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Barycentric Subdivisions of Convex Complexes are Collapsible

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Abstract
A classical question in PL topology, asked among others by Hudson, Lickorish, and Kirby, is whether every linear subdivision of the d-simplex is simplicially collapsible. The answer is known to be positive for $d \leq 3$. We solve the problem up to one subdivision, by proving that any linear subdivision of any polytope is simplicially collapsible after at most one barycentric subdivision. Furthermore, we prove that any linear subdivision of any star-shaped polyhedron in $\mathbb{R}^d$ is simplicially collapsible after $d-2$ derived subdivisions at most. This presents progress on an old question by Goodrick.

1 Introduction

Collapsibility is a combinatorial version of the notion of contractibility, introduced in 1939 by Whitehead. All triangulations of the 2-dimensional ball are collapsible. In contrast, Bing and Goodrick showed how to construct non-collapsible triangulations of the d-ball for each $d \geq 3$ [6], [3, Cor. 4.25]. (See also [4] for an explicit example with 15 vertices.)

In Bing's 3-dimensional examples, the obstruction to collapsibility is the presence of subcomplexes with few facets that are isotopic to knots. This is also an obstruction to admitting a convex geometric realization: For example, if a 3-ball contains a knot realized as subcomplex with ≤ 5 edges, then the 3-ball cannot be embedded in $\mathbb{R}^3$

Dedicated to the memory of Ricky Pollack.

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(since the stick number of the trefoil is 6). In particular, such 3-ball cannot have any convex geometric realization in any $\mathbb{R}^k$.

This led Bing to ask whether simplicial subdivisions of convex 3-polytopes are collapsible. In 1967, Chillingworth answered Bing's question in the affirmative [10]. Chillingworth's proof is based on an elementary induction. Consider the highest vertex v of the complex (according to some generic linear functional). The link of v is necessarily *planar*, and all planar simply connected 2-complexes are collapsible. In particular, the star of v collapses to the link of v. This implies that the complex C collapses to $C - v$, the subcomplex of C consisting of all faces that do not contain v.

Unfortunately, Chillingworth's argument is specific to dimension 3, because vertex links in a 4-ball are no longer planar. In fact, whether all convex d-balls are collapsible represents a long-standing open problem, where little progress has been made since the Sixties. The problem has appeared in the literature in at least three different versions:

Conjecture 1.1 (Lickorish's Conjecture, cf. Kirby [15, Prob. 5.5 (A)]) *Let C be a simplicial complex. If C is a subdivision of the simplex, then C is collapsible.*

Conjecture 1.2 (Goodrick's Conjecture, cf. Kirby [15, Prob. 5.5 (B)]) *Let C be a simplicial complex. If* (*the underlying space of*) *C is star-shaped, then C is collapsible.*

Problem 1.3 (Hudson's Problem [13, Sect. 2, p. 44]) *Let C be a simplicial complex. If C collapses onto some subcomplex C', does every simplicial subdivision D of C collapse to the restriction D' of D to the underlying space of C'?*

In this paper, we show that all three problems above can be solved if we are allowed to modify the complex C by performing a bounded number of barycentric subdivisions. Our bounds are universal, i.e., they do not depend on the complex chosen.

The new idea is to refine Chillingworth's inductive method by expanding the problem into spherical geometry. In fact, if v is the top vertex of a (geometric) simplicial complex $C \subset \mathbb{R}^d$, the vertex link of v has a natural geometric realization as *spherical* simplicial complex, obtained by intersecting C with a small $(d-1)$-sphere centered at v.

Our trick is to find a special subdivision S of the complex C in which the link of the top vertex is a (*geodesically*) *convex* subset of the sphere. It turns out that a subdivision combinatorially equivalent to the barycentric subdivision does the trick. So we aim for a stronger statement, namely, that both convex d-complexes in $\mathbb{R}^d$ and convex spherical d-complexes in S^d become collapsible after one barycentric subdivision. This way we can proceed by induction on the dimension: The inductive assumption will tell us that the subdivided link of the top vertex is collapsible.

After a few technicalities, this idea takes us to the following sequence of results.

Main Theorem I. *Let C be an arbitrary simplicial complex in $\mathbb{R}^d$.*

(1) (Theorem 4.5) *If the underlying space of C in $\mathbb{R}^d$ is convex, then the* (*first*) *barycentric subdivision of C is collapsible.*
(2) (Theorem 3.6) *If the underlying space of C in $\mathbb{R}^d$ is star-shaped, then its $(d-2)$-nd barycentric subdivision is collapsible.*
(3) (Theorem 4.3) *If C collapses simplicially onto some subcomplex C', then for every simplicial subdivision D of C the barycentric subdivision of D collapses to its restriction to the underlying space of C'.*

2 Preliminaries

2.1 Geometric and Intrinsic Polytopal Complexes

By $\mathbb{R}^d$ and S^d we denote the Euclidean d-space and the unit sphere in $\mathbb{R}^{d+1}$, respectively. A (*Euclidean*) *polytope* in $\mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. A face of a polytope is any set which is minimized by some linear function. Examples of faces are the empty set and the whole polytope itself. A *facet* is an inclusion-maximal proper face (that is, not the whole polytope). A *spherical polytope* in S^d is the convex hull of a finite number of points that all belong to some open hemisphere of S^d. Spherical polytopes are in natural one-to-one correspondence with Euclidean polytopes, by taking radial projections.

A *geometric polytopal complex* in $\mathbb{R}^d$ (resp. in S^d) is a finite collection of polytopes in $\mathbb{R}^d$ (resp. S^d) such that the intersection of any two polytopes is a face of both. Two polytopal complexes C, D are *combinatorially equivalent*, denoted by $C \cong D$, if their face posets are isomorphic. Any polytope combinatorially equivalent to the d-simplex, or to the regular unit cube $[0, 1]^d$, shall simply be called a d-*simplex* or a d-*cube*, respectively. A polytopal complex is *simplicial* if all its faces are simplices. (For us simplicial complexes are always "geometric".)

The *underlying space* $|C|$ of a polytopal complex C is the topological space obtained by taking the union of its faces. If two complexes are combinatorially equivalent, their underlying spaces are homeomorphic. We will frequently abuse notation and identify a polytopal complex with its underlying space, as is common in the literature. For instance, we do not distinguish between a polytope and the complex formed by its faces. If C is simplicial, C is sometimes called a *triangulation* of $|C|$ (and of any topological space homeomorphic to $|C|$).

A *subdivision* of a polytopal complex C is a polytopal complex C' with the same underlying space of C, such that for every face F' of C' there is some face F of C for which $F' \subset F$. Two polytopal complexes C and D are called *PL equivalent* if some subdivision C' of C is combinatorially equivalent to some subdivision D' of D. The *star* of σ in C, denoted by $\mathrm{St}\,(\sigma, C)$, is the minimal subcomplex of C that contains all faces of C containing σ. In case $|C|$ is a topological manifold (with or without boundary), we say that C is *PL* (short for Piecewise-Linear) if the star of every face of C is PL equivalent to the simplex of the same dimension.

If C is a polytopal complex, and A is some set, we define the *restriction* $\mathrm{R}\,(C, A)$ *of* C *to* A as the inclusion-maximal subcomplex D of C such that D lies in A. The *deletion* $C - D$ of a face D from C is the subcomplex of C given by $\mathrm{R}\,(C, C\backslash \mathrm{relint}\, D)$, where 'relint' stands for relative interior of D. A *stellar subdivision of* C *at a face* F is defined by first choosing a point v_F (called "stellar center" or "starring vertex") anywhere in the relative interior of F, and then by setting

$$\mathrm{stel}(C, F) = (C - F) \cup \mathrm{conv}\{v_F \cup \sigma : \sigma \in \mathrm{St}\,(F, C) - F\}.$$

A *derived subdivision* $\mathrm{sd}\, C$ of a polytopal complex C is any subdivision of C obtained by stellarly subdividing at all faces in order of decreasing dimension of the faces of C, cf. [13, pp. 8–9]. Different choices of the stellar centers v_F result in (geometric)

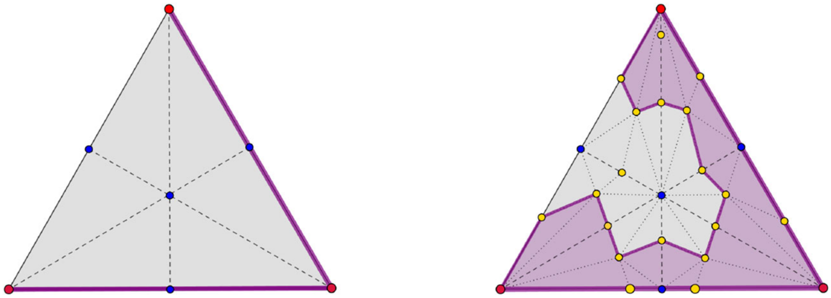

Fig. 1 A purple subcomplex D of the boundary of a disc [LEFT] and its derived neighborhood [RIGHT]. Any subcomplex D is a deformation retract of its derived neighborhood, because the latter collapses onto the former, in the sense of Sect. 2.2

simplicial complexes that are different, but combinatorially equivalent. An example of a derived subdivision is the *barycentric subdivision*, which chooses always as v_F the barycenter of F. Given any derived subdivision $\operatorname{sd} C$ of C, the (*first*) *derived neighborhood* $N(D, C)$ (Fig. 1) of D in C is the simplicial complex

$$N(D, C) := \bigcup_{\sigma \in \operatorname{sd} D} \operatorname{St}(\sigma, \operatorname{sd} C).$$

Next, we define (a geometric realization of) the *link* with a metric approach. We took inspiration from Charney [9] and Davis–Moussong [11, Sect. 2.2]. Let p be any point of a (geometric) simplicial complex X. By $\mathrm{T}_p X$ we denote the "space of directions" in the sense of Burago–Burago–Ivanov [8, Sect. 10.9], which coincides with the tangent space at p when X is (piecewise) smooth. For simplicity, we call the elements of $\mathrm{T}_p X$ "tangent vectors". Let $\mathrm{T}^1_p X$ be the restriction of $\mathrm{T}_p X$ to unit vectors. If Y is any subspace of X, then $\mathrm{N}_{(p,Y)} X$ denotes the subspace of $\mathrm{T}_p X$ spanned by the vectors orthogonal to $\mathrm{T}_p Y$. If p is in the interior of Y, we define $\mathrm{N}^1_{(p,Y)} X := \mathrm{N}_{(p,Y)} X \cap \mathrm{T}^1_p Y$. If τ is any face of a polytopal complex C containing a nonempty face σ of C, then the set $\mathrm{N}^1_{(p,\sigma)} \tau$ of unit tangent vectors in $\mathrm{N}^1_{(p,\sigma)} |C|$ pointing towards τ forms a spherical polytope $P_p(\tau)$, isometrically embedded in $\mathrm{N}^1_{(p,\sigma)} |C|$. The family of all polytopes $P_p(\tau)$ in $\mathrm{N}^1_{(p,\sigma)} |C|$ obtained for all $\tau \supset \sigma$ forms a polytopal complex, called the *link* of C at σ; we will denote it by $\operatorname{Lk}_p(\sigma, C)$. If C is a geometric polytopal complex in $X^d = \mathbb{R}^d$ (or $X^d = S^d$), then $\operatorname{Lk}_p(\sigma, C)$ is naturally realized in $\mathrm{N}^1_{(p,\sigma)} X^d$. Obviously, $\mathrm{N}^1_{(p,\sigma)} X^d$ is isometric to a sphere of dimension $d - \dim \sigma - 1$, and will be considered as such. Up to ambient isometry $\operatorname{Lk}_p(\sigma, C)$ and $\mathrm{N}^1_{(p,\sigma)} \tau$ in $\mathrm{N}^1_{(p,\sigma)} |C|$ or $\mathrm{N}^1_{(p,\sigma)} X^d$ do not depend on p; for this reason, p will be omitted in notation whenever possible. By convention, we define $\operatorname{Lk}(\emptyset, C) = C$.

If C is simplicial and v is a vertex of C, we have the combinatorial equivalence

$$\operatorname{Lk}(v, C) \cong (C - v) \cap \operatorname{St}(v, C) = \operatorname{St}(v, C) - v.$$

If C is a simplicial complex, and σ, τ are faces of C, then $\sigma * \tau$ is the minimal face of C containing both σ and τ (assuming it exists). If σ is a face of C, and τ is a face of $\mathrm{Lk}\,(\sigma, C)$, then $\sigma * \tau$ is the face of C with $\mathrm{Lk}\,(\sigma, \sigma * \tau) = \tau$. In both cases, the operation $*$ is called the *join*.

2.2 Collapsibility and Non-evasiveness

Inside a polytopal complex C, a *free* face σ is a face strictly contained in only one other face of C. An *elementary collapse* is the deletion of a free face σ from a polytopal complex C. We say that C (*elementarily*) *collapses* onto $C - \sigma$, and write $C \searrow_e C - \sigma$. We also say that the complex C *collapses* to a subcomplex C', and write $C \searrow C'$, if C can be reduced to C' by a sequence of elementary collapses. A *collapsible* complex is a complex that collapses onto a single vertex. Collapsibility depends only on the face poset.

Collapsible complexes are contractible. Moreover, collapsible PL manifolds are necessarily balls [18]. Here are a few additional properties:

Lemma 2.1 *Let C be a simplicial complex, and let C' be a subcomplex of C. Then the cone over base C collapses to the cone over C'.*

Lemma 2.2 *Let v be any vertex of any simplicial complex C. If* $\mathrm{Lk}\,(v, C)$ *collapses to some subcomplex S, then C collapses to $(C - v) \cup (v * S)$. In particular, if* $\mathrm{Lk}\,(v, C)$ *is collapsible, then $C \searrow C - v$.*

Lemma 2.3 *Let X be a simplicial complex with subcomplexes C and D such that $X = C \cup D$. If C collapses onto $C \cap D$, then X collapses onto D.*

Proof It is enough to consider the case $C \searrow_e C' = C - \sigma$, where σ is a free face of C. The conclusion follows from the observation that the natural embedding $C \hookrightarrow D \cup C$ takes the free face $\sigma \in C$ to a free face of $D \cup C$. □

Non-evasiveness is a further strengthening of collapsibility that emerged in theoretical computer science [14]. A 0-dimensional simplicial complex is *non-evasive* if and only if it is a point. Recursively, a d-dimensional simplicial complex ($d > 0$) is *non-evasive* if and only if there is some vertex v of the complex whose link and deletion are both non-evasive. Again, non-evasiveness depends only on the face poset.

The notion of non-evasiveness is rather similar to vertex-decomposability, a notion defined only for *pure* simplicial complexes [16]; to avoid confusions, we recall the definition and explain the difference in the lines below. A 0-dimensional simplicial complex is *vertex-decomposable* if and only if it is a nonempty, finite set of points. In particular, not all vertex-decomposable complexes are contractible. Recursively, a d-dimensional simplicial complex ($d > 0$) is *vertex-decomposable* if and only if it is pure and there is some vertex v of the complex whose link and deletion are both vertex-decomposable (so in particular pure). All vertex-decomposable contractible complexes are non-evasive.

An important difference arises when considering cones. It is easy to see that the cone over a simplicial complex C is vertex-decomposable if and only if C is. In contrast,

Lemma 2.4 (cf. Welker [17]) *The cone over any simplicial complex is non-evasive.*

By Lemma 2.2 every non-evasive complex is collapsible. As a partial converse, we also have the following lemma.

Lemma 2.5 (cf. Welker [17]) *The barycentric subdivision of every collapsible complex is non-evasive. In particular, the barycentric subdivision of a non-evasive complex is non-evasive.*

A *non-evasiveness step* is the deletion from a simplicial complex C of a single vertex whose link is non-evasive. Given two simplicial complexes C and C', we write

$$C \searrow_{\mathrm{NE}} C'$$

if there is a sequence of non-evasiveness steps that leads from C to C'. We will need the following lemmas, which are well known and easy to prove.

Lemma 2.6 *Let v be any vertex of any simplicial complex C. Then*

$$(\mathrm{sd}\, C) - v \searrow_{\mathrm{NE}} \mathrm{sd}(C - v).$$

Proof The vertices of $\mathrm{sd}\, C$ correspond to faces of C; the vertices that have to be removed in order to deform $(\mathrm{sd}\, C) - v$ to $\mathrm{sd}\, (C - v)$ correspond to the faces of C strictly containing v. The order in which we remove the vertices of $(\mathrm{sd}\, C) - v$ is by increasing dimension of the associated face. Let τ be a face of C strictly containing v, and let w denote the vertex of $\mathrm{sd}\, C$ corresponding to τ. Assume all vertices corresponding to faces of τ have been removed from $(\mathrm{sd}\, C) - v$ already, and call the remaining complex D. Denote by $\mathrm{L}(\tau, C)$ the set of faces of C strictly containing τ, and let $\mathrm{F}(\tau - v)$ denote the set of nonempty faces of $\tau - v$. Then $\mathrm{Lk}\,(w, D)$ is combinatorially equivalent to the order complex of $\mathrm{L}(\tau, C) \cup \mathrm{F}(\tau - v)$, whose elements are ordered by inclusion. Every maximal chain contains the face $\tau - v$, so $\mathrm{Lk}\,(w, D)$ is a cone, which is non-evasive by Lemma 2.4. Thus, we have $D \searrow_{\mathrm{NE}} D - w$. The iteration of this procedure shows $(\mathrm{sd}\, C) - v \searrow_{\mathrm{NE}} \mathrm{sd}\, (C - v)$, as desired. □

Lemma 2.7 *If $C \searrow_{\mathrm{NE}} C'$, then $\mathrm{sd}^m\, C \searrow_{\mathrm{NE}} \mathrm{sd}^m\, C'$ for all non-negative m.*

Proof Apply Lemma 2.6 to all vertices that are removed in deforming C to C'. □

Lemma 2.8 *Let v be any vertex of any simplicial complex C. Let $m \geq 0$ be an integer. Then*

$$(\mathrm{sd}^m\, C) - v \searrow_{\mathrm{NE}} \mathrm{sd}^m (C - v).$$

In particular, if $\mathrm{sd}^m\, \mathrm{Lk}\,(v, C)$ is non-evasive, then $\mathrm{sd}^m\, C \searrow_{\mathrm{NE}} \mathrm{sd}^m (C - v)$.

Proof We proceed by induction on m, the case $m = 1$ being Lemma 2.6. For $m \geq 2$,

$$(\mathrm{sd}^m\, C) - v = (\mathrm{sd}\,(\mathrm{sd}^{m-1}\, C)) - v \searrow_{\mathrm{NE}} \mathrm{sd}\,((\mathrm{sd}^{m-1}\, C) - v)$$
$$\searrow_{\mathrm{NE}} \mathrm{sd}\,(\mathrm{sd}^{m-1}(C - v)) = \mathrm{sd}^m(C - v),$$

where the first deformation is by inductive assumption (applied twice), and the second deformation follows by Lemma 2.7. □

3 Non-evasiveness of Star-Shaped Complexes

Here we show that *any* subdivision of a star-shaped set becomes collapsible after $d-2$ derived subdivisions (Theorem 3.6); this proves Goodrick's conjecture up to a fixed number of subdivisions.

Definition 3.1 (*Star-shaped sets*) A subset $X \subset \mathbb{R}^d$ is *star-shaped* if there exists a point x in X, a *star-center* of X, such that for each y in X, the segment $[x, y]$ lies in X. Similarly, a subset $X \subset S^d$ is *star-shaped* if X lies in a closed hemisphere of S^d and there exists a *star-center* x of X, in the interior of the hemisphere containing X, such that for each y in X, the segment from x to y lies in X. With abuse of notation, a polytopal complex C (in $\mathbb{R}^d$ or in S^d) is *star-shaped* if its underlying space is star-shaped.

Via central projection, one can see that star-shaped complexes in $\mathbb{R}^d$ are precisely those complexes that have a star-shaped realization in the interior of a hemisphere of S^d. The more delicate situation in which C is star-shaped in S^d but touches the boundary of any closed hemisphere containing it, is addressed by the following lemma.

Lemma 3.2 *Let C be a star-shaped polytopal complex in a closed hemisphere $\overline{H}_+$ of S^d. Let D be the subcomplex of faces of C that lie in the interior of $\overline{H}_+$. Assume that every nonempty face σ of C in $H := \partial\overline{H}_+$ is the facet of a unique face τ of C that intersects both D and H. Then the complex $N(D, C)$ has a star-shaped geometric realization in $\mathbb{R}^d$.*

Proof Let m be the midpoint of $\overline{H}_+$. Let $B_r(m)$ be the closed metric ball in $\overline{H}_+$ with center m and radius r (with respect to the standard Riemannian metric d on S^d). If $C \subset \operatorname{int}\overline{H}_+$, then C has a realization as a star-shaped set in $\mathbb{R}^d$ by central projection, and we are done. Thus, we can assume that C intersects H. Without loss of generality, let us assume that C has a star-center x in the interior of $\overline{H}_+$. Since D and $\{x\}$ are compact and in the interior of $\overline{H}_+$, there is some real number $R < \pi/2$ such that the ball $B_R(m)$ contains both x and D. Let $J := (R, \pi/2]$.

If σ is any nonempty face of C in H, let v_σ be any point in the relative interior of σ. If τ is any face of C intersecting D and H, which exists by assumption, define $\sigma(\tau) := \tau \cap H$. For each $r \in J$, choose a point $w(\tau, r)$ in the relative interior of $\tau \cap \partial B_r(m)$, so that for each τ the point $w(\tau, r)$ depends continuously on r and tends to $v_{\sigma(\tau)}$ as r approaches $\pi/2$. Extend each family $w(\tau, r)$ continuously to $r = \pi/2$ by defining $w(\tau, \pi/2) := v_{\sigma(\tau)}$.

Next, we use these one-parameter families of points to produce a one-parameter family $N_r(D, C)$ of geometric realizations of $N(D, C)$, where $r \in J$. For this, let

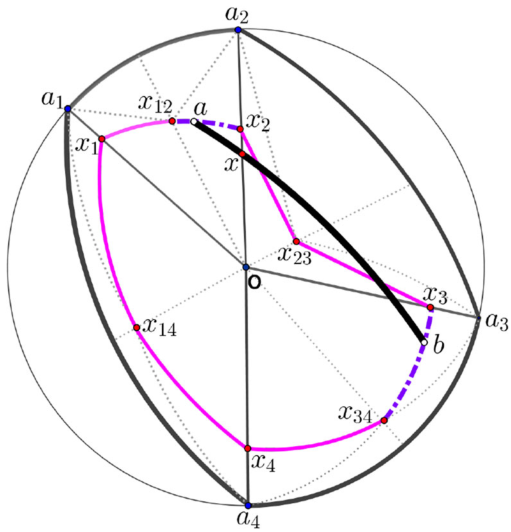

Fig. 2 In the northern hemisphere (here seen from above) consider a quadrilateral $\{a_1, a_2, a_3, a_4\}$ with two opposite edges on the equator. Suppose the North Pole o is in its interior and let C be the stellar subdivision of the quadrilateral from the North Pole. Let $D = \{o\}$. The region $N_r(D, C)$, here delimited by the eight points $x_i = x_i(r)$, is a geometric realization of the derived neighborhood of D in C. A priori this $N_r(D, C)$ is not convex. So it might be that in two extremal faces like $[x_{12}, x_2]$ and $[x_{34}, x_3]$ we can find two points a, b connected by a geodesic that contains a star-center x of C, but does not completely lie in $N_r(D, C)$. If this is the case, we say that $[x_{12}, x_2]$ and $[x_{34}, x_3]$ are "folded"

ϱ be any face of C intersecting D. If ϱ is in D, let $x_{\varrho,r}$ be any point in the relative interior of ϱ (independent of r). If ϱ is not in D, let $x_{\varrho,r} := w(\varrho, r)$. We realize $N_r(D, C) \cong N(D, C)$, for r in J, so that the vertex of $N_r(D, C)$ corresponding to the face ϱ is given the coordinates of $x_{\varrho,r}$. As the coordinates of the vertices determine a simplex entirely, this gives a realization $N_r(D, C)$ of $N(D, C)$, as desired (Fig. 2).

To finish the proof, we claim that if r is close enough to $\pi/2$, then $N_r(D, C)$ is star-shaped with star-center x. Let us prove the claim. First we define the *extremal faces* of $N_r(D, C)$ as the faces all of whose vertices are in $\partial B_r(m)$. We say that a pair $\sigma,\ \sigma'$ of extremal faces is *folded* if there are two points a and b, in σ and σ' respectively, and satisfying $\mathrm{d}(a, b) < \pi$, such that the great circle through a and b contains x, but the (geodesic) segment $[a, b]$ is not contained in $N_r(D, C)$.

When $r = \pi/2$, weakly folded faces do not exist since for every pair of points a and b in extremal faces, $\mathrm{d}(a, b) < \pi$, the great circle through distinct points a and b lies in $\partial B_{\pi/2}$; hence, all such great circles have distance at least $\mathrm{d}(x, H) > 0$ to x.

Alternatively, we can modify the definition of folding: Folding is not a closed condition, but we can strengthen it by saying that $N_r(D, C)$ is *weakly folded* in a face τ of C and at a pair $\sigma,\ \sigma'$ of extremal faces if there are two points a and b, in σ and σ' respectively, and satisfying $\mathrm{d}(a, b) < \pi$, such that the great circle through a and b contains x, but there is no open neighborhood U of (a, b) such that $U \cap \mathrm{relint}\,\tau$ is contained in $N_r(D, C)$. Clearly, folded implies weakly folded. Moreover, it is a closed

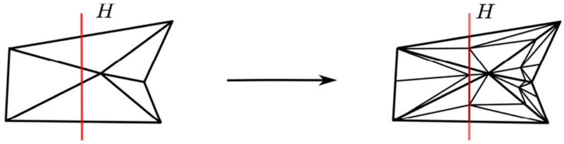

Fig. 3 An example of an H-splitting derived subdivision. (As we are free to choose where to position the new vertices, whenever possible we place them on the red hyperplane H.)

condition as long as we restrict to the r such that faces of $N_r(D, C)$ do not degenerate. This finishes the argument as we have no weak folding at $r = \pi/2$.

Thus, as the set of pairs a, b in extremal faces is compact, and since we chose the vertices of $N_r(D, C)$ to depend continuously on $r \in (R, \pi/2]$, so do above great circles, and we can find a real number $R' \in (R, \pi/2)$ such that for any r in the open interval $J' := \left(R', \pi/2\right)$, the simplicial complex $N_r(D, C)$ contains no folded pair of faces.

But then, for every $r \in J'$, in $N_r(D, C)$ folded pairs of extremal faces are avoided. Hence, for every y in $N_r(D, C)$, the segment $[x, y]$ lies entirely in $N_r(D, C)$, since every part of the segment not in $N_r(D, C)$ must have boundary points in a folded extremal pair. □

In the following, for any simplicial complex C, we denote by $F_i(C)$ the collection of all i-dimensional faces of C. In order to perform collapses on barycentric subdivisions, we now need to introduce an order on $\mathrm{F}_0(\mathrm{sd}\, C)$. Recall that, if $(S, \prec)$ is an arbitrary poset and $S \subset T$, an *extension* of $\prec$ to T is any partial order $\precapprox$ that coincides with $\prec$ when restricted to (pairs of elements in) S.

Definition 3.3 (*Derived order*) Let C be a polytopal complex. Let S denote a subset of mutually disjoint faces of C. Let $\prec$ be a total order on S. We extend this order to an irreflexive partial order $\precapprox$ on C as follows: Let σ be any face of C, and let $\tau \subsetneq \sigma$ be any strict face of σ.

- If τ is the minimal face of σ under $\prec$, then $\tau \precapprox \sigma$.
- If τ is any other face of σ, then $\sigma \precapprox \tau$.

Since we started from a total order, the transitive closure of the relation $\precapprox$ gives an irreflexive partial order on the faces of C. Since faces of C are in bijection with vertices of $\mathrm{sd}\, C$, this gives an irreflexive partial order on $\mathrm{F}_0(\mathrm{sd}\, C)$. Any total order that extends the latter order is a *derived order* of $\mathrm{F}_0(\mathrm{sd}\, C)$ induced by $\prec$.

Definition 3.4 (*H-splitting derived subdivisions*) Let C be a polytopal complex in $\mathbb{R}^d$, and let H be a hyperplane of $\mathbb{R}^d$. An *H-splitting derived subdivision* of C is a derived subdivision, with vertices chosen so that the following property holds: for any face τ of C that intersects the hyperplane H in the relative interior, the vertex of $\mathrm{sd}\, C$ that corresponds to τ in C lies on the hyperplane H (Fig. 3).

Definition 3.5 (*Split link and lower link*) Let C be a simplicial complex in $\mathbb{R}^d$. Let v be a vertex of C. Let $\overline{H}_+$ be a closed halfspace in $\mathbb{R}^d$ that contains v in its boundary

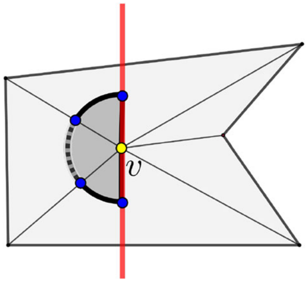

Fig. 4 The lower link of v (dashed) within the split link of v

$H := \partial \overline{H}_+$. Let ν be the outer normal of $\overline{H}_+$ at v. The *split link* (of C at v with respect to ν), denoted by $\mathrm{SLk}^{\,\nu}(v, C)$, is the intersection of $\mathrm{Lk}\,(v, C)$ with the hemisphere $\mathrm{T}^1_v\overline{H}_+$, that is,

$$\mathrm{SLk}^{\,\nu}(v, C) := \{\sigma \cap \mathrm{T}^1_v\overline{H}_+ : \sigma \in \mathrm{Lk}\,(v, C)\}.$$

The *lower link* $\mathrm{LLk}^{\,\nu}(v, C)$ of C at v with respect to the direction ν is the restriction $\mathrm{R}\,(\mathrm{Lk}\,(v, C), \mathrm{int}\,\mathrm{T}^1_v\overline{H}_+)$ of $\mathrm{Lk}\,(v, C)$ to the interior of the hemisphere $\mathrm{T}^1_v\overline{H}_+$ (Fig. 4).

The complex $\mathrm{LLk}^{\,\nu}(v, C)$ is naturally a subcomplex of $\mathrm{SLk}^{\,\nu}(v, C)$: we have as an alternative definition of the lower link the identity

$$\mathrm{LLk}^{\,\nu}(v, C) = \mathrm{R}\,(\mathrm{SLk}^{\,\nu}(v, C), \mathrm{int}\,\mathrm{T}^1_v\overline{H}_+).$$

Finally, for a polytopal complex C and a face τ, we denote by $\mathrm{L}(\tau, C)$ the set of faces of C strictly containing τ.

Theorem 3.6 *Let C be a polytopal complex in $\mathbb{R}^d$, $d \geq 3$, or a simplicial complex in $\mathbb{R}^2$. If C is star-shaped in $\mathbb{R}^d$, then $\mathrm{sd}^{d-2}(C)$ is non-evasive, and in particular collapsible.*

Proof The proof is by induction on the dimension. The case $d = 2$ is easy: Every simply connected planar simplicial complex is non-evasive [1, Lem. 2.3].

Assume now $d \geq 3$. Let ν be generic in $S^{d-1} \subset \mathbb{R}^d$, so that no edge of C is orthogonal to ν. Let H be a hyperplane through a star-center x of C such that H is orthogonal to ν. Throughout this proof, let $\mathrm{sd}\,C$ denote any H-splitting derived subdivision of C. Let $\overline{H}_+$ (resp. $\overline{H}_-$) be the closed halfspace bounded by H in direction ν (resp. $-\nu$), and let H_+ (resp. H_-) denote their respective interiors. We make five claims:

(1) For $v \in \mathrm{F}_0(\mathrm{R}\,(C, H_+))$, the complex $\mathrm{sd}^{d-3}N(\mathrm{LLk}^{\,\nu}(v, C), \mathrm{Lk}\,(v, C))$ is non-evasive.
(2) For $v \in \mathrm{F}_0(\mathrm{R}\,(C, H_-))$, the complex $\mathrm{sd}^{d-3}N(\mathrm{LLk}^{-\nu}(v, C), \mathrm{Lk}\,(v, C))$ is non-evasive.
(3) $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\,C, \overline{H}_+) \searrow_{\mathrm{NE}} \mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\,C, H)$.
(4) $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\,C, \overline{H}_-) \searrow_{\mathrm{NE}} \mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\,C, H)$.

(5) $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\,C, H)$ is non-evasive.

Here are the respective proofs:

(1) Let v be a vertex of C that lies in H_+. The complex $\mathrm{SLk}^{\nu}(v, C)$ is star-shaped in the $(d-1)$-sphere $\mathrm{T}^1_v\mathbb{R}^d$; its star-center is the tangent direction of the segment $[v, x]$ at v. Furthermore, ν is generic, so $\mathrm{SLk}^{\nu}(v, C)$ satisfies the assumptions of Lemma 3.2. The lemma tells us that the complex

$$N(\mathrm{LLk}^{\nu}(v, C), \mathrm{SLk}^{\nu}(v, C)) \cong N(\mathrm{LLk}^{\nu}(v, C), \mathrm{Lk}\,(v, C))$$

has a star-shaped geometric realization in $\mathbb{R}^{d-1}$. So by the inductive assumption, the simplicial complex $\mathrm{sd}^{d-3}N(\mathrm{LLk}^{\nu}(v, C), \mathrm{Lk}\,(v, C))$ is non-evasive.

(2) This is symmetric to (1).

(3) Since ν is generic, the vertices of C are totally ordered according to their value under the functional $\langle\cdot, \nu\rangle$. Let us extend such order to a derived total order on the vertices of $\mathrm{sd}\,C$, as explained in Definition 3.3. Note that the order on the vertices of $\mathrm{sd}\,C$ *does not* have to be induced by $\langle\cdot, \nu\rangle$; it is however easy to arrange the vertices of $\mathrm{sd}\,C$ in such a way that both orders agree.

Let $v_0, v_1, \ldots, v_n$ be the vertices of $\mathrm{R}\,(\mathrm{sd}\,C, H_+) \subset \mathrm{sd}\,C$, labeled according to the derived order (so that the maximal vertex is v_0). Clearly, these vertices form an order filter (i.e., an upward closed subset) for the order on vertices of $\mathrm{sd}\,C$ defined above. Define C_i by restricting $\mathrm{sd}\,C$ to $\overline{H}_+$, and deleting $\{v_0, \ldots, v_{i-1}\}$, i.e.,

$$C_i := \mathrm{R}\,(\mathrm{sd}\,C, \overline{H}_+) - \{v_0, \ldots, v_{i-1}\}$$

and define

$$\Sigma_i := \mathrm{sd}^{d-3}C_i.$$

It remains to show that, for all i, $0 \le i \le n$, we have $\Sigma_i \searrow_{\mathrm{NE}} \Sigma_{i+1}$. We distinguish two cases, according to whether v_i was introduced when subdividing or not.

- If v_i corresponds to a face τ of C of positive dimension, then let w denote the vertex of τ minimizing $\langle\cdot, \nu\rangle$. Following the definition of the derived order, the complex $\mathrm{Lk}\,(v_i, C_i)$ is combinatorially equivalent to the order complex associated to the set of faces $\mathrm{L}(\tau, C) \cup \{w\}$, whose elements are ordered by inclusion.

  Since w is the unique minimum in that order, $\mathrm{Lk}\,(v_i, C_i)$ is combinatorially equivalent to a cone over base $\mathrm{sd}\,\mathrm{Lk}\,(\tau, C)$. But every cone is non-evasive (cf. Lemma 2.4). Thus, $C_i \searrow_{\mathrm{NE}} C_i - v_i = C_{i+1}$. By Lemma 2.7, $\Sigma_i \searrow_{\mathrm{NE}} \Sigma_{i+1}$.

- If v_i corresponds to an original vertex of C, we have by claim (1) that the $(d-3)$-rd derived subdivision of $\mathrm{Lk}\,(v_i, C_i) \cong N(\mathrm{LLk}^{\,\nu}(v_i, C), \mathrm{Lk}\,(v_i, C))$ is non-evasive. With Lemma 2.8, we conclude that

$$\Sigma_i = \mathrm{sd}^{d-3} C_i \searrow_{\mathrm{NE}} \mathrm{sd}^{d-3}(C_i - v_i) = \mathrm{sd}^{d-3} C_{i+1} = \Sigma_{i+1}.$$

Hence in both cases $\Sigma_i \searrow_{\mathrm{NE}} \Sigma_{i+1}$. This means that we can recursively delete one vertex, until the remaining complex has no vertex in H_+. Thus, the complex $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, \overline{H}_+)$ can be deformation retracted to $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, H)$ via non-evasiveness steps.

(4) This is analogous to (3), exploiting claim (2) in place of claim (1).
(5) This follows from the inductive assumption. In fact, $\mathrm{R}\,(\mathrm{sd}\, C, H)$ is star-shaped in the $(d-1)$-dimensional hyperplane H: Hence, the inductive assumption gives that $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, H)$ is non-evasive.

Once our five claims are established, we conclude by showing that claims (3), (4) and (5) imply that $\mathrm{sd}^{d-2} C$ is non-evasive, as desired. First of all, we observe that if A, B and $A \cup B$ are simplicial complexes with the property that $A, B \searrow_{\mathrm{NE}} A \cap B$, then $A \cup B \searrow_{\mathrm{NE}} A \cap B$. Applied to the complexes $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, \overline{H}_+)$, $\mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, \overline{H}_-)$ and $\mathrm{sd}^{d-2} C = \mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, \overline{H}_+) \cup \mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, \overline{H}_-)$, the combination of (3) and (4) shows that $\mathrm{sd}^{d-2} C \searrow_{\mathrm{NE}} \mathrm{sd}^{d-3}\mathrm{R}\,(\mathrm{sd}\, C, H)$, which in turn is non-evasive by (5). □

4 Collapsibility of Convex Complexes

As usual, we say that a polytopal complex C (in $\mathbb{R}^d$ or in S^d) is *convex* if its underlying space is convex. A hemisphere in S^d is in *general position* with respect to a polytopal complex C in S^d if it contains no vertices of C in the boundary.

Since all convex complexes are star-shaped, the results of the previous section immediately imply that Lickorish's conjecture and Hudson's problem admit positive answer up to taking $d-2$ derived subdivisions. In a companion paper [2], we proved that one can do better: up to taking at most *two* barycentric subdivisions, every convex complex is shellable. All contractible shellable complexes are collapsible, cf. e.g. [12, Lem. 17].

In this section, we improve the previous results even further, by establishing that for Lickorish and Hudson's problems a positive answer can be reached after only *one* derived subdivision (Theorems 4.3 and 4.5). For this, we rely on the following Theorem 4.1, the proof of which is very similar to the proof of Theorem 3.6.

Theorem 4.1 *Let C be a convex polytopal d-complex in S^d and let $\overline{H}_+$ be a closed hemisphere of S^d in general position with respect to C. Then we have the following:*

(A) *If C intersects the interior of $\overline{H}_+$ non-trivially, then $N(\mathrm{R}\,(C, \overline{H}_+), C)$ is collapsible.*
(B) *If C intersects $\overline{H}_+$ in its boundary, then $N(\mathrm{R}\,(C, \overline{H}_+), C)$ collapses to the subcomplex $N(\mathrm{R}\,(\partial C, \overline{H}_+), \partial C)$.*

(C) *If* C *lies in* $\overline{H}_+$ *there exists a facet* σ *of* $\operatorname{sd}\partial C$ *such that* $\operatorname{sd} C$ *collapses to* $C_\sigma := \operatorname{sd}\partial C - \sigma$.

Part (C) of Theorem 4.1 can be equivalently rephrased as follows: "If C lies in $\overline{H}_+$, for *any* facet σ of $\operatorname{sd}\partial C$ the complex $\operatorname{sd} C$ collapses to $C_\sigma := \operatorname{sd}\partial C - \sigma$". In fact, for any d-dimensional simplicial ball B, the following statements are equivalent [5, Prop. 3.7 & Lem. 3.9]:

(i) $B \searrow \partial B - \sigma$ for some facet σ of ∂B;
(ii) $B - \sigma \searrow \partial B$ for any facet σ of B.

Thus, we have the following corollary:

Corollary 4.2 *Let* C *be a convex polytopal complex in* $\mathbb{R}^d$. *Then for any facet* σ *of* $\operatorname{sd} C$ *we have that* $\operatorname{sd} C - \sigma$ *collapses to* $\operatorname{sd}\partial C$.

Proof of Theorem 4.1 Claims (A), (B) and (C) can be proved analogously to the proof of Theorem 3.6, by induction on the dimension. Let us adopt convenient shortenings. We denote:

– by $(\mathrm{A})_d$, the statement "(A) is true for complexes of dimension $\leq d$";
– by $(\mathrm{B})_d$, the claim "(B) is true for complexes of dimension $\leq d$"; and finally
– by $(\mathrm{C})_d$, the claim "(C) is true for complexes of dimension $\leq d$".

Clearly $(\mathrm{A})_0$ and $(\mathrm{B})_0$ are true. $(\mathrm{C})_0$ is also true, because in this case ∂C is the empty set, so σ is the empty face. We assume, from now on, that $d > 0$ and that $(\mathrm{A})_{d-1}$, $(\mathrm{B})_{d-1}$ and $(\mathrm{C})_{d-1}$ are proven already. We then proceed to prove $(\mathrm{A})_d$, $(\mathrm{B})_d$ and $(\mathrm{C})_d$. Recall that $\mathrm{L}(\tau, C)$ denotes the faces of C strictly containing a face τ of C. We will make use of the notions of derived order (Definition 3.3) and lower link (Definition 3.5) from the previous section.

Proving $(\mathrm{A})_d$ **and** $(\mathrm{B})_d$: Let m denote the center of $\overline{H}_+$.

If $\partial C \cap \overline{H}_+ \neq \emptyset$, then C is a polyhedron that intersects $S^d \backslash \overline{H}_+$ in its interior since $\overline{H}_+$ is in general position with respect to C. Thus, $\overline{H}_+ \backslash C$ is star-shaped, and for every p in $\operatorname{int} C \backslash \overline{H}_+$, the point $-p \in \operatorname{int} \overline{H}_+$ is a star-center for it. In particular, the set of star-centers of $\overline{H}_+ \backslash C$ has non-trivial interior. If we choose a star-center x of $\overline{H}_+ \backslash C$ generically, and apply a linear transformation φ of the corresponding fan (the collection of cones over the faces of C) that takes the halfspace corresponding to $\overline{H}_+$ to itself and identifies x and m. If we then denote by $\mathrm{d}(y)$ the distance from x of a point $y \in S^d$ with respect to the canonical metric on S^d, we can assume that the distance function d to faces induces a total order on the faces. Indeed, up to moving the midpoint, and the whole hemisphere with it, by a small amount in any direction, we do not change $N(\mathrm{R}\,(C, \overline{H}_+), C)$ combinatorially (specifcally, as long as the boundary of the hemisphere does not pass vertices of C). But generically, the distance from the midpoint will now distinguish faces of C. This is obvious: The distance to a totally geodesic subspace is real analytic in polar coordinates, and as such, having an open set where the distance to two polyhedra is the same implies that they generate a common subspace by geodesic closure, and have the projection points in their relative interior in common. Hence, they are the same polyhedron in C.

Let $\mathrm{M}(C, \overline{H}_+)$ denote the faces σ of $\mathrm{R}\,(C, \overline{H}_+)$ for which the function d attains its minimum in the relative interior of σ. In particular, $\mathrm{M}(C, \overline{H}_+)$ contains all vertices of $\mathrm{R}\,(C, \overline{H}_+)$. With this, we order the elements of $\mathrm{M}(C, \overline{H}_+)$ strictly by defining $\sigma \prec \sigma'$ whenever $\min_{y\in\sigma} \mathrm{d}(y) < \min_{y\in\sigma'} \mathrm{d}(y)$.

This allows us to induce an associated derived order on the vertices of $\mathrm{sd}\, C$, which we restrict to the vertices of $N(\mathrm{R}\,(C, \overline{H}_+), C)$. Let $v_0, v_1, v_2, \ldots, v_n$ denote the vertices of $N(\mathrm{R}\,(C, \overline{H}_+), C)$ labeled according to the latter order, starting with the maximal element v_0. Let C_i denote the complex $N(\mathrm{R}\,(C, \overline{H}_+), C) - \{v_0, v_1, \ldots, v_{i-1}\}$, and define

$$\Sigma_i := C_i \cup N(\mathrm{R}\,(\partial C, \overline{H}_+), \partial C).$$

We will prove that $\Sigma_i \searrow \Sigma_{i+1}$ for all i, $0 \le i \le n-1$; from this $(\mathrm{A})_d$ and $(\mathrm{B})_d$ will follow. There are four cases to consider here.

(1) v_i is in the interior of $\mathrm{sd}\, C$ and corresponds to an element of $\mathrm{M}(C, \overline{H}_+)$.
(2) v_i is in the interior of $\mathrm{sd}\, C$ and corresponds to a face of C not in $\mathrm{M}(C, \overline{H}_+)$.
(3) v_i is in the boundary of $\mathrm{sd}\, C$ and corresponds to an element of $\mathrm{M}(C, \overline{H}_+)$.
(4) v_i is in the boundary of $\mathrm{sd}\, C$ and corresponds to a face of C not in $\mathrm{M}(C, \overline{H}_+)$.

We need some notation to prove these four cases. Recall that we can define N, N^1 and Lk with respect to a basepoint; we shall need this notation in cases (1) and (3). Furthermore, let us denote by τ the face of C corresponding to v_i in $\mathrm{sd}\, C$, and let u denote the point $\arg\min_{y\in\tau} \mathrm{d}(y)$. Finally, define the ball B_u as the set of points y in S^d with $\mathrm{d}(y) \le \mathrm{d}(u)$.
Case (1): The complex $\mathrm{Lk}\,(v_i, \Sigma_i)$ is combinatorially equivalent to $N(\mathrm{LLk}_u(\tau, C), \mathrm{Lk}_u(\tau, C))$, where

$$\mathrm{LLk}_u(\tau, C) := \mathrm{R}\,(\mathrm{Lk}_u(\tau, C),\ \ \mathrm{int}\ \mathrm{N}^1_{(u,\tau)} B_u)$$

is the restriction of $\mathrm{Lk}_u(\tau, C)$ to the interior of the hemisphere $\mathrm{N}^1_{(u,\tau)} B_u$ of $\mathrm{N}^1_{(u,\tau)} S^d$. Since the transformation φ was chosen to be generic, $\mathrm{N}^1_{(u,\tau)} B_u$ is in general position with respect to $\mathrm{Lk}_u(\tau, C)$[1] and

$$\mathrm{LLk}_u(\tau, C) \cong \mathrm{R}\,(\mathrm{Lk}_u(\tau, C), \mathrm{N}^1_{(u,\tau)} B_u).$$

Hence, by assumption $(\mathrm{A})_{d-1}$, the complex

$$N(\mathrm{LLk}_u(\tau, C), \mathrm{Lk}_u(\tau, C)) \cong \mathrm{Lk}\,(v_i, \Sigma_i)$$

is collapsible. Consequently, Lemma 2.2 proves $\Sigma_i \searrow \Sigma_{i+1} = \Sigma_i - v_i$.
Case (2): If τ is not an element of $\mathrm{M}(C, \overline{H}_+)$, let σ denote the face of τ containing u in its relative interior. Then, $\mathrm{Lk}\,(v_i, \Sigma_i) = \mathrm{Lk}\,(v_i, C_i)$ is combinatorially equivalent to

[1] This is true because any movement of star-center x normal to the geodesic span of τ and x induces a motion of the hemisphere $\mathrm{N}^1_{(u,\tau)} B_u$ in $\mathrm{N}^1_{(u,\tau)} S^d$, specifically, by moving the midpoint of the hemisphere in the same direction.

the order complex of the union $\mathrm{L}(\tau, C) \cup \sigma$, whose elements are ordered by inclusion. Since σ is a unique global minimum of the poset, the complex $\mathrm{Lk}\,(v_i, \Sigma_i)$ is a cone, and in fact combinatorially equivalent to a cone over base $\mathrm{sd}\,\mathrm{Lk}\,(\tau, C)$. But all cones are non-evasive (Lemma 2.1), so $\mathrm{Lk}\,(v_i, \Sigma_i)$ is collapsible. Consequently, Lemma 2.2 gives $\Sigma_i \searrow \Sigma_{i+1} = \Sigma_i - v_i$.

Case (3): This time, v_i is in the boundary of $\mathrm{sd}\,C$. As in case (1), $\mathrm{Lk}\,(v_i, C_i)$ is combinatorially equivalent to the complex

$$N(\mathrm{LLk}_u(\tau, C), \mathrm{Lk}_u(\tau, C)) \cong \mathrm{LLk}_u(\tau, C) \cong \mathrm{R}\,(\mathrm{Lk}_u(\tau, C), \mathrm{N}^1_{(u,\tau)} B_u)$$

in the sphere $\mathrm{N}^1_{(u,\tau)} S^d$. Recall that $\overline{H}_+ \backslash C$ is star-shaped with star-center x and that τ is not the face of C that minimizes $\mathrm{d}(y)$ since $v_i \neq v_n$, so that $\mathrm{N}^1_{(m,\tau)} B_u \cap \mathrm{N}^1_{(u,\tau)} \partial C$ is nonempty. Since furthermore $\mathrm{N}^1_{(m,\tau)} B_u$ is a hemisphere in general position with respect to the complex $\mathrm{Lk}_u(\tau, C)$ in the sphere $\mathrm{N}^1_{(u,\tau)} S^d$, the inductive assumption $(\mathrm{B})_{d-1}$ applies: The complex $N(\mathrm{LLk}_u(\tau, C), \mathrm{Lk}_u(\tau, C))$ collapses to

$$N(\mathrm{LLk}_u(\tau, \partial C), \mathrm{Lk}_u(\tau, \partial C)) \cong \mathrm{Lk}\,(v_i, C'_i),$$
$$C'_i := C_{i+1} \cup (C_i \cap N(\mathrm{R}\,(\partial C, \overline{H}_+), \partial C)).$$

Consequently, Lemma 2.2 proves that C_i collapses to C'_i. Since

$$\begin{aligned}\Sigma_{i+1} \cap C_i &= (C_{i+1} \cup N(\mathrm{R}\,(\partial C, \overline{H}_+), \partial C)) \cap C_i\\ &= C_{i+1} \cup (C_i \cap N(\mathrm{R}\,(\partial C, \overline{H}_+), \partial C)) = C'_i,\end{aligned}$$

Lemma 2.3, applied to the union $\Sigma_i = C_i \cup \Sigma_{i+1}$ of complexes C_i and Σ_{i+1} gives that Σ_i collapses onto Σ_{i+1}.

Case (4): As observed in case (2), the complex $\mathrm{Lk}\,(v_i, C_i)$ is combinatorially equivalent to a cone over base $\mathrm{sd}\,\mathrm{Lk}\,(\tau, C)$, which collapses to the cone over the subcomplex $\mathrm{sd}\,\mathrm{Lk}\,(\tau, \partial C)$ by Lemma 2.1. Thus, the complex C_i collapses to $C'_i := C_{i+1} \cup (C_i \cap N(\mathrm{R}\,(\partial C, \overline{H}_+), \partial C))$ by Lemma 2.2. Now, we have $\Sigma_{i+1} \cap C_i = C'_i$ as in case (3), so that Σ_i collapses onto Σ_{i+1} by Lemma 2.3.

This finishes the proof of $(\mathrm{A})_d$ and $(\mathrm{B})_d$ of Theorem 4.1. It remains to prove the notationally simpler case $(\mathrm{C})_d$.

Proving $(\mathrm{C})_d$: Since C is contained in the open hemisphere $\mathrm{int}\,\overline{H}_+$ (by the general position of $\overline{H}_+$), we may assume, by central projection, that C is actually a convex polytopal complex in $\mathbb{R}^d$. Let ν be generic in $S^{d-1} \subset \mathbb{R}^d$.

The vertices of C are totally ordered according to the decreasing value of $\langle \cdot, \nu\rangle$ on them. Let us extend this order to a total order on the vertices of $\mathrm{sd}\,C$, using the derived order. Let v_i denote the i-th vertex of $\mathrm{F}_0(\mathrm{sd}\,C)$ in the derived order, starting with the maximal vertex v_0 and ending up with the minimal vertex v_n.

The complex $\mathrm{Lk}\,(v_0, C) = \mathrm{LLk}^{\,\nu}(v_0, C)$ is a subdivision of the convex polytope $\mathrm{T}^1_{v_0} C$ in the sphere $\mathrm{T}^1_{v_0} \mathbb{R}^d$ of dimension $d-1$. By assumption $(\mathrm{C})_{d-1}$, the complex

$\mathrm{Lk}\,(v_0, \mathrm{sd}\,C) \cong \mathrm{sd}\,\mathrm{Lk}\,(v_0, C)$ collapses onto $\partial \mathrm{sd}\,\mathrm{Lk}\,(v_0, C) - \sigma'$, where σ' is some facet of $\partial \mathrm{Lk}\,(v_0, \mathrm{sd}\,C)$. By Lemma 2.2, the complex $\mathrm{sd}\,C$ collapses to

$$\Sigma_1 := (\mathrm{sd}\,C - v_0) \cup (\partial \mathrm{sd}\,C - \sigma) = (\mathrm{sd}\,C - v_0) \cup C_\sigma,$$

where $\sigma := v_0 * \sigma'$ and $C_\sigma = \partial \mathrm{sd}\,C - \sigma$.

We proceed removing the vertices one-by-one, according to their position in the order we defined. More precisely, set $C_i := \mathrm{sd}\,C - \{v_0, \ldots, v_{i-1}\}$, and set $\Sigma_i := C_i \cup C_\sigma$. We shall now show that $\Sigma_i \searrow \Sigma_{i+1}$ for all i, $1 \le i \le n-1$; this in particular implies $(\mathrm{C})_d$. There are four cases to consider:

(1) v_i corresponds to an interior vertex of C.
(2) v_i is in the interior of $\mathrm{sd}\,C$ and corresponds to a face of C of positive dimension.
(3) v_i corresponds to a boundary vertex of C.
(4) v_i is in the boundary of $\mathrm{sd}\,C$ and corresponds to a face of C of positive dimension.

Case (1): In this case, the complex $\mathrm{Lk}\,(v_i, \Sigma_i)$ is combinatorially equivalent to the simplicial complex $N(\mathrm{LLk}^{\,\nu}(v_i, C), \mathrm{Lk}\,(v_i, C))$ in the $(d-1)$-sphere $\mathrm{T}^1_{v_i}\mathbb{R}^d$. By assumption $(\mathrm{A})_{d-1}$, the complex

$$N(\mathrm{LLk}^{\,\nu}(v_i, C), \mathrm{Lk}\,(v_i, C)) \cong \mathrm{Lk}\,(v_i, \Sigma_i)$$

is collapsible. Consequently, by Lemma 2.2, the complex Σ_i collapses onto $\Sigma_{i+1} = \Sigma_i - v_i$.
Case (2): If v_i corresponds to a face τ of C of positive dimension, let w denote the vertex of τ minimizing $\langle \cdot, \nu \rangle$. The complex $\mathrm{Lk}\,(v_i, \Sigma_i)$ is combinatorially equivalent to the order complex of the union $\mathrm{L}(\tau, C) \cup w$, whose elements are ordered by inclusion. Since w is a unique global minimum of this poset, the complex $\mathrm{Lk}\,(v_i, \Sigma_i)$ is a cone (with a base naturally combinatorially equivalent to $\mathrm{sd}\,\mathrm{Lk}\,(\tau, C)$). Thus, $\mathrm{Lk}\,(v_i, \Sigma_i)$ is collapsible since every cone is collapsible (Lemma 2.1). Consequently, Lemma 2.2 gives $\Sigma_i \searrow \Sigma_{i+1} = \Sigma_i - v_i$.
Case (3): Similary to case (1), $\mathrm{Lk}\,(v_i, C_i)$ is combinatorially equivalent to the derived neighborhood $N(\mathrm{LLk}^{\,\nu}(v_i, C), \mathrm{Lk}\,(v_i, C))$ in the $(d-1)$-sphere $\mathrm{T}^1_{v_i}\mathbb{R}^d$. By assumption $(\mathrm{B})_{d-1}$, the complex

$$N(\mathrm{LLk}^{\,\nu}(v_i, C), \mathrm{Lk}\,(v_i, C)) \cong \mathrm{Lk}\,(v_i, C_i)$$

collapses to

$$N(\mathrm{LLk}^{\,\nu}(v_i, \partial C), \mathrm{Lk}\,(v_i, \partial C)) \cong \mathrm{Lk}\,(v_i, C_i'), \quad C_i' := C_{i+1} \cup (C_i \cap C_\sigma).$$

Consequently, Lemma 2.2 proves that C_i collapses to C_i'. Now,

$$\Sigma_{i+1} \cap C_i = (C_{i+1} \cup C_\sigma) \cap C_i = C_{i+1} \cup (C_i \cap C_\sigma) = C_i'.$$

If we apply Lemma 2.3 to the union $\Sigma_i = C_i \cup \Sigma_{i+1}$, we obtain that Σ_i collapses to the subcomplex Σ_{i+1}.

Case (4): As in Case (2) above, $\mathrm{Lk}\,(v_i, C_i)$ is naturally combinatorially equivalent to a cone over base $\mathrm{sd}\,\mathrm{Lk}\,(\tau, C)$, which collapses to the cone over the subcomplex $\mathrm{sd}\,\mathrm{Lk}\,(\tau, \partial C)$ by Lemma 2.1. Thus, the complex C_i collapses to $C_i' := C_{i+1} \cup (C_i \cap C_\sigma)$ by Lemma 2.2.

Now, we have $\Sigma_{i+1} \cap C_i = C_i'$ as in Case (3), so that Σ_i collapses onto Σ_{i+1} by Lemma 2.3. □

Lickorish's Conjecture and Hudson's Problem

In this section, we provide the announced partial answers to Lickorish's conjecture (Theorem 4.5) and Hudson's Problem (Theorem 4.3).

Theorem 4.3 *Let C, C' be polytopal complexes such that $C' \subset C$ and $C \searrow C'$. Let D denote any subdivision of C, and define $D' := \mathrm{R}\,(D, C')$. Then, $\mathrm{sd}\,D \searrow \mathrm{sd}\,D'$.*

Proof It suffices to prove the claim for the case where C' is obtained from C by a single elementary collapse; the claim then follows by induction on the number of elementary collapses. Let σ denote the free face deleted in the collapsing, and let Σ denote the unique face of C that strictly contains it.

Let (δ, Δ) be any pair of faces of $\mathrm{sd}\,D$, such that δ is a facet of $\mathrm{R}\,(\mathrm{sd}\,D, \sigma)$, Δ is a facet of $\mathrm{R}\,(\mathrm{sd}\,D, \Sigma)$, and δ is a codimension-one face of Δ. With this, the face δ is a free face of $\mathrm{sd}\,D$. Now, by Corollary 4.2, $\mathrm{R}\,(\mathrm{sd}\,D, \Sigma) - \Delta$ collapses onto $\mathrm{R}\,(\mathrm{sd}\,D, \partial\Sigma)$. Thus, $\mathrm{sd}\,D - \Delta$ collapses onto $\mathrm{R}\,(\mathrm{sd}\,D, \mathrm{sd}\,D \backslash \mathrm{relint}\,\Sigma)$, or equivalently,

$$\mathrm{sd}\,D - \delta \searrow \mathrm{R}\,(\mathrm{sd}\,D, \mathrm{sd}\,D \backslash \mathrm{relint}\,\Sigma) - \delta.$$

Now, $\mathrm{R}\,(\mathrm{sd}\,D, \sigma) - \delta$ collapses onto $\mathrm{R}\,(\mathrm{sd}\,D, \partial\sigma)$ by Corollary 4.2, and thus

$$\mathrm{R}\,(\mathrm{sd}\,D, \mathrm{sd}\,D \backslash \mathrm{relint}\,\Sigma) - \delta \searrow \mathrm{R}\,(\mathrm{sd}\,D, \mathrm{sd}\,D \backslash (\mathrm{relint}\,\Sigma \cup \mathrm{relint}\,\sigma)).$$

To summarize, if C can be collapsed onto $C - \sigma$, then

$$\begin{aligned} \mathrm{sd}\,D &\searrow_e \mathrm{sd}\,D - \delta \\ &\searrow \mathrm{R}\,(\mathrm{sd}\,D, \mathrm{sd}\,D \backslash (\mathrm{relint}\,\Sigma \cup \mathrm{relint}\,\sigma)) = \mathrm{R}\,(\mathrm{sd}\,D, C - \sigma) \\ &= \mathrm{R}\,(\mathrm{sd}\,D, C'). \end{aligned}$$ □

Lemma 4.4 (Bruggesser–Mani [7]) *Let σ be any nonempty face of a d-dimensional polytope P. As polytopal complex, P collapses onto $\mathrm{St}\,(\sigma, \partial P)$, which collapses onto σ, which is collapsible.*

Proof By induction on d. When $d = 1$ the claim boils down to the obvious fact that a segment collapses onto any of its endpoints. When $d \geq 2$, let us perform a rocket shelling of ∂P with a generic line through σ, as explained in [19, Cor. 8.13] (for the case where σ is a vertex). This yields a shelling of ∂P in which $\mathrm{St}\,(\sigma, \partial P)$ is shelled first. Let τ be the last facet of such a shelling. Now, any contractible shellable complex is collapsible; the collapsing sequence of the facets is given by the inverse shelling order, cf. e.g. [12, Lem. 17]. Therefore $\partial P - \tau$ collapses onto St $(\sigma, \partial P)$. But as a polytopal complex, the complex P collapses polyhedrally onto $\partial P - \tau$; so P collapses also onto St $(\sigma, \partial P)$. But by the inductive assumption each $(d-1)$-dimensional polytope Q in St $(\sigma, \partial P)$ collapses down to St $(\sigma, \partial Q)$. Repeatedly applying collapses St $(\sigma, \partial P)$ to σ. By inductive assumption, σ is collapsible. □

Theorem 4.5 *Let C denote any subdivision of a d-dimensional polytope. Then* sd C *is collapsible.*

Proof By Lemma 4.4, the polytope is collapsible. Now for any subdivision C of the polytope, the facets of C are all polytopes (not necessarily simplicial). Hence sd C is collapsible by Theorem 4.3. □

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