

Real Analytic Local Well-Posedness for the Triple Deck

SAMEER IYER
Princeton University

VLAD VICOL
Courant Institute

Abstract

The triple-deck model is a classical high-order boundary layer model that has been proposed to describe flow regimes where the Prandtl theory is expected to fail. At first sight the model appears to lose two derivatives through the pressure-displacement relation that links pressure to the tangential slip. In order to overcome this, we split the triple-deck system into two coupled equations: a Prandtl-type system on \mathbb{H} and a Benjamin-Ono-type equation on \mathbb{R} . This splitting enables us to extract a crucial leading-order cancellation at the top of the lower deck. We develop a functional framework to subsequently extend this cancellation into the interior of the lower deck, which enables us to prove the local well-posedness of the model in tangentially real analytic spaces. © 2019 Wiley Periodicals, Inc.

1 Introduction

A fundamental challenge in fluid mechanics is to describe the vanishing viscosity limit ($\nu \rightarrow 0$) of the Navier-Stokes equations on domains with a solid boundary. In this paper we consider the fluid domain to be the two-dimensional half-space \mathbb{H} . The main difficulty is due to the incompatibility between the no-slip boundary condition for the Navier-Stokes velocity field ($\mathbf{U}^\nu|_{\partial\Omega} = 0$) and the slip boundary condition for the Euler velocity field ($\mathbf{U}^E|_{\partial\Omega} \cdot (0, -1) = 0$), which makes it difficult to obtain uniform-in- ν estimates for norms of \mathbf{U}^ν that are stronger than $L_t^\infty L_x^2$.

1.1 Historical Overview

In order to rectify this mismatch, Prandtl [62] proposed the existence of a thin, $\mathcal{O}(\nu^{1/2})$, fluid layer near the boundary through which the Navier-Stokes velocity field transitions from an outer Euler flow in the bulk to the no-slip condition on the solid wall. Mathematically, this corresponds to a formal asymptotic expansion of the viscous incompressible flow \mathbf{U}^ν as

$$(1.1) \quad \mathbf{U}^\nu(t, x, y) = \mathbf{U}^E(t, x, y) + \mathbf{U}^{BL}\left(t, x, \frac{y}{\sqrt{\nu}}\right) + \mathcal{O}(\nu^{1/2})$$

where \mathbf{U}^E is the Euler flow typically assumed to be known a priori, and \mathbf{U}^{BL} is known as the Prandtl boundary layer corrector. The boundary layer unknown $\mathbf{u}^P = [u^P, v^P] := \mathbf{U}^{BL} + \mathbf{U}^E|_{Y=0}$ is a function of the tangential variable x and the normal fast variable $\bar{Y} = y/\sqrt{v}$, and is governed by the famous Prandtl boundary layer equations

$$(1.2a) \quad \partial_t u^P + u^P \partial_x u^P + v^P \partial_{\bar{Y}} u^P - \partial_{\bar{Y}}^2 u^P = -\partial_x P^E(t, x, 0),$$

$$(1.2b) \quad \partial_x u^P + \partial_{\bar{Y}} v^P = 0,$$

$$(1.2c) \quad [u^P, v^P]|_{\bar{Y}=0} = 0, \quad u^P|_{\bar{Y} \rightarrow 0} = u^E(t, x, 0),$$

posed in the half-space $\mathbb{H} = \{(x, \bar{Y}): \bar{Y} > 0\}$. The system treats the Euler pressure trace P^E and the Euler wall slip velocity u^E as known, and is supplemented with an initial condition $u^P|_{t=0} = u_0^P(x, \bar{Y})$.

A first step towards establishing the validity or the invalidity of the Prandtl expansion (1.1) is a detailed understanding of the Prandtl system (1.2) itself. The well- and ill-posedness of the Prandtl equations has a long history of which we only provide a very brief summary (see the reviews [2, 52] for further references). Under the monotonicity assumption $\partial_y u^P|_{t=0} > 0$, Oleinik [59, 60] obtained global-in-time, regular solutions on the domain $[0, L] \times \mathbb{R}_+$ for small L , and local-in-time regular solutions for arbitrary finite L . The aforementioned results rely on the Crocco transform, which is available from the monotonicity hypothesis. See also the global-in-time existence result of weak solutions obtained in [76] under the additional assumption of a favorable pressure gradient $\partial_x P^E(t, x) \leq 0$. Without using the Crocco transform, local existence was established in the works of [39, 53] using energy methods and [1] using a Nash-Moser iteration. When the monotonicity assumption is removed, local well-posedness results for (1.2) were first established assuming tangential real analyticity of the initial datum [41, 50, 63] (see also [36] for an almost global existence result for small datum), and more recently assuming only tangential Gevrey-class regularity [23, 47]. The sharp Gevrey-2 result without any structural assumptions was recently established in [13]. On the other hand, in Sobolev spaces without monotonicity, the Prandtl equations are ill-posed, as was shown in [20, 25, 34, 49].

Concerning the validity of Prandtl ansatz (1.1), in the unsteady setting the expansion has been verified locally in time assuming the initial datum is real-analytic [58, 64, 75], under the assumption that the initial vorticity is supported away from the boundary [16, 17, 51], in the Gevrey setting for initial data close to certain stable shear flows [22], or assuming that the initial vorticity is analytic only near the boundary of the half space [43]. In contrast, for initial datum in Sobolev spaces the ansatz (1.1) has been proven to be *invalid* [27–30], with the recent result [31] proving that the expansion is not valid in the L^∞ topology.

A notable success of the Prandtl theory is in the steady regime, where it was in fact derived in [62]. For steady flows in (1.2a), the initial datum (1.2c) is typically replaced by in-flow data at $\{x = 0\}$, which represents, for instance, the leading

edge of a flat plate. Shortly after Prandtl's original work, Blasius [3] discovered the self-similar solution to the steady Prandtl equations

$$(1.3a) \quad [u^P, v^P] := \left[f'(\eta), \frac{1}{\sqrt{x}} \{ \eta f'(\eta) - f(\eta) \} \right] \quad \text{where } \eta = \frac{y}{\sqrt{x}},$$

$$(1.3b) \quad ff'' + f''' = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \quad \frac{f(\eta)}{\eta} \xrightarrow{\eta \rightarrow \infty} 1.$$

Experiments have confirmed the accuracy of (1.1) for steady flow over a plate to a remarkable degree of precision [65], especially for the Blasius self-similar boundary layers (1.3). Mathematically, in the steady case, the ansatz has been recently verified in [21] for shear boundary layer flows that arise from forced Navier-Stokes equations, and [32, 33] for a general class of x -dependent boundary layer flows, which arise from homogeneous Navier-Stokes flows, and which include the Blasius solution. See also [35] for related results on a moving plate, and [37, 66] for results concerning the large- x convergence of the steady Prandtl solution to Blasius.

In spite of the success of the Prandtl theory in the steady regime, the phenomenon of boundary layer separation remains mostly unsolved, both in the steady and the unsteady regimes [10, 65, 69, 72]. In the unsteady case, the van Dommelen and Shen singularity [74], which was recently proven to occur rigorously [8, 9, 15, 42], may be seen as a diagnostic of separation [19]: an adverse Euler pressure gradient causes a finite-time singularity in the displacement thickness, and so the flow is detached from the flat plate. The vorticity generated at the boundary is ejected into the bulk of the flow, where it rolls up and is considered as one of the factors responsible for the anomalous dissipation of energy. In the steady case the detachment of the boundary layer from the flat plate was predicted by Goldstein [26] and has been proven recently in [12]. This breakdown of the assumptions on which Prandtl equations are derived signals the limitations of the classical Prandtl boundary layer theory, and new, higher-order theories are required in order to model the inviscid-boundary layer coupling near points of separation [7, 10].

Two well-known higher-order models are the *prescribed displacement thickness* (PDT) model [6] and the *interactive boundary layer* (IBL) model [5, 44, 45]. For instance, in the IBL model the Euler flow and boundary layer flow are strongly coupled through a boundary condition of the type

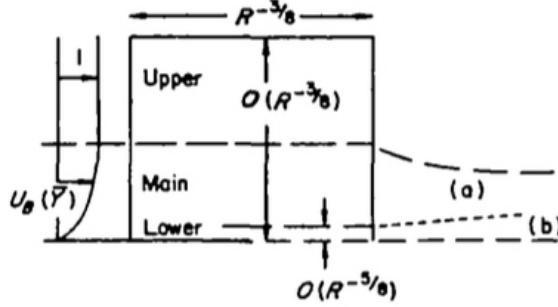
$$(1.4) \quad v^E|_{y=0} = \sqrt{v} \partial_x \{ \kappa u^E|_{y=0} \},$$

$$\kappa := \int_{\mathbb{R}_+} \left(1 - \frac{u^P}{u^E|_{Y=0}} \right) dy, \quad u^P|_{y \rightarrow \infty} = u^E(t, x, 0).$$

This model has been studied rigorously in [11], where it is shown to be *linearly ill-posed even in analytic spaces*. Similar dramatic ill-posedness results are shown in [11] to hold for the PDT model. These severe instabilities in the PDT and IBL higher-order boundary layer models lead us to consider the triple-deck system, which is the main purpose of this paper.

1.2 Triple-Deck Equations

In order to describe the triple-deck system, it is useful to keep in mind the following diagram, taken from [68, p. 220, figure 4], which describes the steady flow past a finite plate whose boundary is at $\{y = 0\}$, with a leading edge to the left and a trailing in the bottom center of the figure:



Here R denotes the Reynolds number. Near the leading edge of the plate, the flow is accurately described by the Prandtl theory, and in particular by the self-similar Blasius profile (1.3). The trailing edge of the plate creates a disturbance, and the flow undergoes the so-called Goldstein singularity. The triple-deck theory describes specifically the transition from the Blasius profile on the left of the plate to the Goldstein, which occurs after the trailing edge of the plate. This was formalized in the works of [54, 57, 70, 71], who proposed the *three deck* structure and introduced the horizontal $\mathcal{O}(R^{-5/8})$ and vertical $\mathcal{O}(R^{-5/8}), \mathcal{O}(R^{-3/8})$ length scales that are not present in the Prandtl theory. The notion of introducing different scales at the point of boundary layer separation was introduced earlier in [48]. We refer the reader to the works [38, 44, 55, 56, 68, 73] for an overview of the ideas and history behind the triple-deck model, and include a formal derivation of the unsteady triple-deck model (cf. system (1.5)–(1.7) below) in Appendix 4.4 of this paper.

Specifically, see, e.g. [68, sec. 3], [73, sec. 2], or [14, sec. 4], we consider the *unsteady triple-deck system* posed in the half-space $\mathbb{H} = \{(x, y): y > 0\}$, which is given by

$$(1.5a) \quad \partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \partial_y^2 u,$$

$$(1.5b) \quad \partial_x u + \partial_y v = 0,$$

$$(1.5c) \quad \partial_y p = 0,$$

supplemented with the boundary conditions

$$(1.6a) \quad u(x, 0, t) = v(x, 0, t) = 0,$$

$$(1.6b) \quad u(x, y, t) - y \rightarrow 0 \quad \text{as } x \rightarrow -\infty$$

$$(1.6c) \quad u(x, y, t) - y \rightarrow A(x, t) \quad \text{as } y \rightarrow +\infty$$

and with the *pressure-displacement relation*

$$(1.7) \quad p(x, t) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{(\partial_x A)(\bar{x}, t)}{x - \bar{x}} d\bar{x} = |\partial_x| A(x, t)$$

characteristic of incompressible flows. Note that other pressure-displacement relations may be specified in the case of supersonic and jetlike compressible boundary layer flows (see, e.g., [73, eq. (2.4a-e)]). The system (1.5)–(1.7) is supplemented with a compatible initial condition

$$(1.8) \quad u|_{t=0} = u_0$$

on \mathbb{H} . Note that while equations (1.5) look the same as the classical Prandtl equations, the main difference is that p is not given in advance, and neither is the value of u at the top of the lower deck. Instead, these are coupled by the relation (1.7) above.

The triple-deck and the IBL models share the common feature that u and u^P , respectively, converge as $y \rightarrow \infty$ to a function that is not given a priori and must be determined through the evolution. However, in contrast to (1.4), in which $u^P|_{y \rightarrow \infty}$ is governed ultimately by the Euler equations, the behavior of $u|_{y \rightarrow \infty}$ in (1.5) is governed by the Benjamin-Ono equations, as is shown in Section 2.1. It has been alluded to in [11] (see also [14, 67]) that the triple-deck has favorable stability features relative to the IBL model, but to our knowledge this had not been studied mathematically until the present work. In fact, it is not known whether the system (1.5)–(1.8) is well-posed, even locally in time.

The unsteady triple-deck model poses significant mathematical difficulties, because the map $u \mapsto \partial_x p$ loses two derivatives in x (in view of (1.7)) and half a derivative in y (due to the restriction to the boundary $\{y = \infty\}$). The *two-derivative loss* in x seems to preclude the well-posedness of the system, even in spaces of analytic functions. Our goal is to show that due to a certain cancellation in L_x^2 , the loss is only of one derivative in x , and hence the system admits local-in-time real-analytic solutions with respect to x , which are Sobolev smooth in y . Our main result is Theorem 1.1 below, which may be stated informally as: assume that $A_0(x)$ is real-analytic and that the function $u_0(x, y) - y - A_0(x)$ is tangentially real-analytic and lies in a weighted L^2 space with respect to the normal variable; then there exists locally in time a unique solution in this class. We discuss the main difficulties and the main ideas of the proof in Section 1.3 below. Prior to this, we introduce the functional setting of the paper and the decomposition (1.13)–(1.14) of the solution.

1.3 Main Result and Functional Setting

Analytic Norms

In order to measure decay in y , we introduce the Gaussian y -weight given by

$$(1.9) \quad \rho(y, t) = e^{\frac{y^2}{8(1+t/\varepsilon)}}.$$

Note that ρ does not depend on x . The parameter ε , appearing in (1.9), will be selected small, based only on the initial datum, according to the relation (3.25a). The time scale over which we prove existence will be, without loss of generality, restricted to $t \in [0, T_*]$, where $T_* \leq \varepsilon$, so that in particular the quotient t/ε appearing in the weight above is bounded. It would be interesting to determine whether one can relax the growth rate as $y \rightarrow \infty$ of ρ , for instance, by following the strategy of [13] or of [41].

We denote the Fourier transform a function f in the x -variable only, at frequency $\xi \in \mathbb{R}$, as $f_\xi = f_\xi(y, t)$. Since f is real-valued, we automatically have that $f_{-\xi} = \overline{f_\xi}$. For $\tau = \tau(t) > 0$, $r > 2$, and a function $f(x, y, t)$ we use Plancherel to define

$$(1.10) \quad \|f\|_{\tau, r}^2 = \|\rho e^{\tau|\partial_x|} f\|_{H_x^r L_y^2}^2 = \int_{\mathbb{R}} \int_0^\infty \rho^2(y) |f_\xi(y)|^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi$$

where $\langle \xi \rangle^2 = 1 + |\xi|^2$, and we have suppressed the time dependence of τ, ρ , and f_ξ . Associated to this norm, it is convenient to also define the inner product

$$\langle f, g \rangle_{\tau, r} = \int_{\mathbb{R}} \int_0^\infty \rho^2(y) f_\xi(y) \overline{g_\xi(y)} e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi$$

where the time dependence is suppressed. The idea to use real-analytic norms of the type (1.10) goes back to the work of Foias-Temam [18] in the context of the Navier-Stokes equations, and to [46] in the context of the Euler equations. See also [4, 40, 61] and references therein.

Notice that by definition of the $\|\cdot\|_{\tau, r}$ norm, we have the identity

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|f\|_{\tau, r}^2 + (-i) \|\partial_x|^{1/2} f\|_{\tau, r}^2 \\ &= \int_{\mathbb{R}} \left(\frac{1}{2} \frac{d}{dt} \|\rho f_\xi\|_{L_y^2([0, \infty))}^2 \right) e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi \\ &= \langle \partial_t f + f \partial_t(\log \rho), f \rangle_{\tau, r}. \end{aligned}$$

Therefore, a decrease in the analyticity radius yields a $|\partial_x|^{1/2}$ -dissipative term.

We introduce similar analytic norms for functions $g(x, t)$, which are independent of y . Here we let

$$(1.11) \quad \|g\|_{\widetilde{\tau}, r}^2 = \|e^{\tau|\partial_x|} g\|_{H_x^r}^2 = \int_{\mathbb{R}} |g_\xi|^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi$$

with associated inner product

$$\langle f, g \rangle_{\widetilde{\tau}, r} = \int_{\mathbb{R}} f_\xi \overline{g_\xi} e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi.$$

As in (1.11), we have

$$(1.12) \quad \frac{1}{2} \frac{d}{dt} \|g\|_{\widetilde{\tau},r}^2 + (-\dot{\tau}) \|\partial_x|^{1/2}g\|_{\widetilde{\tau},r}^2 = \int_{\mathbb{R}} \left(\frac{1}{2} \frac{d}{dt} |g_{\xi}|^2 \right) e^{2\tau|\xi|} \langle \xi \rangle^{2r} = \langle \partial_t g, g \rangle_{\widetilde{\tau},r}.$$

Having defined the basic norms, we turn to the definition of the total norms used in this paper and the corresponding unknowns that we measure using these norms.

Representation of the Solution and the Total Energy

We shall work with the following decomposition of the solution $u(x, y, t)$ of (1.5)–(1.6). We write

$$(1.13) \quad u(x, y, t) = y + w(x, y, t)$$

where the function $w(x, y, t)$ is defined in terms of its tangential (i.e., with respect to x) Fourier transform coefficients, $w_{\xi}(y, t) = \int_{\mathbb{R}} w(x, y, t) e^{-ix\xi} dx$, given by

$$(1.14) \quad w_{\xi}(y, t) = \bar{w}_{\xi}(y, t) + A_{\xi}(t) \theta_{\xi}(y, t).$$

The Gaussian weight function $\theta_{\xi}(y, t)$ is defined *explicitly* in (2.11) below. The coefficients $A_{\xi}(t)$ are nothing but the Fourier coefficients in x of the function $A(x, t) = \lim_{y \rightarrow \infty} w(x, y, t)$. In Section 2.1 we show that A obeys a *forced Benjamin-Ono equation* (cf. (2.6)), which arises as a compatibility equation for (1.5)–(1.6). On the other hand, the main unknowns $w_{\xi}(y, t)$ are shown in Section 2.2 to solve an evolution equation (cf. (2.9)–(2.10)), which has a very similar structure to the classical Prandtl system, with the addition of certain singular coupling terms to the evolution for A . The point is that the original function u may be reconstructed explicitly from knowledge of the Fourier coefficients \bar{w}_{ξ} and A_{ξ} . Accordingly, our total norms measure the analytic regularity of \bar{w} and A .

Throughout the paper fix a value for $r > 2$ and a smooth cutoff function $\chi(y)$ approximating $\mathbf{1}_{\{y \geq 2\}}$ (defined in (3.11) below). For a function $\tau(t) > 0$ to be defined later, for a parameter $\delta > 1$ to be chosen precisely later, and with the norms $\|\cdot\|_{\tau,r}$ and $\|\cdot\|_{\widetilde{\tau},r}$ defined in (1.10) respectively (1.11), we let

$$(1.15a) \quad \|(\bar{w}, A)\|_{X_{\tau}} = \|\bar{w}(t)\|_{\tau(t),r} + \frac{1}{\delta} \|\chi \partial_y \bar{w}\|_{\tau(t),r-1/2} + \|A(t)\|_{\widetilde{\tau(t),r}},$$

$$(1.15b) \quad \|(\bar{w}, A)\|_{Y_{\tau}} = \|\partial_x|^{1/2} \bar{w}(t)\|_{\tau(t),r} + \frac{1}{\delta} \|\chi |\partial_x|^{1/2} \partial_y \bar{w}\|_{\tau(t),r-1/2} + \|\partial_x|^{1/2} A(t)\|_{\widetilde{\tau(t),r}},$$

$$(1.15c) \quad \|\bar{w}\|_{Z_{\tau}} = \|\partial_y \bar{w}(t)\|_{\tau(t),r} + \frac{1}{\delta} \|\chi \partial_{yy} \bar{w}\|_{\tau(t),r-1/2}$$

$$(1.15d) \quad \|\bar{w}\|_{H_{\tau}} = \|y \bar{w}(t)\|_{\tau(t),r} + \frac{1}{\delta} \|y \chi \partial_y \bar{w}\|_{\tau(t),r-1/2}.$$

The X_{τ} norm is the main analytic-in- x and weighted L^2 -in- y norm used in this paper. The Y_{τ} norm quantifies dissipation in the x -variable due to a shrinking

analyticity radius, the Z_τ norm quantifies dissipation in the y -variable due to the ∂_{yy} terms present in the equation, while the H_τ norm encodes a gain of a y -weight, which is important due to the unboundedness of the domain $[0, \infty)$. We define the total analytic energy associated to these norms via

$$(1.16) \quad \begin{aligned} E(T) = & \sup_{t \in [0, T]} \|(\bar{w}, A)\|_{X_{\tau(t)}}^2 + \int_0^T \|(\bar{w}, A)\|_{Y_\tau}^2 dt \\ & + \frac{1}{16} \int_0^T \|\bar{w}\|_{Z_{\tau(t)}}^2 dt + \frac{1}{64\varepsilon} \int_0^T \|\bar{w}\|_{H_{\tau(t)}}^2 dt. \end{aligned}$$

Main Theorem and Overview of Proof

We are now ready to state the main result.

THEOREM 1.1 (Main Theorem). *Fix an initial radius of analyticity $\tau_0 > 0$ and any $r > 2$, where r is the analytic weight parameter appearing in (1.10). We decompose the initial data in the following form, written on the Fourier side in the tangential variable*

$$u_\xi^{(0)}(y) = y + \theta_\xi(y, 0)A^{(0)}(\xi) + \bar{w}_\xi^{(0)}(y),$$

where θ_ξ is defined in (2.11). Assume that $\bar{w}^{(0)}$ and $A^{(0)}$ satisfy

$$E_0 := \|(\bar{w}^{(0)}, A^{(0)})\|_{X_{10\tau_0}} < \infty,$$

where the analytic energy is defined in (1.16). Then there exists a $T_* > 0$ depending on τ_0, r, E_0 , and there exists a unique solution (\bar{w}, A) to the coupled system (2.9), (2.14) with initial datum $(\bar{w}, A)|_{t=0} = (\bar{w}^{(0)}, A^{(0)})$ such that the total analytic energy $E(t)$ is defined (1.16) is bounded as

$$\sup_{t \in [0, T_*]} E(t) \leq 2E_0.$$

Equivalently, this defines a unique tangentially analytic solution

$$u_\xi = y + \theta_\xi(t, y)A_\xi(t) + \bar{w}_\xi(t, y)$$

to the original system (1.5)–(1.7).

The central difficulty in establishing Theorem 1.1 is the apparent loss of two x -derivatives in the coupled equations (1.5a) and (1.7). Indeed, replacing $-\partial_x p$ on the right-hand side of (1.5a) with $\partial_x |\partial_x| A(x, t)$ according to (1.7), and subsequently replacing A with $u(x, y) - y|_{y \rightarrow \infty}$, we see that, in terms of a formal derivative count, we have

$$(1.17) \quad \partial_t u + u \partial_x u + v \partial_y u = -\partial_x |\partial_x| u(x, \infty).$$

This loss of two x -derivatives precludes the well-posedness of the system even in analytic spaces. Our starting point is the observation of skew-adjointness of

the loss term on the right-hand side of (1.17). Indeed, for any smooth decaying function $g(x)$ one has

$$(1.18) \quad \int_{\mathbb{R}} g \partial_x |\partial_x| g = 0.$$

The cancellation (1.18) holds because we have $|\partial_x| = -H \partial_x$, where H is the Hilbert transform, and both H and ∂_x are skew-adjoint operators on $L^2(\mathbb{R})$.

This motivates our main reformulation of the system and the extraction of the unknowns we analyze. First, we notice that according to (1.6c), u grows like y as $y \rightarrow \infty$, while $\partial_x u = -\partial_y v$ converges to $\partial_x A(t, x)$, a bounded function as $y \rightarrow \infty$. Hence $v = -I_y[\partial_x u]$ also grows like y at ∞ . Here and throughout the paper we write

$$(1.19) \quad I_y[f] = \int_0^y f(y') dy'.$$

We are thus led to introduce the expansion of v at ∞ :

$$v = y v_1(t, x) + v_0(t, x) + \mathcal{O}(y^{-1}) \quad \text{as } y \rightarrow \infty.$$

The coefficient $v_0(t, x)$ will play a crucial role in the analysis, and is given by the nonlocal integral $I_{\infty}[\partial_x u - \partial_x A]$. We thus reinterpret (1.5) as giving three relations simultaneously, corresponding to the orders of growth as $y \rightarrow \infty$. First, collecting the contributions from (1.5a), which are $\mathcal{O}(y)$ (arising from the terms $u \partial_x u$ and $v \partial_y u$), we obtain the asymptotic information that $v_1 = -\partial_x A$. Second, we collect the terms that contribute $\mathcal{O}(1)$ terms at $y = \infty$. This yields a forced Benjamin-Ono equation for the unknown $A(t, x)$:

$$(1.20) \quad \partial_t A + A \partial_x A + \partial_x |\partial_x| A = -v_0 \quad \text{on } \mathbb{R}.$$

The cancellation alluded to earlier in (1.18) is now readily apparent upon computing the inner product of A against the Benjamin-Ono equation.

Having extracted the $\mathcal{O}(y)$ and $\mathcal{O}(1)$ contributions, the third step is to extract the functions in (1.5a) that decay as $y \rightarrow \infty$. The relevant unknown, \bar{w} , is then a homogenized version of u , and it obeys a Prandtl-type equation (see (2.9a) below). This procedure gives rise to the start of our analysis: we analyze simultaneously a Benjamin-Ono equation for A , forced by $v_0 = v_0(\bar{w})$, as well as a Prandtl-type equation for \bar{w} , forced by A related quantities. Summarizing, the simultaneous system of equations we extract are (leaving F an unspecified forcing term for now)

$$(1.21a) \quad \partial_t A + A \partial_x A + \partial_x |\partial_x| A = -v_0(\bar{w}) \quad \text{on } \mathbb{R} \quad (\text{Benjamin-Ono}),$$

$$(1.21b) \quad \partial_t \bar{w} - \partial_y^2 \bar{w} + y \partial_x \bar{w} = F(\bar{w}, A) \quad \text{on } \mathbb{H} \quad (\text{Prandtl-type}).$$

The cancellation (1.18) applies for the quantity A , which describes u at $y = \infty$, and thus (1.18) should be interpreted as solving the derivative loss problem at $y = \infty$. We now must continue exploiting this cancellation for values of $y < \infty$. Indeed, the two-derivative loss is still lurking for finite y through the forcing term in (1.21b). Specifically, the reader should consult $\mathcal{B}_{\xi}(\bar{w}, A)$, defined in (2.15),

and in particular the most singular contributions arise from the $\partial_t A_\xi$ term, which in turn create a $i\xi|\xi|A_\xi$ contribution, again yielding a two-derivative loss. Our observation is that such a term is accompanied by a factor of $(1 - \theta_\xi)$. By selecting the lift function θ_ξ in a frequency-dependent manner, we are able to gain back $(\partial_x)^{3/2}$. The idea of tangential-frequency-dependent boundary layer lifts was also successfully used in [24] in the context of the hydrostatic Navier-Stokes equations. For us, the selection of a ξ -dependent lift, coupled with Hardy-type inequalities with the homogeneous weights of y , enables us to gain back enough regularity *near* $\{y = 0\}$.

A further difficulty that arises in our analysis is the loss of one y -weight. This occurs due to the nonlocal integral in (2.10c), which forces the \bar{w} evolution. In order to handle the loss of a y -weight, we control the quantity $y\partial_y \bar{w}$ in L^2 , which is seen in the specification of the $\|\cdot\|_{H_\tau}$ in (1.15d). To control this component of the H_τ norm, we in turn need to commute the vector field $y dy$ with the Prandtl system, which necessitates an analysis of the vorticity equation that governs the evolution of $\partial_y \bar{w}$. To successfully analyze the vorticity equation, we capitalize on two essential features. First, we only require this enhanced vector field for values of $y \geq 1$, so we do not see the boundary effect of the vorticity. Second, we can control the $y dy$ in a *weaker norm* in terms of x -regularity, which is the reason that the second terms in (1.15c) and (1.15d) are measured on the Sobolev scale $r - \frac{1}{2}$. This type of *lagging norm* structure is essential for our scheme of estimates to close, and in particular prevents a further loss of y -weight in the vorticity equation.

Remark 1.2 (Notation). We use heavily the notation \lesssim to suppress universal constants. It is important to emphasize that these universal constants are *independent* of small values of t, ε, δ , where ε is the weight parameter in (1.9) and δ is the parameter appearing in our norms, (1.15a)–(1.15d).

2 The Prandtl-Benjamin-Ono Splitting

2.1 Benjamin-Ono Evolution for A

We need to understand the asymptotic behavior at $y = \infty$ a bit more carefully. First, from (1.6) we obtain

$$u \sim y + A(x, t), \quad \partial_x u \rightarrow \partial_x A, \quad \partial_t u \rightarrow \partial_t A, \quad \partial_y u \rightarrow 1 \quad \text{as } y \rightarrow \infty.$$

The function $v(x, y, t) = -I_y[\partial_x u](x, t) = -\int_0^y \partial_x u(x, z, t) dz$ is expected to grow like y at ∞ , so we let

$$(2.1) \quad v \sim v_0(x, t) + v_1(x, t)y \quad \text{as } y \rightarrow \infty.$$

We now evaluate the original equation (1.5a) at $y = \infty$ and use the above information to obtain

$$(2.2) \quad \partial_t A + (y + A)\partial_x A + (v_0 + v_1 y) + \partial_x |\partial_x| A = 0.$$

Due to the superexponential weights in $y, \rho(t, y)$, appearing in our norm (1.10), we guarantee that the remaining terms in (1.5a) vanish sufficiently rapidly as $y \rightarrow \infty$ so as to not contribute towards (2.2). From here, we extract two equations by matching the orders of y for $y \rightarrow \infty$:

$$(2.3a) \quad \partial_x A + v_1 = 0,$$

$$(2.3b) \quad \partial_t A(x, t) + A \partial_x A + v_0 + \partial_x |\partial_x| A = 0.$$

We now compute the function v_0 in a different fashion:

$$(2.4) \quad v = -I_y[\partial_x u] = -I_y[(\partial_x u - \partial_x A) + \partial_x A] = -y \partial_x A - I_y[\partial_x u - \partial_x A].$$

Here we use the notation in (1.19) for $I_y[\cdot]$. From (2.1) and (2.4) we deduce that $v_1 = -A_x$ and that

$$(2.5) \quad v_0(x, t) = -I_\infty[\partial_x u(x, y, t) - \partial_x A(x, t)].$$

Thus, v_0 can be expressed in terms of u and A . To emphasize this, we will write $v_0 = v_0(u, A)$. Note that we need to understand u (or $\partial_x u$) for all y in order to understand v_0 (it is nonlocal). Inserting back into (2.3b), we obtain the evolution equation for A :

$$(2.6) \quad \partial_t A + A \partial_x A + \partial_x |\partial_x| A = -v_0(u, A) \quad \text{for } x \in \mathbb{R}.$$

2.2 Prandtl-Type Evolution for \bar{w}

The first step towards homogenizing the boundary conditions for u in the equation (1.5a) is to remove the linear profile y and introduce the unknown

$$w = u - y$$

so that (1.6a)–(1.6c) yield

$$w|_{y=0} = w|_{x=-\infty} = w|_{x=\infty} = 0, \quad w|_{y \rightarrow \infty} = A(x, t).$$

We do not need to change v here, as it is given by $-I_y(\partial_x u) = -I_y(\partial_x w)$. It follows that the evolution equation for w is

$$\partial_t w + w \partial_x w + (y \partial_x w + v) + v \partial_y w - \partial_y^2 w + \partial_x |\partial_x| A(x, t) = 0.$$

Summarizing, the unknowns w and v take the place of the usual Prandtl unknowns, and the equation obeyed by w is nothing but the usual Prandtl system with a few extra linear terms:

$$(2.7a) \quad \partial_t w - \partial_{yy} w + w \partial_x w + v \partial_y w + (y \partial_x w + v) + \partial_x |\partial_x| A = 0,$$

$$(2.7b) \quad w|_{y=0} = w|_{x=-\infty} = w|_{x=+\infty} = 0, \quad w|_{y \rightarrow \infty} = A(x, t),$$

$$(2.7c) \quad \partial_x w + \partial_y v = 0, \quad v|_{y=0} = 0 \quad \Rightarrow \quad v = -I_y[\partial_x w],$$

The system (2.7) is of course coupled to the evolution equation for A given in (2.6).

In order to analyze the system (2.7), it is convenient to homogenize the boundary condition of w as $y \rightarrow \infty$. For this purpose we introduce a tangential-frequency-dependent lift of the normal boundary condition, so that we need to write the system obeyed by the Fourier transform in the x -variable of (2.7a)–(2.7c). This yields

$$\begin{aligned} \partial_t w_\xi - \partial_{yy} w_\xi + (w \partial_x w + v \partial_y w)_\xi + i \xi (y w_\xi - I_y[w_\xi]) + i \xi |\xi| A_\xi &= 0, \\ w_\xi|_{y=0} &= 0, \quad w_\xi|_{y \rightarrow \infty} = A_\xi, \quad i \xi w_\xi + \partial_y v_\xi = 0, \\ v_\xi|_{y=0} &= 0 \Rightarrow v_\xi = -i \xi I_y[w_\xi]. \end{aligned}$$

For each $\xi \in \mathbb{R}$ we introduce a lift function θ_ξ (given explicitly in (2.11) below) and define new unknowns,

$$\begin{aligned} \bar{w}_\xi &:= w_\xi(y, t) - A_\xi(t) \theta_\xi(y, t), \\ \bar{v}_\xi &:= v_\xi + i \xi A_\xi I_y[\theta_\xi]. \end{aligned}$$

We derive from (2.7) the evolution for \bar{w}_ξ , which reads

$$(2.9a) \quad \begin{aligned} \partial_t \bar{w}_\xi - \partial_{yy} \bar{w}_\xi + i \xi y \bar{w}_\xi + \mathcal{N}_\xi(\bar{w}, \bar{w}) \\ + \mathcal{L}_\xi(\bar{w}, A) + \mathcal{M}_\xi(\bar{w}, A) + \mathcal{B}_\xi(\bar{w}, A) &= 0, \end{aligned}$$

$$(2.9b) \quad \bar{v}_\xi = -i \xi I_y[\bar{w}_\xi],$$

$$(2.9c) \quad \bar{w}_\xi|_{y=0} = \bar{w}_\xi|_{y \rightarrow \infty} = 0,$$

where in (2.9a) above we have defined

$$(2.10a) \quad \mathcal{N}_\xi(\bar{w}, \bar{w}) := i \int_{\mathbb{R}} (\bar{w}_\eta(\xi - \eta) \bar{w}_{\xi-\eta} - \eta I_y[\bar{w}_\eta] \partial_y \bar{w}_{\xi-\eta}) d\eta$$

$$(2.10b) \quad \begin{aligned} \mathcal{L}_\xi(\bar{w}, A) := i \int_{\mathbb{R}} (\bar{w}_\eta(\xi - \eta) A_{\xi-\eta} \theta_{\xi-\eta} + A_\eta \theta_\eta(\xi - \eta) \bar{w}_{\xi-\eta}) d\eta \\ - i \int_{\mathbb{R}} \eta I_y[\bar{w}_\eta] A_{\xi-\eta} \partial_y \theta_{\xi-\eta} d\eta \end{aligned}$$

$$(2.10c) \quad \mathcal{M}_\xi(\bar{w}, A) := -i \int_{\mathbb{R}} \eta A_\eta I_y[\theta_\eta] \partial_y \bar{w}_{\xi-\eta} d\eta$$

$$(2.10d) \quad \begin{aligned} \mathcal{B}_\xi(\bar{w}, A) := A_\xi(\partial_t - \partial_{yy}) \theta_\xi + (\theta_\xi - 1) \partial_t A_\xi + (\partial_t A_\xi + i \xi |\xi| A_\xi) \\ + i \xi (A_\xi (y \theta_\xi - I_y[\theta_\xi]) - I_y[\bar{w}_\xi]) \\ + i \int_{\mathbb{R}} (A_\eta \theta_\eta(\xi - \eta) A_{\xi-\eta} \theta_{\xi-\eta} \\ - \eta A_\eta I_y[\theta_\eta] A_{\xi-\eta} \partial_y \theta_{\xi-\eta}) d\eta. \end{aligned}$$

At this stage, we make the following choice for the lift function:

$$(2.11) \quad \theta_\xi(y, t) = 1 - e^{-\frac{y^2(\xi)^2}{2(1+t/\varepsilon)}},$$

where $\varepsilon > 0$ is a parameter to be chosen later. We emphasize here that $\theta_\xi(y, 0)$ does not depend on ε , which is crucial for the proof. Informally, ε will be selected small

relative to universal constants, and relative to the size of the initial data (which is independent of ε). The time of existence T_* will be selected small relative to ε , and in particular we restrict ourselves to $T_* \leq \varepsilon$, so that the quotient t/ε is always bounded by 1.

It is also useful to denote

$$(2.12) \quad \begin{aligned} c_{\theta,\xi}(t) &:= I_\infty[1 - \theta_\xi](t) = \int_0^\infty (1 - \theta_\xi(y, t)) dy \\ &= \int_0^\infty e^{-\frac{y^2 \langle \xi \rangle^2}{1+t/\varepsilon}} dy = \frac{\sqrt{\pi(1+t/\varepsilon)}}{2\langle \xi \rangle}. \end{aligned}$$

With θ_ξ as defined by (2.11), we identify the function $v_0(\bar{w}, A)$ from (2.5) as

$$(2.13) \quad \begin{aligned} (v_0(\bar{w}, A))_\xi(t) &= -i\xi \int_0^\infty (\bar{w}_\xi(y, t) - A_\xi(t)(1 - \theta_\xi(y, t))) dy \\ &= -i\xi I_\infty[\bar{w}_\xi](x, t) + i\xi c_{\theta,\xi}(t) A_\xi(t). \end{aligned}$$

With this notation, we return to (2.6), which in view of (2.13) becomes

$$(2.14) \quad \partial_t A_\xi + i\xi c_{\theta,\xi} A_\xi - i\xi I_\infty[\bar{w}_\xi] + i\xi |\xi| A_\xi = -i \int_{\mathbb{R}} A_\eta(\xi - \eta) A_{\xi-\eta} d\eta.$$

We notice that A enters the evolution equation for \bar{w} only through the coefficients of \mathcal{B} , \mathcal{L} , and \mathcal{M} , whereas $\partial_x \bar{w}$ enters the evolution equation for A only through its vertical mean encoded in $v_0(\bar{w}, A)$.

Lastly, using that A obeys the Benjamin-Ono equation (2.14), and using that $c_{\theta,\xi} = I_\infty[1 - \theta_\xi]$, we may rewrite the forcing term $\mathcal{B}_\xi(\bar{w}, A)$ given in (2.10d) as

$$\begin{aligned} \mathcal{B}_\xi(\bar{w}, A) &= A_\xi(\partial_t - \partial_{yy})\theta_\xi + (\theta_\xi - 1)\partial_t A_\xi + i\xi(I_\infty[\bar{w}_\xi] - I_y[\bar{w}_\xi]) \\ &\quad + i\xi A_\xi(y(\theta_\xi - 1) - (I_\infty[1 - \theta_\xi] - I_y[1 - \theta_\xi])) \\ &\quad + i \int_{\mathbb{R}} ((\xi - \eta) A_\eta A_{\xi-\eta}(\theta_\eta \theta_{\xi-\eta} - 1) - \eta A_\eta A_{\xi-\eta} I_y[\theta_\eta] \partial_y \theta_{\xi-\eta}) \end{aligned}$$

Because of our choice of θ_ξ , every single term in $\mathcal{B}_\xi(\bar{v}, A)$ decays to 0 as $y \rightarrow \infty$. Throughout the rest of the paper, we use the formulation (2.15) of the \mathcal{B}_ξ term (instead of (2.10d)).

3 Energy Estimates and Proof of the Main Theorem

In this section we give the energy estimates that prove Theorem 1.1 under the assumption that the nonlinear terms may be bounded suitably (cf. Lemma 3.1). These terms are then estimated in Section 4.

3.1 Energy Inequality for A

In view of (1.12) we take the product of equation (2.14) with the complex conjugate \overline{A}_ξ and integrate in ξ against $e^{2\tau|\xi|}\langle\xi\rangle^{2r}\langle\xi\rangle$ to obtain

$$\begin{aligned}
 \langle \partial_t A, A \rangle_{\tilde{\tau}, r} &= -i \int_{\mathbb{R}} e^{2\tau|\xi|} \langle\xi\rangle^{2r} \\
 &\quad \times \left(\xi(c_{\theta,\xi} + |\xi|) |A_\xi|^2 - \xi I_\infty[\bar{w}_\xi] \overline{A}_\xi \right. \\
 &\quad \left. + \overline{A}_\xi \int_{\mathbb{R}} A_\eta(\xi - \eta) A_{\xi-\eta} d\eta \right) d\xi \\
 &= T_{\mathcal{A},1} - T_{\mathcal{A},2},
 \end{aligned} \tag{3.1}$$

where we have defined

$$T_{\mathcal{A},1} = \langle I_\infty[\partial_x \bar{w}], A \rangle_{\tilde{\tau}, r} = \int_{\mathbb{R}} i\xi I_\infty[\bar{w}_\xi] \overline{A}_\xi e^{2\tau|\xi|} \langle\xi\rangle^{2r} d\xi \tag{3.2a}$$

$$T_{\mathcal{A},2} = \langle A \partial_x A, A \rangle_{\tilde{\tau}, r} = \int_{\mathbb{R}} \int_{\mathbb{R}} i(\xi - \eta) A_\eta A_{\xi-\eta} \overline{A}_\xi e^{2\tau|\xi|} \langle\xi\rangle^{2r} d\eta d\xi. \tag{3.2b}$$

In (3.3) we have used that A is real-valued, so that $A_{-\xi} = \overline{A}_\xi$ and that $c_{\theta,\xi} = c_{\theta,-\xi} \in \mathbb{R}$ (cf. (2.12)). Combining (1.12) with (3.1) we arrive at

$$\frac{d}{dt} \|A\|_{\tilde{\tau}, r}^2 + (-\dot{\tau}) \|\partial_x A\|_{\tilde{\tau}, r}^{1/2} \leq |T_{\mathcal{A},1}| + |T_{\mathcal{A},2}|, \tag{3.3}$$

which is the desired energy inequality for the analytic norm of A . The terms on the right side of (3.3) are estimated in Lemma 3.1, bounds (3.15).

3.2 Energy Inequality for \bar{w}

In view of (1.11) we need to compute $\langle \partial_t \bar{w} + \bar{w}(\partial_t \log \rho), \bar{w} \rangle_{\tau, r}$. Note that by definition (1.9) we have

$$\partial_t(\log \rho) = -\frac{y^2}{8\varepsilon(1+t/\varepsilon)^2}, \tag{3.4}$$

and thus we obtain the damping weight-gaining term

$$\langle \bar{w} \partial_t(\log \rho), \bar{w} \rangle_{\tau, r} = -\frac{1}{8\varepsilon(1+t/\varepsilon)^2} \|y \bar{w}\|_{\tau, r}^2. \tag{3.5}$$

In order to compute $\langle \partial_t \bar{w}, \bar{w} \rangle_{\tau, r}$, we multiply (2.9a) by $\rho^2 \bar{w}_\xi e^{2\tau|\xi|} \langle \xi \rangle^{2r}$ and integrate over $(\xi, y) \in \mathbb{R} \times [0, \infty)$ to obtain

$$\begin{aligned}
& \langle \partial_t \bar{w}, \bar{w} \rangle_{\tau, r} \\
&= -\|\partial_y \bar{w}\|_{\tau, r}^2 - \frac{1}{2(1+t/\varepsilon)} \int_{\mathbb{R}} \int_0^\infty \partial_y \bar{w}_\xi y \bar{w}_\xi \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi \\
(3.6) \quad & - i \int_{\mathbb{R}} \int_0^\infty y \xi |\bar{w}_\xi|^2 \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi - T_{\mathcal{N}} - T_{\mathcal{L}} - T_{\mathcal{M}} - T_{\mathcal{B}} \\
&\leq -\frac{1}{2} \|\partial_y \bar{w}\|_{\tau, r}^2 + \frac{1}{8(1+t/\varepsilon)^2} \|y \bar{w}\|_{\tau, r}^2 + |T_{\mathcal{N}}| \\
&\quad + |T_{\mathcal{L}}| + |T_{\mathcal{M}}| + |T_{\mathcal{B}}|
\end{aligned}$$

where we have used that by oddness in ξ we have

$$i \int_{\mathbb{R}} \int_0^\infty \xi |\bar{w}_\xi|^2 \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\eta d\xi = 0,$$

and we have denoted

$$(3.7a) \quad T_{\mathcal{N}} = \int_{\mathbb{R}} \int_0^\infty \mathcal{N}_\xi(\bar{w}, \bar{w}) \bar{w}_\xi \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi$$

$$(3.7b) \quad T_{\mathcal{L}} = \int_{\mathbb{R}} \int_0^\infty \mathcal{L}_\xi(\bar{w}, A) \bar{w}_\xi \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi$$

$$(3.7c) \quad T_{\mathcal{M}} = \int_{\mathbb{R}} \int_0^\infty \mathcal{M}_\xi(\bar{w}, A) \bar{w}_\xi \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi$$

$$(3.7d) \quad T_{\mathcal{B}} = \int_{\mathbb{R}} \int_0^\infty \mathcal{B}_\xi(\bar{w}, A) \bar{w}_\xi \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi$$

with \mathcal{N}_ξ , \mathcal{L}_ξ , \mathcal{M}_ξ , and \mathcal{B}_ξ as defined in (2.10). Combining (1.11) with (3.5)–(3.6) we arrive at

$$\begin{aligned}
(3.8) \quad & \frac{d}{2dt} \|\bar{w}\|_{\tau, r}^2 + (-i) \|\partial_x|^{1/2} \bar{w}\|_{\tau, r}^2 + \frac{1}{2} \|\partial_y \bar{w}\|_{\tau, r}^2 + \frac{1-\varepsilon}{8\varepsilon(1+t/\varepsilon)^2} \|y \bar{w}\|_{\tau, r}^2 \\
& \leq |T_{\mathcal{N}}| + |T_{\mathcal{L}}| + |T_{\mathcal{M}}| + |T_{\mathcal{B}}|
\end{aligned}$$

which is the desired energy inequality for the analytic norm of \bar{w} . The four error terms on the right side of (3.8) are estimated in Lemma 3.1, bounds (3.16).

3.3 Energy Inequality for $\partial_y \bar{w}$

In order to overcome a loss of y weight in the term $T_{\mathcal{M}}$, we need to also consider the evolution of the normalized vorticity $\partial_y \bar{w}$. We apply ∂_y to (2.9a) to obtain

$$\begin{aligned}
(3.9) \quad & \partial_t \partial_y \bar{w}_\xi - \partial_y^2 \partial_y \bar{w}_\xi + i \xi \bar{w}_\xi + i y \xi \partial_y \bar{w}_\xi \\
&= -(\partial_y \mathcal{N}_\xi + \partial_y \mathcal{L}_\xi + \partial_y \mathcal{M}_\xi + \partial_y \mathcal{B}_\xi).
\end{aligned}$$

Note that some of the terms on the right side of (3.9) have cancelations in them. Using that $I_y[\partial_y \bar{w}_\xi] = \bar{w}_\xi = \partial_y I_y[\bar{w}_\xi]$, and upon noting symmetries $\eta \leftrightarrow \xi - \eta$ in the below integrals, we may rewrite

$$(3.10a) \quad \partial_y \mathcal{N}_\xi(\bar{w}, \bar{w}) = \mathcal{N}_\xi(\bar{w}, \partial_y \bar{w}) \\ = i \int_{\mathbb{R}} (\bar{w}_\eta(\xi - \eta) \partial_y \bar{w}_{\xi-\eta} - \eta I_y[\bar{w}_\eta] \partial_{yy} \bar{w}_{\xi-\eta}) d\eta$$

$$(3.10b) \quad \partial_y \mathcal{L}_\xi(\bar{w}, \bar{w}) = i \int_{\mathbb{R}} (\partial_y \bar{w}_\eta(\xi - \eta) A_{\xi-\eta} \theta_{\xi-\eta} + \bar{w}_\eta(\xi - \eta) A_{\xi-\eta} \partial_y \theta_{\xi-\eta} \\ + A_\eta \theta_\eta(\xi - \eta) \partial_y \bar{w}_{\xi-\eta} - \eta I_y[\bar{w}_\eta] A_{\xi-\eta} \partial_{yy} \theta_{\xi-\eta}) d\eta$$

$$(3.10c) \quad \partial_y \mathcal{M}_\xi(\bar{w}, A) = -i \int_{\mathbb{R}} (\eta A_\eta \theta_\eta \partial_y \bar{w}_{\xi-\eta} + \eta A_\eta I_y[\theta_\eta] \partial_{yy} \bar{w}_{\xi-\eta}) d\eta$$

$$(3.10d) \quad \partial_y \mathcal{B}_\xi(\bar{w}, A) = A_\xi(\partial_t - \partial_{yy}) \partial_y \theta_\xi \\ + \partial_y \theta_\xi \partial_t A_\xi - i \xi \bar{w}_\xi + i \xi A_\xi y \partial_y \theta_\xi \\ + i \int_{\mathbb{R}} ((\xi - \eta) A_\eta A_{\xi-\eta} \theta_\eta \partial_y \theta_{\xi-\eta} \\ - \eta A_\eta A_{\xi-\eta} I_y[\theta_\eta] \partial_{yy} \theta_{\xi-\eta}).$$

It turns out that we only need information on the vorticity $\partial_y \bar{w}$ away from $\partial\mathbb{H} = \{y = 0\}$, and for this purpose we introduce a cutoff function, $\chi = \chi(y)$, such that $0 \leq \chi' \leq 1$, satisfying

$$(3.11) \quad \chi(y) = \begin{cases} 0 & \text{on } y \in [0, 1), \\ 1 & \text{on } y \geq 6. \end{cases}$$

Note that χ is independent of time. Our goal is to estimate $\|\chi \partial_y \bar{w}\|_{\tau, r-1/2}$. The shift in Sobolev regularity of for the vorticity norm, i.e., the change $r \mapsto r - 1/2$, is essential for the energy estimate to close.

Using (1.11) with $f = \chi \partial_y \bar{w}$, property (3.4) of the weight ρ , and the evolution equation (3.9), we obtain

$$(3.12) \quad \begin{aligned} & \frac{d}{2dt} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2}^2 + (-\dot{t}) \|\chi |\partial_x|^{1/2} \partial_y \bar{w}\|_{\tau, r-1/2}^2 \\ &= \langle \chi \partial_t \partial_y \bar{w}, \chi \partial_y \bar{w} \rangle_{\tau, r-1/2} + \langle \chi \partial_y \bar{w} \partial_t (\log \rho), \chi \partial_y \bar{w} \rangle_{\tau, r-1/2} \\ &= -\frac{1}{8\varepsilon(1+t/\varepsilon)^2} \|y \chi \partial_y \bar{w}\|_{\tau, r-1/2}^2 - \|\chi \partial_{yy} \bar{w}\|_{\tau, r-1/2}^2 \\ & \quad - \int_{\mathbb{R}} \int_0^\infty \partial_{yy} \bar{w}_\xi \partial_y (\chi^2 \rho^2) \overline{\partial_y \bar{w}_\xi} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} dy d\xi - \end{aligned}$$

$$\begin{aligned}
& -i \int_{\mathbb{R}} \int_0^\infty \xi \bar{w}_\xi \chi^2 \rho^2 \bar{\partial}_y \bar{w}_\xi \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} dy d\xi \\
& - T_{\partial_y \mathcal{N}} - T_{\partial_y \mathcal{L}} - T_{\partial_y \mathcal{M}} - T_{\partial_y \mathcal{B}}.
\end{aligned}$$

Here we have used the cancellation property

$$i \int_{\mathbb{R}} \int_0^\infty y \xi |\partial_y \bar{w}_\xi|^2 \chi^2 \rho^2 \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} dy d\xi = 0$$

and have denoted

$$(3.13a) \quad T_{\partial_y \mathcal{N}} = \int_{\mathbb{R}} \int_0^\infty \partial_y \mathcal{N}_\xi(\bar{w}, \bar{w}) \bar{\partial}_y \bar{w}_\xi \chi^2 \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r-1} dy d\xi,$$

$$(3.13b) \quad T_{\partial_y \mathcal{L}} = \int_{\mathbb{R}} \int_0^\infty \partial_y \mathcal{L}_\xi(\bar{w}, A) \bar{\partial}_y \bar{w}_\xi \chi^2 \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r-1} dy d\xi,$$

$$(3.13c) \quad T_{\partial_y \mathcal{M}} = \int_{\mathbb{R}} \int_0^\infty \partial_y \mathcal{M}_\xi(\bar{w}, A) \bar{\partial}_y \bar{w}_\xi \chi^2 \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r-1} dy d\xi,$$

$$(3.13d) \quad T_{\partial_y \mathcal{B}} = \int_{\mathbb{R}} \int_0^\infty \partial_y \mathcal{B}_\xi(\bar{w}, A) \bar{\partial}_y \bar{w}_\xi \chi^2 \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r-1} dy d\xi,$$

with $\partial_y \mathcal{N}_\xi$, $\partial_y \mathcal{L}_\xi$, $\partial_y \mathcal{M}_\xi$, and $\partial_y \mathcal{B}_\xi$, as defined in (3.10). From (3.12), the Cauchy-Schwarz inequality, and the definitions of χ and ρ we obtain

$$\begin{aligned}
& \frac{d}{dt} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2}^2 + (-\dot{\tau}) \|\chi |\partial_x|^{1/2} \partial_y \bar{w}\|_{\tau, r-1/2}^2 \\
& + \frac{1-\varepsilon}{8\varepsilon(1+t/\varepsilon)^2} \|y \chi \partial_y \bar{w}\|_{\tau, r-1/2}^2 + \frac{1}{4} \|\chi \partial_{yy} \bar{w}\|_{\tau, r-1/2}^2 \\
& \leq \|\partial_y \bar{w}\|_{\tau, r}^2 + \|\bar{w}\|_{\tau, r} \|\partial_y \bar{w}\|_{\tau, r} \\
& + |T_{\partial_y \mathcal{N}}| + |T_{\partial_y \mathcal{L}}| + |T_{\partial_y \mathcal{M}}| + |T_{\partial_y \mathcal{B}}|.
\end{aligned} \tag{3.14}$$

Here we have used that $\langle \xi \rangle \geq 1$. The remaining four error terms on the right side of (3.14) are bounded in Lemma 3.1, estimate (3.17).

3.4 Nonlinear Estimates

The following lemma summarizes the available estimates for the error terms in (3.3), (3.8), and (3.14).

LEMMA 3.1 (Main nonlinear lemma). *Assume that $r > 2$ and that $t \leq \varepsilon$. For the error terms in the A energy estimate (3.3), we have*

$$(3.15a) \quad |T_{\mathcal{A},1}| \lesssim \|\partial_x|^{1/2} \bar{w}\|_{\tau, r} \|\partial_x|^{1/2} A\|_{\tilde{\tau}, r},$$

$$(3.15b) \quad |T_{\mathcal{A},2}| \lesssim \|\partial_x|^{1/2} A\|_{\tilde{\tau}, r}^2 \|A\|_{\tilde{\tau}, r} + \|\partial_x|^{1/2} A\|_{\tilde{\tau}, r} \|A\|_{\tilde{\tau}, r}^2,$$

for the error terms in the \bar{w} energy estimate (3.8); it holds that

$$(3.16a) \quad |T_{\mathcal{N}}| \lesssim \|\partial_x|^{1/2}\bar{w}\|_{\tau,r}\|\langle\partial_x\rangle^{1/2}\bar{w}\|_{\tau,r}\|\partial_y\bar{w}\|_{\tau,r},$$

$$(3.16b) \quad |T_{\mathcal{L}}| \lesssim \|\partial_x|^{1/2}\bar{w}\|_{\tau,r}\|\bar{w}\|_{\tau,r}\|\partial_x|^{1/2}A\|_{\tilde{\tau},r} \\ + (\|\bar{w}\|_{\tau,r}^2 + \|\partial_x|^{1/2}\bar{w}\|_{\tau,r}^2)\|A\|_{\tilde{\tau},r},$$

$$(3.16c) \quad |T_{\mathcal{M}}| \lesssim \|\langle\partial_x\rangle^{1/2}\bar{w}\|_{\tau,r}\|\langle\partial_x\rangle^{1/2}A\|_{\tilde{\tau},r}(\|\partial_y\bar{w}\|_{\tau,r} + \|y\chi\partial_y\bar{w}\|_{\tau,r-1/2}),$$

$$(3.16d) \quad |T_{\mathcal{B}}| \lesssim \frac{1}{\varepsilon}\|\bar{w}\|_{\tau,r}\|A\|_{\tilde{\tau},r} + \|\langle\partial_x\rangle^{1/2}A\|_{\tau,r}\|\partial_y\bar{w}\|_{\tau,r} + \|\partial_x|^{1/2}\bar{w}\|_{\tau,r}^2 \\ + \|\partial_x|^{1/2}\bar{w}\|_{\tau,r}\|\bar{w}\|_{\tau,r} + \|\langle\partial_x\rangle^{1/2}\bar{w}\|_{\tau,r}\|\partial_x|^{1/2}A\|_{\tilde{\tau},r}\|A\|_{\tilde{\tau},r},$$

while for the error terms in the energy estimate (3.14) for $\partial_y\bar{w}$ the estimates

$$(3.17a) \quad |T_{\partial_y\mathcal{N}}| \lesssim \|\langle\partial_x\rangle^{1/2}\chi\partial_y\bar{w}\|_{\tau,r-1/2}^2\|\partial_y\bar{w}\|_{\tau,r} \\ + \|\chi\partial_{yy}\bar{w}\|_{\tau,r-1/2}\|\chi\partial_y\bar{w}\|_{\tau,r-1/2}\|\langle\partial_x\rangle^{1/2}\bar{w}\|_{\tau,r},$$

$$(3.17b) \quad |T_{\partial_y\mathcal{L}}| \lesssim \|\chi\partial_y\bar{w}\|_{\tau,r-1/2}^2\|\partial_x|^{1/2}A\|_{\tilde{\tau},r} \\ + \|\chi\partial_y\bar{w}\|_{\tau,r-1/2}\|\langle\partial_x\rangle^{1/2}\bar{w}\|_{\tau,r}\|A\|_{\tilde{\tau},r} \\ + \|\chi\partial_x|^{1/2}\partial_y\bar{w}\|_{\tau,r-1/2}\|\chi\langle\partial_x\rangle^{1/2}\partial_y\bar{w}\|_{\tau,r-1/2}\|A\|_{\tilde{\tau},r},$$

$$(3.17c) \quad |T_{\partial_y\mathcal{M}}| \lesssim \|\chi\partial_y\bar{w}\|_{\tau,r-1/2}^2\|\partial_x|^{1/2}A\|_{\tilde{\tau},r} \\ + \|y\chi\partial_y\bar{w}\|_{\tau,r-1/2}\|\chi\partial_{yy}\bar{w}\|_{\tau,r-1/2}\|\partial_x|^{1/2}A\|_{\tilde{\tau},r},$$

$$(3.17d) \quad |T_{\partial_y\mathcal{B}}| \lesssim \frac{1}{\varepsilon}\|A\|_{\tilde{\tau},r}\|\chi\partial_y\bar{w}\|_{\tau,r-1/2} \\ + (\|\langle\partial_x\rangle^{1/2}\bar{w}\|_{\tau,r} + \|A\|_{\tilde{\tau},r}^2)\|\chi\partial_y\bar{w}\|_{\tau,r-1/2} \\ + \|\langle\partial_x\rangle^{1/2}A\|_{\tilde{\tau},r}\|\chi\partial_y\bar{w}\|_{\tau,r-1/2} \\ + \|\partial_x|^{1/2}A\|_{\tilde{\tau},r}\|A\|_{\tilde{\tau},r}\|\chi\partial_y\bar{w}\|_{\tau,r-1/2},$$

hold. The implicit constants in the above estimates are independent of τ , t , and ε (they depend solely on r).

The proof of Lemma 3.1 is given in Section 4 below. Assuming that this lemma is established, we continue with the proof of the main theorem. Before doing so, we summarize the bounds proven in Lemma 3.1 using the total norms defined in (1.15) above. Estimate (3.15) shows that

$$(3.18) \quad (|T_{\mathcal{A},1}| + |T_{\mathcal{A},2}|) \lesssim (1 + \|(\bar{w}, A)\|_{X_\tau})\|(\bar{w}, A)\|_{Y_\tau}^2 + \|(\bar{w}, A)\|_{X_\tau}^3.$$

The bounds (3.16) and an ε -Young inequality for the second term in (3.16d) yields

$$(3.19) \quad |T_{\mathcal{N}}| + |T_{\mathcal{L}}| + |T_{\mathcal{M}}| + |T_{\mathcal{B}}| \\ \lesssim \|(\bar{w}, A)\|_{Y_\tau}^2(\varepsilon^{-1} + \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta\|\bar{w}\|_{H_\tau}) \\ + \|(\bar{w}, A)\|_{X_\tau}^2(\varepsilon^{-1} + \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta\|\bar{w}\|_{H_\tau}) + \varepsilon\|\bar{w}\|_{Z_\tau}^2,$$

while inequality (3.17) implies

$$\begin{aligned}
 (3.20) \quad & \delta^{-2}(|T_{\partial_y \mathcal{N}}| + |T_{\partial_y \mathcal{L}}| + |T_{\partial_y \mathcal{M}}| + |T_{\partial_y \mathcal{B}}|) \\
 & \lesssim \|(\bar{w}, A)\|_{Y_\tau}^2 (\delta^{-1} + \|\bar{w}\|_{Z_\tau} + (1 + \delta^{-1}) \|(\bar{w}, A)\|_{X_\tau}) \\
 & \quad + \|\bar{w}\|_{Z_\tau} \|\bar{w}\|_{H_\tau} \|(\bar{w}, A)\|_{Y_\tau} \\
 & \quad + \|(\bar{w}, A)\|_{X_\tau}^2 (\varepsilon^{-1} \delta^{-1} + \delta^{-1} + \|\bar{w}\|_{Z_\tau} + (1 + \delta^{-1}) \|(\bar{w}, A)\|_{X_\tau}).
 \end{aligned}$$

The implicit constants in (3.18), (3.19), and (3.20) only depend on r , since we have assumed $t, \varepsilon \leq 1$.

3.5 Proof of the Main Theorem

In order to prove Theorem 1.1 we couple together the energy estimates (3.3), (3.8), and (3.14) multiplied by the small factor $\delta^{-2} \leq 1$, together with the error estimates (3.18), (3.19), and (3.20), to obtain, for universal constants C_0, \tilde{C}_0 ,

$$\begin{aligned}
 (3.21) \quad & \frac{d}{dt} \|(\bar{w}, A)\|_{X_\tau}^2 + (-\dot{\tau}) \|(\bar{w}, A)\|_{Y_\tau}^2 + \frac{1}{8} \|\bar{w}\|_{Z_\tau}^2 + \frac{1-\varepsilon}{8\varepsilon(1+t/\varepsilon)^2} \|\bar{w}\|_{H_\tau}^2 \\
 & \leq C_0 (\delta^{-2} \|\bar{w}\|_{Z_\tau}^2 + \|\bar{w}\|_{Z_\tau} \|\bar{w}\|_{H_\tau} \|(\bar{w}, A)\|_{Y_\tau} \\
 & \quad + \|(\bar{w}, A)\|_{Y_\tau}^2 \\
 & \quad \times ((\varepsilon^{-1} + \delta^{-1}) + \|\bar{w}\|_{Z_\tau} + (1 + \delta^{-1}) \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau})) \\
 & \quad + \|(\bar{w}, A)\|_{X_\tau}^2 \\
 & \quad \times ((\varepsilon^{-1} + \delta^{-1}) + \|\bar{w}\|_{Z_\tau} + (1 + \delta^{-1}) \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau}) \\
 & \leq \tilde{C}_0 \delta^{-2} \|\bar{w}\|_{Z_\tau}^2 + \frac{1}{100} \|\bar{w}\|_{Z_\tau}^2 + \tilde{C}_0 \|\bar{w}\|_{H_\tau}^2 \|(\bar{w}, A)\|_{Y_\tau}^2 \\
 & \quad + \tilde{C}_0 \|(\bar{w}, A)\|_{Y_\tau}^2 ((\varepsilon^{-1} + \delta^{-1}) + \|\bar{w}\|_{Z_\tau} + (1 + \delta^{-1}) \\
 & \quad \times \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau}) \\
 & \quad + \tilde{C}_0 \|(\bar{w}, A)\|_{X_\tau}^2 \\
 & \quad \times ((\varepsilon^{-1} + \delta^{-1}) + \|\bar{w}\|_{Z_\tau} + (1 + \delta^{-1}) \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau}).
 \end{aligned}$$

To go from the second inequality to the final inequality, we have simply used Young's inequality for products to split the trilinear term and denoted by \tilde{C}_0 the resulting (universal) constant. The constant \tilde{C}_0 is independent of the parameters δ and ε . We now take $\delta \gg 1$ so as to ensure that

$$(3.22) \quad \frac{1}{100} + \frac{\tilde{C}_0}{\delta^2} \leq \frac{1}{16},$$

upon which the first two $\|\bar{w}\|_{Z_\tau}^2$ terms in (3.21) can be absorbed to the left-hand side. This yields the bound

$$\begin{aligned} \frac{d}{dt} \|(\bar{w}, A)\|_{X_\tau}^2 + (-\dot{\tau}) \|(\bar{w}, A)\|_{Y_\tau}^2 + \frac{1}{16} \|\bar{w}\|_{Z_\tau}^2 + \frac{1-\varepsilon}{8\varepsilon(1+t/\varepsilon)^2} \|\bar{w}\|_{H_\tau}^2 \\ \leq \widetilde{C}_1 \|(\bar{w}, A)\|_{Y_\tau}^2 (\|\bar{w}\|_{H_\tau}^2 + (\varepsilon^{-1} + \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau})) \\ + \widetilde{C}_1 \|(\bar{w}, A)\|_{X_\tau}^2 (\varepsilon^{-1} + \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau}), \end{aligned}$$

for another universal constant \widetilde{C}_1 , which is again independent of ε , and large values of δ .

By multiplying through by a sufficiently large universal constant and taking $\varepsilon \leq 1/64$, we obtain

$$\begin{aligned} \frac{d}{dt} \|(\bar{w}, A)\|_{X_\tau}^2 + (-\dot{\tau}) \|(\bar{w}, A)\|_{Y_\tau}^2 + \frac{1}{16} \|\bar{w}\|_{Z_\tau}^2 + \frac{1}{64\varepsilon} \|\bar{w}\|_{H_\tau}^2 \\ \leq \Gamma_1(t) \|(\bar{w}, A)\|_{Y_\tau}^2 + \Gamma_2(t) \|(\bar{w}, A)\|_{X_\tau}^2, \end{aligned}$$

where we have defined

$$(3.23a) \quad \Gamma_1(t) := C_1 \left(\|\bar{w}\|_{H_\tau}^2 + \left(\varepsilon^{-1} + \frac{1}{4} \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau} \right) \right)$$

$$(3.23b) \quad \Gamma_2(t) := C_2 \left(\varepsilon^{-1} + \frac{1}{4} \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau} \right).$$

for some universal constants $C_1, C_2 \geq 1$. We now make the selection of

$$\dot{\tau} = -\Gamma_1 - 1$$

from which the following integral identity and inequality follow:

$$(3.24) \quad \begin{aligned} \tau(t) &= \tau_0 - t - \int_0^t \Gamma_1(s) ds, \\ E(t) &\leq E(0) + \left| \int_0^t \Gamma_2(s) \|(\bar{w}, A)\|_{X_\tau(s)}^2 ds \right|, \end{aligned}$$

where the total analytic energy, $E(t)$, has been defined in (1.16). The main result will now follow from the following:

LEMMA 3.2. *Fix the parameter δ according to (3.22). There exist universal constants C_1, C_2 so that if the parameters ε and the time of existence T_* satisfy simultaneously the inequalities*

$$(3.25a) \quad \frac{3}{2} C_1 \varepsilon E_0 \leq \frac{\tau_0}{8},$$

$$(3.25b) \quad T_*^{\frac{1}{4}} C_2 (1 + \delta) (T_*^{\frac{1}{2}} \varepsilon^{-1} + 1) \leq 1,$$

$$(3.25c) \quad T_*^{\frac{1}{2}} C_1 (1 + \delta) (T_*^{\frac{1}{2}} + 1) \left(\frac{3}{2} E_0 \right)^{\frac{1}{2}} \leq \frac{\tau_0}{8},$$

$$(3.25d) \quad T_* \leq \frac{\tau_0}{4},$$

$$(3.25e) \quad T_*^{\frac{1}{4}} (1 + E_0^{\frac{1}{4}}) \leq \frac{1}{16}.$$

Then $|E(t)| \leq \frac{3}{2}E(0)$ for all $t \in [0, T_*]$ and $\tau(t) \geq \frac{\tau_0}{2}$ for all $t \in [0, T_*]$. Moreover, it is possible to select the parameter ε and the time of existence T_* so as to achieve the inequalities (3.25).

PROOF OF LEMMA 3.2. We first establish, using the definition of Γ_2 in (3.23b), the following estimate:

$$\begin{aligned} \left| \int_0^{T_*} \Gamma_2(s) ds \right| &\leq C_2 \int_0^{T_*} (\varepsilon^{-1} + \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau}) ds \\ &\leq C_2 T_* \varepsilon^{-1} + C_2 T_*^{\frac{1}{2}} \|\bar{w}\|_{Z_\tau} \|_{L^2(0, T_*)} \\ &\quad + C_2 T_* \sup_{t \in [0, T_*]} \|(\bar{w}, A)\|_{X_\tau} + \delta T_*^{\frac{1}{2}} \|\bar{w}\|_{H_\tau} \|_{L^2(0, T_*)} \\ &\leq C_2 (1 + \delta) (T_* \varepsilon^{-1} + T_*^{\frac{1}{2}}) (1 + E(T_*)^{\frac{1}{2}}). \end{aligned}$$

Next, we establish using (3.23a),

$$\begin{aligned} (3.26) \quad &\left| \int_0^{T_*} \Gamma_1(s) ds \right| \\ &\leq C_1 \left| \int_0^{T_*} \left(\|\bar{w}\|_{H_\tau}^2 \right. \right. \\ &\quad \left. \left. + \left(\varepsilon^{-1} + \frac{1}{4} \|\bar{w}\|_{Z_\tau} + \|(\bar{w}, A)\|_{X_\tau} + \delta \|\bar{w}\|_{H_\tau} \right) \right) ds \right| \\ &\leq C_1 \varepsilon \left\| \frac{1}{64\varepsilon} \|\bar{w}\|_{H_\tau} \right\|_{L^2(0, T_*)}^2 \\ &\quad + C_1 \left(T_* \varepsilon^{-1} + T_*^{\frac{1}{2}} \|\bar{w}\|_{Z_\tau} \|_{L^2(0, T_*)} + T_* \|(\bar{w}, A)\|_{X_\tau} \|_{L^\infty(0, T_*)} \right. \\ &\quad \left. + \delta T_*^{\frac{1}{2}} \|\bar{w}\|_{H_\tau} \|_{L^2(0, T_*)} \right) \\ &\leq C_1 \varepsilon E(T_*) + C_1 (1 + \delta) (T_* \varepsilon^{-1} + T_*^{\frac{1}{2}}) E(T_*)^{\frac{1}{2}}. \end{aligned}$$

We now select ε via

$$(3.27) \quad \frac{3}{2} C_1 \varepsilon E_0 < \frac{\tau_0}{8}.$$

Once ε has been selected (*depending only on the initial datum*) through (3.27), we pick T_* depending on δ and ε in order to satisfy simultaneously the two inequalities

$$(3.28) \quad \begin{aligned} C_2(1 + \delta)(T_*^{\frac{3}{4}}\varepsilon^{-1} + T_*^{\frac{1}{4}}) &\leq 1, \\ C_1(1 + \delta)(T_*\varepsilon^{-1} + T_*^{\frac{1}{2}})\left(\frac{3}{2}E_0\right)^{\frac{1}{2}} &< \frac{\tau_0}{8}. \end{aligned}$$

Such a choice is possible because every parameter other than T_* in (3.28) has been fixed already, so we can take T_* small enough so as to achieve (3.28).

Inserting the first inequality in (3.28) into the second integral inequality in (3.24), we obtain the nonlinear inequality

$$\begin{aligned} E(T_*) &\leq E_0 + \sup_{t \in [0, T_*]} \|(\bar{w}, A)\|_{X_{\tau(s)}}^2 \int_0^{T_*} \Gamma_2(s) ds \\ &\leq E_0 + C_2(1 + \delta)(T_*\varepsilon^{-1} + T_*^{\frac{1}{2}})(1 + E(T_*^{\frac{1}{2}})E(T_*)) \\ &\leq E_0 + T_*^{\frac{1}{4}}E(T_*) + T_*^{\frac{1}{4}}E(T_*^{\frac{3}{2}}), \end{aligned}$$

which implies the desired bound, $E(T_*) \leq \frac{3}{2}E_0$ by selecting T_* small enough to obey

$$T_*^{\frac{1}{4}}(1 + E_0^{\frac{1}{4}}) \leq \frac{1}{16}.$$

We subsequently insert the inequality $E(T_*) \leq \frac{3}{2}E_0$ together with the two inequalities (3.27) and the second inequality of (3.28) into (3.26) so as to ensure for all $t \in [0, T_*]$

$$(3.29) \quad |\tau(t)| \geq \tau_0 - |t| - \left| \int_0^t \Gamma_1(s) ds \right| \geq \tau_0 - T_* - \frac{\tau_0}{8} - \frac{\tau_0}{8} \geq \frac{\tau_0}{2},$$

where we have appealed to the last inequality on T_* , (3.25d), to establish the final inequality in (3.29). In summary, one first chooses δ , then ε , and T_* is picked last. \square

Remark 3.3 (Construction of the solution). The proof of Theorem 1.1 presented above only consists of a priori estimates. Since the spaces in which the solution is shown to remain bounded decrease with time, the construction of solutions is not trivial (once one shows that solutions exist, it is however easy to show that they are unique). The existence of solutions with finite energy $E(T)$, as defined in (1.16) above, follows by approximating the Prandtl equation with a fully parabolic problem (by adding an $\varepsilon|\xi|^2\bar{w}_\xi$ term to the left side of (2.9)), by adding a dissipative term to the forced Benjamin-Ono equation (by adding $\varepsilon|\xi|^2A_\xi$ term to the left side of (2.14)), and by establishing bounds that are independent of ε . Since our estimates are fully L^2 -based, and since the initial datum assumption in Theorem 1.1 is for a space with analyticity radius $10\tau_0 > \tau_0$, all our a priori estimates carry over to this regularized system, yielding ε -independent bounds.

Passing $\varepsilon \rightarrow 0$, the existence of solutions follows, by using the detailed argument presented in [36, pp. 834–846] or in [24, p. 33].

4 Proof of the Main Nonlinear Lemma

Before turning to the proof, we recall a few technical results that are used in the proof of Lemma 3.1.

4.1 Properties of the Weight and the Lift Function

Let us record the straightforward inequality

$$(4.1) \quad |I_y[\theta_\xi]| = \int_0^y \theta_\xi(y', t) dy' \leq \int_0^y 1 dy' \leq y,$$

and emphasize that (4.1) is *independent* of the parameter ε . Recalling (1.9), we note that θ_ξ obeys

$$(4.2) \quad (1 - \theta_\xi)\rho = e^{-\frac{y^2(\xi)^2}{2(1+\frac{t}{\varepsilon})}} e^{\frac{y^2}{8(1+\frac{t}{\varepsilon})}} = e^{-\frac{y^2(4(\xi)^2-1)}{8(1+\frac{t}{\varepsilon})}} \leq e^{-\frac{3y^2(\xi)^2}{8(1+\frac{t}{\varepsilon})}}$$

from which we may deduce the the inequality

$$(4.3) \quad \|(1 - \theta_\xi)\rho\|_{L_y^2} \lesssim \frac{(1 + t/\varepsilon)^{1/4}}{\langle \xi \rangle^{1/2}} \lesssim \frac{1}{\langle \xi \rangle^{1/2}},$$

which will be used frequently below.

It will be convenient to appeal to the bound

$$(4.4) \quad \|I_y[f]\|_{L_y^\infty} \lesssim (1 + t/\varepsilon)^{1/4} \|\rho f\|_{L^2([0, \infty))} \lesssim \|\rho f\|_{L^2([0, \infty))},$$

which is a consequence of the estimate $\|\rho^{-1}\|_{L^2([0, \infty))} \lesssim (1 + t/\varepsilon)^{1/4}$ and

$$\|I_y[f]\|_{L_y^\infty} \leq \int_0^\infty \rho(y) f(y) \rho(y)^{-1} dy \leq \|\rho f\|_{L^2([0, \infty))} \|\rho^{-1}\|_{L^2([0, \infty))}.$$

As a consequence of the proof of (4.4), the fundamental theorem of calculus, we have that whenever $f|_{y=0} = 0$, the estimate

$$(4.5) \quad \begin{aligned} \|f\|_{L_y^\infty} &\leq \|I_y[\partial_y f]\|_{L_y^\infty} \lesssim (1 + t/\varepsilon)^{1/4} \|\rho \partial_y f\|_{L^2([0, \infty))} \\ &\lesssim \|\rho \partial_y f\|_{L^2([0, \infty))} \end{aligned}$$

also holds. The following weighted Poincaré/Hardy inequality will be useful for our proof.

LEMMA 4.1. *Let the weight function ρ be as defined in (1.9). Assume that f is such that $\rho \partial_y f \in L^2$. Then we have*

$$\|\rho f\|_{L^2}^2 + \frac{1}{4(1 + t/\varepsilon)} \|y \rho f\|_{L^2}^2 \leq 4(1 + t/\varepsilon) \|\rho \partial_y f\|_{L^2}^2.$$

PROOF OF LEMMA 4.1.

$$\begin{aligned} \int_0^\infty \rho^2 f^2 &= \int_0^\infty e^{\frac{y^2}{4(1+t/\varepsilon)}} f^2 = \int_0^\infty \partial_y \{y\} e^{\frac{y^2}{4(1+t/\varepsilon)}} f^2 \\ &= \frac{-1}{2(1+t/\varepsilon)} \int_0^\infty y^2 e^{\frac{y^2}{4(1+t/\varepsilon)}} f^2 - 2 \int_0^\infty y e^{\frac{y^2}{4(1+t/\varepsilon)}} f \partial_y f. \end{aligned}$$

Due to the monotone increasing and superexponential nature of our weight, the negative term on the right side of the above is the key contribution. Rearranging yields

$$\begin{aligned} \|\rho f\|_{L^2}^2 + \frac{1}{2(1+t/\varepsilon)} \|y\rho f\|_{L^2}^2 &\leq 2 \left| \int_0^\infty y \rho^2 f \partial_y f \right| \leq 2 \|y\rho f\|_{L^2} \|\rho \partial_y f\|_{L^2} \\ &\leq \frac{1}{4(1+t/\varepsilon)} \|y\rho f\|_{L^2}^2 + 4(1+t/\varepsilon) \|\rho \partial_y f\|_{L^2}^2, \end{aligned}$$

which concludes the proof of the lemma. \square

4.2 Error Terms in the A Energy

First, using (4.4) and the Cauchy-Schwarz inequality it follows that the term $T_{\mathcal{A},1}$ defined in (3.2a) may be bounded as

$$\begin{aligned} |T_{\mathcal{A},1}| &\leq \||\xi|^{1/2} \langle \xi \rangle^r e^{\tau|\xi|} I_\infty[\bar{w}_\xi]\|_{L_\xi^2} \||\xi|^{1/2} \langle \xi \rangle^r e^{\tau|\xi|} A_\xi\|_{L_\xi^2} \\ &\lesssim \||\partial_x|^{1/2} \bar{w}\|_{\tau,r} \||\partial_x|^{1/2} A\|_{\tau,\tilde{r}}. \end{aligned}$$

In order to estimate the term $T_{\mathcal{A},2}$ defined in (3.2b), we use that for any $r \geq 0$ we have

$$\langle \xi \rangle^r \lesssim \langle \xi - \eta \rangle^r + \langle \eta \rangle^r$$

where the implicit constant depends solely on r and $\langle \xi \rangle^{1/2} \leq \langle \eta \rangle^{1/2} \langle \xi - \eta \rangle^{1/2}$ to conclude that

$$\begin{aligned} |T_{\mathcal{A},2}| &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi - \eta| (\langle \xi - \eta \rangle^{r-1/2} + \langle \eta \rangle^{r-1/2}) \\ &\quad \times |A_\eta| e^{\tau|\eta|} |A_{\xi-\eta}| e^{\tau|\xi-\eta|} \langle \xi \rangle^{r+1/2} |A_\xi| e^{\tau|\xi|} d\eta d\xi \\ &\lesssim \||\partial_x|^{1/2} A\|_{\tau,\tilde{r}} \|A_\eta e^{\tau|\eta|}\|_{L_\eta^1} \langle \partial_x \rangle^{1/2} \|A\|_{\tau,\tilde{r}} \\ &\quad + \||\xi - \eta| \langle \xi - \eta \rangle^{1/2} A_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1} \|A\|_{\tau,\tilde{r}}^2 \\ &\lesssim \||\partial_x|^{1/2} A\|_{\tau,\tilde{r}}^2 \|A\|_{\tau,\tilde{r}} + \||\partial_x|^{1/2} A\|_{\tau,\tilde{r}} \|A\|_{\tau,\tilde{r}}^2. \end{aligned}$$

In the last inequality above we have used that for $r > 3/2$ we have $\langle \xi \rangle^{-r+1} \in L_\xi^2$. This proves (3.15).

4.3 Error Terms in the \bar{w} Energy

The $T_{\mathcal{N}}$ Term

According to (2.10a) and (3.7), we decompose $T_{\mathcal{N}}$ as

$$\begin{aligned}
 (4.6) \quad T_{\mathcal{N}} &= i \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \bar{w}_\eta(\xi - \eta) \bar{w}_{\xi-\eta} \bar{w}_\xi \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\eta d\xi \\
 &\quad - i \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \eta I_y[\bar{w}_\eta] \partial_y \bar{w}_{\xi-\eta} \bar{w}_\xi \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\eta d\xi \\
 &=: T_{\mathcal{N}}^{(1)} + T_{\mathcal{N}}^{(2)}.
 \end{aligned}$$

For the first term above we appeal to the inequality

$$|\xi - \eta|^{1/2} \langle \xi \rangle^r \lesssim (|\eta|^{1/2} + |\xi|^{1/2}) (\langle \eta \rangle^r + \langle \xi - \eta \rangle^r),$$

to the triangle inequality of the exponential term and to the bound (4.5) to conclude

$$\begin{aligned}
 (4.7) \quad |T_{\mathcal{N}}^{(1)}| &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta|^{1/2} \|\rho \bar{w}_\eta\|_{L_y^2} |\xi - \eta|^{1/2} \|\rho \bar{w}_{\xi-\eta}\|_{L_y^2} \\
 &\quad \times (\langle \eta \rangle^r + \langle \xi - \eta \rangle^r) \langle \xi \rangle^r \|\bar{w}_\xi\|_{L_y^\infty} e^{2\tau|\xi|} d\eta d\xi \\
 &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \|\bar{w}_\eta\|_{L_y^\infty} |\xi - \eta|^{1/2} \|\rho \bar{w}_{\xi-\eta}\|_{L_y^2} \\
 &\quad \times (\langle \eta \rangle^r + \langle \xi - \eta \rangle^r) |\xi|^{1/2} \langle \xi \rangle^r \|\rho \bar{w}_\xi\|_{L_y^2} e^{2\tau|\xi|} d\eta d\xi \\
 &\lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \|\partial_y \bar{w}\|_{\tau, r} \| |\xi|^{1/2} \rho \bar{w}_\xi e^{\tau|\xi|} \|_{L_\xi^1 L_y^2} \\
 &\quad + \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r}^2 \|\rho \partial_y \bar{w}_\xi e^{\tau|\xi|}\|_{L_\xi^1 L_y^2} \\
 &\lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r}^2 \|\partial_y \bar{w}\|_{\tau, r}
 \end{aligned}$$

where in the last inequality we have used that $r > \frac{1}{2}$.

For the second term in (4.6) we proceed similarly, but appeal to the bound (4.4), which yields

$$\begin{aligned}
 (4.8) \quad |T_{\mathcal{N}}^{(2)}| &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta|^{1/2} \langle \eta \rangle^r \|I_y[\bar{w}_\eta]\|_{L_y^\infty} |\xi - \eta|^{1/2} \|\rho \partial_y \bar{w}_{\xi-\eta}\|_{L_y^2} \\
 &\quad \times \langle \xi \rangle^r \|\rho \bar{w}_\xi\|_{L_y^2} e^{2\tau|\xi|} d\eta d\xi \\
 &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta|^{1/2} \langle \eta \rangle^r \|I_y[\bar{w}_\eta]\|_{L_y^\infty} \|\rho \partial_y \bar{w}_{\xi-\eta}\|_{L_y^2} |\xi|^{1/2} \\
 &\quad \times \langle \xi \rangle^r \|\rho \bar{w}_\xi\|_{L_y^2} e^{2\tau|\xi|} d\eta d\xi +
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta| \|I_y[\bar{w}_\eta]\|_{L_y^\infty} \langle \xi - \eta \rangle^r \|\rho \partial_y \bar{w}_{\xi-\eta}\|_{L_y^2} \\
& \quad \times \langle \xi \rangle^r \|\rho \bar{w}_\xi\|_{L_y^2} e^{2\tau|\xi|} d\eta d\xi \\
& \lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \|\bar{w}\|_{\tau, r} \|\langle \xi - \eta \rangle^{1/2} \rho \partial_y \bar{w}_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1 L_y^2} \\
& \quad + \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r}^2 \|\rho \partial_y \bar{w}_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1 L_y^2} \\
& \quad + \|\bar{w}\|_{\tau, r} \|\partial_y \bar{w}\|_{\tau, r} \|\langle \eta \rangle \rho \bar{w}_\eta e^{\tau|\eta|}\|_{L_\eta^1 L_y^2} \\
& \lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau, r} \|\partial_y \bar{w}\|_{\tau, r}
\end{aligned}$$

since $r > 1$. Combining the estimates (4.7)–(4.8) yields the desired bound (3.16a).

The $T_{\mathcal{L}}$ Term

Recall that the term $T_{\mathcal{L}}$ is defined in (3.7), via (2.10), as

$$\begin{aligned}
T_{\mathcal{L}} &= i \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} \bar{w}_\xi \eta \\
& \quad \times (\bar{w}_{\xi-\eta} A_\eta \theta_\eta + A_{\xi-\eta} \theta_{\xi-\eta} \bar{w}_\eta \\
& \quad - I_y[\bar{w}_\eta] A_{\xi-\eta} \partial_y \theta_{\xi-\eta}) dy d\eta d\xi \\
&= T_{\mathcal{L}}^{(1)} + T_{\mathcal{L}}^{(2)} + T_{\mathcal{L}}^{(3)}.
\end{aligned}$$

We estimate each of the above terms individually. Using that $0 \leq \theta_\xi \leq 1$ pointwise in t , y , and ξ , and that $|\eta|^{1/2} \leq |\xi|^{1/2} + |\xi - \eta|^{1/2}$ and $\langle \xi \rangle^r \lesssim \langle \eta \rangle^r + \langle \xi - \eta \rangle^r$, for the $T_{\mathcal{L}}^{(1)}$ term we have

$$\begin{aligned}
(4.9) \quad |T_{\mathcal{L}}^{(1)}| &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \rho^2 e^{\tau|\xi|} e^{\tau|\xi-\eta|} e^{\tau|\eta|} \langle \xi \rangle^{2r} |\bar{w}_\xi| \\
& \quad \times |\eta|^{1/2} (|\xi|^{1/2} + |\xi - \eta|^{1/2}) |\bar{w}_{\xi-\eta}| |A_\eta| dy d\eta d\xi \\
&\lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \| |\partial_x|^{1/2} A \|_{\tilde{\tau}, \tilde{r}} \|\rho e^{\tau|\xi-\eta|} |\bar{w}_{\xi-\eta}|\|_{L_{\xi-\eta}^1 L_y^2} \\
& \quad + \|\bar{w}\|_{\tau, r} \| |\partial_x|^{1/2} A \|_{\tilde{\tau}, \tilde{r}} \|\rho |\xi - \eta|^{1/2} e^{\tau|\xi-\eta|} |\bar{w}_{\xi-\eta}|\|_{L_{\xi-\eta}^1 L_y^2} \\
& \quad + \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \|\bar{w}\|_{\tau, r} \|\langle \eta \rangle^{1/2} e^{\tau|\eta|} |A_\eta|\|_{L_\eta^1} \\
&\lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \|\bar{w}\|_{\tau, r} \| |\partial_x|^{1/2} A \|_{\tilde{\tau}, \tilde{r}}
\end{aligned}$$

since $r > 1$. Similarly, for the $T_{\mathcal{L}}^{(2)}$ term we have

$$\begin{aligned}
|T_{\mathcal{L}}^{(2)}| &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \rho^2 e^{\tau|\xi|} e^{\tau|\xi-\eta|} e^{\tau|\eta|} \langle \xi \rangle^{2r} |\bar{w}_\xi| \\
&\quad \times |\eta|^{1/2} (|\xi|^{1/2} + |\xi - \eta|^{1/2}) |\bar{w}_\eta| |A_{\xi-\eta}| dy d\eta d\xi \\
&\lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r}^2 \| e^{\tau|\xi-\eta|} |A_{\xi-\eta}| \|_{L_{\xi-\eta}^1} \\
(4.10) \quad &\quad + \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \| \bar{w} \|_{\tau, r} \| |\xi - \eta|^{1/2} e^{\tau|\xi-\eta|} |A_{\xi-\eta}| \|_{L_{\xi-\eta}^1} \\
&\quad + (\| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \| A \|_{\tilde{\tau}, r} + \| \bar{w} \|_{\tau, r} \| |\partial_x|^{1/2} A \|_{\tilde{\tau}, r}) \\
&\quad \times \| \rho |\eta|^{1/2} e^{\tau|\eta|} |\bar{w}_\eta| \|_{L_\eta^1 L_y^2} \\
&\lesssim \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r}^2 \| A \|_{\tilde{\tau}, r} + \| |\partial_x|^{1/2} \bar{w} \|_{\tau, r} \| \bar{w} \|_{\tau, r} \| |\partial_x|^{1/2} A \|_{\tilde{\tau}, r}.
\end{aligned}$$

The $T_{\mathcal{L}}^{(3)}$ term is treated slightly differently due to the presence of $\partial_y \theta_{\xi-\eta}$. Here we use that

$$\begin{aligned}
\rho \langle \xi \rangle^{-1/2} \partial_y \theta_\xi &= \frac{\rho(y, t) \langle \xi \rangle^{3/2} y}{(1 + t/\varepsilon)} e^{-\frac{y^2 \langle \xi \rangle^2}{2(1+t/\varepsilon)}} \\
&\lesssim \frac{\langle \xi \rangle^{1/2}}{(1 + t/\varepsilon)^{1/2}} e^{\frac{y^2}{8(1+t/\varepsilon)}} e^{-\frac{y^2 \langle \xi \rangle^2}{4(1+t/\varepsilon)}} \\
&\lesssim \frac{\langle \xi \rangle^{1/2}}{(1 + t/\varepsilon)^{1/2}} e^{-\frac{y^2 \langle \xi \rangle^2}{8(1+t/\varepsilon)}} \lesssim \langle \xi \rangle^{1/2} e^{-\frac{y^2 \langle \xi \rangle^2}{8(1+t/\varepsilon)}}
\end{aligned}$$

from which it follows upon taking an L^2 norm in y and a supremum over $\xi \in \mathbb{R}$ that

$$(4.11) \quad \|\rho \langle \xi \rangle^{-1/2} \partial_y \theta_\xi\|_{L_\xi^\infty L_y^2} \lesssim 1.$$

Appealing also to (4.4) with $f = \bar{w}_\eta$, and to the inequality

$$\begin{aligned}
|\eta| \langle \xi - \eta \rangle^{1/2} \langle \xi \rangle^r &\lesssim |\eta| \langle \xi - \eta \rangle^{r+1/2} + |\eta|^{1/2} \langle \eta \rangle^r |\xi|^{1/2} \langle \xi - \eta \rangle^{1/2} \\
&\quad + |\eta|^{1/2} \langle \eta \rangle^r |\xi - \eta|^{1/2} \langle \xi - \eta \rangle^{1/2},
\end{aligned}$$

we obtain

$$\begin{aligned}
(4.12) \quad |T_{\mathcal{L}}^{(3)}| &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \rho e^{\tau|\xi|} e^{\tau|\xi-\eta|} e^{\tau|\eta|} \langle \xi \rangle^{2r} |\bar{w}_\xi| \\
&\quad \times |\eta| \langle \xi - \eta \rangle^{1/2} |A_{\xi-\eta}| |I_y[\bar{w}_\eta]| \frac{\rho |\partial_y \theta_{\xi-\eta}|}{\langle \xi - \eta \rangle^{1/2}} dy d\eta d\xi
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|\rho \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^r e^{\tau|\xi|} |A_{\xi-\eta}| e^{\tau|\xi-\eta|} \|\rho \bar{w}_\eta\|_{L_y^2} e^{\tau|\eta|} \\
&\quad \times |\eta| \langle \xi - \eta \rangle^{1/2} \langle \xi \rangle^r d\eta d\xi \lesssim \\
&\lesssim \|\bar{w}\|_{\tau,r} \|\langle \partial_x \rangle^{1/2} A\|_{\tilde{\tau},r} \|\eta\| \|\rho \bar{w}_\eta\|_{L_y^2} e^{\tau|\eta|} \|_{L_\eta^1} \\
&\quad + \||\partial_x|^{1/2} \bar{w}\|_{\tau,r}^2 \|\langle \xi - \eta \rangle^{1/2} A_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1} \\
&\quad + \|\bar{w}\|_{\tau,r} \||\partial_x|^{1/2} \bar{w}\|_{\tau,r} \|\langle \xi - \eta \rangle A_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1} \\
&\lesssim \|\bar{w}\|_{\tau,r}^2 \|\langle \partial_x \rangle^{1/2} A\|_{\tilde{\tau},r} + \||\partial_x|^{1/2} \bar{w}\|_{\tau,r}^2 \|A\|_{\tilde{\tau},r} \\
&\quad + \|\bar{w}\|_{\tau,r} \||\partial_x|^{1/2} \bar{w}\|_{\tau,r} \|A\|_{\tilde{\tau},r},
\end{aligned}$$

since $r > 3/2$. Summing (4.9), (4.10), and (4.12), and massaging the resulting terms we arrive at (3.16b).

The $T_{\mathcal{M}}$ Term

This term has the distinguished feature of losing a y -weight, which is why we have introduced the vorticity $\partial_y \bar{w}$ in the first place. Recall from (2.10c) and (3.7) that

$$T_{\mathcal{M}} = -i \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} \bar{w}_\xi \eta A_\eta I_y[\theta_\eta] \partial_y \bar{w}_{\xi-\eta} dy d\eta d\xi.$$

Recall from (4.1) that $|I_y[\theta_\eta]| \leq y$, and thus, with the cutoff χ defined in (3.11) we have that

$$|(1 - \chi(y)) I_y[\theta_\eta]| \lesssim 1$$

pointwise in η . Therefore, the contribution to $T_{\mathcal{M}}$ coming from the support of $1 - \chi(y)$ may be bounded as

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} \bar{w}_\xi \eta A_\eta I_y[\theta_\eta] \partial_y \bar{w}_{\xi-\eta} (1 - \chi(y)) dy d\eta d\xi \right| \\
&\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty e^{\tau|\xi|} e^{\tau|\eta|} e^{\tau|\xi-\eta|} \langle \xi \rangle^{2r} \rho |\bar{w}_\xi| |\eta| |A_\eta| \rho |\partial_y \bar{w}_{\xi-\eta}| dy d\eta d\xi \\
&\lesssim \||\partial_x|^{1/2} \bar{w}\|_{\tau,r} \||\partial_x|^{1/2} A\|_{\tilde{\tau},r} \|\rho \partial_y \bar{w}_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1 L_y^2} \\
&\quad + \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau,r} \|\partial_y \bar{w}\|_{\tau,r} \|\eta\|^{1/2} A_\eta e^{\tau|\eta|} \|_{L_\eta^1} \\
&\quad + \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau,r} \|A\|_{\tilde{\tau},r} \|\rho |\xi - \eta|^{1/2} \partial_y \bar{w}_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1 L_y^2} \\
&\lesssim \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau,r} \|\partial_y \bar{w}\|_{\tau,r} \|\langle \partial_x \rangle^{1/2} A\|_{\tilde{\tau},r}.
\end{aligned} \tag{4.13}$$

On the other hand, the contribution to $T_{\mathcal{M}}$ from the support of $\chi(y)$ requires us to use the third term on the left side of (3.14), but with r replaced by $r - 1/2$. More precisely, we use that $\chi(y)|I_y[\theta_\eta]| \leq y\chi(y)$ pointwise in η , and the inequality

$$|\eta|\langle\xi\rangle^{2r} \lesssim |\eta|\langle\eta\rangle^{r-1/2}\langle\xi\rangle^{r+1/2} + |\eta|\langle\xi-\eta\rangle^{r-1/2}\langle\xi\rangle^{r+1/2},$$

to deduce that

$$\begin{aligned} & \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \rho^2 e^{2\tau|\xi|} \langle\xi\rangle^{2r} \overline{w_\xi} \eta A_\eta I_y[\theta_\eta] \partial_y \overline{w}_{\xi-\eta} \chi(y) dy d\eta d\xi \right| \\ & \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\tau|\xi|} e^{\tau|\eta|} e^{\tau|\xi-\eta|} \langle\xi\rangle^{2r} \|\rho \overline{w}_\xi\|_{L_y^2} |\eta| \|A_\eta\| \|\rho y \chi(y) \partial_y \overline{w}_{\xi-\eta}\|_{L_y^2} d\eta d\xi \\ & \lesssim \|\langle\partial_x\rangle^{1/2} \overline{w}\|_{\tau,r} \|\partial_x\|^{1/2} A\|_{\tau,\tilde{r}} \|\rho y \chi(y) \partial_y \overline{w}_{\xi-\eta} e^{\tau|\xi-\eta|}\|_{L_{\xi-\eta}^1 L_y^2} \\ & \quad + \|\langle\partial_x\rangle^{1/2} \overline{w}\|_{\tau,r} \|y \chi \partial_y \overline{w}\|_{\tau,r-1/2} \|\eta| A_\eta e^{\tau|\eta|}\|_{L_\eta^1} \\ & \lesssim \|\langle\partial_x\rangle^{1/2} \overline{w}\|_{\tau,r} \|y \chi \partial_y \overline{w}\|_{\tau,r-1/2} \|\partial_x\|^{1/2} A\|_{\tau,\tilde{r}}, \end{aligned}$$

since r was taken to be sufficiently large. Combining (4.13) and (4.14) we obtain the estimate (3.16c).

The $T_{\mathcal{B}}$ Term

According to (2.15) and (3.7), we decompose the $T_{\mathcal{B}}$ term as

$$\begin{aligned} (4.14) \quad T_{\mathcal{B}} &= \int_{\mathbb{R}} \int_0^\infty A_\xi (\partial_t - \partial_{yy}) \theta_\xi \overline{w}_\xi \rho^2 e^{2\tau|\xi|} \langle\xi\rangle^{2r} dy d\xi \\ &+ \int_{\mathbb{R}} \int_0^\infty (\theta_\xi - 1) \partial_t A_\xi \overline{w}_\xi \rho^2 e^{2\tau|\xi|} \langle\xi\rangle^{2r} dy d\xi \\ &+ i \int_{\mathbb{R}} \int_0^\infty \xi (I_\infty[\overline{w}_\xi] - I_y[\overline{w}_\xi]) \overline{w}_\xi \rho^2 e^{2\tau|\xi|} \langle\xi\rangle^{2r} dy d\xi \\ &+ i \int_{\mathbb{R}} \int_0^\infty \xi A_\xi (y(\theta_\xi - 1) - (I_\infty[1 - \theta_\xi] - I_y[1 - \theta_\xi])) \\ &\quad \times \overline{w}_\xi \rho^2 e^{2\tau|\xi|} \langle\xi\rangle^{2r} dy d\xi \\ &+ i \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \xi A_\eta A_{\xi-\eta} (\theta_\eta \theta_{\xi-\eta} - 1) \overline{w}_\xi \rho^2 e^{2\tau|\xi|} \langle\xi\rangle^{2r} dy d\eta d\xi \\ &- i \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \eta A_\eta A_{\xi-\eta} I_y[\theta_\eta] \partial_y \theta_{\xi-\eta} \overline{w}_\xi \rho^2 e^{2\tau|\xi|} \langle\xi\rangle^{2r} dy d\eta d\xi \\ &= T_{\mathcal{B}}^{(1)} + T_{\mathcal{B}}^{(2)} + T_{\mathcal{B}}^{(3)} + T_{\mathcal{B}}^{(4)} + T_{\mathcal{B}}^{(5)} + T_{\mathcal{B}}^{(6)}. \end{aligned}$$

We bound the six terms above individually, and note that the second term, $T_{\mathcal{B}}^{(2)}$, is the most involved one, as it involves analyzing the four terms arising from the A_ξ evolution in (2.14).

We bound the most difficult term first. Combining (4.14) and (2.14) we rewrite

$$\begin{aligned}
(4.15) \quad T_{\mathcal{B}}^{(2)} &= -i \int_{\mathbb{R}} \int_0^\infty (\theta_\xi - 1) \xi c_{\theta, \xi} A_\xi \overline{w_\xi} \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi \\
&\quad - i \int_{\mathbb{R}} \int_0^\infty (\theta_\xi - 1) \xi |\xi| A_\xi \overline{w_\xi} \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi \\
&\quad + i \int_{\mathbb{R}} \int_0^\infty (\theta_\xi - 1) \xi I_\infty[\overline{w_\xi}] \overline{w_\xi} \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi \\
&\quad - i \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty (\theta_\xi - 1) A_\eta A_{\xi-\eta} (\xi - \eta) \overline{w_\xi} \rho^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\eta d\xi \\
&= T_{\mathcal{B}}^{(2,1)} + T_{\mathcal{B}}^{(2,2)} + T_{\mathcal{B}}^{(2,3)} + T_{\mathcal{B}}^{(2,4)}.
\end{aligned}$$

Using the definition of $c_{\theta, \xi}$ in (2.12) and the bound (4.3) for the L_y^2 norm of $\rho(\theta_\xi - 1)$, we obtain

$$(4.16) \quad |T_{\mathcal{B}}^{(2,1)}| \lesssim \|\overline{w}\|_{\tau, r} \|A\|_{\widetilde{\tau}, r}.$$

For the second term, we use the one-dimensional Hardy inequality $\|f/y\|_{L_y^2} \lesssim \|\partial_y f\|_{L_y^2} \lesssim \|\rho \partial_y f\|_{L_y^2}$, valid for f such that $f|_{y=0} = 0$ since $\rho \geq 1$, and the bound

$$\begin{aligned}
(4.17) \quad \langle \xi \rangle^{3/2} \|y(\theta_\xi - 1) \rho^2\|_{L_y^2} &\lesssim \langle \xi \rangle^{3/2} \|y e^{-\frac{y^2 \langle \xi \rangle^2}{4(1+t/\varepsilon)}}\|_{L_y^2} \\
&\lesssim \langle \xi \rangle^{1/2} \|e^{-\frac{y^2 \langle \xi \rangle^2}{8(1+t/\varepsilon)}}\|_{L_y^2} \lesssim 1,
\end{aligned}$$

which follows similarly to (4.3), and we obtain

$$\begin{aligned}
(4.18) \quad |T_{\mathcal{B}}^{(2,2)}| &\leq \int_{\mathbb{R}} \int_0^\infty |\xi|^{3/2} |y \rho^2 (\theta_\xi - 1)| |\xi|^{1/2} |A_\xi| \left| \frac{\overline{w_\xi}}{y} \right| e^{2\tau|\xi|} \langle \xi \rangle^{2r} dy d\xi \\
&\lesssim \||\partial_x|^{1/2} A\|_{\tau, r} \|\partial_y \overline{w}\|_{\tau, r}.
\end{aligned}$$

For the third term on the right side of (4.15) we simply appeal to (4.3) and (4.4) to obtain

$$\begin{aligned}
(4.19) \quad |T_{\mathcal{B}}^{(2,3)}| &\lesssim \int_{\mathbb{R}} |\xi| \|\rho(\theta_\xi - 1)\|_{L_y^2} |I_\infty[\overline{w_\xi}]| \|\rho \overline{w_\xi}\|_{L_y^2} e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi \\
&\lesssim \int_{\mathbb{R}} |\xi|^{1/2} \|\rho \overline{w_\xi}\|_{L_y^2}^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi \\
&\lesssim \||\partial_x|^{1/2} \overline{w}\|_{\tau, r} \|\overline{w}\|_{\tau, r}.
\end{aligned}$$

Lastly, for the nonlinear term in (4.15) we similarly have

$$\begin{aligned}
& |T_{\mathcal{B}}^{(2,4)}| \\
& \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi - \eta| \|\rho(\theta_{\xi} - 1)\|_{L_y^2} \|A_{\eta}\| \|A_{\xi-\eta}\| \|\rho \bar{w}_{\xi}\|_{L_y^2} e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\eta d\xi \\
(4.20) \quad & \lesssim \int_{\mathbb{R}} |\xi - \eta| (\langle \eta \rangle^{r-1/2} + \langle \xi - \eta \rangle^{r-1/2}) \\
& \quad \times \|A_{\eta}\| \|A_{\xi-\eta}\| \|\rho \bar{w}_{\xi}\|_{L_y^2} e^{2\tau|\xi|} \langle \xi \rangle^{r+1/2} d\xi \\
& \lesssim \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau,r} \|\langle \partial_x \rangle^{1/2} A\|_{\tilde{\tau},r} \|A\|_{\tilde{\tau},r}.
\end{aligned}$$

Combined, the bounds (4.16)–(4.20) yield an estimate for the $T_{\mathcal{B}}^{(2)}$ term. Next, we estimate the remaining five terms on the right side of (4.14).

In order to bound $T_{\mathcal{B}}^{(1)}$ we note from (2.11) that

$$|\partial_t \theta_{\xi}| = \frac{y^2 \langle \xi \rangle^2}{2\varepsilon(1+t/\varepsilon)^2} e^{-\frac{y^2 \langle \xi \rangle^2}{2(1+t/\varepsilon)}} \leq \frac{1}{\varepsilon(1+t/\varepsilon)} e^{-\frac{y^2 \langle \xi \rangle^2}{3(1+t/\varepsilon)}} \lesssim \frac{1}{\varepsilon} e^{-\frac{y^2 \langle \xi \rangle^2}{3(1+t/\varepsilon)}}$$

and

$$|y \partial_{yy} \theta_{\xi}| \leq \frac{y \langle \xi \rangle^2}{(1+t/\varepsilon)} e^{-\frac{y^2 \langle \xi \rangle^2}{2(1+t/\varepsilon)}} + \frac{y^3 \langle \xi \rangle^4}{(1+t/\varepsilon)^2} e^{-\frac{y^2 \langle \xi \rangle^2}{2(1+t/\varepsilon)}} \lesssim \langle \xi \rangle e^{-\frac{y^2 \langle \xi \rangle^2}{3(1+t/\varepsilon)}}$$

from which we deduce

$$(4.21a) \quad \|\rho \partial_t \theta_{\xi}\|_{L_y^2} \lesssim \frac{1}{\varepsilon \langle \xi \rangle^{1/2}},$$

$$(4.21b) \quad \|y \rho^2 \partial_{yy} \theta_{\xi}\|_{L_y^2} \lesssim \langle \xi \rangle^{1/2}.$$

With estimates (4.21b)–(4.21a), the Hardy inequality, and the fact that $\rho \geq 1$, we can estimate

$$\begin{aligned}
|T_{\mathcal{B}}^{(1)}| & \leq \int_{\mathbb{R}} |A_{\xi}| \|\rho \partial_t \theta_{\xi}\|_{L_y^2} \|\rho \bar{w}_{\xi}\|_{L_y^2} e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi \\
& \quad + \int_{\mathbb{R}} |A_{\xi}| \|y \rho^2 \partial_{yy} \theta_{\xi}\|_{L_y^2} \left\| \frac{\bar{w}_{\xi}}{y} \right\|_{L_y^2} e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi \\
& \lesssim \|\bar{w}\|_{\tau,r} \|A\|_{\tilde{\tau},r} + \|\partial_y \bar{w}\|_{\tau,r} \|\langle \partial_x \rangle^{1/2} A\|_{\tilde{\tau},r}.
\end{aligned}$$

To estimate the $T_{\mathcal{B}}^{(3)}$ term in (4.14), we note that upon applying Lemma 4.1 to $f = I_{\infty}[\bar{w}_{\xi}] - I_y[\bar{w}_{\xi}]$, we have that

$$(4.22) \quad \|\rho(y) (I_{\infty}[\bar{w}_{\xi}] - I_y[\bar{w}_{\xi}])\|_{L_y^2} \lesssim \|\rho \bar{w}_{\xi}\|_{L_y^2}$$

and thus

$$(4.23) \quad |T_{\mathcal{B}}^{(3)}| \lesssim \int_{\mathbb{R}} |\xi| \|\rho \bar{w}_{\xi}\|_{L_y^2}^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi \lesssim \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau,r}^2.$$

For the $T_{\mathcal{B}}^{(4)}$ term in (4.14), we use (4.17) to obtain that

$$\|y\rho(\theta_{\xi} - 1)\|_{L_y^2} \lesssim (1 + t/\varepsilon)^{3/4} \langle \xi \rangle^{-3/2}$$

and the estimate

$$(4.24) \quad \|\rho(I_{\infty}[1 - \theta_{\xi}] - I_y[1 - \theta_{\xi}])\|_{L_y^2} \lesssim \left\| \int_y^{\infty} e^{-\frac{(y')^2 \langle \xi \rangle^2}{2(1+t/\varepsilon)}} dy' \right\|_{L_y^2} \lesssim \frac{1}{\langle \xi \rangle^{3/2}}$$

to conclude

$$\begin{aligned} (4.25) \quad & |T_{\mathcal{B}}^{(4)}| \\ & \leq \int_{\mathbb{R}} |\xi| |A_{\xi}| (\|\rho y(1 - \theta_{\xi})\|_{L_y^2} + \|\rho(I_{\infty}[1 - \theta_{\xi}] - I_y[1 - \theta_{\xi}])\|_{L_y^2}) \\ & \quad \times \rho \bar{w}_{\xi} L_y^2 e^{2\tau|\xi|} \langle \xi \rangle^{2r} d\xi \\ & \lesssim \|\bar{w}\|_{\tau,r} \|A\|_{\tilde{\tau},r}. \end{aligned}$$

Lastly, we turn to the two nonlinear terms in (4.14). First, we use (4.3) to estimate

$$\|\rho|\theta_{\eta}\theta_{\xi-\eta} - 1|\|_{L_y^2} \leq \|\rho|\theta_{\xi-\eta} - 1|\|_{L_y^2} + \|\rho|\theta_{\eta} - 1|\|_{L_y^2} \lesssim \frac{1}{\langle \xi - \eta \rangle^{1/2}} + \frac{1}{\langle \eta \rangle^{1/2}}$$

and then use the bound

$$\langle \xi \rangle^r |\xi|^{1/2} \lesssim \langle \eta \rangle^r |\eta|^{1/2} + \langle \xi - \eta \rangle^r |\xi - \eta|^{1/2}$$

to conclude

$$\begin{aligned} (4.26) \quad & |T_{\mathcal{B}}^{(5)}| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi|^{1/2} \langle \xi \rangle^r |A_{\eta}| |A_{\xi-\eta}| \|\rho(\theta_{\eta}\theta_{\xi-\eta} - 1)\|_{L_y^2} \\ & \quad \times |\xi|^{1/2} \langle \xi \rangle^r \|\rho \bar{w}_{\xi}\|_{L_y^2} e^{2\tau|\xi|} d\eta d\xi \\ & \lesssim \|\partial_x\|^{1/2} \bar{w} \|A\|_{\tilde{\tau},r} \|\partial_x\|^{1/2} A \|_{\tilde{\tau},r}. \end{aligned}$$

For the last term in (4.14), we use that

$$\begin{aligned} \|\rho I_y[\theta_{\eta}]\partial_y\theta_{\xi-\eta}\|_{L_y^2} & \leq \|\rho y\partial_y\theta_{\xi-\eta}\|_{L_y^2} \lesssim \|e^{\frac{y^2}{8(1+t/\varepsilon)}} e^{-\frac{y^2(\xi-\eta)^2}{4(1+t/\varepsilon)}}\|_{L_y^2} \\ & \lesssim \frac{1}{\langle \xi - \eta \rangle^{1/2}} \end{aligned}$$

and the triangle inequality $|\eta|^{1/2} \leq |\xi - \eta|^{1/2} + |\xi|^{1/2}$ to conclude that

$$\begin{aligned} (4.27) \quad & |T_{\mathcal{B}}^{(6)}| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta|^{1/2} (|\xi - \eta|^{1/2} + |\xi|^{1/2}) \langle \xi \rangle^r |A_{\eta}| |A_{\xi-\eta}| \\ & \quad \times \|\rho I_y[\theta_{\eta}]\partial_y\theta_{\xi-\eta}\|_{L_y^2} \langle \xi \rangle^r \|\rho \bar{w}_{\xi}\|_{L_y^2} e^{2\tau|\xi|} d\eta d\xi \\ & \lesssim \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau,r} \|A\|_{\tilde{\tau},r} \|\partial_x\|^{1/2} A \|_{\tilde{\tau},r}. \end{aligned}$$

Upon collecting the bounds (4.16)–(4.20), (4.22)–(4.23), and (4.25)–(4.27), we conclude the proof of (3.16d).

4.4 Error Terms in the $\partial_y \bar{w}$ Energy

The estimates in this section are very similar to those in Section 4.3; however, several modifications are in order: we are testing the equation with the conjugate of $\partial_y \bar{w}$, $\langle \xi \rangle^{2r}$ becomes $\langle \xi \rangle^{2r-1}$, and we may use that the cutoff χ vanishes near $\{y = 0\}$.

The $T_{\partial_y \mathcal{N}}$ Term

From (3.10a) we see that $\partial_y \mathcal{N}_\xi(\bar{w}, \bar{w}) = \mathcal{N}_\xi(\bar{w}, \partial_y \bar{w})$, and thus the estimates are very similar to the $T_{\mathcal{N}}$ term. From (3.10a) and (3.13a) we have

$$\begin{aligned} |T_{\partial_y \mathcal{N}}| &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|\rho \partial_y \bar{w}_\eta\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_{\xi-\eta}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \\ &\quad \times |\xi - \eta| (\langle \eta \rangle^{r-1/2} + \langle \xi - \eta \rangle^{r-1/2}) \langle \xi \rangle^{r-1/2} e^{2\tau|\xi|} d\eta d\xi \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \|\rho \bar{w}_\eta\|_{L_y^2} \|\chi \rho \partial_{yy} \bar{w}_{\xi-\eta}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \\ &\quad \times |\eta| (\langle \eta \rangle^{r-1/2} + \langle \xi - \eta \rangle^{r-1/2}) \langle \xi \rangle^{r-1/2} e^{2\tau|\xi|} d\eta d\xi \lesssim \\ &\lesssim \|\langle \partial_x \rangle^{1/2} \chi \partial_y \bar{w}\|_{\tau, r-1/2}^2 \|\partial_y \bar{w}\|_{\tau, r} \\ &\quad + \|\chi \partial_{yy} \bar{w}\|_{\tau, r-1/2} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} \|\langle \partial_x \rangle^{1/2} \bar{w}\|_{\tau, r} \end{aligned}$$

where we have used that $r > 2$. The above estimate gives the proof of (3.17a).

The $T_{\partial_y \mathcal{L}}$ Term

The term $T_{\partial_y \mathcal{L}}$ is defined via (3.10b) and (3.13b) and may be split into four terms $T_{\partial_y \mathcal{L}}^{(j)}$ with $j \in \{1, \dots, 4\}$, according to the four terms in the integrand of (3.10b). For the first term we have

$$\begin{aligned} (4.28) \quad |T_{\partial_y \mathcal{L}}^{(1)}| &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|\chi \rho \partial_y \bar{w}_\eta\|_{L_y^2} |A_{\xi-\eta}| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \\ &\quad \times |\xi - \eta| \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\ &\lesssim \|\chi \partial_y \bar{w}\|_{\tau, r-1/2}^2 \|\partial_x|^{1/2} A\|_{\tilde{\tau}, r}. \end{aligned}$$

Using $\chi \equiv 0$ on $[0, 1]$, we have that $\chi(y) \lesssim y^2 \chi(y)$, which we may combine with the pointwise bound $|y^2 \partial_y \theta_\xi| \lesssim \langle \xi \rangle^{-1}$ to estimate

$$\begin{aligned}
 (4.29) \quad & |T_{\partial_y \mathcal{L}}^{(2)}| \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|\rho \bar{w}_\eta\|_{L_y^2} \|\chi y^2 \partial_y \theta_{\xi-\eta}\|_{L_y^\infty} \\
 & \quad \times |A_{\xi-\eta}| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} |\xi - \eta| \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\
 & \lesssim \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} \|\bar{w}\|_{\tau, r} \|A\|_{\tilde{\tau}, r}.
 \end{aligned}$$

For the third term in the definition of $T_{\partial_y \mathcal{L}}$ we have

$$\begin{aligned}
 (4.30) \quad & |T_{\partial_y \mathcal{L}}^{(3)}| \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|\chi \rho \partial_y \bar{w}_{\xi-\eta}\|_{L_y^2} |A_\eta| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \\
 & \quad \times |\xi - \eta| \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\
 & \lesssim \|\chi |\partial_x|^{1/2} \partial_y \bar{w}\|_{\tau, r-1/2} \|\chi \langle \partial_x \rangle^{1/2} \partial_y \bar{w}\|_{\tau, r-1/2} \|A\|_{\tilde{\tau}, r}.
 \end{aligned}$$

For the last term, we use again that $\chi(y) \lesssim y^2 \chi(y)$, and that similarly to (4.21b) (and the equation two lines above it) we have

$$|\rho^2 y^2 \partial_{yy} \theta_\xi| \lesssim (y^2 \langle \xi \rangle^2 + y^4 \langle \xi \rangle^4) e^{\frac{y^2}{4(1+t/\varepsilon)}} e^{-\frac{y^2 \langle \xi \rangle^2}{2(1+t/\varepsilon)}} \lesssim e^{-\frac{y^2 \langle \xi \rangle^2 - 1}{4(1+t/\varepsilon)}} \lesssim 1,$$

which implies

$$\begin{aligned}
 (4.31) \quad & |T_{\partial_y \mathcal{L}}^{(4)}| \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|I_y[\bar{w}_\eta]\|_{L_y^\infty} |A_{\xi-\eta}| \|\rho^{-1}\|_{L_y^2} \|\chi y^2 \rho^2 \partial_{yy} \theta_{\xi-\eta}\|_{L_y^\infty} \\
 & \quad \times \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} |\eta| \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\
 & \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|\rho \bar{w}_\eta\|_{L_y^2} |A_{\xi-\eta}| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} |\eta| \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\
 & \lesssim \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} \|\partial_x|^{1/2} \bar{w}\|_{\tau, r} \|A\|_{\tilde{\tau}, r}.
 \end{aligned}$$

Combining the bounds (4.28), (4.29), (4.30), and (4.31), we obtain the proof of (3.17b).

The $T_{\partial_y \mathcal{M}}$ Term

The $T_{\partial_y \mathcal{M}}$ terms is defined via (3.10c) and (3.13c) as $T_{\partial_y \mathcal{M}}^{(1)} + T_{\partial_y \mathcal{M}}^{(2)}$, where the decomposition is between the two terms in the integrand of (3.10c). For the first term we use that $|\theta_\eta| \leq 1$ to obtain

$$\begin{aligned}
 (4.32) \quad & |T_{\partial_y \mathcal{M}}^{(1)}| \\
 & \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta| |A_\eta| \|\chi \rho \partial_y \bar{w}_{\xi-\eta}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\
 & \lesssim \|\chi \partial_y \bar{w}\|_{\tau, r-1/2}^2 \|\partial_x|^{1/2} A\|_{\tilde{\tau}, r}
 \end{aligned}$$

while for the second term we appeal to $|\theta_\eta| \leq y$, which gives

$$(4.33) \quad \begin{aligned} & |T_{\partial_y \mathcal{M}}^{(2)}| \\ & \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta| |A_\eta| \|\chi \rho \partial_{yy} \bar{w}_{\xi-\eta}\|_{L_y^2} \|y \chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\ & \lesssim \|y \chi \partial_y \bar{w}\|_{\tau, r-1/2} \|\chi \partial_{yy} \bar{w}\|_{\tau, r-1/2} \||\partial_x|^{1/2} A\|_{\tilde{\tau}, r}. \end{aligned}$$

Combining (4.32) and (4.33) we obtain the proof of (3.17c).

The $T_{\partial_y \mathcal{B}}$ Term

According to (3.10d) and (3.13d) we write

$$T_{\partial_y \mathcal{B}} = \sum_{j=1}^6 T_{\partial_y \mathcal{B}}^{(j)},$$

where the decomposition is according to the six terms in (3.10d).

Since we are doing estimates on the support of χ , i.e., for $y \geq 1$, y -derivatives of the lift function θ_ξ can be made arbitrarily small on this region, resulting in simpler estimates. For instance, similarly to (4.21a)–(4.21b), we can show that

$$\begin{aligned} \|\chi \rho^2 \partial_t \partial_y \theta_\xi\|_{L_y^\infty} & \lesssim \|\rho^2 y \partial_t \partial_y \theta_\xi\|_{L_y^\infty} \lesssim \frac{1}{\varepsilon}, \\ \|\chi \rho^2 \partial_{yyy} \theta_\xi\|_{L_y^\infty} & \lesssim \|\rho^2 y^3 \partial_{yyy} \theta_\xi\|_{L_y^\infty} \lesssim 1, \end{aligned}$$

and therefore

$$(4.34) \quad \begin{aligned} & |T_{\partial_y \mathcal{B}}^{(1)}| \\ & \lesssim \int_{\mathbb{R}} |A_\xi| \|\chi \rho^2 \partial_t \partial_y \theta_\xi\|_{L_y^\infty} \|\rho^{-1}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi \\ & + \int_{\mathbb{R}} |A_\xi| \|\chi \rho^2 \partial_{yyy} \theta_\xi\|_{L_y^\infty} \|\rho^{-1}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi \\ & \lesssim \frac{1}{\varepsilon} \|A\|_{\tilde{\tau}, r} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} \end{aligned}$$

where we have used that $\varepsilon \lesssim 1$ and $t \lesssim 1$. We may also directly estimate

$$(4.35) \quad \begin{aligned} |T_{\partial_y \mathcal{B}}^{(3)}| & \lesssim \int_{\mathbb{R}} |\xi| \|\rho \bar{w}_\xi\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi \\ & \lesssim \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} \||\partial_x|^{1/2} \bar{w}\|_{\tau, r} \end{aligned}$$

and similarly to (4.34) we have

$$\begin{aligned} & |T_{\partial_y \mathcal{B}}^{(4)}| \\ & \lesssim \int_{\mathbb{R}} |A_\xi| |\xi| \|\chi \rho^2 y \partial_y \theta_\xi\|_{L_y^\infty} \|\rho^{-1}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi \\ & \lesssim \||\partial_x|^{1/2} A\|_{\tilde{\tau}, r} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2}. \end{aligned}$$

It remains to treat the $\partial_t A_\xi$ and the nonlinear terms in (3.10d).

For the $\partial_t A_\xi$ contribution, namely $T_{\partial_y \mathcal{B}}^{(2)}$, we need to use a decomposition that is analogous to (4.15). The main difference is that the $\theta_\xi - 1$ are now replaced by $\partial_y \theta_\xi$, and as mentioned earlier, $\chi(y)|\partial_y \theta_\xi|$ can be made arbitrarily small. In particular, we may use the bound

$$(4.36) \quad \langle \xi \rangle^j \|\chi \rho^2 \partial_y \theta_\xi\|_{L_y^\infty} \lesssim \langle \xi \rangle^j \|\rho^2 y^{1+j} \partial_y \theta_\xi\|_{L_y^\infty} \lesssim 1$$

combined with (2.14) to estimate

$$(4.37) \quad \begin{aligned} |T_{\mathcal{B}}^{(2)}| &\lesssim \int_{\mathbb{R}} |\partial_t A_\xi| \|\chi \rho^2 \partial_y \theta_\xi\|_{L_y^\infty} \|\rho^{-1}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi \\ &\lesssim \int_{\mathbb{R}} |\xi| c_{\theta, \xi} |A_\xi| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi \\ &\quad + \int_{\mathbb{R}} \frac{1}{\langle \xi \rangle} |\xi|^2 |A_\xi| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi + \\ &\quad + \int_{\mathbb{R}} \frac{|\xi|}{\langle \xi \rangle^{1/2}} |I_\infty[\bar{w}_\xi]| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\xi \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\xi|}{\langle \xi \rangle} |A_\eta| |A_{\xi-\eta}| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\ &\lesssim \|\langle \partial_x \rangle^{1/2} A\|_{\tilde{\tau}, r} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} + \|\bar{w}\|_{\tau, r} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} \\ &\quad + \|A\|_{\tilde{\tau}, r}^2 \|\chi \partial_y \bar{w}\|_{\tau, r-1/2}. \end{aligned}$$

Lastly, for the two nonlinear contributions, arising due to the last two terms in (3.10d), we again appeal to (4.36) and estimate

$$(4.38) \quad \begin{aligned} |T_{\mathcal{B}}^{(5)}| &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi - \eta| |A_\eta| |A_{\xi-\eta}| \|\chi \rho^2 \partial_y \theta_{\xi-\eta}\|_{L_y^\infty} \\ &\quad \times \|\rho^{-1}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |A_\eta| |A_{\xi-\eta}| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\ &\lesssim \|A\|_{\tilde{\tau}, r}^2 \|\chi \partial_y \bar{w}\|_{\tau, r-1/2} \end{aligned}$$

and

$$(4.39) \quad \begin{aligned} |T_{\mathcal{B}}^{(6)}| &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta| |A_\eta| |A_{\xi-\eta}| \|\chi \rho^2 y \partial_{yy} \theta_{\xi-\eta}\|_{L_y^\infty} \\ &\quad \times \|\rho^{-1}\|_{L_y^2} \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta| |A_\eta| |A_{\xi-\eta}| \|\chi \rho \partial_y \bar{w}_\xi\|_{L_y^2} \langle \xi \rangle^{2r-1} e^{2\tau|\xi|} d\eta d\xi \\ &\lesssim \|\langle \partial_x \rangle^{1/2} A\|_{\tilde{\tau}, r} \|A\|_{\tilde{\tau}, r} \|\chi \partial_y \bar{w}\|_{\tau, r-1/2}. \end{aligned}$$

By summing the bounds (4.34)–(4.36) and (4.37)–(4.39), we obtain the bound (3.17d).

Appendix: Review of the Incompressible Triple Deck over a Flat 2D Plate

We roughly follow the presentation from [68]. We first introduce the variables

$$(A.1) \quad X = \frac{x-1}{\nu^{3/8}}, \quad \bar{Y} = \frac{y}{\nu^{1/2}}, \quad Y = \frac{y}{\nu^{5/8}}, \quad \widetilde{Y} = \frac{y}{\nu^{3/8}}, \quad T = \frac{t}{\nu^{1/4}}.$$

Here \bar{Y} , Y , and \widetilde{Y} are the fast vertical variables in the main deck, lower deck, and upper deck, respectively. The X -variable is the fast variable in the vicinity of the trailing edge, situated at $x = 1, y = 0$. On the fast time scale T , to leading order only perturbances in the lower deck are active, while in the other decks the fast time dependence does not enter the momentum equation. Throughout this section we abuse notation and write ν instead of an inverse Reynolds number; i.e., we treat ν as if it were dimensionless.

A.1 Main Deck

The ansatz on the solution of the 2D Navier-Stokes equation in this region is

$$(A.2) \quad \begin{aligned} & (u_M, v_M, p_M) \\ &= (U_B(\bar{Y}) + \nu^{1/8}u_1(X, \bar{Y}, T), \nu^{1/4}v_1(X, \bar{Y}, T), \nu^{1/4}p_1(X, \bar{Y}, T)) \\ &+ \text{lower-order terms.} \end{aligned}$$

Above, U_B is defined to be the Blasius boundary layer, introduced in (1.3a) as $U_B := f'(\bar{Y}/\sqrt{x})$, where f solves (1.3b). To ease notation, we suppress the x -dependence of U_B , and denote by $U'_B = \partial_{\bar{Y}}U_B$, since our scaling ensures that we are very close to $x = 1$. Inserting ansatz (A.2) into the 2D Navier-Stokes equations and collecting only the leading-order terms we obtain the *inviscid type system*

$$(A.3a) \quad U_B \partial_X u_1 + v_1 U'_B = 0,$$

$$(A.3b) \quad \partial_{\bar{Y}} p_1 = 0,$$

$$(A.3c) \quad \partial_X u_1 + \partial_{\bar{Y}} v_1 = 0.$$

Note that both the time derivative and the dissipation term in the tangential momentum equation drop out, as they are lower order in ν . The system (A.3) has as a solution

$$(A.4) \quad u_1 = A(X, T)U'_B(\bar{Y}), \quad v_1 = -\partial_X A(X, T)U_B(\bar{Y}), \quad p_1 = P(X, T),$$

for some unknown functions $A(X, T)$ and $P(X, T)$. Note that the solution (A.4) satisfies the boundary condition $u_1|_{\bar{Y} \rightarrow \infty} = 0$. This type of matching condition enforces, from (A.2), that the horizontal velocity in the main deck converges rapidly to the ambient Blasius flow, which is what is observed when the boundary layer separates.

The boundary condition

$$A(X, T) \rightarrow 0 \quad \text{as } X \rightarrow -\infty$$

ensures that, approaching from the left, i.e., as $x \rightarrow 1^-$, the main deck profile matches with the Blasius boundary layer profile $U_B(\bar{Y}/\sqrt{x})$. Therefore, at the lateral boundary $x = 1^-$ in original variables, which is the same as the boundary $X \rightarrow -\infty$ in rescaled variables (as $\nu \rightarrow 0$).

A.2 Lower Deck

Notice now that the main deck, (A.3), contributes a nonzero trace onto $\bar{Y} = 0$, which needs to be adjusted. Thus, it does not suffice to take the main deck as the full flow, as it does not satisfy the no-slip boundary condition. This is the purpose of introducing the lower deck. More precisely, we compute

$$\begin{aligned} u_M|_{\bar{Y} \rightarrow 0} &= U_B(\bar{Y}) + \nu^{\frac{1}{8}} u_1(X, \bar{Y}, T) \sim \bar{Y} U'_B(0) + \nu^{\frac{1}{8}} A(X, T) U'_B(0) \\ &\sim \nu^{\frac{1}{8}} U'_B(0)(\nu^{-\frac{1}{8}} \bar{Y} + A(X, T)) \sim \nu^{\frac{1}{8}} U'_B(0)(Y + A(X, T)), \end{aligned}$$

where we have used the scaling $Y = \nu^{-1/8} \bar{Y}$, which relates the lower deck scaling and main deck/Prandtl scaling. This then suggests that in order to correct for the boundary trace, (A.2), we need to seek a lower deck expansion of magnitude $\nu^{1/8}$. The ansatz on the solution (u_L, v_L, p_L) of the 2D Navier-Stokes equation in this region is thus

$$\begin{aligned} (A.5) \quad (u_L, v_L, p_L) &= \left(\nu^{\frac{1}{8}} U(X, Y, T), \nu^{\frac{3}{8}} V(X, Y, T), \nu^{\frac{1}{4}} P(X, Y, T) \right) \\ &\quad + \text{lower-order terms.} \end{aligned}$$

Inserting ansatz (A.5) into the 2D Navier-Stokes equations and collecting only the leading-order terms, we obtain the *boundary layer type system*

$$(A.6a) \quad \partial_T U + U \partial_X U + V \partial_Y U + \partial_X P - \partial_Y^2 U = 0,$$

$$(A.6b) \quad \partial_Y P = 0,$$

$$(A.6c) \quad \partial_X U + \partial_Y V = 0.$$

Matching the tangential velocity as $Y \rightarrow \infty$ in (A.5) with the tangential velocity as $\bar{Y} \rightarrow 0$ in (A.2), we obtain the boundary condition (for simplicity, take $U'_B(0) = 1$)

$$(A.7) \quad \lim_{Y \rightarrow \infty} (U(X, Y, T) - Y) = A(X, T).$$

On the other hand, at the boundary of the plate we impose Dirichlet boundary conditions

$$(A.8) \quad U(X, 0, T) = V(X, 0, T) = 0.$$

A.3 Upper Deck

The flow in this region is to leading order of steady potential inviscid type. That is, the leading order is an Euler flow, which takes as arguments the unscaled variables, (t, x, y) , from (A.1). In comparison to the perturbations, which in this deck are functions of (X, \tilde{Y}) , every Euler flow fluctuates slowly, and so, without loss of generality, we take the outer Euler flow to be the constant shear flow $(1, 0)$. This yields the ansatz

$$(A.9) \quad (u_U, v_U, p_U) = (1 + \nu^{\frac{1}{4}} u_2(X, \tilde{Y}, T), \nu^{\frac{1}{4}} v_2(X, \tilde{Y}, T), \nu^{\frac{1}{4}} p_2(X, \tilde{Y}, T)) + \text{lower-order terms.}$$

Inserting this ansatz into the 2D Navier-Stokes equations and collecting only the leading-order terms, we obtain the *potential-type system*

$$(A.10a) \quad \partial_X u_2 + \partial_X p_2 = 0,$$

$$(A.10b) \quad \partial_X v_2 + \partial_{\tilde{Y}} p_2 = 0,$$

$$(A.10c) \quad \partial_X u_2 + \partial_{\tilde{Y}} v_2 = 0.$$

The matching condition at $\tilde{Y} = 0$ with the flow in the main deck as $\tilde{Y} \rightarrow \infty$ requires that

$$(A.11) \quad p_2(X, 0, T) = P(X, T), \quad v_2(X, 0, T) = -\partial_X A(X, T),$$

which, according to (A.3), cancels out the normal velocity component from the main deck. Matching the upper deck velocity with the outer Euler solution as $\tilde{Y} \rightarrow \infty$ yields

$$\lim_{\tilde{Y} \rightarrow +\infty} p_2(X, \tilde{Y}, T) = \lim_{\tilde{Y} \rightarrow +\infty} v_2(X, \tilde{Y}, T) = 0.$$

A.4 The Closed Coupled System

From (A.10) we deduce that the pressure p_2 and the normal velocity v_2 are harmonic in the variables X and \tilde{Y} . But one can say more: the functions p_2 and v_2 are harmonic conjugates. Therefore, we may view v_2 as the real part of an analytic function, and p_2 as its imaginary part. Hence their traces at the boundary of the half-space are related via the Hilbert transform

$$(A.12) \quad \begin{aligned} P(X, T) &= p_2(X, 0, T) = H v_2(X, 0, T) = -(H \partial_X) A(X, T) \\ &= \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\partial_{X'} A(X')}{X - X'} dX' \end{aligned}$$

which concludes the proof of (1.7). Recall that $-H \partial_X = |\partial_X|$. Therefore, the system (A.6) together with the boundary conditions (A.7), (A.8), and (A.12) form a closed evolution system. This is precisely the triple-deck model that we used in Section 1.2. Once the solution in the lower deck is determined, we derive from (A.4) the leading-order solution in the main deck, while from (A.11) and harmonic extension in the upper half-space, we determine the leading-order solution in the upper deck.

A.5 The Scalings

Now that we have presented the derivation of the model, we briefly discuss the idea behind the scalings (A.1). We roughly follow the exposition in [44]. A priori, one wants to rescale the x -variable near the point of separation, which may physically correspond to the trailing edge of a flat plate or if a plate were to have a disturbance. In our presentation, $x = 1$ is this point. To achieve this, one introduces a fast, horizontal variable. Next, one scales the magnitude of the deviation from Blasius in the main deck, which represents the separation effect. Summarizing the starting point:

$$(A.13) \quad \begin{aligned} X &= \frac{x-1}{L}, & u_M &= U_B(\bar{Y}) + \ell u_1(X, \bar{Y}, T), \\ v_M &= \frac{\sqrt{\nu} \ell}{L} v_1(X, \bar{Y}, T). \end{aligned}$$

for scalings L and ℓ to be determined in terms of ν . An inspection of (A.1) shows that $L = \nu^{3/8}$ and $\ell = \nu^{1/8}$.

In the next step, one introduces a lower deck to cancel out the boundary contribution from the main deck. This contribution, the content of (A.2), is now of the form

$$u_M|_{\bar{Y} \rightarrow 0} \sim \bar{Y} U'_B(0) + \ell A(X, T) U'_B(0) \sim \ell U'_B(0)(Y + A(X, T)),$$

where the lower deck fast variable and magnitude are now

$$Y := \ell^{-1} \bar{Y}, \quad u_L := \ell U.$$

In the lower deck step, equating the convection $u_L \partial_x$ and the viscosity $\nu \partial_{yy}$ just as in the standard Prandtl theory gives the relation

$$\ell^2 L^{-1} \sim u_L \partial_x u_L = \nu \partial_{yy} u_L \sim \nu \frac{1}{\ell^2} \ell,$$

which gives $\ell^3 = L$. The final physical determination comes from the upper deck. One realizes that the contribution of v_M at the top of the main deck needs to match the order of the pressure, which is ℓ^2 . Thus,

$$\frac{\sqrt{\nu}}{\ell^2} = \ell^2 \Rightarrow \ell = \nu^{\frac{1}{8}}, \quad L = \nu^{\frac{3}{8}},$$

which concludes the scale analysis.

Acknowledgments. The authors are grateful to Stephen Childress for pointing them to a number of references concerning the physical origins of the triple deck. We also thank Klaus Widmayer for discussions concerning the Benjamin-Ono equation. We also thank the anonymous referee for useful remarks. The work of S.I. was supported by the National Science Foundation Grant DMS-1802940. The work of V.V. was partially supported by the National Science Foundation CAREER Grant DMS-1911413.

Bibliography

[1] Alexandre, R.; Wang, Y.-G.; Xu, C.-J.; Yang, T. Well-posedness of the Prandtl equation in Sobolev spaces. *J. Amer. Math. Soc.* **28** (2015), no. 3, 745–784. doi:10.1090/S0894-0347-2014-00813-4

[2] Bardos, C. W.; Titi, E. S. Mathematics and turbulence: where do we stand? *J. Turbul.* **14** (2013), no. 3, 42–76. doi:10.1080/14685248.2013.771838

[3] Blasius, H. Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.* **56** (1908), 1–37.

[4] Bona, J. L.; Grujić, Z.; Kalisch, H. A KdV-type Boussinesq system: from the energy level to analytic spaces. *Discrete Contin. Dyn. Syst.* **26** (2010), no. 4, 1121–1139. doi:10.3934/dcds.2010.26.1121

[5] Carter, J. Solutions for laminar boundary layers with separation and reattachment. *7th Fluid and Plasma Dynamics Conference* (1974), 583. doi:10.2514/6.1974-583

[6] Catherall, D.; Mangler, K. The integration of the two-dimensional laminar boundary-layer equations past the point of vanishing skin friction. *J. Fluid Mech.* **26** (1966), no. 1, 163–182. doi:10.1017/S0022112066001150

[7] Cebeci, T.; Cousteix, J. *Modeling and computation of boundary layer flows*. Springer, Cambridge, 2005.

[8] Collot, C.; Ghoul, T.; Ibrahim, S.; Masmoudi, N. On singularity formation for the two dimensional unsteady Prandtl's system. Preprint, 2018. arXiv:1808.05967 [math.AP]

[9] Collot, C.; Ghoul, T.; Masmoudi, N. Singularity formation for Burgers equation with transverse viscosity. Preprint, 2018. arXiv:1803.07826 [math.AP]

[10] Cousteix, J.; Mauss, J. *Asymptotic analysis and boundary layers*. Scientific Computation. Springer, Berlin, 2007.

[11] Dalibard, A.-L.; Dietert, H.; Gérard-Varet, D.; Marbach, F. High frequency analysis of the unsteady interactive boundary layer model. *SIAM J. Math. Anal.* **50** (2018), no. 4, 4203–4245. doi:10.1137/17M1157477

[12] Dalibard, A.-L.; Masmoudi, N. Separation for the stationary Prandtl equation. *Publ. Math. Inst. Hautes Études Sci.* **130** (2019), 187–297. doi:10.1007/s10240-019-00110-z

[13] Dietert, H.; Gérard-Varet, D. Well-posedness of the Prandtl equations without any structural assumption. *Ann. PDE* **5** (2019), no. 1, Art. 8, 51 pp. doi:10.1007/s40818-019-0063-6

[14] Duck, P. Unsteady triple-deck flows leading to instabilities. *Boundary-Layer Separation*, 297–312. Springer, Berlin-Heidelberg, 1987. doi:10.1007/978-3-642-83000-6_17

[15] E, W.; Engquist, B. Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.* **50** (1997), no. 12, 1287–1293. doi:10.1002/(SICI)1097-0312(199712)50:12<1287::AID-CPA4>3.0.CO;2-4

[16] Fei, M.; Tao, T.; Zhang, Z. On the zero-viscosity limit of the Navier-Stokes equations in the half-space. Preprint, 2016. arXiv:1609.03778 [math.AP]

[17] Fei, N.; Tao, T.; Zhang, Z. On the zero-viscosity limit of the Navier-Stokes equations in R_+^3 without analyticity. *J. Math. Pures Appl. (9)* **112** (2018), 170–229. doi:10.1016/j.matpur.2017.09.007

[18] Foias, C.; Temam, R. Gevrey class regularity for the solutions of the Navier-Stokes equations. *J. Funct. Anal.* **87** (1989), no. 2, 359–369. doi:10.1016/0022-1236(89)90015-3

[19] Gargano, F.; Sammartino, M.; Sciacca, V. Singularity formation for Prandtl's equations. *Phys. D* **238** (2009), no. 19, 1975–1991. doi:10.1016/j.physd.2009.07.007

[20] Gérard-Varet, D.; Dormy, E. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.* **23** (2010), no. 2, 591–609. doi:10.1090/S0894-0347-09-00652-3

[21] Gérard-Varet, D.; Maekawa, Y. Sobolev stability of Prandtl expansions for the steady Navier-Stokes equations. *Arch. Ration. Mech. Anal.* **223** (2019), no. 3, 1319–1382. doi:10.1007/s00205-019-01380-x

Please update any entries that are now in print.

- [22] Gérard-Varet, D.; Maekawa, Y.; Masmoudi, N. Gevrey stability of Prandtl expansions for 2-dimensional Navier-Stokes flows. *Duke Math. J.* **167** (2018), no. 13, 2531–2631. doi:10.1215/00127094-2018-0020
- [23] Gérard-Varet, D.; Masmoudi, N. Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Sci. Éc. Norm. Supér. (4)* **48** (2015), no. 6, 1273–1325. doi:10.24033/asens.2270
- [24] Gerard-Varet, D.; Masmoudi, N.; Vicol, V. Well-posedness of the hydrostatic Navier-Stokes equations. Preprint, 2018. arXiv:1804.04489 [math.AP]
- [25] Gérard-Varet, D.; Nguyen, T. Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* **77** (2012), no. 1-2, 71–88.
- [26] Goldstein, S. On laminar boundary-layer flow near a position of separation. *Quart. J. Mech. Appl. Math.* **1** (1948), 43–69. doi:10.1093/qjmam/1.1.43
- [27] Grenier, E. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53** (2000), no. 9, 1067–1091. doi:10.1002/1097-0312(200009)53:9<1067::AID-CPA1>3.3.CO;2-H
- [28] Grenier, E.; Guo, Y.; Nguyen, T. Spectral stability of Prandtl boundary layers: an overview. *Analysis (Berlin)* **35** (2015), no. 4, 343–355. doi:10.1515/anly-2015-0001
- [29] Grenier, E.; Guo, Y.; Nguyen, T. Spectral instability of characteristic boundary layer flows. *Duke Math. J.* **165** (2016), no. 16, 3085–3146. doi:10.1215/00127094-3645437
- [30] Grenier, E.; Nguyen, T. On nonlinear instability of Prandtl’s boundary layers: the case of Rayleigh’s stable shear flows. Preprint, 2017. arXiv:1706.01282 [math.AP]
- [31] Grenier, E.; Nguyen, T. L^∞ instability of Prandtl layers. Preprint, 2018. arXiv:1803.11024 [math.AP]
- [32] Guo, Y.; Iyer, S. Steady Prandtl layer expansions with external forcing. Preprint, 2018. arXiv:1810.06662 [math.AP]
- [33] Guo, Y.; Iyer, S. Validity of steady Prandtl layer expansions. Preprint, 2018. arXiv:1805.05891 [math.AP]
- [34] Guo, Y.; Nguyen, T. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.* **64** (2011), no. 10, 1416–1438. doi:10.1002/cpa.20377
- [35] Guo, Y.; Nguyen, T. Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. *Ann. PDE* **3** (2017), no. 1, Art. 10, 58 pp. doi:10.1007/s40818-016-0020-6
- [36] Ignatova, M.; Vicol, V. Almost global existence for the Prandtl boundary layer equations. *Arch. Ration. Mech. Anal.* **220** (2016), no. 2, 809–848. doi:10.1007/s00205-015-0942-2
- [37] Iyer, I. Global-in- x stability of Blasius profiles. arXiv:1812.03906 [math.AP] *Arch. Ration. Mech. Anal.*, forthcoming.
- [38] Klingenberg, C. On the stability of triple deck flow. Ph.D. thesis, New York University, 1983.
- [39] Kukavica, I.; Masmoudi, N.; Vicol, V.; Wong, T. On the local well-posedness of the Prandtl and hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.* **46** (2014), no. 6, 3865–3890. doi:10.1137/140956440
- [40] Kukavica, I.; Vicol, V. On the radius of analyticity of solutions to the three-dimensional Euler equations. *Proc. Amer. Math. Soc.* **137** (2009), no. 2, 669–677. doi:10.1090/S0002-9939-08-09693-7
- [41] Kukavica, I.; Vicol, V. On the local existence of analytic solutions to the Prandtl boundary layer equations. *Commun. Math. Sci.* **11** (2013), no. 1, 269–292. doi:10.4310/CMS.2013.v11.n1.a8
- [42] Kukavica, I.; Vicol, V.; Wang, F. The van Dommelen and Shen singularity in the Prandtl equations. *Adv. Math.* **307** (2017), 288–311. doi:10.1016/j.aim.2016.11.013
- [43] Kukavica, I.; Vicol, V.; Wang, F. The inviscid limit for the Navier-Stokes equations with data analytic only near the boundary. Preprint, 2019. arXiv:1904.04983 [math.AP] *Arch. Ration. Mech. Anal.*, forthcoming.

- [44] Lagrée, P.-Y. Interactive boundary layer (IBL). *Asymptotic methods in fluid mechanics: survey and recent advances*, 247–286. CISM International Centre for Mechanical Sciences. Courses and Lectures, 523. Springer, Vienna, 2010. doi:10.1007/978-3-7091-0408-8_8
- [45] Le Balleur, J.-C. New possibilities of viscous-inviscid numerical techniques for solving viscous flow equations with massive separation. *Numerical and Physical Aspects of Aerodynamic Flows IV*, 71–96. Springer, Berlin-Heidelberg, 1990. doi:10.1007/978-3-662-02643-4_5
- [46] Levermore, C. D.; Oliver, M. Analyticity of solutions for a generalized Euler equation. *J. Differential Equations* **133** (1997), no. 2, 321–339. doi:10.1006/jdeq.1996.3200
- [47] Li, W.-X.; Yang, T. Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points. Preprint, 2016. arXiv:1609.08430 [math.AP]
- [48] Lighthill, M. On boundary-layer and upstream influence: II. supersonic flows without separation. *Proc. R. Soc., Ser. A* (1953), no. 217, 478–507.
- [49] Liu, C.-J.; Yang, T. Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay. *J. Math. Pures Appl. (9)* **108** (2017), no. 2, 150–162. doi:10.1016/j.matpur.2016.10.014
- [50] Lombardo, M. C.; Cannone, M.; Sammartino, M. Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.* **35** (2003), no. 4, 987–1004. doi:10.1137/S0036141002412057
- [51] Maekawa, Y. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.* **67** (2014), no. 7, 1045–1128. doi:10.1002/cpa.21516
- [52] Maekawa, Y.; Mazzucato, A. The inviscid limit and boundary layers for Navier-Stokes flows. *Handbook of mathematical analysis in mechanics of viscous fluids*, 781–828. Springer, Cham, 2018.
- [53] Masmoudi, N.; Wong, T. K. Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.* **68** (2015), no. 10, 1683–1741. doi:10.1002/cpa.21595
- [54] Messiter, A. Boundary layer flow near the trailing edge of a flat plate. *SIAM J. Appl. Math.* **18** (1970), no. 1, 241–257. doi:10.1137/0118020
- [55] Meyer, R. Stewartson's Triple Deck. U. Wisconsin-Madison MRC Technical Summary Report, 2391. 1982.
- [56] Meyer, R. E. A view of the triple deck. *SIAM J. Appl. Math.* **43** (1983), no. 4, 639–663. doi:10.1137/0143044
- [57] Neiland, V. Y. Propagation of perturbation upstream with interaction between a hypersonic flow and a boundary layer. *Mekh. Zhid. Gaz.* **4** (1969), 53–57.
- [58] Nguyen, T. T.; Nguyen, T. T. The inviscid limit of Navier-Stokes equations for analytic data on the half-space. *Arch. Ration. Mech. Anal.* **230** (2018), no. 3, 1103–1129. doi:10.1007/s00205-018-1266-9
- [59] Oleinik, O. A. On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid. *Prikl. Mat. Meh.* **30** 801–821 (Russian); translated as *J. Appl. Math. Mech.* **30** (1966), 951–974 (1967). doi:10.1016/0021-8928(66)90001-3
- [60] Oleinik, O. A.; Samokhin, V. N. *Mathematical models in boundary layer theory*. Applied Mathematics and Mathematical Computation, 15. Chapman & Hall/CRC, Boca Raton, Fla., 1999.
- [61] Oliver, M.; Titi, E. S. Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in \mathbf{R}^n . *J. Funct. Anal.* **172** (2000), no. 1, 1–18. doi:10.1006/jfan.1999.3550
- [62] Prandtl, L. Über Flüssigkeitsbewegung bei sehr kleiner Reibung. *Verh. III Intern. Math. Kongr. Heidelberg, Teuber, Leipzig* (1904), 485–491.
- [63] Sammartino, M.; Caflisch, R. Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* **192** (1998), no. 2, 433–461.

- [64] Sammartino, M.; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.* **192** (1998), no. 2, 463–491. doi:10.1007/s002200050305
- [65] Schlichting, H. *Boundary layer theory*. McGraw-Hill Series in Mechanical Engineering. McGraw-Hill, New York–Toronto–London; Verlag G. Braun, Karlsruhe, 1960.
- [66] Serrin, J. Asymptotic behaviour of velocity profiles in the Prandtl boundary layer theory. *Proc. R. Soc. Lond. A* **299** (1967), 491–507. doi:10.1098/rspa.1967.0151
- [67] Smith, F. T. On the non-parallel flow stability of the Blasius boundary layer. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **366** (1979), no. 1724, 91–109. doi:10.1098/rspa.1979.0041
- [68] Smith, F. On the high Reynolds number theory of laminar flows. *IMA Journal of Applied Mathematics* **28** (1982), no. 3, 207–281.
- [69] Smith, F.; Brown, S. N. *Boundary-Layer Separation: Proceedings of the IUTAM Symposium London, August 26–28, 1986*. Springer, Berlin-Heidelberg, 2012. doi:10.1007/978-3-642-83000-6
- [70] Stewartson, K. On the flow near the trailing edge of a flat plate. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **306** (1968), no. 1486, 275–290. doi:10.1098/rspa.1968.0150
- [71] Stewartson, K. On the flow near the trailing edge of a flat plate II. *Mathematika* **16** (1969), no. 1, 106–121. doi:10.1112/S0025579300004678
- [72] Sychev, V. V.; Ruban, A. I.; Sychev, V. V.; Korolev, G. L. *Asymptotic theory of separated flows*. Cambridge University Press, Cambridge, 1998. doi:10.1017/CBO9780511983764
- [73] Tutty, O. R.; Cowley, S. J. On the stability and the numerical solution of the unsteady interactive boundary-layer equation. *J. Fluid Mech.* **168** (1986), 431–456. doi:10.1017/S0022112086000459
- [74] van Dommelen, L. L.; Shen, S. F. The spontaneous generation of the singularity in a separating laminar boundary layer. *J. Comput. Phys.* **38** (1980), no. 2, 125–140. doi:10.1016/0021-9991(80)90049-2
- [75] Wang, C.; Wang, Y.; Zhang, Z. Zero-viscosity limit of the Navier-Stokes equations in the analytic setting. *Arch. Ration. Mech. Anal.* **224** (2017), no. 2, 555–595. doi:10.1007/s00205-017-1083-6
- [76] Xin, Z.; Zhang, L. On the global existence of solutions to the Prandtl’s system. *Adv. Math.* **181** (2004), no. 1, 88–133. doi:10.1016/S0001-8708(03)00046-X

SAMEER IYER
 Department of Mathematics
 Princeton University
 Fine Hall
 Washington Road
 Princeton, NJ 08544
 USA
 E-mail: ssiyer@math.princeton.edu

VLAD VICOL
 Courant Institute
 251 Mercer St.
 New York, NY 10012
 USA
 E-mail: vicol@cims.nyu.edu

Received May 2019.