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Time-asymptotic convergence rates towards discrete steady states of a nonlocal selection-mutation model

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This paper is concerned with large time behavior of solutions to a semi-discrete model involving nonlinear competition that describes the evolution of a trait-structured population. Under some threshold assumptions, the steady solution is shown unique and strictly positive, and also globally stable. The exponential convergence rate to the steady state is also established. These results are consistent with the results in [P.-E. Jabin, H. L. Liu. *Nonlinearity* **30** (2017) 4220–4238] for the continuous model.

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1. Introduction

The evolution by natural selection is the most ubiquitous and well-understood process of evolution in living systems. Adaptive dynamics (see Refs. 13, 14, 19 and 31) is a branch of evolutionary ecology, that aims at describing the Darwinian evolution

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of populations along a phenotypic trait x, which characterizes each individual. We are interested in the population dynamics subject to mutations and selection due to competition between individuals. In such setting probabilistic models are usually considered as the most realistic. We refer to Ref. 30 and the references therein for a nice introduction to the probabilistic approach. When the total number of individuals is too large (it can easily reach $10^{10} \sim 10^{12}$ for some microorganisms), it becomes prohibitive to compute numerically the solution to this process. In that case one expects to be able to derive a deterministic model as a limit of large populations. Such of derivation was proved in Refs. 9 and 10 and one obtains integro-differential equations like

$$\partial_t u(t,x) = M(u)(t,x) + u(t,x)\left(a(x) - \int b(x,y)u(t,y)dy\right), \tag{1.1}$$

where the mutation kernel is for instance

$$M(v)(x) = \int K(z)(a(x+z)v(x+z) - a(x)v(x))dz.$$

Although mutation cannot be fully understood without molecular knowledge, it can be replaced by diffusion for qualitative analysis so that we have

$$\partial_t u(t,x) = \Delta u + u(t,x)\left(a(x) - \int b(x,y)u(t,y)dy\right). \tag{1.2}$$

For rare mutations, the above models become the well known

$$\partial_t u(t,x) = u(t,x)\left(a(x) - \int b(x,y)u(t,y)dy\right). \tag{1.3}$$

These competition models or their variants when arising in ecology with individuals competing for resources, x denotes a location variable, see, e.g. Refs. 5, 16 and 20.

Of special interest are the steady states of population models, which may represent some biological patterns emerged as intrinsic properties of such models. For instance, with model (1.3), one expects that the population density concentrates at large times, see, e.g. Refs. 2, 6, 12, 24 and 37. The singular steady-state solutions of such competition model correspond to highly concentrated population densities of the form of well-separated Dirac masses. This density concentration phenomena has been shown to happen only asymptotically in the model with mutations (see Refs. 3, 11, 27, 32, 33, 35 and 36). The emergence of patterns also occurs in other mathematical models that mimic dynamics of populations (see for example, Amaral and Meyer,[1] Ji and Li,[25] Murray,[34] Tokita[39]). Hence, links are often made between patterns from models and in nature because of their similarity. Through inquiries into the dynamics of evolution, one may understand why we observe such patterns in real world.

Generally speaking, the emergence of patterns follows an optimization process, and that explains why game theory has served as a powerful framework for the construction of evolution models of natural selection (see for example the review paper Ref. 29). A central concept in the game-theoretic approach to ecology and evolution is that of evolutionary stable strategies (ESS), pioneered by Marynard Smith and Price in 1973[28]. For example, this concept has been used as a guide to determine a favorable density distribution among infinitely many steady states to (1.3), see, e.g. Ref. 24.

In this paper, we focus on a nonlocal selection-mutation model with a gradient flow structure. Such structure facilitates the analysis of the underlying dynamics (see Refs. 8 and 22) and fulfills criteria that are relevant for optimization; see, e.g. Ref. 21 for dynamics in gradient systems. Our objective is to investigate issues on time-asymptotic convergence rates towards discrete steady states. This is related to recent results in Refs. 8 and 22.

1.1. *The model and its properties*

Our model equation is the nonlocal selection-mutation equation

$$\partial_t f(t,x) = \Delta f(t,x) + \frac{1}{2} f(t,x)\left(a(x) - \int_X b(x,y) f^2(t,y)dy\right), \quad t>0, \ \ x \in X, \tag{1.4a}$$

$$f(0,x) = f^0(x) \geq 0, \quad x \in X, \tag{1.4b}$$

$$\frac{\partial f}{\partial \nu} = 0, \quad x \in \partial X, \tag{1.4c}$$

this is an intergo-differential equation that describes the evolution of a population of density $f(t,x)$ structured with respect to a continuous trait x, where X is a subset of $\mathbb{R}^m$, and ν is the unit outward normal at a point x on the boundary ∂X. Delicate competition of selection and mutation leads to mathematical and numerical challenges in solving such problems.

In this model, the diffusion term plays certain role of mutations in the population dynamics. Coefficient $a(x)$ is the intrinsic growth rate of individuals with trait x, and $b(x,y) > 0$ represents the competitive interaction between individuals. Such a trait dependent competition appears in many population balance models of Lotka–Volterra type, see, e.g. Refs. 4, 12, 17 and 18. In particular, a competition model for fish species population is introduced in Ref. 38 to study the effect of exploitation on these species. Compared to competition term of classical models $-\int_X b(x,y)f(t,y)dy$. The main difference between (1.4) and classical models is its nonlinear competition term

$$-\int_X b(x,y) f^2(t,y)dy.$$

From the biological point of view, such competition terms may represent direct predation. More commonly though, it is typically a simplified way of modeling competition for resources between subpopulations. Classical linear competition terms like $-\int_X b(x,y)f(t,y)dy$ have long been used in that context, e.g. Lotka–Volterra systems. They can simply be seen as assuming that each subpopulation will consume an amount of resource proportional to their headcount; the kernel $b(x,y)$ then models how close the resources used by the subpopulation with trait x are to the ones used by the subpopulation with trait y.

Nonlinear terms $-\int_X b(x,y)\, f^2(t,y)dy$, as we consider here, are used where modeling more complex effects between population and resources. At the level of microorganisms, resource consumption results from a chain of biochemical reactions where various subpopulations may each play a role; when those subpopulations are small, their interactions may in fact look more collaborative than competitive (see for example Ref. 23). On larger animal populations, such as the fish species model mentioned above, resources will usually include over animal population in complex interaction. In particular one may observe some drastic effects on the resource populations with almost complete collapse when the predatory population becomes too large. This type of effects are poorly represented by linear competition terms with respect to the quadratic term which increases the impact of large populations.

From a mathematical point of view, the main attractive feature of model (1.4a) is its gradient flow structure in the sense that (1.4a) can be written as

$$\partial_t f = -\frac{1}{2}\frac{\delta F}{\delta f}, \tag{1.5}$$

where the corresponding energy functional is

$$\begin{aligned} F[f] = {} & \frac{1}{4}\iint b(x,y) f^2(t,x) f^2(t,y)dxdy \\ & - \frac{1}{2}\int a(x) f^2(t,x)dx + \int |\nabla_x f(t,x)|^2 dx, \end{aligned} \tag{1.6}$$

so that the energy dissipation law $\frac{d}{dt}F[f] = -2\int |\partial_t f|^2 dx \leq 0$ holds for all $t > 0$, at least for classical solutions.

Such a gradient flow structure is preserved by the finite volume scheme recently proposed in Ref. 8, in which we show that both semi-discrete and fully discrete schemes satisfy the two desired properties: positivity of numerical solutions and energy dissipation. These ensure that the positive steady state is asymptotically stable. A rigorous analysis on global existence of (1.4), and time-asymptotic convergence to the positive steady state is further provided in Ref. 22, complementing with the numerical results in Ref. 8.

The objective of this paper is to study the time asymptotic convergence rates towards discrete steady states of the semi-discrete scheme in Ref. 8. Our starting point is Ref. 22, and we refer the reader to it for more references to earlier results on this and related models.

It is of course natural to conjecture that steady states of the system describes its permanent behavior. For this reason, calculating steady states and obtaining rates of convergence is especially useful. In models without mutations like (1.3), there can actually exist several steady states: u is a steady state iff

$$a(x) = \int b(x,y)u(y)dy \quad \forall\, x \in \operatorname{supp} u.$$

Many of those steady states can actually be unstable, for example if for some $x_0 \notin \operatorname{supp} u$,

$$a(x_0) > \int b(x_0, y)\, u(y)\, dy$$

then a mutant with trait x_0 may invade and u is unstable. This leads to the definition of ESS which also satisfies

$$a(x) \leq \int b(x,y)u(y)dy, \quad \forall\, x.$$

The ESS may be interpreted as a Nash equilibrium with appropriately defined payoff. Note that a strict inequality is sometimes required instead (or some other condition as in the famous Ref. 28), in which case a Nash equilibrium may not be an ESS.

With mutations and diffusion in the model however, one expects any steady state to have full support and hence not to require any additional assumption for stability. Moreover in our present case the nonlinear dependence with respect to u in the competition term prevents any straightforward interpretation in terms of game theory or strategy.

To compare the results in Ref. 22 with what we do here we need to recall some conventions. In Ref. 22, we make the following basic assumptions:

$$a \in L^\infty(X), \quad |\{x \,|\, a(x) > 0\}| \neq 0; \tag{1.7a}$$

$$b \in L^\infty(X \times X), \quad \operatorname{essinf}_{x,x' \in X} b(x,x') > 0; \tag{1.7b}$$

$$b(x,y) = b(y,x), \quad \forall\, g \in L^1(X)\backslash\{0\}, \quad \iint b(x,y)g(x)g(y)dxdy > 0. \tag{1.7c}$$

From the biological point of view, the first assumption on a in (1.7a) simply means that the growth rate of the population is necessarily limited. The second assumption ensures that there are at least some phenotypic traits where the population grows so as to avoid extinction. Assumption (1.7b) simply again guarantees that competition exists between any subpopulations but cannot be unlimited. Assumption (1.7c) is more fundamental for the structure of the interactions between sub-populations. The equality $b(x,y) = b(y,x)$ forces a strict symmetry in the competition: sub-population x has exactly the same negative impact on sub-population y that y has on x. The second part of (1.7c) is intimately connected to the stability of the dynamics. It imposes a strong constraint on how small perturbation in the population density will increase competition.

With these assumptions, in Ref. 22, the steady state g is shown to be strictly positive if $\lambda_1 < 1/2$, where λ_1 is uniquely determined by

$$\lambda_1 = \frac{\int |\nabla_x \psi|^2 dx}{\int a\psi^2 dx} = \inf\left\{ \frac{\int |\nabla_x v|^2 dx}{\int a v^2 dx} : v \in D(L_1) \text{ and } \int a v^2 dx > 0 \right\}, \tag{1.8}$$

where the positive function $\psi \in D(L_1)$ with $\int a\psi^2 dx > 0$ and $D(L_1) = \{u \in H^2(X) : \partial_n u|_{\partial X} = 0\}$ is the domain of the Laplace operator $L_1 u = -\Delta u$. And the principal result of Ref. 22 becomes

Theorem 1.1. (Theorem 1.3 in Ref. 22) *Assume both a and b satisfy* (1.7). *Consider any nonnegative $f^0 \in L^1(X) \cap L^\infty(X)$. Then the corresponding solution $f(t,\cdot)$ of* (1.4) *converges to the steady state g*

$$\lim_{t\to\infty} \|f(t,\cdot) - g(\cdot)\|_{L^2(X)} = 0. \tag{1.9}$$

And moreover, there exists C depending on initial data f^0 and $g \geq 0$ such that

$$\int |f(t,x) - g(x)|^2 dx \leq Ce^{-rt} \quad \forall t > 0,$$

for $\int a dx \geq 0$ or $\int a dx < 0$ with $\lambda_1 \neq \frac{1}{2}$, where of course $g = 0$ if $\lambda_1 > 1/2$.

For $\int a dx < 0$ and $\lambda_1 = \frac{1}{2}$,

$$\int |f(t,x)|^2 dx \leq \frac{C}{1+t} \quad \forall t > 0.$$

In this paper, we establish a discrete version of Theorem 1.1 and related results.

1.2. *Assumptions and main results*

For simplicity of presentation, we restrict ourselves to only one-dimensional setting for $X = [0, L]$. We partition X into subcells $I_j = [x_{j-1/2}, x_{j+1/2}]$, $j = 1 \cdots N$, for a uniform mesh of size $h = L/N$ so that $x_{j-1/2} = x_{1/2} + (j-1)h$ with $x_{1/2} = 0$, $x_{N+1/2} = L$. We consider the following semi-discrete scheme:

$$\frac{d}{dt} f_j = \frac{f_{j-1} - 2f_j + f_{j+1}}{h^2} + \frac{1}{2} f_j \left(\bar{a}_j - h \sum_{i=1}^{N} \bar{b}_{ji} f_i^2 \right), \quad 1 \leq j \leq N, \tag{1.10}$$

where

$$f_0 = f_1, \quad f_{N+1} = f_N,$$
$$\bar{a}_j = \frac{1}{h} \int_{I_j} a(x) dx, \quad \bar{b}_{ji} = \frac{1}{h^2} \int_{I_i} \int_{I_j} b(x,y) dx dy, \tag{1.11}$$

and the numerical solution $f_j(t)$ approximates the average of the exact solution f on I_j. From the basic assumptions (1.7) one may derive similar assumptions at the discrete level:

$$|\bar{a}_j| \leq \|a\|_\infty, \quad \{1 \leq j \leq N, \bar{a}_j > 0\} \neq \emptyset; \tag{1.12a}$$

$$0 < b_m \leq \bar{b}_{ji} \leq \|b\|_\infty \quad \text{for } 1 \leq i, j \leq N; \tag{1.12b}$$

$$\bar{b}_{ji} = \bar{b}_{ij}; \sum_{j=1}^{N} \sum_{i=1}^{N} \bar{b}_{ji} v_i v_j > 0 \quad \text{for any } v_j \text{ such that } \sum_{j=1}^{N} |v_j|^2 \neq 0. \tag{1.12c}$$

Under additional assumptions, (1.12) ensures the existence and uniqueness of positive steady state g satisfying

$$\frac{-g_1+g_2}{h^2}+\frac{1}{2}g_1\left(\bar{a}_1-h\sum_{i=1}^{N}\bar{b}_{1i}g_i^2\right)=0, \tag{1.13a}$$

$$\frac{g_{j-1}-2g_j+g_{j+1}}{h^2}+\frac{1}{2}g_j\left(\bar{a}_j-h\sum_{i=1}^{N}\bar{b}_{ji}g_i^2\right)=0, \quad j=2,\ldots,N-1, \tag{1.13b}$$

$$\frac{g_{N-1}-g_N}{h^2}+\frac{1}{2}g_N\left(\bar{a}_N-h\sum_{i=1}^{N}\bar{b}_{Ni}g_i^2\right)=0. \tag{1.13c}$$

Note that (1.12b)–(1.12c) imply that $B=(\bar{b}_{ij})_{N\times N}$ is a positive definite matrix, so it defines a scalar product

$$\langle w,v\rangle = h^2\sum_{i,j=1}^{N}\bar{b}_{ij}w_iv_j$$

with corresponding norm

$$\|w\|_b=\left(h^2\sum_{i,j=1}^{N}\bar{b}_{ji}w_jw_i\right)^{\frac{1}{2}}. \tag{1.14}$$

We will also use the discrete l^p norm

$$\|w\|_p=\left(\sum_{j=1}^{N}|w_j|^p h\right)^{1/p}.$$

In order to characterize the conditions on $\bar{a}_j$ for which the steady state g is strictly positive, we set

$$A=\left\{v\in\mathbb{R}^N | v\neq 0 \text{ and } \sum_{j=1}^{N}\bar{a}_j{v_j}^2h>0\right\}, \tag{1.15}$$

and

$$K[v]=\frac{\sum_{j=1}^{N-1}(v_{j+1}-v_j)^2h^{-1}}{\sum_{j=1}^{N}\bar{a}_j{v_j}^2h}, \quad v\in A. \tag{1.16}$$

We then have the following result.

Theorem 1.2. *There exists $u\in A$, such that*

$$\lambda_1=K[u]=\min_{v\in A}K[v]. \tag{1.17}$$

Moreover,

(i) *If $h\sum_{j=1}^{N}\bar{a}_j\geq 0$, then $\lambda_1=0$;*
(ii) *If $h\sum_{j=1}^{N}\bar{a}_j<0$, then $\lambda_1>0$.*

We can show that the steady state is strictly positive if $\lambda_1 < 1/2$. More precisely, we have the following result.

Theorem 1.3. *There exists* $g \in S = \{f \in \mathbb{R}^N \mid f \geq 0\}$ *satisfying* (1.13). *Moreover,*

(i) *if* $0 \leq \lambda_1 < 1/2$, *then there exists a unique positive solution such that*

$$0 < g_{\min} \leq g_j \leq g_{\max} < \infty \quad \text{for } 1 \leq j \leq N;$$

(ii) *if* $\lambda_1 \geq 1/2$, *there is no positive steady state.*

Thanks to this result, we can show the convergence of $f(t)$ to g as $t \to \infty$.

Theorem 1.4. *Assume both* $\bar{a}_j$ *and* $\bar{b}_{ij}$ *satisfy* (1.12). *Let* $f_j(t)$ *be the solution to* (1.10) *subject to initial data* f_j^0, *and* g *the steady state. Then there exists* λ *independent of* f^0, *and* C *depending on the initial data* f^0 *such that*

$$\|f(t) - g\|_2 \leq C \exp(-\lambda t), \quad \forall t > 0,$$

for $\sum_{j=1}^N \bar{a}_j h \geq 0$ *or* $\sum_{j=1}^N \bar{a}_j h < 0$ *with* $\lambda_1 \neq 1/2$, *where* $g = 0$ *if* $\sum_{j=1}^N \bar{a}_j h < 0$ *with* $\lambda_1 > 1/2$.

For $\sum_{j=1}^N \bar{a}_j h < 0$ *with* $\lambda_1 = 1/2$,

$$\|f\|_2 \leq \frac{C}{1+t}, \quad \forall t > 0.$$

The proofs of these theorems rely on a careful use of the competition assumption and are given in Sec. 2. The main techniques in the proof of Theorem 1.4 mimic those in Ref. 22, yet we overcome a number of new difficulties arising from the spatial discretization.

Let us remark that under the transformation $u = f^2$, the resulting equation from model (1.4) becomes

$$\partial_t u(t,x) = \Delta u - \frac{|\nabla u|^2}{2u} + u(t,x)\left(a(x) - \int_X b(x,y)u(t,y)dy\right).$$

Therefore there does not seem to be any simple way to reduce (1.4) to an already studied case.

Numerical schemes with similar methodology to that in Ref. 8 have been proposed and analyzed for the linear selection dynamics governed by (1.3) in Refs. 7 and 26. The schemes in Ref. 26 feature two nice properties: positivity preserving and entropy satisfying, and numerical solutions are proved to asymptotically converge to the Evolutionary Stable Distribution (ESD). The exponential convergence rates towards the special ESD and the algebraic convergence rate towards the general ESD are obtained in Ref. 7.

The rest of this paper is devoted to the proof of Theorems 1.2–1.4, as presented above.

2. Proofs of the Results

2.1. *Proof of Theorem* 1.2

We recall that the minimum value of a multivariate continuous function in a closed domain can be reached (the discrete problem is finite-dimensional). Hence from

$$\min_{v\in A} K[v] = \min_{w\in\mathbb{R}^N, h\sum_{j=1}^N \bar{a}_j {w_j}^2=1} \sum_{j=1}^{N-1}(w_{j+1}-w_j)^2 h^{-1}, \quad w_j = \frac{v_j}{\sqrt{\sum_{j=1}^N \bar{a}_j {v_j}^2 h}},$$

it follows that there exists $u \in A$, such that

$$\lambda_1 = K[u] = \min_{v\in A} K[v]. \tag{2.1}$$

Note that we always have $\lambda_1 \geq 0$. Introduce

$$Q_\lambda(v) = \sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1} - \lambda\sum_{j=1}^N \bar{a}_j {v_j}^2 h, \tag{2.2}$$

clearly

$$Q_{\lambda_1}(u) = 0,$$

and $Q_{\lambda_1}(v) \geq 0$ for all $v \in A$.

(i) In order to show $\lambda_1 = 0$ when $h\sum_{j=1}^N \bar{a}_j \geq 0$, it suffices to show for any $\lambda > 0$ there exists $v \in A$ such that $Q_\lambda(v) < 0$.

If $\sum_{j=1}^N \bar{a}_j h > 0$, we choose $v = (1, 1, \ldots, 1) \in \mathbb{R}^N$, so that

$$Q_\lambda(v) = -\lambda\sum_{j=1}^N \bar{a}_j h < 0.$$

If $\sum_{j=1}^N \bar{a}_j h = 0$, we choose any $v \in A$ such that $\sum_{j=1}^N \bar{a}_j v_j h > 0$ and $s \in \mathbb{R}$, so that for $s > 0$ sufficiently small we have

$$\begin{aligned} Q_\lambda(1+sv) &= s^2\sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1} - \lambda\sum_{j=1}^N \bar{a}_j\left(1+2sv_j+s^2v_j^2\right)h \\ &= s^2 Q_\lambda(v) - 2s\lambda\sum_{j=1}^N \bar{a}_j v_j h < 0. \end{aligned}$$

(ii) If $\sum_{j=1}^N \bar{a}_j h < 0$, the situation is different. The key to the situation is the following lemma, whose continuous analog is stated by Fleming in Ref. 15.

Lemma 2.1. *If* $\sum_{j=1}^N \bar{a}_j h < 0$, *then there exists* $\epsilon > 0, \eta > 0$ *such that*

$$\sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1} \geq \epsilon\sum_{j=1}^N v_j^2 h$$

for all $v \in \mathbb{R}^N$ *such that* $\sum_{j=1}^N \bar{a}_j v_j^2 h > -\eta\sum_{j=1}^N v_j^2 h$.

Assume this lemma for the time being. If $\sum_{j=1}^{N} \bar{a}_j v_j^2 h > 0$, then

$$K[v] \geq \frac{\sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1}}{\max\limits_{1\leq j\leq N} |\bar{a}_j| \sum_{j=1}^{N} {v_j}^2 h} \geq \epsilon/\|a\|_\infty > 0,$$

and so $\lambda_1 \geq \epsilon/\|a\|_\infty > 0$.

We now return to prove Lemma 2.1 by contradiction. Suppose a sequence $\{v^n\} \subset \mathbb{R}^N$ such that

$$\sum_{j=1}^{N}(v_j^n)^2 h = 1, \quad \sum_{j=1}^{N} \bar{a}_j (v_j^n)^2 h > -1/n \quad \text{and} \quad \sum_{j=1}^{N-1}(v_{j+1}^n - v_j^n)^2 h^{-1} < 1/n,$$

for all n. Then $\{v^n\}$ is bounded in $\mathbb{R}^N$ and so has a subsequence $\{v^{n_k}\}$ converging to v in $\mathbb{R}^N$ (by the Bolzano–Weierstrass theorem). Then

$$\sum_{j=1}^{N}(v_j)^2 h = 1, \quad \sum_{j=1}^{N} \bar{a}_j (v_j)^2 h \geq 0 \quad \text{and} \quad \sum_{j=1}^{N-1}(v_{j+1} - v_j)^2 h^{-1} = 0.$$

Since $\sum_{j=1}^{N}(v_j)^2 h = 1$, and $\sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1} = 0$, we must have that $v_j \equiv c = 1/\sqrt{L}$ for $j = 1, 2, \ldots, N$. But this implies

$$\sum_{j=1}^{N} \bar{a}_j (v_j)^2 h = c^2 \sum_{j=1}^{N} \bar{a}_j h < 0$$

and this is a contradiction. Thus, we have finished the proof of Lemma 2.1.

Finally, we relate the sign of $Q_\lambda(v)$ to the value of $\lambda < \lambda_1$. This result will be used to prove exponential convergence to zero steady state in Theorem 1.4.

Lemma 2.2. *Assume $h\sum_{j=1}^{N} \bar{a}_j < 0$ and $0 < \lambda < \lambda_1$, then there exists $\nu > 0$ (ν depends on λ) such that $Q_\lambda(v) \geq \nu\|v\|^2$ for all $v \in \mathbb{R}^N$.*

Proof. Let $\lambda = (1-s)\lambda_1$ with $0 < s < 1$. Then,

$$\begin{aligned} Q_\lambda(v) &= \sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1} - \lambda \sum_{j=1}^{N} \bar{a}_j v_j^2 h \\ &= \frac{\lambda}{\lambda_1} Q_{\lambda_1}(v) + \left(1 - \frac{\lambda}{\lambda_1}\right) \sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1} \\ &\geq s \sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1}, \end{aligned}$$

since, by the definition of λ_1, $Q_{\lambda_1} \geq 0$.

Let ϵ and η be the constants described in Lemma 2.1. If $\sum_{j=1}^{N} \bar{a}_j v_j^2 h > -\eta \sum_{j=1}^{N} v_j^2 h$, we have

$$Q_\lambda(v) \geq s \sum_{j=1}^{N-1}(v_{j+1}-v_j)^2 h^{-1} \geq s\epsilon \sum_{j=1}^{N} v_j^2 h,$$

and if $\sum_{j=1}^N \bar{a}_j v_j^2 h \le -\eta \sum_{j=1}^N v_j^2 h$, we have

$$Q_\lambda(v) = \sum_{j=1}^{N-1} (v_{j+1} - v_j)^2 h^{-1} - \lambda \sum_{j=1}^{N} \bar{a}_j v_j^2 h \ge \lambda\eta \sum_{j=1}^{N} v_j^2 h. \qquad \square$$

2.2. *Proof of Theorem* 1.3

Here we give a variational construction of a positive solution of (1.13). The uniqueness of positive steady state has been proven in Ref. 8. First, we claim that having a nonnegative solution to (1.13) is equivalent to the nonzero critical point of the functional

$$F(w) = \frac{1}{4} \sum_{j,i=1}^{N} \bar{b}_{ji} (w_j)^2 (w_i)^2 h^2 - \frac{1}{2} \sum_{j=1}^{N} \bar{a}_j (w_j^+)^2 h + \sum_{j=1}^{N-1} (w_{j+1} - w_j)^2 h^{-1}, \tag{2.3}$$

where $w^+ = \max\{w, 0\}$. Indeed, a nonnegative solution of (1.13) is obviously a critical point of $F(w)$ since $\partial_w F(w)|_{w=g} = 0$ is exactly the system (1.13). Conversely, if g is a critical point of $F(w)$, then

$$\begin{aligned} 0 = \sum_{j=1}^{N} \partial_{w_j} F(g) g_j^- &= h^2 \sum_{i,j=1}^{n} \bar{b}_{i,j} g_i^2 (g_j^-)^2 + \frac{2}{h} \sum_{j=1}^{N-1} (g_{j+1} - g_j)(g_{j+1}^- - g_j^-) \\ &= h^2 \sum_{i,j=1}^{n} \bar{b}_{i,j} g_i^2 (g_j^-)^2 + \frac{2}{h} \sum_{j=1}^{N-1} (g_{j+1}^- - g_j^-)^2 - \frac{2}{h} \sum_{j=1}^{N-1} (g_{j+1}^- g_j^+ + g_{j+1}^+ g_j^-) \\ &\ge h^2 \sum_{i,j=1}^{n} \bar{b}_{i,j} g_i^2 (g_j^-)^2 + \frac{2}{h} \sum_{j=1}^{N-1} (g_{j+1}^- - g_j^-)^2, \end{aligned}$$

where $g_j^- = \min\{g_j, 0\}$. We see that $g^- = 0$, which implies $g \ge 0$. Hence g is a nonnegative solution of (1.13).

We next prove the existence of a minimizer for the variational problem F. By Young's inequality, we have

$$\begin{aligned} F(w) &\ge \frac{1}{4} b_m \|w\|_2^4 - \frac{1}{2} \|a\|_\infty \|w\|_2^2 + \sum_{j=1}^{N-1} (w_{j+1} - w_j)^2 h^{-1} \\ &\ge -\frac{\|a\|_\infty^2}{4b_m}. \end{aligned}$$

This says that $F(w)$ is bounded from below. Hence $m = \inf_{w \in S} F(w)$ is finite. Select a minimizing sequence $\{g^k\}_{k=1}^\infty$, so that

$$\lim_{k\to\infty} F(g^k) = m.$$

Set $C = \sup_k F(g^k)$, then

$$\frac{1}{4} b_m \|g^k\|_2^4 - \frac{1}{2}\|a\|_\infty \|g^k\|_2^2 \leq C,$$

which implies that

$$\|g^k\|_2^2 \leq \frac{\|a\|_\infty}{b_m} + \frac{1}{b_m}\sqrt{\|a\|_\infty^2 + 4b_m C} < \infty.$$

Hence, $\{g^k\}$ is bounded sequence in $\mathbb{R}^N$. There exist $g \in \mathbb{R}^N$ and a subsequence of $\{g^k\}$ (still denoted by g^k) converging to g such that for any $v \in \mathbb{R}^N$,

$$h \sum_{j=1}^N v_j (g_j^k - g_j) = 0 \quad \text{as } k \to \infty.$$

Note that

$$|w_{j+1}^+ - w_j^+| \leq |w_{j+1} - w_j|,$$

therefore

$$F(g^+) \leq F(g),$$

one may replace g^k by its positive part and as a consequence we may assume $g^k \geq 0$. Hence $g \geq 0$. A direct calculation shows that

$$h \sum_{j=1}^N |g_j^k|^2 - h \sum_{j=1}^N |g_j|^2 - h \sum_{j=1}^N |g_j^k - g_j|^2 = 2h \sum_{j=1}^N g_j (g_j^k - g_j) \to 0, \quad \text{as } k \to \infty$$

and

$$\begin{aligned} &h \sum_{j=1}^{N-1} |g_{j+1}^k - g_j^k|^2 - h \sum_{j=1}^{N-1} |g_{j+1} - g_j|^2 - h \sum_{j=1}^{N-1} |g_{j+1}^k - g_j^k - (g_{j+1} - g_j)|^2 \\ &\quad = 2h \sum_{j=1}^{N-1} (g_{j+1} - g_j)\big[g_{j+1}^k - g_j^k - (g_{j+1} - g_j)\big] \to 0, \quad \text{as } k \to \infty. \end{aligned}$$

Note also that

$$\begin{aligned} &\sum_{i,j=1}^N \bar{b}_{ji} (g_i^k)^2 (g_j^k)^2 h^2 - \sum_{i,j=1}^N \bar{b}_{ji} (g_i^k - g_i)^2 (g_j^k - g_j)^2 h^2 \\ &\quad = \sum_{i,j=1}^N \bar{b}_{ji} [(g_i^k)^2 - (g_i^k - g_i)^2][(g_j^k)^2 - (g_j^k - g_j)^2] h^2 \\ &\qquad + 2 \sum_{i,j=1}^N \bar{b}_{ji} [(g_i^k)^2 - (g_i^k - g_i)^2](g_j^k - g_j)^2 h^2 \\ &\quad \to \sum_{i,j=1}^N \bar{b}_{ji} (g_i)^2 (g_j)^2 h^2, \quad \text{as } k \to \infty \end{aligned}$$

and

$$\lim_{k\to\infty}\sum_{j=1}^{N}\bar{a}_j(g_j^k)^2h=\lim_{k\to\infty}\sum_{j=1}^{N}\bar{a}_j(g_j)^2h.$$

These together ensure that

$$F(g^k)-\sum_{j=1}^{N-1}|g_{j+1}-g_j|^2-\sum_{j=1}^{N-1}|g_{j+1}^k-g_j^k-(g_{j+1}-g_j)|^2$$
$$-\frac{1}{4}\sum_{i,j=1}^{N}\bar{b}_{ji}(g_i^k-g_i)^2(g_j^k-g_j)^2h^2\to F(g),\quad \text{as } k\to\infty.$$

Then

$$m=\lim_{k\to\infty}F(g^k)\geq F(g)\geq m.$$

By $g\in S$, it follows that

$$F(g)=m=\min_{f\in S}F(f).$$

This proves the existence of a nonnegative minimizer.

To prove that g is not identically 0, when $\sum_{j=1}^N\bar{a}_jh\geq 0$ or $\sum_{j=1}^N\bar{a}_jh<0$ with $\lambda_1<1/2$, we discuss case by case, keeping in mind that $F(0)=0$.

(1) If $\sum_{j=1}^N\bar{a}_jh>0$, choose $w\equiv\epsilon$, then

$$F(\epsilon)=\frac{\epsilon^4}{4}\sum_{j,i=1}^{N}\bar{b}_{ji}h^2-\frac{\epsilon^2}{2}\sum_{j=1}^{N}\bar{a}_jh<0$$

for ϵ small enough.

(2) If $\sum_{j=1}^N\bar{a}_jh=0$, we take $w=\epsilon(1+\delta v)$ satisfying $\sum_{j=1}^N\bar{a}_jv_jh>0$, so that

$$\begin{aligned}F(w)&=\frac{\epsilon^4}{4}\sum_{i,j=1}^{N}\bar{b}_{ji}(1+\delta v_j)^2(1+\delta v_i)^2h^2\\&\quad-\frac{\epsilon^2}{2}\sum_{j=1}^{N}\bar{a}_j(1+\delta v_j)^2h+\epsilon^2\delta^2\sum_{j=1}^{N-1}(v_{j+1}-v_j)^2h^{-1}\\&=\frac{\epsilon^2}{2}\left[\frac{\epsilon^2}{2}\sum_{i,j=1}^{N}\bar{b}_{ji}(1+\delta v_j)^2(1+\delta v_i)^2h^2\right.\\&\quad\left.+\delta^2\left(2\sum_{j=1}^{N-1}(v_{j+1}-v_j)^2h^{-1}-\sum_{j=1}^{N}\bar{a}_jv_j^2h\right)-2\delta\sum_{j=1}^{N}\bar{a}_jv_jh\right]<0\end{aligned}$$

for ϵ,δ suitably small.

(3) If $\sum_{j=1}^N \bar{a}_j h < 0$ with $\lambda_1 < 1/2$, we take $w = \tau u$ with $\tau > 0$ and u satisfying (2.1) so that

$$F(\tau u) = \frac{\tau^4}{4}\sum_{i,j=1}^N \bar{b}_{ji} u_j^2 u_i^2 h^2 + \tau^2\left(\lambda_1 - \frac{1}{2}\right)\sum_{j=1}^N \bar{a}_j u_j^2 h$$
$$\leq \frac{\tau^2}{4}\|u^2\|_b^2\left[\tau^2 + 4\left(\lambda_1 - \frac{1}{2}\right)\frac{\sum_{j=1}^N \bar{a}_j u_j^2 h}{\|u^2\|_b^2}\right] < 0$$

for $\tau > 0$ sufficiently small. Hence, in all these three cases the minimizer cannot be 0.

(4) Finally, we show 0 is the only minimizer if $\sum_{j=1}^N \bar{a}_j h < 0$ and $\lambda_1 \geq 1/2$. Note for any $v \in \mathbb{R}^N$ with $\sum_{j=1}^N \bar{a}_j v_j^2 h \leq 0$, we have

$$F(v) \geq \frac{1}{4}\sum_{i,j=1}^N \bar{b}_{ji} v_j^2 v_i^2 h^2 + \sum_{j=1}^{N-1}(v_{j+1} - v_j)^2 h^{-1} \geq 0.$$

If $\sum_{j=1}^N \bar{a}_j v_j^2 h > 0$, then

$$F(v) = \frac{1}{4}\|v^2\|_b^2 + \left(\lambda_1 - \frac{1}{2}\right)\sum_{j=1}^N \bar{a}_j v_j^2 h + Q_{\lambda_1}(v) \geq 0 = F(0).$$

That is, we have $F(v) \geq 0 = F(0)$, in such case 0 is the only minimizer.

Finally, we show the $0 < g_{\min} \leq g_j \leq g_{\max} < \infty$. Assume $g_{\min} = g_{j_0} = 0$, then from (1.13b), we have

$$g_{j_0+1} + g_{j_0-1} = 0,$$

leading to $g_{j_0 \pm 1} = 0$, hence $g_j = 0$ for all $1 \leq j \leq N$, leading to a contradiction. Assume that $g_{\max} = g_{j_0}$, then again from (1.13b), we have

$$\|a\|_\infty \geq \bar{a}_{j_0} \geq b_m \|g\|_2^2.$$

Hence, $g_{\max} \leq \sqrt{\frac{\|a\|_\infty}{b_m h}} < \infty$.

2.3. *Proof of Theorem* 1.4

As it will prove convenient later, we first give a uniform l^2-bound of the numerical solution when $\bar{b}_{ji} \geq b_m > 0$. Let $\gamma = \|a\|_\infty / b_m$ which will be used to quantity the uniform bound.

Lemma 2.3. *Assume* (1.12) *holds. Let f be the solution to* (1.10) *with nonnegative initial data $f^0 \geq 0$, and $\|f^0\|_2 < \infty$. Then,*

$$\|f\|_2 \leq \max\left\{\|f^0\|_2, \sqrt{\gamma}\right\} = M. \tag{2.4}$$

Proof. Let $Q(t) = \sum_{j=1}^N f_j^2 h = \|f\|_2^2$. Then using (1.10), we have

$$\frac{d}{dt}Q(t) = 2\sum_{j=1}^N \frac{f_j f_{j-1} - 2f_j^2 + f_j f_{j+1}}{h} + \sum_{j=1}^N \left[f_j^2 h \left(\bar{a}_j - h\sum_{i=1}^N \bar{b}_{ji} f_i^2 \right) \right]. \tag{2.5}$$

The first term on the right-hand side of (2.5) is nonnegative since

$$\frac{2}{h}\left[2\sum_{j=2}^N f_j f_{j-1} - 2\sum_{j=1}^N f_j^2 + f_1^2 + f_N^2 \right] \leq \frac{2}{h}\left[\sum_{j=2}^N f_j^2 + \sum_{j=2}^N f_{j-1}^2 - 2\sum_{j=1}^N f_j^2 + f_1^2 + f_N^2 \right] = 0,$$

where we have used $f_0 = f_1$ and $f_{N+1} = f_N$, and the Cauchy–Schwarz inequality.

The second term on the right-hand side of (2.5) is bounded above by

$$\max_{1\leq j\leq N} \left(\bar{a}_j - h\sum_{i=1}^N \bar{b}_{ji} f_i^2 \right) Q(t) \leq (\|a\|_\infty - b_m Q(t))Q(t).$$

Combining the above estimates, we obtain

$$\frac{d}{dt}Q \leq (\|a\|_\infty - b_m Q(t))Q(t).$$

Hence, $Q(t) \leq \max\{Q(0), \gamma\}$. This yields the claimed estimate (2.4). □

Before going further, we provide a discrete version of the one-dimensional Poincare inequality, which will be used twice in the proof of Theorem 1.4.

Lemma 2.4. *For any* $v = (v_1, v_2, \ldots, v_N) \in \mathbb{R}^N$, *then*

$$\sum_{j=1}^{N-1} |v_{j+1} - v_j|^2 \geq \frac{6}{N^2} \inf_{c\in\mathbb{R}} \sum_{j=1}^N |v_j - c|^2, \tag{2.6}$$

where the minimum is achieved at $c^* = \frac{1}{N}\sum_{j=1}^N v_j$.

Proof. The quadratic form in c in (2.6) implies that the minimum point must be $c^* = \frac{1}{N}\sum_{j=1}^N v_j$. To prove (2.6), it suffices to show

$$\sum_{j=1}^N |Nv_j - (v_1 + v_2 + \cdots + v_N)|^2 \leq \frac{N^4}{6} \sum_{j=1}^{N-1} |v_{j+1} - v_j|^2. \tag{2.7}$$

The left-hand side of (2.7) can be estimated using the Cauchy–Schwartz inequality as

$$\begin{aligned}\sum_{j=1}^{N}\left|\sum_{i=1}^{N}(v_j - v_i)\right|^2 &= \sum_{j=1}^{N}\left|\sum_{i=1}^{j-1} i(v_{i+1} - v_i) - \sum_{i=j}^{N-1}(N-i)(v_{i+1} - v_i)\right|^2 \\ &\le \sum_{j=1}^{N}\left(\sum_{i=1}^{j-1} i^2 + \sum_{i=j}^{N-1}(N-i)^2\right)\sum_{j=1}^{N-1}|v_{j+1} - v_j|^2 \\ &= \frac{1}{3}\sum_{i=1}^{N-1} i(i+1)(2i+1)\sum_{j=1}^{N-1}|v_{j+1} - v_j|^2 \\ &= \frac{N^2(N^2-1)}{6}\sum_{j=1}^{N-1}|v_{j+1} - v_j|^2.\end{aligned}$$

This leads to (2.7). □

We proceed to prove Theorem 1.4. For a positive steady state $g > 0$, we introduce the auxiliary functional

$$G = h\sum_{j=1}^{N}\left[\frac{f_j^2 - g_j^2}{2} - g_j^2 \ln\left(\frac{f_j}{g_j}\right)\right].$$

After rewriting, we have

$$\begin{aligned}G &= \frac{h}{2}\sum_{j=1}^{N}\left[f_j^2 - g_j^2 - g_j^2 \ln\left(\frac{f_j^2}{g_j^2}\right)\right] \\ &= \frac{h}{2}\sum_{j=1}^{N}\left[(f_j^2 - g_j^2) - g_j^2\int_0^1 \frac{f_j^2 - g_j^2}{sf_j^2 + (1-s)g_j^2}ds\right] \\ &= \frac{h}{2}\sum_{j=1}^{N} K_j(f_j - g_j)^2,\end{aligned}$$

where

$$K_j = \int_0^1 \frac{s(f_j + g_j)^2}{sf_j^2 + (1-s)g_j^2}ds.$$

We thus can estimate both the upper and lower bound of G.

For some $\eta \in (0, \eta_0]$ with $\eta_0 < 1$,

$$G = \frac{h}{2}\left[\sum_{f_j \ge \eta g_j} K_j(f_j - g_j)^2 + \sum_{f_j < \eta g_j} K_j(f_j - g_j)^2\right] =: I + II.$$

If $f_j \geq \eta g_j$, then

$$K_j \leq \int_0^1 \frac{s(f_j+g_j)^2}{sf_j^2} ds \leq \left(1+\frac{g_j}{f_j}\right)^2 \leq (1+\eta^{-1})^2.$$

If $f_j < \eta g_j$, then for $\theta \in (0,1)$,

$$\begin{aligned} K_j &\leq 4\int_0^1 \frac{sg_j^2}{sf_j^2+(1-s)g_j^2} ds \\ &\leq 4\left[\int_0^{1-\theta} \frac{sg_j^2}{(1-s)g_j^2} ds + \int_{1-\theta}^1 \frac{sg_j^2}{sf_j^2} ds\right] \\ &= 4\left(-\ln\theta - 1 + \theta + \theta\frac{g_j^2}{f_j^2}\right) \quad \text{taking } \theta = f_j^2/g_j^2 \\ &= 4\left[f_j^2/g_j^2 + 2\ln(g_j/f_j)\right] \\ &\leq 4 + 8\ln(g_j/f_j) \leq C\ln(g_j/f_j), \end{aligned}$$

where $C = 8 + \frac{4}{|\ln\eta_0|}$. This implies that for any $0 < \eta \leq \eta_0 < 1$,

$$\begin{aligned} G &\leq 2\eta^{-2}\|f-g\|_2^2 + C\frac{h}{2}\sum_{f_j\leq\eta g_j} \ln(g_j/f_j)(f_j-g_j)^2 \\ &\leq 2\eta^{-2}\|f-g\|_2^2 + Chg_{\max}^2 \sum_{f_j\leq\eta g_j} \ln(g_j/f_j). \end{aligned} \tag{2.8}$$

On the other hand,

$$\begin{aligned} G &= \frac{h}{2}\sum_{j=1}^N \left[(f_j-g_j)^2 + 2g_j^2\left(\frac{f_j}{g_j} - 1 - \ln(f_j/g_j)\right)\right] \\ &= \frac{1}{2}\|f-g\|_2^2 + h\sum_{j=1}^N g_j^2\left(\frac{f_j}{g_j} - 1 + \ln(g_j/f_j)\right) \\ &\geq \frac{1}{2}\|f-g\|_2^2 + \frac{g_{\min}^2 h}{C}\sum_{f_j\leq\eta g_j} \ln(g_j/f_j) \end{aligned} \tag{2.9}$$

for any $\eta \leq \eta_0 < 1$, where C depends only on η_0.

A direct estimate now gives

$$\begin{aligned} \frac{d}{dt}G &= h\sum_{j=1}^N \left(f_j f_{jt} - g_j^2\frac{f_{jt}}{f_j}\right) = h\sum_{j=1}^N (f_j^2-g_j^2)\frac{f_{jt}}{f_j} \\ &= h\sum_{j=1}^N (f_j^2-g_j^2)\left[\frac{f_{j+1}-2f_j+f_{j-1}}{h^2 f_j} + \frac{1}{2}\left(\bar{a}_j - h\sum_{i=1}^N \bar{b}_{ji} f_i^2\right)\right] \end{aligned}$$

$$= h\sum_{j=1}^{N}(f_j^2 - g_j^2)$$

$$\times \left[\frac{f_{j+1} - 2f_j + f_{j-1}}{h^2 f_j} - \frac{g_{j+1} - 2g_j + g_{j-1}}{h^2 g_j} - \frac{h}{2}\sum_{i=1}^{N}\bar{b}_{ji}(f_i^2 - g_i^2)\right]$$

$$= \frac{1}{h}\sum_{j=1}^{N}(f_j^2 - g_j^2)\left(\frac{f_{j+1} + f_{j-1}}{f_j} - \frac{g_{j+1} + g_{j-1}}{g_j}\right)$$

$$- \frac{h^2}{2}\sum_{i,j=1}^{N}\bar{b}_{ji}(f_i^2 - g_i^2)(f_j^2 - g_j^2)$$

$$= I_1 + I_2 - \frac{1}{2}\|f^2 - g^2\|_b^2,$$

where we have used $f_{N+1} = f_N$, $f_0 = f_1$, $g_{N+1} = g_N$, and $g_0 = g_1$. For I_1, we have

$$I_1 = \frac{1}{h}\sum_{j=1}^{N} f_j^2\left(\frac{f_{j+1} + f_{j-1}}{f_j} - \frac{g_{j+1} + g_{j-1}}{g_j}\right)$$

$$= \frac{1}{h}\sum_{j=1}^{N}\frac{f_j}{g_j}\left[g_j(f_{j+1} + f_{j-1}) - f_j(g_{j+1} + g_{j-1})\right]$$

$$= \frac{1}{h}\sum_{j=1}^{N-1}\frac{f_j}{g_j}(g_j f_{j+1} - f_j g_{j+1}) + \frac{1}{h}\sum_{j=2}^{N}\frac{f_j}{g_j}(g_j f_{j-1} - f_j g_{j-1})$$

$$= \frac{1}{h}\sum_{j=1}^{N-1}\frac{f_j}{g_j}(g_j f_{j+1} - f_j g_{j+1}) + \frac{1}{h}\sum_{j=1}^{N-1}\frac{f_{j+1}}{g_{j+1}}(g_{j+1} f_j - f_{j+1} g_j)$$

$$= \frac{1}{h}\sum_{j=1}^{N-1}\left(\frac{f_j}{g_j} - \frac{f_{j+1}}{g_{j+1}}\right)(g_j f_{j+1} - f_j g_{j+1})$$

$$= -\frac{1}{h}\sum_{j=1}^{N-1} g_j g_{j+1}\left|\frac{f_{j+1}}{g_{j+1}} - \frac{f_j}{g_j}\right|^2.$$

Similarly,

$$I_2 = \frac{1}{h}\sum_{j=1}^{N} g_j^2\left(\frac{g_{j+1} + g_{j-1}}{g_j} - \frac{f_{j+1} + f_{j-1}}{f_j}\right) = -\frac{1}{h}\sum_{j=1}^{N-1} f_j f_{j+1}\left|\frac{g_{j+1}}{f_{j+1}} - \frac{g_j}{f_j}\right|^2.$$

Collecting the above relations, we have

$$\frac{d}{dt}G = -D(f,g) - \frac{1}{h}\sum_{j=1}^{N-1} f_j f_{j+1}\left|\frac{g_{j+1}}{f_{j+1}} - \frac{g_j}{f_j}\right|^2,$$

where

$$D(f,g) = \frac{1}{2}\|f^2 - g^2\|_b^2 + \frac{1}{h}\sum_{j=1}^{N-1} g_j g_{j+1} \left|\frac{f_{j+1}}{g_{j+1}} - \frac{f_j}{g_j}\right|^2.$$

We claim that there exists a constant ν such that for every $P > 0$, we can choose $\eta_P < 1$ to obtain

$$h \sum_{f_j \le \eta g_j} \ln(g_j/f_j) \le \frac{\nu}{P}\frac{1}{h}\sum_{j=1}^{N-1} f_j f_{j+1}\left|\frac{g_{j+1}}{f_{j+1}} - \frac{g_j}{f_j}\right|^2 + \frac{\nu G(0)}{P} G(t), \quad \forall\, \eta \le \eta_P, \tag{2.10}$$

and that there exists $\mu > 0$ such that

$$D(f,g) \ge \mu \|f/g - 1\|_2^2. \tag{2.11}$$

Assuming (2.10) and (2.11) are correct for the time being, this gives for any $0 < \epsilon \le 1$,

$$\begin{aligned}\frac{dG}{dt} &\le -D(f,g) - \frac{\epsilon}{h}\sum_{j=1}^{N-1} f_j f_{j+1}\left|\frac{g_{j+1}}{f_{j+1}} - \frac{g_j}{f_j}\right|^2 \\ &\le -\mu\|f/g - 1\|_2^2 - \frac{\epsilon P}{\nu} h \sum_{f_j \le \eta g_j} \ln(g_j/f_j) + \epsilon G(0) G(t). \end{aligned}\tag{2.12}$$

From the upper bound of G in (2.8), we have

$$\begin{aligned}\|f/g - 1\|_2^2 &\ge \frac{h}{g_{\max}^2}\sum_{j=1}^{N}(f_j - g_j)^2 \\ &\ge \frac{\eta^2}{2g_{\max}^2}\left[G(t) - C g_{\max}^2 h \sum_{f_j \le \eta g_j} \ln(g_j/f_j)\right] \\ &= \frac{\eta^2}{2g_{\max}^2} G(t) - \frac{C\eta^2}{2} h \sum_{f_j \le \eta g_j} \ln(g_j/f_j). \end{aligned}\tag{2.13}$$

Substituting this into (2.12) gives

$$\frac{dG}{dt} \le -\left(\frac{\mu\eta^2}{2g_{\max}^2} - \epsilon G(0)\right) G(t) + \left(\frac{C\mu\eta^2}{2} - \frac{\epsilon P}{\nu}\right) h \sum_{f_j \le \eta g_j} \ln(g_j/f_j).$$

Choose $P = 2C\nu g_{\max}^2 G(0)$ (and hence η_P accordingly), and choose $\epsilon = \frac{\mu\eta^2}{4g_{\max}^2 G(0)}$, which can be made smaller than 1 for some $\eta \le \eta_P$.

This implies that there exists $\lambda = \frac{\mu\eta^2}{4g_{\max}^2}$ such that

$$\frac{dG}{dt} \le -\lambda G(t),$$

which yields $G(t) \leq G(0)\exp(-\lambda t)$. This when combined this with the lower bounded G gives

$$\|f-g\|_2 \leq \sqrt{2G(t)} \leq \sqrt{2G(0)}\exp(-\lambda t/2).$$

We now return to prove (2.10) and (2.11), respectively.

In order to prove (2.10), we recall for any $\xi, \eta > 0$

$$|\xi-\eta|^2 \geq \xi\eta\,|\ln\xi - \ln\eta|^2,$$

with which we proceed to estimate

$$\begin{aligned}
&\frac{1}{h}\sum_{j=1}^{N-1} f_j f_{j+1}\left|\frac{g_{j+1}}{f_{j+1}} - \frac{g_j}{f_j}\right|^2 \\
&\quad\geq \frac{1}{h}\sum_{j=1}^{N-1} g_j g_{j+1}\left|\ln\frac{g_{j+1}}{f_{j+1}} - \ln\frac{g_j}{f_j}\right|^2 \\
&\quad\geq \frac{1}{h}g_{\min}^2\sum_{j=1}^{N-1}\left|\ln\frac{g_{j+1}}{f_{j+1}} - \ln\frac{g_j}{f_j}\right|^2 \\
&\quad\geq \frac{1}{h}g_{\min}^2 Ch^2\inf_c\sum_{j=1}^{N}\left|\ln_+\frac{g_j}{f_j} - c\right|^2,
\end{aligned}$$

where $\ln_+ x = \ln x\,|_{x\geq\eta^{-1}}$ for $0<\eta<1$, and $C = \frac{6}{L^2}$ from (2.6). The optimal constant c is given by

$$c = \frac{h}{L}\sum_{f_j\leq\eta g_j}\ln\frac{g_j}{f_j}. \tag{2.14}$$

By using the lower bound on G, we estimate

$$c \leq \frac{1}{L}\frac{CG(t)}{g_{\min}^2} \leq \frac{CG(0)}{Lg_{\min}^2}.$$

Then using (2.14), we have

$$\begin{aligned}
h\sum_{f_j\leq\eta g_j}\ln\frac{g_j}{f_j} &\leq h\sum_{f_j\leq\eta g_j}\frac{1}{|\ln\eta|}\left(\ln_+\frac{g_j}{f_j}\right)^2 \\
&\leq h\sum_{j=1}^{N}\frac{1}{|\ln\eta|}\left[\left|\ln_+\frac{g_j}{f_j} - c + c\right|^2\right] \\
&\leq \frac{1}{|\ln\eta|}\left[h\sum_{j=1}^{N}\left|\ln_+\frac{g_j}{f_j} - c\right|^2 + c^2L\right]
\end{aligned}$$

$$\leq \frac{1}{|\ln \eta|}\frac{1}{Cg_{\min}^2}\left[\frac{1}{h}\sum_{j=1}^{N-1} f_j f_{j+1}\left|\frac{g_{j+1}}{f_{j+1}} - \frac{g_j}{f_j}\right|^2\right] + \frac{L}{|\ln \eta|}\frac{C^2}{L^2 g_{\min}^4} G(0)G(t),$$

which leads to (2.10) with $\eta_P = e^{-P}$, and

$$\nu = \max\left\{\frac{1}{Cg_{\min}^2}, \frac{C^2}{Lg_{\min}^4}\right\}.$$

Next, we prove (2.11). Using the lower bound of g and (2.6), we obtain

$$\begin{aligned}\frac{1}{h}\sum_{j=1}^{N-1} g_j g_{j+1}\left|\frac{f_{j+1}}{g_{j+1}} - \frac{f_j}{g_j}\right|^2 &\geq \frac{g_{\min}^2}{h}\sum_{j=1}^{N-1}\left|\frac{f_{j+1}}{g_{j+1}} - \frac{f_j}{g_j}\right|^2\\ &\geq Cg_{\min}^2 h \inf_{c\geq 0}\sum_{j=1}^{N}\left|\frac{f_j}{g_j} - c\right|^2,\end{aligned}$$

where the minimum is achieved at $c^* = \frac{1}{N}\sum_{j=1}^{N}\frac{f_j}{g_j}$ and $C = \frac{6}{L^2}$.

As a consequence it suffices to find μ independent of c such that

$$Cg_{\min}^2 h\sum_{j=1}^{N}\left|\frac{f_j}{g_j} - c\right|^2 + \frac{1}{2}\|f^2 - g^2\|_b^2 \geq \mu\left\|\frac{f}{g} - 1\right\|_2^2. \tag{2.15}$$

If $c = 1$, the inequality is obvious for $\mu \leq Cg_{\min}^2$.

For $c \neq 1$, we estimate

$$\begin{aligned}\|f^2 - g^2\|_b^2 &= \|f^2 - c^2g^2 + (c^2-1)g^2\|_b^2\\ &= (c^2-1)^2\|g^2\|_b^2 + 2(c^2-1)\langle f^2 - c^2g^2, g^2\rangle_b + \|f^2 - c^2g^2\|_b^2\\ &\geq \delta(c^2-1)^2\|g^2\|_b^2 - \frac{\delta}{1-\delta}\|f^2 - c^2g^2\|_b^2\end{aligned}$$

for any $0 < \delta < 1$ by using Young's inequality.

Note that for $\|f\| \leq M$, we have

$$\begin{aligned}\|f^2 - c^2g^2\|_b^2 &\leq h^2\|b\|_\infty\left[\sum_{j=1}^{N}(f_j^2 - c^2g_j^2)\right]^2\\ &\leq h^2\|b\|_\infty\left[\sum_{j=1}^{N}(f_j + cg_j)^2\right]\left[\sum_{j=1}^{N}(f_j - cg_j)^2\right]\\ &\leq 2h^2\|b\|_\infty\left(\sum_{j=1}^{N} f_j^2 + c^2\sum_{j=1}^{N} g_j^2\right) g_{\max}^2\sum_{j=1}^{N}(f_j/g_j - c)^2\end{aligned}$$

$$\leq 2\|b\|_\infty g_{\max}^2 \left(\|f\|_2^2 + c^2\|g\|_2^2\right) h \sum_{j=1}^N (f_j/g_j - c)^2$$

$$\leq 2\|b\|_\infty g_{\max}^2 (1+c^2)\tilde{M}^2 \|f/g - c\|_2^2, \quad \tilde{M} := \max\{M, \|g\|\}.$$

Therefore, the left-hand side of (2.15) satisfies

$$\text{LHS} \geq \left(C g_{\min}^2 - \frac{\delta}{2(1-\delta)} 2\|b\|_\infty g_{\max}^2 (1+c^2)\tilde{M}^2\right) \|f/g - c\|_2^2 + \frac{\delta}{2}(c^2-1)^2 \|g^2\|_b^2$$

$$= \frac{C g_{\min}^2}{2} \|f/g - c\|_2^2 + \frac{\delta}{2}(c^2-1)^2 \|g^2\|_b^2$$

by taking $\delta = \frac{C g_{\min}^2}{C g_{\min}^2 + 2(1+c^2)\|b\|_\infty g_{\max}^2 \tilde{M}^2}$.

Furthermore, we use the Young inequality for any $\tau \in (0,1)$ so that

$$\|f/g - c\|_2^2 = \|f/g - 1 + 1 - c\|_2^2 \geq \tau \|f/g - 1\|_2^2 - \frac{\tau L}{1-\tau}(c-1)^2.$$

This finally gives

$$\text{LHS} \geq \frac{C g_{\min}^2}{2} \left[\tau \|f/g - 1\|_2^2 - \frac{\tau L}{1-\tau}(c-1)^2\right] + \frac{\delta}{2}(c^2-1)^2 \|g^2\|_b^2$$

$$= \frac{C g_{\min}^2 \tau}{2} \|f/g - 1\|_2^2 + (c-1)^2 \left[\frac{\delta}{2}(c+1)^2 \|g^2\|_b^2 - \frac{C g_{\min}^2 \tau L}{2(1-\tau)}\right].$$

Therefore, it is enough to take τ such that

$$\frac{\delta}{2}(c+1)^2 \|g^2\|_b^2 - \frac{C g_{\min}^2 \tau L}{2(1-\tau)} \geq 0,$$

that is for $c_1 = CG(0)/(L g_{\min}^2)$,

$$\tau = \min_{0 \leq c \leq c_1} \frac{\delta (c+1)^2 \|g^2\|_b^2}{\delta (c+1)^2 \|g^2\|_b^2 + C g_{\min}^2 L} = \frac{\|g^2\|_b^2}{\|g^2\|_b^2 + L(C g_{\min}^2 + 2\|b\|_\infty g_{\max}^2 \tilde{M}^2)},$$

which leads to (2.15) for $\mu = \frac{C g_{\min}^2 \tau}{2}$. Hence, (2.11) is also proved.

Finally, we investigate the case when 0 is the only steady state, which is the case when $\sum_{j=1}^N \bar{a}_j h < 0$ and $\lambda_1 \geq 1/2$. In such case, the convergence rate can also be established, by introducing

$$G(t) = \frac{1}{2} \sum_{j=1}^N f_j(t)^2 h.$$

We find that

$$\frac{d}{dt} G(t) = \sum_{j=1}^N f_j \left[(f_{j+1} - 2f_j + f_{j-1})h^{-2} + \frac{1}{2} f_j \left(\bar{a}_j - h \sum_{i=1}^N \bar{b}_{ji} f_i^2\right)\right] h$$

$$= -\sum_{j=1}^{N-1} (f_{j+1} - f_j)^2 h^{-1} + \frac{1}{2} \sum_{j=1}^N \bar{a}_j f_j^2 h - \frac{1}{2} \sum_{i,j=1}^N \bar{b}_{ji} f_i^2 f_j^2 h^2.$$

If $\lambda_1 > \frac{1}{2}$, then by Lemma 2.2, it follows that there exists $\nu > 0$ such that

$$Q_{1/2}(v) = \sum_{j=1}^{N-1} (v_{j+1} - v_j)^2 h^{-1} - \frac{1}{2} \sum_{j=1}^{N} \bar{a}_j v_j^2 h \geq \nu \sum_{j=1}^{N} v_j^2 h$$

for any $v \in \mathbb{R}^N$. Hence, we have

$$\frac{d}{dt} G(t) \leq -\nu \sum_{j=1}^{N} f_j^2 h - \frac{1}{2} \sum_{i,j=1}^{N} \bar{b}_{ji} f_i^2 f_j^2 h^2 \leq -2\nu G.$$

This leads to $G(t) \leq G(0)e^{-2\nu t}$, hence

$$\|f\|_2^2 = 2G(t) \leq 2G(0)e^{-2\nu t}.$$

If $\lambda_1 = \frac{1}{2}$, we have

$$\begin{aligned} \frac{d}{dt} G(t) &= -\sum_{j=1}^{N-1} (f_{j+1} - f_j)^2 h^{-1} + \lambda_1 \sum_{j=1}^{N} \bar{a}_j f_j^2 h - \frac{1}{2} \sum_{i,j=1}^{N} \bar{b}_{ji} f_i^2 f_j^2 h^2 \\ &\leq -\frac{1}{2} \sum_{i,j=1}^{N} \bar{b}_{ji} f_i^2 f_j^2 h^2, \end{aligned}$$

where we have used the definition for λ_1 when $\sum_{j=1}^{N} \bar{a}_j f_j^2 h > 0$, and the inequality remains valid when $\sum_{j=1}^{N} \bar{a}_j f_j^2 h \leq 0$. Hence,

$$\frac{d}{dt} G(t) \leq -\frac{1}{2} \sum_{i,j=1}^{N} \bar{b}_{ji} f_i^2 f_j^2 h^2 \leq -\frac{b_m}{2} \left[\sum_{j=1}^{N} f_j^2 h \right]^2 = -2b_m G^2.$$

This upon integration over $[0, t]$ using $\|f\|_2^2 = 2G$ gives

$$\|f\|_2^2 \leq \|f^0\|_2^2 (1 + b_m \|f^0\|_2^2 t)^{-1}$$

for arbitrary $t > 0$. This completes the proof of Theorem 1.4.

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