (6.50), we bound the two sums as given in (6.51) and (6.54) and apply the estimates (6.52) and (6.53) on the coefficients $A_0, \tilde{A}_0$ and $A_1, \tilde{A}_1$. With the final application of discrete Young's inequality on the sums (6.51), (6.54), we reach the estimate

$$\sum_{\substack{|(\ell,n,m)|=2\\ n,m\geq 1}} B_{\ell,n,m} \lesssim T\phi_0 + T^2\bar{\phi}(u) + T\phi_0(u) + T^3\bar{\phi}(u)$$

and putting this estimate on (6.48), we get

$$\tilde{M}_{132} \lesssim T\phi_0(u)^2 + T^{5/2}\phi(u)^2. \tag{6.55}$$

Combining (6.43) and (6.55) and selecting the maximal prefactors in T, we conclude that

$$M_{13} \lesssim \phi_0(u)^2 + T\phi(u)^2. \tag{6.56}$$

Adding the estimates (6.37), (6.39), (6.42), and (6.56), we conclude the proof of Lemma 6.4. □

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