FREE BOUNDARY REGULARITY FOR A MULTIPHASE SHAPE OPTIMIZATION PROBLEM

LUCA SPOLAOR, BAPTISTE TREY, BOZHIDAR VELICHKOV

Abstract. In this paper we prove a $C^{1,\alpha}$ regularity result in dimension two for almost-minimizers of the constrained one-phase Alt-Caffarelli and the two-phase Alt-Caffarelli-Friedman functionals for an energy with variable coefficients. As a consequence, we deduce the complete regularity of solutions of a multiphase shape optimization problem for the first eigenvalue of the Dirichlet Laplacian, up to the boundary of a fixed domain that acts as a geometric inclusion constraint. One of the main ingredient is a new application of the epiperimetric inequality of [22] up to the boundary of the constraint. While the framework that leads to this application is valid in every dimension, the epiperimetric inequality is known only in dimension two, thus the restriction on the dimension.

1. Introduction

In this paper we prove the regularity of the free boundary of solutions to variational one-phase and two-phase free boundary problems in dimension two. In particular, we consider the case when the support of the solution is constrained in certain region $D \subset \mathbb{R}^2$ with smooth ($C^{1,\beta}$-regular, for some $\beta > 0$) boundary and we show that the free boundaries must be $C^{1,\alpha}$-regular up to the points of contact with the boundary of the region D. Our arguments strongly rely on the epiperimetric inequality for the one-phase and the two-phase Bernoulli free boundary functionals (see [22]). The only obstruction to the generalization of our results to any dimension comes from the fact that the epiperimetric inequality is only known in dimension two. Since our methods are of purely variational nature, we are able to obtain the regularity of the free boundaries of almost-minimizers (even in the presence of a geometric constraint D). Precisely, we prove the following results.

(OP) The $C^{1,\alpha}$-regularity of the boundaries of the almost-minimizers of the one-phase Alt-Caffarelli functional for an operator with variable coefficients, which may also satisfy a further geometric inclusion constraint (Theorem 1.1 and Corollary 1.3);

(TP) The $C^{1,\alpha}$-regularity of the boundaries (of each phase) of the almost-minimizers of the two-phase Alt-Caffarelli-Friedman functional for an operator with variable coefficients (Theorem 1.5). In particular, this is a problem left open in [12].

In the second part of the paper, we apply our regularity results for almost-minimizers to the solutions of multiphase shape optimization problems, where the variables are n-uples of different disjoint domains (phases). Precisely, we consider the model case in which the variational cost functional is given by the sum of the first eigenvalue (of the Dirichlet Laplacian) and the area of each domain. In Theorem 1.11, we show that if the family of domains $(\Omega_1, \dots, \Omega_n)$ is a solution to such a multiphase problem, then the boundaries $\partial\Omega_i$ are $C^{1,\alpha}$-regular; thus we solve the (regularity) problem left open in [3].

The regularity of the free boundaries arising in the context of shape optimization problems involving Dirichlet eigenvalues is a topic that received a lot of attention recently (we refer for instance to the recent papers [20, 18, 19, 10]). The multiphase shape optimization problem we

1991 *Mathematics Subject Classification.* 35R35, 49Q10, 47A75.

Key words and phrases. Regularity of free boundaries, the one-phase problem, Alt-Caffarelli, the two-phase problem, Alt-Caffarelli-Friedman, almost-minimizers, epiperimetric inequality, shape optimization, Dirichlet eigenvalues, multiphase problems.

consider is part of this general framework and was studied in [8] and [3]. It is related to the regularity of the optimal sets (in a box) for the second Dirichlet eigenvalue, which is expected to have common qualitative properties with the solutions to the multiphase problem in the case $n = 2$. We also notice that our almost-minimality conditions naturally arise in shape optimization problems involving Dirichlet eigenvalues (see for instance [20], [7] and Section 7). The geometric inclusion constraints are also very common in this framework (see for instance [6], [4] and the books [17], [5]). For instance, they are often used to provide the compactness necessary for the existence of optimal shapes. Thus, our regularity results and techniques can be applied not only to the solutions of the multiphase problem studied in [8] and [3], but to a variety of shape optimization problems, for instance, on manifolds or for operators with variable coefficients.

The regularity of the one-phase free boundaries was first studied by Alt and Caffarelli in [1] who prove that in any dimension the (local) minimizers of the one-phase functional have smooth free boundaries up to a small singular set, while in dimension two, they show that the entire free boundaries are C^∞ smooth. The regularity of almost-minimizers of the one-phase functional was addressed by David and Toro in [13] and by David, Engelstein and Toro in [12], where they prove the $C^{1,\alpha}$ regularity of the free boundary up to a singular set, which is empty in dimension two; the same result was recently obtained by De Silva and Savin [15] by a different method based on a non-infinitesimal notion of viscosity solution. In [**?**] and [15], the authors consider only the case of the Laplacian, but it is clear that their proof can be generalized to an operator with variable coefficients.

With our approach, working with a functional with variable coefficients, instead of constant ones, leads only to minor variations in the proof. We also stress that there is no way to reduce the non-constant-coefficients case to the constant-coefficients one. In fact, if u is a minimizer (or almost-minimizer) of a functional with variable coefficients, then for any point x, there is a change of coordinates, for which the new function becomes an almost-minimizer for a functional involving only the Dirichlet energy (see Lemma 3.2). This change of coordinates, on the other hand, depends on the point. It is also responsible for the anisotropic optimality condition on the free boundary (see (1.5)).

The regularity of minimizers for the one-phase functional subjected to a geometric inclusion constraint was studied by Chang-Lara and Savin in [11], where, using the approach of De Silva [14], the authors prove that in a neighborhood of the contact set (with the boundary of the constraint) the free boundary is $C^{1,\alpha}$ regular in any dimension. In Theorem 1.1 we consider almost-minimizers satisfying an inclusion constraint, which combines the difficulties from [12] and [11]: the lack of equation ([12]) and the presence of a geometric constraint ([11]). In fact, our approach allows to treat these two situations at once.

The two-phase problem was first studied by Alt, Caffarelli and Friedman in [2] (see Remark 1.6) but the regularity of the free boundary (for minimizers) was achieved only recently in [22]. In [12], David, Engelstein and Toro address the question of the regularity of almost-minimizers of the two-phase functional and prove the rectifiability of the free boundary in any dimension. In Theorem 1.5 we prove the $C^{1,\alpha}$ regularity of the free boundaries of almost-minimizers for two-phase functional. We notice that, since each phase acts as a geometric obstacle for the other one, and in view of the Chang-Lara-Savin result ([11]), the $C^{1,\alpha}$-regularity is optimal even for minimizers, the best α being $1/2$. We also notice that, in the two-phase case, an additional (smooth) geometric inclusion constraint does not affect the regularity of the free boundaries. In fact, a two-phase free boundary cannot touch the boundary of the constraint. This is a simple consequence of the three-phase monotonicity formula introduced in [23] and [8], just as in the case of the multiphase shape optimization problem (1.10) (see [3] and Section 7).

1.1. **Regularity for almost-minimizers.** Throughout this paper we will use the following notations.

Let Sym_k^+ be the family of the real positive symmetric $k \times k$ matrices. We fix a matrix-valued function $A = (a_{ij})_{ij} : B_2 \to Sym_2^+$, for which there are constants $\delta_{\rm A}, C_{\rm A}, M_{\rm A} > 0$ such that

$$|a_{ij}(x) - a_{ij}(y)| \le C_{\rm A}|x-y|^{\delta_{\rm A}} \quad \text{for every} \quad i,j \quad \text{and} \quad x,y \in B_2\,;$$

$$M_{\rm A}^{-1}|\xi|^2 \le \xi \cdot A(x)\xi = \sum_{i,j=1}^{2} \xi_i\xi_j a_{ij}(x) \le M_{\rm A}|\xi|^2 \quad \text{for every} \quad x \in B_2 \quad \text{and} \quad \xi \in \mathbb{R}^2. \tag{1.1}$$

We fix $Q_{\rm OP}$, $Q_{\rm TP}^+$ and $Q_{\rm TP}^-$ to be Hölder continuous functions on B_2, for which there are constants $\delta_{\rm Q}$, $C_{\rm Q}$, $M_{\rm Q} > 0$ such that for any $Q = Q_{\rm OP}$, $Q_{\rm TP}^+$ or $Q_{\rm TP}^-$, we have

$$|Q(x) - Q(y)| \le C_{\rm Q}|x-y|^{\delta_{\rm Q}} \quad \text{for every} \quad x,y \in B_2\,;$$

$$M_{\rm Q}^{-1} \le Q(x) \le M_{\rm Q} \quad \text{for every} \quad x \in B_2\,.$$

Finally, for every function $u : \mathbb{R}^2 \to \mathbb{R}$, we will use the following standard notations

$$u_\pm(x) := \max\{\pm u(x), 0\}\,, \quad \Omega_u := \{u \neq 0\}\,, \quad \Omega_u^+ := \{u > 0\} \quad \text{and} \quad \Omega_u^- := \{u < 0\}.$$

We are now in position to state our main free boundary regularity results.

The one-phase free boundaries. For every $u \in H^1(B_2)$, $x_0 \in B_1$ and $r \in (0,1)$, we define the one-phase functional

$$J_{\rm OP}(u, x_0, r) = \int_{B_r(x_0)} \Big(\sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + Q_{\rm OP}(x)\mathbb{1}_{\{u>0\}} \Big)\, dx.$$

Here and after $B_r(x)$ denotes the ball with center $x \in \mathbb{R}^2$ and radius $r > 0$ and we will write $B_r := B_r(0)$. Let $\mathcal{A}^+(B_r)$ be the admissible set

$$\mathcal{A}^+(B_r) = \big\{ u \in H^1(B_r) \ : \ u \ge 0 \text{ in } B_r\,,\ u = 0 \text{ on } B_r \setminus B_r^+ \big\},$$

where H stands for the upper half-plane $H := \big\{(x,y) \in \mathbb{R}^2 \ : \ y > 0\big\}$ and $B_r^+ := B_r \cap H$. We say that the nonnegative function $u : B_2 \to \mathbb{R}$ is a *almost-minimizer of the one-phase functional* $J_{\rm OP}$ *in the upper half-disk* B_2^+, if $u \in \mathcal{A}^+(B_2)$ and there are constants $r_1 > 0$, $C_1 > 0$ and $\delta_1 > 0$ such that, for every $x_0 \in B_1 \cap \partial\Omega_u$ and $r \in (0, r_1)$, we have

$$\begin{aligned} J_{\rm OP}(u, x_0, r) \le \big(1 + C_1 r^{\delta_1}\big) J_{\rm OP}(v, x_0, r) + C_1 r^{2+\delta_1}, \\ \text{for every } v \in \mathcal{A}^+(B_2) \text{ such that } u = v \text{ on } B_2 \setminus B_r(x_0). \end{aligned} \tag{1.2}$$

We have the following result for the almost-minimizers of the one-phase Alt-Caffarelli functional $J_{\rm OP}$ constrained in the upper half-disk B_2^+.

Theorem 1.1 (Regularity of the constrained one-phase free boundaries)**.** *Let $B_2 \subset \mathbb{R}^2$ and $u : B_2 \to \mathbb{R}$ be a non-negative and Lipschitz continuous function. If u is a almost-minimizer of the functional $J_{\rm OP}$ in $\mathcal{A}^+(B_2)$, then the free boundary $B_1 \cap \partial\Omega_u$ is locally the graph of a $C^{1,\alpha}$ function. Moreover, u satisfies the optimality condition*

$$\begin{cases} |A_{x_0}^{1/2}\nabla u|(x_0) = Q_{\rm OP}^{1/2}(x_0) & \text{for every} \quad x_0 \in \partial\Omega_u^+ \cap B_2 \cap \{x_2 > 0\}, \\ |A_{x_0}^{1/2}\nabla u|(x_0) \ge Q_{\rm OP}^{1/2}(x_0) & \text{for every} \quad x_0 \in \partial\Omega_u^+ \cap B_2 \cap \{x_2 = 0\}. \end{cases} \tag{1.3}$$

Remark 1.2. The Hölder continuity of the (exterior) normal vector n_Ω is the best regularity result that one can expect. Indeed, recently Chang-Lara and Savin [11] showed that even for minimizers the regularity of the constrained free boundaries cannot exceed $C^{1,1/2}$. Moreover, we notice that the result analogous to Theorem 1.1 was proved in any dimension in [11], by a viscosity approach, but only for minimizers of the functional $J_{\rm OP}$.

Analogously, we say that the nonnegative function $u : B_2 \to \mathbb{R}$ is a *almost-minimizer of the one-phase functional* J_{OP} *in* B_2, if $u \in H^1(B_2)$ and there are constants $r_1 > 0$, $C_1 > 0$ and $\delta_1 > 0$ such that, for every $x_0 \in B_1 \cap \partial\Omega_u$ and $r \in (0, r_1)$, we have

$$J_{\text{OP}}(u, x_0, r) \le \big(1 + C_1 r^{\delta_1}\big) J_{\text{OP}}(v, x_0, r) + C_1 r^{2+\delta_1}, \qquad \text{for every } v \in H^1(B_2) \text{ such that } u = v \text{ on } B_2 \setminus B_r(x_0). \tag{1.4}$$

The regularity of the unconstrained one-phase free boundary $\partial\Omega_u$ follows directly by Theorem 1.1. For the sake of completeness, we give the precise statement in Corollary 1.3 below.

Corollary 1.3 (Regularity of the unconstrained one-phase free boundaries). *Let* $B_2 \subset \mathbb{R}^2$ *and* $u : B_2 \to \mathbb{R}$ *be a non-negative and Lipschitz continuous function. If* u *is a almost-minimizer of the functional* J_{OP} *in* B_2, *then the free boundary* $B_1 \cap \partial\Omega_u$ *is locally the graph of a* $C^{1,\alpha}$ *function. Moreover,* u *satisfies the optimality condition*

$$|A_{x_0}^{1/2} \nabla u|(x_0) = Q_{\text{OP}}^{1/2}(x_0) \quad \textit{for every} \quad x_0 \in \partial\Omega_u^+ \cap B_2. \tag{1.5}$$

Remark 1.4. The regularity of the free boundaries of the one-phase (unconstrained) almost-minimizers was proved in [12] in every dimension, by a different approach.

The two-phase free boundaries. For every $u \in H^1(B_2)$, $x_0 \in B_1$ and $r \in (0,1)$, we define the two-phase functional

$$J_{\text{TP}}(u, x_0, r) = \int_{B_r(x_0)} \Big(\sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + Q_{\text{TP}}^+(x) \mathbb{1}_{\{u>0\}} + Q_{\text{TP}}^-(x) \mathbb{1}_{\{u<0\}} \Big)\, dx.$$

We say that the function $u \in H^1(B_2)$ is a *almost-minimizer of the two-phase functional* J_{TP} *in* B_2, if there are constants $r_2 > 0$, $C_2 > 0$ and $\delta_2 > 0$ such that, for every $x_0 \in B_1 \cap \partial\Omega_u$ and $r \in (0, r_2)$, we have

$$J_{\text{TP}}(u, x_0, r) \le \big(1 + C_2 r^{\delta_2}\big) J_{\text{TP}}(v, x_0, r) + C_2 r^{2+\delta_2}, \qquad \text{for every } v \in H^1(B_2) \text{ such that } u = v \text{ on } B_2 \setminus B_r(x_0). \tag{1.6}$$

Then, we have the following result:

Theorem 1.5 (Regularity of the two-phase free boundaries). *Let* $B_2 \subset \mathbb{R}^2$ *and let* $u : B_2 \to \mathbb{R}$ *be Lipschitz continuous and such that the functions* $u_\pm$ *are solutions of the PDEs*

$$-\text{div}\,(A\nabla u_\pm) = f_\pm \quad \textit{in} \quad \Omega_u^\pm \cap B_2, \tag{1.7}$$

where:

(a) *the functions* $f_\pm : \overline{\Omega}_u^\pm \to \mathbb{R}$ *are bounded and continuous;*
(b) *the matrix-valued function* $A : B_2 \to Sym_2^+$ *satisfies* (1.1) *and has* C^1*-regular coefficients.*

Under these conditions, if u *is a almost-minimizer of the two-phase functional* J_{TP} *in* B_2, *then the free boundaries* $B_1 \cap \partial\Omega_u^+$ *and* $B_1 \cap \partial\Omega_u^-$ *are locally graphs of* $C^{1,\alpha}$ *functions, for some* $\alpha > 0$. *Moreover,* u *satisfies the optimality condition (on* $\partial\Omega_u^+$*)*

$$\begin{cases} |A_{x_0}^{1/2} \nabla u_+|(x_0) = \big(Q_{\text{OP}}^+(x_0)\big)^{1/2} & \textit{for every} \quad x_0 \in \partial\Omega_u^+ \setminus \partial\Omega_u^- \cap B_2, \\ |A_{x_0}^{1/2} \nabla u_+|(x_0) \ge \big(Q_{\text{OP}}^+(x_0)\big)^{1/2} & \textit{for every} \quad x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_2. \end{cases} \tag{1.8}$$

Remark 1.6 (Regularity of the two-phase free boundaries for minimizers). Alt, Caffarelli and Friedman were the first to study the two-phase free boundaries in (see [2]). Precisely, they studied the local minimizers of the functional

$$J_{\text{ACF}}(u, x_0, r) = \int_{B_r(x_0)} \Big(|\nabla u|^2 + q^2(x)\lambda^2(u(x)) \Big)\, dx,$$

where q is Hölder continuous and bounded away from zero and λ is the function

$$\lambda^2(u) = \lambda_1^2 \quad \text{if} \quad u > 0 \qquad \text{and} \qquad \lambda^2(u) = \lambda_2^2 \quad \text{if} \quad u < 0.$$

In the case, $\lambda(0) = \lambda_1$, they prove that the free boundary $\partial\{u > 0\}$ is C^1 and that the two free boundaries $\partial\{u > 0\}$ and $\partial\{u < 0\}$ coincide (this means that the set $\{u = 0\}$ has empty interior). On the other hand, the case

$$0 \le \lambda^2(0) < \min\{\lambda_1^2, \lambda_2^2\}, \tag{1.9}$$

where branching points may appear, was left completely open in [2]. We notice that our Theorem 1.5 covers this case, by setting

$$Q_+(x) = q^2(x)\big(\lambda_1^2 - \lambda^2(0)\big) \qquad \text{and} \qquad Q_-(x) = q^2(x)\big(\lambda_2^2 - \lambda^2(0)\big).$$

Precisely, Theorem 1.5 implies that, if u is a minimizer of J_{ACF} in a ball B_r and (1.9) holds, then the free boundaries $\partial\Omega_u^+ \cap B_r$ and $\partial\Omega_u^- \cap B_r$ are $C^{1,\alpha}$-regular. This result (for minimizers in the constant-coefficient case and in dimension two) was first proved by the first and the third author in [22]. In particular, it concludes the analysis of the free boundary for minimizers of the functional J_{ACF}, which was started in [2].

Remark 1.7 (Remark on the Lipschitz continuity of u)*.* In Theorem 1.3 and Theorem 1.5 we assume that the function u is Lipschitz continuous. In the case of the Laplacian, David and Toro [13] proved that the Lipschitz continuity is a consequence of the the almost-minimality condition. It is, of course, natural to expect that the same will hold if the operator involved has variable coefficients. We will not address this question in the present paper since our main motivation comes from the application to shape optimization problems as (1.10), for which the Lipschitz continuity is often already known. Actually, in the case of (1.10), the Lipschitz continuity of the eigenfunctions is used to deduce the almost-minimality (see Section 7).

We notice that, just from the fact that u is almost-minimizer of the two-phase functional J_{TP}, without using the additional assumption (1.7), we can still deduce that u is differentiable at points of the free boundary and that the optimality condition (1.8) holds. We summarize the regularity properties, that can be obtained just from the almost-minimality, in the following proposition.

Proposition 1.8. *Let $B_2 \subset \mathbb{R}^2$ and let $u : B_2 \to \mathbb{R}$ be Lipschitz continuous almost-minimizer of the two-phase functional J_{TP} in B_2. Then, for every free boundary point $x_0 \in \partial\Omega_u^+ \cap B_1$ (the case $x_0 \in \partial\Omega_u^- \cap B_1$ is symmetric), the following claims hold true.*

(i) *For every $x_0 \in \partial\Omega_u^+$, there is a unique blow-up limit $u_{x_0}^+ = \lim_{r\to 0} u_{r,x_0}^+$, where u_{r,x_0}^+ is the rescaling $u_{r,x_0}^+(x) = \frac{1}{r}u_+(x_0 + rx)$ and the convergence is uniform on every ball $B_R \subset \mathbb{R}^2$.*

(ii) *The blow-up limit is of the form*

$$u_{x_0}^+(x) = \mu_+(x_0)\max\big\{0, x \cdot A_{x_0}^{-1/2}[\widetilde{\nu}_{x_0}]\big\},$$

where $\mu_+(x_0)$ is a positive real number and $\widetilde{\nu}_{x_0} \in \mathbb{R}^2$ is a unit vector.

(iii) *There are universal constants $C > 0$ and $\alpha > 0$, and a radius $r(x_0) > 0$, such that*

$$\|u_{r,x_0}^+ - u_{x_0}^+\|_{L^\infty(B_1)} \le Cr^\alpha \quad \text{for every} \quad 0 < r < r(x_0).$$

(iv) *u is differentiable at x_0, up to the boundary of Ω_u^+, that is,*

$$u(x) = (x - x_0)\cdot\nabla u_+(x_0) + O\big(|x - x_0|^\alpha\big) \qquad \text{for every} \qquad x \in \Omega_u^+,$$

and the gradient at x_0 is given by

$$\nabla u_+(x_0) = \mu_+(x_0)A_{x_0}^{-1/2}[\widetilde{\nu}_{x_0}],$$

where $\widetilde{\nu}_{x_0}$ is the unit vector from (ii)*.*

(v) *If x_0 is a one-phase point, $x_0 \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$, then*

$$\mu_+(x_0)^2 = Q_{\text{TP}}^+(x_0) \qquad \text{and} \qquad r(x_0) \text{ depends on the point } x_0.$$

(vi) *If x_0 is a two-phase point, $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, then*

$$\mu_+(x_0)^2 \ge Q_{\text{TP}}^+(x_0) \qquad \text{and} \qquad r(x_0) = r_0,$$

where r_0 may depend on u, but not on x_0.

This statement is contained in Proposition 4.3, Lemma 6.1, Lemma 6.2 and Lemma 6.3. As a corollary of Proposition 1.8, we obtain the following corollary, whose proof is standard and is, in fact, similar to the proof of Theorem 1.1.

Corollary 1.9. *Let $B_2 \subset \mathbb{R}^2$ and let $u : B_2 \to \mathbb{R}$ be Lipschitz continuous almost-minimizer of the two-phase functional J_{TP} in B_2. Then, for every $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, there exists a neighborhood $\mathcal{U}_{x_0}$ such that the two-phase free boundary $\partial\Omega_u^+ \cap \partial\Omega_u^- \cap \mathcal{U}_{x_0}$ is contained in a single $C^{1,\alpha}$-regular embedded curve.*

We notice that the result from Theorem 1.5 is stronger (but requires the additional technical assumption (1.7)). In fact, if the set Ω_u^+, then the two-phase boundary $\partial\Omega_u^+ \cap \partial\Omega_u^-$ is (obviously) contained in a $C^{1,\alpha}$ curve. We believe that the $C^{1,\alpha}$ regularity of the sets Ω_u^+ and Ω_u^- still holds without the additional assumption (1.7), but at the moment, we cannot remove it from our argument. This is mainly due to the fact that we use a combination of viscosity and variational techniques. We give more details on the use of (1.7) in the following remark.

Remark 1.10 (On the role of the additional assumption (1.7) in Theorem 1.5). In order to conclude the proof of Theorem 1.5, we will use Proposition 1.8 and the assumption (1.7). Indeed, the $C^{1,\alpha}$ regularity of $\partial\Omega_u^+ \cap B_1$ follows from the following two claims:

(1) The function $\mu_+ : \partial\Omega_u \to \mathbb{R}$ is Hölder continuous.
(2) Suppose that $u_+ : B_1 \to \mathbb{R}$ is also a (viscosity) solution to

$$-\text{div}\,(A\nabla u_+) = f \quad \text{in} \quad \Omega_u^+ \cap B_1, \qquad |A^{1/2}\nabla u_+| = Q \quad \text{on} \quad \partial\Omega_u^+ \cap B_1,$$

where f is bounded, A is C^1 and Q is Hölder continuous and bounded from below and above. Then $\partial\Omega_u^+ \cap B_1$ is $C^{1,\alpha}$ regular for some $\alpha > 0$.

The first claim is proved in Lemma 6.4. We notice that in order to prove the continuity of μ_+, we use an argument by contradiction, in which we consider a sequence of the form $u_n := u^+_{x_n,r_n}$, which converges to some function u_∞. At this point, we need that the boundary condition on $\partial\{u_n > 0\}$ passes to the limit (in viscosity sense). It is not at the present clear how to overcome this difficulty if u is just an almost-minimizer. The second claim follows by the De Silva ε-regularity theorem (see Theorem A.1). Also in this case we need that u solves an elliptic equation inside the positivity set Ω_u^+.

1.2. **Multiphase shape optimization problem for the first eigenvalue.** As a consequence of Theorem 1.1 and Theorem 1.5, we prove a regularity result for the solutions of the following multiphase shape optimization problem:

$$\min\Big\{\sum_{i=1}^n \big(\lambda_1(\Omega_i) + q_i|\Omega_i|\big) \ :\ \Omega_1,\dots,\Omega_n \ \text{ are disjoint open subsets of } \ \mathcal{D}\Big\}, \tag{1.10}$$

where, we will use the following notations:

- $1 \le n \in \mathbb{N}$, and $0 < q_i \in \mathbb{R}$, for every $i = 1,\dots,n$;
- $\mathcal{D} \subset \mathbb{R}^2$ is a bounded open set with $C^{1,\beta}$-regular boundary, for some $\beta > 0$;
- $|\Omega|$ denotes the Lebesgue measure of Ω;
- $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian on Ω.

Theorem 1.11. *Let $(\Omega_1,\dots,\Omega_n)$ be a solution of (1.10). Then, each set Ω_i, $i = 1,\dots,n$, is a bounded open set with $C^{1,\alpha}$ regular boundary, for some $\alpha > 0$.*

We notice that, in the above theorem, we prove that the *entire* boundary $\partial\Omega_i$, $i = 1,\dots,n$, is $C^{1,\alpha}$-regular. In particular, this holds at the contact points of Ω_i with the other phases Ω_j, $j \ne i$, and also at the contact points of $\partial\Omega_i$ with the boundary of the box $\partial\mathcal{D}$.

In the special case $n = 1$, (1.10) reduces to the classical (one-phase) shape optimization problem

$$\min\big\{\lambda_1(\Omega) + \Lambda|\Omega| \ :\ \Omega \text{ open},\ \Omega \subset \mathcal{D}\big\}. \tag{1.11}$$

The existence of a solution in the class of open sets and the regularity of the free boundary (precisely, of the part contained in the open set $\mathcal{D}$) was proved by Briançon and Lamboley in [4]. As a direct corollary of our Theorem 1.11, we obtain that the entire boundary is $C^{1,\alpha}$ regular.

Corollary 1.12 (Regularity of the optimal sets for the first eigenvalue)**.** *Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded open set of class $C^{1,\beta}$, for some $\beta > 0$, and let $\Lambda > 0$. Then, there is $\alpha \in (0,1)$ such that every solution $\Omega \subset D$ of (1.11) is $C^{1,\alpha}$ regular.*

1.3. **Organization of the paper.** In Section 2 we recall the definitions of the Weiss' boundary adjusted energies and the statements of the epiperimetric inequalities. Moreover, we show how to use the monotonicity formula and the epiperimetric inequality to deduce the rate of convergence of the blow-up sequences and the uniqueness of the blow-up limits. In Section 3 we prove a technical lemma that reduces the one-phase and two-phase problems to the case of the Laplacian, which allows us to apply the results of Section 2. Section 4 is dedicated to the classification of the blow-up limits for the one-phase and the two-phase problems. In Section 5 and Section 6 we prove Theorem 1.1 and Theorem 1.5, respectively. In Section 7 we prove that the (eigenfunctions associated to the) solutions of the multiphase problem (1.10) are locally almost-minimizers of the one-phase or the two-phase problems, and we prove Theorem 1.11.

2. Boundary adjusted energy and epiperimetric inequality

All the arguments in this section hold in every dimension $d \geq 2$, except the epiperimetric inequalities Theorem 2.2 and Theorem 2.3, which are known only in dimension two.

2.1. **One-homogeneous rescaling and excess.** Let $d \geq 2$ and $u \in H^1_{loc}(\mathbb{R}^d)$. For $r > 0$ and $x_0 \in \mathbb{R}^d$, we define the one-homogeneous rescaling of u as

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r} \quad \text{for every} \quad x \in \mathbb{R}^d. \tag{2.1}$$

Then, $u_{x_0,r} \in H^1_{loc}(\mathbb{R}^d)$ and for almost every $r > 0$, $E(u_{x_0,r})$ is well defined, where we set

$$E(v) := \int_{\partial B_1} |x \cdot \nabla v - v|^2 \, d\mathcal{H}^{d-1}, \tag{2.2}$$

where $x \in \partial B_1$ is the exterior normal derivative to ∂B_1 at the point $x \in \mathbb{R}^d$ and $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure. The excess function $e(r) = E(u_{x_0,r})$ controls the asymptotic behavior, as $r \to 0^+$, of the one parameter family $u_{x_0,r} \in L^2(\partial B_1)$. Precisely, we have the following elementary estimate.

Lemma 2.1. *Let $u \in H^1_{loc}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$. Suppose that there are constants $r_0 > 0$, $\gamma \in (0,1)$ and $I > 0$ such that*

$$\int_0^{r_0} \frac{E(u_{x_0,r})}{r^{1+\gamma}} \, dr \leq I. \tag{2.3}$$

Then, there is a unique function $u_{x_0} \in L^2(\partial B_1)$ such that

$$\|u_{r,x_0} - u_{x_0}\|^2_{L^2(\partial B_1)} \leq \gamma^{-1} I \, r^\gamma \quad \textit{for every} \quad r \in (0, r_0).$$

Proof. We set for simplicity, $x_0 = 0$ and $u_r := u_{x_0,r}$. Let $0 < r < R \leq r_0$. Notice that, for any $x \in \partial B_1$, we have

$$\frac{u(Rx)}{R} - \frac{u(rx)}{r} = \int_r^R \left(\frac{x \cdot (\nabla u)(sx)}{s} - \frac{u(sx)}{s^2} \right) ds = \frac{1}{s} \int_r^R \big(x \cdot \nabla u_s(x) - u_s(x) \big) \, ds.$$

Thus, by the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\int_{\partial B_1}|u_R-u_r|^2\,d\mathcal{H}^{d-1}&\le\int_{\partial B_1}\left(\int_r^R\frac{1}{s}|x\cdot\nabla u_s-u_s|\,ds\right)^2\,d\mathcal{H}^{d-1}\\&\le\int_{\partial B_1}\left(\int_r^R s^{\gamma-1}ds\right)\left(\int_r^R\frac{1}{s^{1+\gamma}}|x\cdot\nabla u_s-u_s|^2ds\right)\,d\mathcal{H}^{d-1}\\&\le\frac{R^\gamma-r^\gamma}{\gamma}\int_r^R\frac{E(u_s)}{s^{1+\gamma}}\,ds\le\frac{R^\gamma}{\gamma}\int_0^{r_0}\frac{E(u_s)}{s^{1+\gamma}}\,ds,\end{aligned}$$

which implies the claim by a standard argument. □

2.2. **The one-phase boundary adjusted energy.** Let $d\ge 2$ and $u\in H^1(B_1)$. For any $\Lambda>0$, we define the one-phase Weiss' boundary adjusted energy as

$$W_{\text{OP}}(u):=\int_{B_1}|\nabla u|^2\,dx-\int_{\partial B_1}u^2\,d\mathcal{H}^{d-1}+\Lambda\big|\{u>0\}\cap B_1\big|. \tag{2.4}$$

Let $r>0$, $x_0\in\mathbb{R}^2$ and $u\in H^1_{loc}(\mathbb{R}^d)$. The relation between W_{OP} and the excess E is given by the following formula, which holds for any function u and can be obtained by a direct computation (see [24] and [20]).

$$\frac{\partial}{\partial r}W_{\text{OP}}(u_{x_0,r})=\frac{d}{r}\big(W_{\text{OP}}(z_{x_0,r})-W_{\text{OP}}(u_{x_0,r})\big)+\frac{1}{r}E(u_{x_0,r}), \tag{2.5}$$

where $z_{x_0,r}$ denotes the one-homogeneous extension of the trace of $u_{x_0,r}$ in B_1, that is,

$$z_{x_0,r}(x):=|x|\,u_{x_0,r}\left({}^{x}\!/\!_{|x|}\right)=\frac{|x|}{r}\,u\left({}^{rx}\!/\!_{|x|}\right)\,,\quad\text{for every}\quad x\in B_1. \tag{2.6}$$

In [22], the first and the third author proved the following *epiperimetric inequality* for the Weiss' energy W_{OP}.

Theorem 2.2 (Epiperimetric inequality for W_{OP})**.** *Let $d=2$. Let $C_0>0$ be a given constant. There exists a constant $\varepsilon>0$ such that: for every non-negative $c\in H^1(\partial B_1)$ satisfying $\int_{\partial B_1}c\,d\mathcal{H}^1\ge C_0$, there exists a non-negative function $h\in H^1(B_1)$ such that $h=c$ on ∂B_1 and*

$$W_{\text{OP}}(h)-\Lambda\frac{\pi}{2}\le(1-\varepsilon)\left(W_{\text{OP}}(z)-\Lambda\frac{\pi}{2}\right), \tag{2.7}$$

where W_{OP} is given by (2.4) and $z\in H^1(B_1)$ denotes the one-homogeneous extension of c into B_1. Moreover, the competitor h has the following properties:

(a) There is a universal numerical constant $C>0$ such that $\|h\|_{H^1(B_1)}\le C\|c\|_{H^1(\partial B_1)}$.
(b) If $H_{x_0,\nu}:=\big\{x\in\mathbb{R}^2\,:\,(x-x_0)\cdot\nu\ge 0\big\}$, for some $x_0\in\mathbb{R}^2$ and $\nu\in\partial B_1$, is a half-plane such that

$$0\in H_{x_0,\nu}\qquad\textit{and}\qquad z=0\quad\textit{on}\quad\mathbb{R}^2\setminus H_{x_0,\nu}\,, \tag{2.8}$$

then we can choose the competitor $h:B_1\to\mathbb{R}$ such that $h=0$ on $\mathbb{R}^2\setminus H_{x_0,\nu}$.

2.3. **The two-phase boundary adjusted energy.** For every $\Lambda_1,\Lambda_2>0$ and $v\in H^1(B_1)$, we define the two-phase Weiss' boundary adjusted energy as

$$W_{\text{TP}}(v)=\int_{B_1}|\nabla v|^2\,dx-\int_{\partial B_1}v^2\,d\mathcal{H}^{d-1}+\Lambda_1\big|\{v>0\}\cap B_1\big|+\Lambda_2\big|\{v<0\}\cap B_1\big|. \tag{2.9}$$

As in the one phase case, we have

$$\frac{\partial}{\partial r}W_{\text{TP}}(u_{x_0,r})=\frac{d}{r}\big(W_{\text{TP}}(z_{x_0,r})-W_{\text{TP}}(u_{x_0,r})\big)+\frac{1}{r}E(u_{x_0,r}), \tag{2.10}$$

where $z_{x_0,r}$ is given by (2.6).

Theorem 2.3 (Epiperimetric inequality for W_{TP})**.** *Let $d = 2$. For every $C_0 > 0$ there is $\varepsilon > 0$ such that: for every $c \in H^1(\partial B_1)$ such that $\int_{\partial B_1} c^+ \, d\mathcal{H}^1 \geq C_0$ and $\int_{\partial B_1} c^- \, d\mathcal{H}^1 \geq C_0$, there exists a function $h \in H^1(B_1)$ with $h = c$ on ∂B_1 such that*

$$W_{\text{TP}}(h) - (\Lambda_1 + \Lambda_2)\frac{\pi}{2} \leq (1-\varepsilon)\left(W_{\text{TP}}(z) - (\Lambda_1 + \Lambda_2)\frac{\pi}{2}\right), \tag{2.11}$$

where $z \in H^1(B_1)$ is the one-homogeneous extension of the trace of c to B_1. Moreover, there is a universal numerical constant $C > 0$ such that $\|h\|_{H^1(B_1)} \leq C\|c\|_{H^1(\partial B_1)}$.

2.4. **Almost-monotonicity and almost-minimality.** Let $u \in H^1_{loc}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$. For any $r > 0$, the function $u_{x_0,r}$ and $z_{x_0,r}$ are defined as in (2.1) and (2.6), respectively. In the next lemma we will show that a almost-minimality of u, with respect to radial perturbations, implies that the function $r \mapsto W_\square(u_{x,r})$ is monotone up to a small error term ($\square$ stands for OP or TP).

Lemma 2.4 (Monotonicity of $W_\square$)**.** *Let $u \in H^1_{loc}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$. Suppose that there are constants $r_0 > 0$, $C > 0$ and $\delta > 0$ such that*

$$W_\square(u_{x_0,r}) \leq W_\square(z_{x_0,r}) + Cr^\delta \quad \textit{for every} \quad r \in (0, r_0), \tag{2.12}$$

where $\square$ stands for OP or TP. Then, the function

$$r \mapsto W_\square(u_{x_0,r}) + \frac{Cd}{\delta} r^\delta, \tag{2.13}$$

is non-decreasing on the interval $(0, r_0)$.

Proof. Using (2.5) for $\square =$OP (resp. (2.10) for $\square =$TP), and the condition (2.12) we get

$$\frac{\partial}{\partial r} W_\square(u_{x_0,r}) \geq \frac{d}{r}\big(W_\square(z_{x_0,r}) - W_\square(u_{x_0,r})\big) \geq Cd\, r^{\delta-1},$$

which gives (2.13). $\square$

2.5. **Epiperimetric inequality and energy decay.** In this section we show how to use the epiperimetric inequality to obtain at once the decay for the energy $W_\square(u_{x_0,r})$ and the convergence of $u_{x_0,r}$ in $L^2(\partial B_1)$. The argument is very general and we treat the cases $\square = $ OP and $\square = $ TP simultaneously.

Lemma 2.5. *Let $u \in H^1_{loc}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and $W_\square$ be as in (2.4), if $\square = $ OP, and (2.9), if $\square = $ TP. Suppose that there are constants $r_0 \in (0,1)$, $C > 0$, $\delta > 0$ and $\varepsilon \in (0, \frac{\delta}{2d+\delta})$ such that:*
(a) *(2.12) holds and the limit $\Theta_\square := \lim_{r\to 0} W_\square(u_{x_0,r})$ (which exists due to Lemma 2.4) is finite;*
(b) *for every $r \in (0, r_0)$ there is a function $h_{x_0,r} \in H^1(B_1)$ such that*

$$W_\square(u_{x_0,r}) \leq W_\square(h_{x_0,r}) + Cr^\delta, \tag{2.14}$$

and we have the epiperimetric inequality

$$W_\square(h_{x_0,r}) - \Theta_\square \leq (1-\varepsilon)\big(W_\square(z_{x_0,r}) - \Theta_\square\big). \tag{2.15}$$

Then, there is a unique function $u_{x_0} \in L^2(\partial B_1)$ such that

$$\|u_{r,x_0} - u_{x_0}\|^2_{L^2(\partial B_1)} \leq \gamma^{-1} I\, r^\gamma \quad \textit{for every} \quad r \in (0, r_0),$$

where $\gamma = \frac{d\varepsilon}{1-\varepsilon}$ and $I = r_0^{-\gamma}\big(W_\square(u_{x_0,r_0}) - \Theta_\square\big) + \dfrac{dC}{\delta - \gamma} r_0^{\delta-\gamma}$.

Proof. We use (2.5) for $\square = \text{OP}$ (resp. (2.10) for $\square = \text{TP}$), then the epiperimetric inequality (2.15) and the almost-minimality condition (2.14).

$$\begin{aligned}\frac{\partial}{\partial r}\big(W_\square(u_{x_0,r})-\Theta_\square\big) &\geq \frac{d}{r}\Big(\big(W_\square(z_{x_0,r})-\Theta_\square\big)-\big(W_\square(u_{x_0,r})-\Theta_\square\big)\Big)\\ &\geq \frac{d}{r}\Big(\frac{1}{1-\varepsilon}\big(W_\square(h_{x_0,r})-\Theta_\square\big)-\big(W_\square(u_{x_0,r})-\Theta_\square\big)\Big)\\ &\geq \frac{d}{r}\Big(\frac{\varepsilon}{1-\varepsilon}\big(W_\square(u_{x_0,r})-\Theta_\square\big)-Cr^\delta\Big),\end{aligned}$$

which implies that the function

$$f(r)=\frac{W_\square(u_{x_0,r})-\Theta_\square}{r^\gamma}+\frac{dC}{\delta-\gamma}r^{\delta-\gamma}$$

is non-decreasing on $(0,r_0)$ for $\gamma=\frac{d\varepsilon}{1-\varepsilon}$, where we notice that $\gamma\leq\frac{\delta}{2}$ due to the choice $\varepsilon\leq\frac{\delta}{2d+\delta}$. In particular, using again (2.5) (resp. (2.10)), we get

$$f'(r)\geq\frac{1}{r^{\gamma+1}}E(u_{x_0,r}),$$

which integrated gives

$$f(r_0)-f(s)\geq\int_s^{r_0}\frac{1}{r^{\gamma+1}}E(u_{x_0,r})\,dr,$$

for every $s\in(0,r_0)$. Now, notice that, up to choosing a bigger constant C in (2.14), Lemma 2.4 implies that $f(s)\geq 0$ for every $s>0$. Thus, we get

$$f(r_0)\geq\int_0^{r_0}\frac{1}{r^{\gamma+1}}E(u_{x_0,r})\,dr,$$

which is precisely (2.3) with $I:=f(r_0)$. □

3. Change of variables and freezing of the coefficients

The arguments of the previous section, the monotonicity formula and the decay of the blow-up sequences, can be applied only in the case when the operator in J_{OP} (resp. J_{TP}) is the identity. Thus, in order to prove the regularity results Theorem 1.1 and Theorem 1.5 we need to change the coordinates and reduce to the case $A=Id$. We prove the main estimate of this section in Lemma 3.2 below, but before we will introduce several notations.

Let $A=(a_{ij})_{ij}:B_2\to Sym_2^+$ and $Q_{\text{OP}},Q_{\text{TP}}^+,Q_{\text{TP}}^-:B_2\to\mathbb{R}^+$ be as in the Introduction and note that we have

$$\|A_x^{1/2}\|\leq M_{\text{A}}^{1/2}\quad\text{and}\quad\|A_x^{-1/2}\|\leq M_{\text{A}}^{1/2}\quad\text{for every}\quad x\in B_2,$$

where $\|A\|=\sup\big\{|Au|\,:\,u\in\mathbb{R}^2,\ |u|=1\big\}$ and M_{A} is a constant (as in Subsection 1.1).

Remark 3.1. We recall that if the (real) matrix A is symmetric and positive ($A\in Sym_d^+$), then there is an orthogonal matrix P such that $PAP^t=\text{diag}(\lambda_1,\dots,\lambda_d)$, where P^t is the transpose of P and $\text{diag}(\lambda_1,\dots,\lambda_d)$ is the diagonal matrix with eigenvalues $\lambda_1,\dots,\lambda_d$. We set $D=\text{diag}(\sqrt{\lambda_1},\dots,\sqrt{\lambda_d})$ and define $A^{1/2}:=P^tDP$.

We now fix $x_0\in B_2$ and, for any $r>0$, we define the functionals

$$J_{\text{OP}}^{x_0}(v,r):=\int_{B_r}\Big(|\nabla v|^2+Q_{\text{OP}}(x_0)\mathbb{1}_{\{v>0\}}\Big)\,dx\,;$$

$$J_{\text{TP}}^{x_0}(v,r):=\int_{B_r}\Big(|\nabla v|^2+Q_{\text{TP}}^+(x_0)\mathbb{1}_{\{v>0\}}+Q_{\text{TP}}^-(x_0)\mathbb{1}_{\{v<0\}}\Big)\,dx\,.$$

For every $x_0\in B_1$, we define the function

$$F_{x_0}(x):=x_0+A_{x_0}^{1/2}(x)\tag{3.1}$$

and the half-plane $\mathcal{H}_{x_0}:=\big\{x\in\mathbb{R}^2\,:\,F_{x_0}(x)\cdot e_2>0\big\}$, where $e_2=(0,1)$.

Lemma 3.2. *Let $L>0$. There are constants $C>0$ and $r_0\in(0,1)$ (depending only on the constants $C_{\rm A}$, $C_{\rm Q}$, $M_{\rm A}$, $M_{\rm Q},\delta_{\rm A},\delta_{\rm Q},\delta_1,C_1$ and L defined in Subsection 1.1) and $\delta=\min\{\delta_{\rm A},\delta_{\rm Q},\delta_1\}$ such that: if $u\in H^1(B_1)$ is a nonnegative L-Lipschitz continuous function and a almost-minimizer of $J_{\rm OP}$ in B_2^+, $x_0\in B_{r_0}\cap\partial\Omega_u$ and $\bar u=u\circ F_{x_0}$ (F_{x_0} is defined in (3.1) above), then for every $r\in(0,r_0)$,*

$$J_{\rm OP}^{x_0}(\bar u,r)\le(1+Cr^\delta)J_{\rm OP}^{x_0}(\bar v,r)+Cr^{2+\delta},\tag{3.2}$$
$$\text{for every } \bar v\in H^1(B_r) \text{ such that } \bar u-\bar v\in H_0^1(B_r) \text{ and } \bar v=0 \text{ on } \mathbb{R}^2\setminus\mathcal{H}_{x_0}.$$

Moreover, there is a numerical constant $C_0>0$, such that

$$W_{\rm OP}(\bar u_r)\le\begin{cases}W_{\rm OP}(\bar z_r)+C_0(M_{\rm A}L^2+M_Q)Cr^\delta,\\ W_{\rm OP}(\bar h_r)+C_0(M_{\rm A}L^2+M_Q)Cr^\delta,\end{cases}$$

for every $r\in(0,r_0)$, where C is the constant from (3.2), $\bar u_r(x):=\frac1r\bar u(rx)$, $\bar z_r$ is the 1-homogeneous extension of $\bar u_r$ in B_1, $\bar h_r$ is the competitor from Theorem 2.2 and $\Lambda=Q_{\rm OP}(x_0)$, as in (2.4).

Proof. Let $x_0\in\partial\Omega_u\cap B_1$ be fixed. For $r>0$, we set $\rho=M_{\rm A}^{1/2}r$. Notice that we have the inclusion $F_{x_0}(B_r)\subset B_\rho(x_0)$. Let $\bar u=u\circ F_{x_0}$ and $\bar v=v\circ F_{x_0}$. Then, using the Hölder continuity of A and $Q:=Q_{\rm OP}$, and the ellipticity of A, we estimate

$$\begin{aligned}\tilde J_{\rm OP}(u,x_0,\rho):=\int_{B_\rho(x_0)}\Big(a_{ij}(x_0)\,\partial_i u\,\partial_j u+Q(x_0)\mathbb{1}_{\{u>0\}}\Big)\,dx\le J_{\rm OP}(u,x_0,\rho)\\ +C_{\rm A}M_{\rm A}\rho^{\delta_{\rm A}}\int_{B_\rho(x_0)}a_{ij}(x)\,\partial_i u\,\partial_j u\,dx+C_{\rm Q}M_{\rm Q}\rho^{\delta_{\rm Q}}\int_{B_\rho(x_0)}Q(x)\mathbb{1}_{\{u>0\}}\,dx\\ \le(1+Cr^\delta)J_{\rm OP}(u,x_0,\rho),\end{aligned}$$

for some positive constant $C>0$. Analogously, we get the following estimate from below:

$$\tilde J_{\rm OP}(v,x_0,\rho)\ge(1-Cr^\delta)J_{\rm OP}(v,x_0,\rho).\tag{3.3}$$

Putting the two estimates together and using the almost-minimality of u, we get

$$\tilde J_{\rm OP}(u,x_0,\rho)\le\frac{1+Cr^\delta}{1-Cr^\delta}(1+C_1\rho^{\delta_1})\tilde J_{\rm OP}(v,x_0,\rho)+C_1(1+Cr^\delta)\rho^{2+\delta_1}.$$

Now, notice that by the choice of the function F_{x_0} we have the identity

$$|\nabla\bar u|^2(x)=a_{ij}(x_0)\,\partial_i u(F_{x_0}(x))\,\partial_j u(F_{x_0}(x))\quad\text{for every}\quad x\in B_{\mathrm{M}_{\rm A}^{-1/2}}.$$

Therefore, a change of coordinates and the estimate (3.3) give

$$\begin{aligned}\int_{F_{x_0}^{-1}(B_\rho(x_0))}\Big(|\nabla\bar u|^2+Q(x_0)\mathbb{1}_{\{\bar u>0\}}\Big)\,dx=\det\big(A_{x_0}^{-1/2}\big)\tilde J_{\rm OP}(u,x_0,\rho)\\ \le(1+Cr^\delta)\int_{F_{x_0}^{-1}(B_\rho(x_0))}\Big(|\nabla\bar v|^2+Q(x_0)\mathbb{1}_{\{\bar v>0\}}\Big)\,dx+Cr^{2+\delta},\end{aligned}$$

for some other positive constant $C>0$. Finally, since $B_r\subset F_{x_0}^{-1}(B_\rho(x_0))$ and $\bar u=\bar v$ outside B_r, we can rearrange the terms of the above estimate to obtain

$$J_{\rm OP}^{x_0}(\bar u,r)\le(1+Cr^\delta)J_{\rm OP}^{x_0}(\bar v,r)+Cr^{2+\delta}+Cr^\delta J_{\rm OP}^{x_0}(\bar u,\mathrm{M}_{\rm A}^{-1/2}r),$$

which gives (3.2) since $\bar u$ is Lipschitz continuous with Lipschitz constant $\|\nabla\bar u\|_{L^\infty}=M_{\rm A}^{1/2}L$.

We next notice that we have the scaling

$$J_{\rm OP}^{x_0}(\bar u_r,1)=\frac1{r^2}J_{\rm OP}^{x_0}(\bar u,r).$$

Thus, the almost-minimality inequality (3.2) translates in

$$J_{\rm OP}^{x_0}(\bar u_r,1)\le(1+Cr^\delta)J_{\rm OP}^{x_0}(\bar v_r,1)+Cr^\delta.\tag{3.4}$$

Let $C_{\mathrm{E}} > 0$ be the constant from Theorem 2.2. Then, since $\bar{u}$ is Lipschitz continuous, we have

$$\int_{B_1} |\nabla \bar{h}_r|^2\, dx \le C_{\mathrm{E}} \int_{\partial B_1} \left(|\nabla \bar{u}_r|^2 + \bar{u}_r^2\right) dx \le C_0 M_{\mathrm{A}} L^2,$$

where C_0 is a numerical constant and $\bar{h}_r$ is the competitor from Theorem 2.2. Taking $\bar{h}_r$ as a competitor in (3.4), we obtain

$$\begin{aligned} J_{\mathrm{OP}}^{x_0}(\bar{u}_r, 1) &\le J_{\mathrm{OP}}^{x_0}(\bar{h}_r, 1) + Cr^{\delta}\Big(\int_{B_1} |\nabla \bar{h}_r|^2\, dx + Q(x_0)|B_1|\Big) + Cr^{\delta} \\ &\le J_{\mathrm{OP}}^{x_0}(\bar{h}_r, 1) + Cr^{\delta}\Big(C_0 M_{\mathrm{A}} L^2 + M_{\mathrm{Q}}|B_1|\Big) + Cr^{\delta}, \end{aligned}$$

which concludes the proof, the case $\bar{v}_r = \bar{z}_r$ being analogous. □

An analogous result, with essentially the same proof holds in the two-phase case.

Lemma 3.3. *Let $L > 0$. There are constants $C > 0$ and $r_0 \in (0,1)$ (depending only on C_{A}, C_{Q}, M_{A}, $M_{\mathrm{Q}}, \delta_{\mathrm{A}}, \delta_{\mathrm{Q}}, \delta_1, C_2$ and L) and $\delta = \min\{\delta_{\mathrm{A}}, \delta_{\mathrm{Q}}, \delta_2\}$ such that: if $u \in H^1(B_1)$ is a L-Lipschitz continuous function and a almost-minimizer of J_{TP} in B_2, $x_0 \in B_{r_0} \cap \partial\Omega_u$ and $\bar{u} = u \circ F_{x_0}$, then we have that for every $r \in (0, r_0)$,*

$$J_{\mathrm{TP}}^{x_0}(\bar{u}, r) \le (1 + Cr^{\delta}) J_{\mathrm{TP}}^{x_0}(\bar{v}, r) + Cr^{2+\delta}, \tag{3.5}$$

$$\text{for every } \bar{v} \in H^1(B_r) \text{ such that } \bar{u} - \bar{v} \in H^1_0(B_r).$$

Moreover, there is a numerical constant $C_0 > 0$ such that

$$W_{\mathrm{TP}}(\bar{u}_r) \le \begin{cases} W_{\mathrm{TP}}(\bar{z}_r) + C_0(M_{\mathrm{A}}L^2 + M_Q)Cr^{\delta}, \\ W_{\mathrm{TP}}(\bar{h}_r) + C_0(M_{\mathrm{A}}L^2 + M_Q)Cr^{\delta}, \end{cases}$$

for every $r \in (0, r_0)$, where C is the constant from (3.5), $\bar{u}_r(x) := \frac{1}{r}\bar{u}(rx)$, $\bar{z}_r$ is the one homogeneous extension of $\bar{u}_r$ in B_1, $\bar{h}_r$ is the competitor given by Theorem 2.3 and $\Lambda_1 = Q^+_{\mathrm{TP}}(x_0)$, $\Lambda_2 = Q^-_{\mathrm{TP}}(x_0)$ are as in (2.9).

Remark 3.4 (On the non-degeneracy). In [13] David and Toro proved that Lipschitz continuous almost-minimizers to the one-phase and the two-phase functionals for the Laplacian are non-degenerate (see [13, Theorem 10.1]). Note that their definition of almost-minimizer is slightly different from ours. However, their proof still holds in our case with small changes which come from the additional term $Cr^{2+\delta}$ of our definition. It follows from Lemma 3.2 and Lemma 3.3 that if u is a almost-minimizer of the functional J_{OP} (resp. u is a almost-minmizer of J_{TP}) then u (resp. $u_{\pm}$) is non-degenerate with respect to A in the sense of the following definition.

Definition 3.5 (Non-degeneracy)**.** *Let $d \ge 2$ and $A : \mathbb{R}^d \to Sym_d^+$ be a given function. We say that the non-negative function $u \in H^1(B_2)$ is non-degenerate (with respect to A), if there are constants $\eta > 0$, $\varepsilon \in (0,1)$ and $r_0 > 0$ such that, for every $x_0 \in B_1$ and $r \in (0, r_0)$, the following implication holds:*

$$\int_{\partial B_r} u \circ F_{x_0}\, d\mathcal{H}^{d-1} < \eta\, r^d \qquad \Rightarrow \qquad u \circ F_{x_0} \equiv 0 \quad \text{in} \quad B_{\varepsilon r}(x_0),$$

where $F_{x_0}(x) := x_0 + A_{x_0}^{1/2}(x)$.

4. Blow-up sequences and blow-up limits

Let $u \in H^1(B_2)$ be a Lipschitz continuous function. Let $(x_n)_{n\ge1}$ be a sequence of points in $B_1 \cap \partial\Omega_u$ converging to some $x_0 \in B_1 \cap \partial\Omega_u$, and $(r_n)_{n\ge1}$ be an infinitesimal sequence in $(0,1)$. Then, the sequence u_{x_n,r_n} is uniformly Lipschitz in every compact subset of $\mathbb{R}^2$. Thus, up to extracting a subsequence, there is a Lipschitz continuous function $u_0 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{n\to\infty} u_{x_n,r_n} = u_0, \tag{4.1}$$

where u_{r_n,x_n} is defined in (2.1) and the convergence is uniform on every compact subset of $\mathbb{R}^2$.

Definition 4.1. *If (4.1) holds, we will say that u_{x_n,r_n} is a blow up sequence (with fixed center, if $x_n = x_0$, for every $n \geq 1$). If the center is fixed, we will say that u_0 is a blow-up limit at x_0.*

We summarize the main properties of the blow-up sequences and the blow-up limits in the following two propositions. We notice that Proposition 4.2 below holds in every dimension $d \geq 2$, while Proposition 4.3 is known to hold only for $2 \leq d \leq 4$.

Proposition 4.2 (Convergence of the blow-up sequences)**.** *Let $u \in H^1(B_2)$ be as in Theorem 1.1 or Theorem 1.5 and let $u_n := u_{r_n,x_n}$ be a blow-up sequence converging to some $u_0 \in H^1_{loc}(\mathbb{R}^2)$. Then:*

(i) the sequence u_n converges to u_0 strongly in $H^1_{loc}(\mathbb{R}^2)$;
(ii) the sequences of characteristic functions $\mathbb{1}_{\{u_n>0\}}$ and $\mathbb{1}_{\{u_n<0\}}$ converge strongly in $L^1_{loc}(\mathbb{R}^2)$ to the characteristic functions $\mathbb{1}_{\{u_0>0\}}$ and $\mathbb{1}_{\{u_0<0\}}$, respectively.

Proposition 4.3 (Classification of the blow-up limits)**.** *Let $u \in H^1(B_2)$ be as in Theorem 1.1 or Theorem 1.5. Let $x_0 \in \partial\Omega_u \cap B_1$ and $u_0 \in H^1_{loc}(\mathbb{R}^2)$ be a blow-up limit of u at x_0.*

(OP) *If u is as in Theorem 1.1 and $x_0 \in \partial\Omega_u \cap B_1^+$, then u_0 is of the form*

$$u_0(x) = Q_{\text{OP}}^{1/2}(x_0) \max\big\{0, x \cdot A_{x_0}^{-1/2}[\nu]\big\}, \quad \text{where} \quad \nu \in \partial B_1. \tag{4.2}$$

(OP-c) *If u is as in Theorem 1.1 and $x_0 \in \partial\Omega_u \cap \partial H \cap B_1$, then u_0 is of the form*

$$u_0(x) = \mu \max\big\{0, x \cdot A_{x_0}^{-1/2}[\nu]\big\}, \tag{4.3}$$

where $\mu \geq Q_{\text{OP}}^{1/2}(x_0)$ and $\nu \in \partial B_1$ is such that $A_{x_0}^{-1/2}[\nu]$ is the (interior) normal to ∂H.

(TP) *If u is as in Theorem 1.5 and $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, then u_0 is of the form*

$$u_0(x) = \mu_+ \max\big\{0, x \cdot A_{x_0}^{-1/2}[\nu]\big\} + \mu_- \min\big\{0, x \cdot A_{x_0}^{-1/2}[\nu]\big\}, \tag{4.4}$$

for some $\nu \in \partial B_1$ and some $\mu_+, \mu_- > 0$ such that

$$\mu_+^2 \geq Q_{\text{TP}}^+(x_0) \quad \mu_-^2 \geq Q_{\text{TP}}^-(x_0) \quad \text{and} \quad \mu_+^2 - \mu_-^2 = Q_{\text{TP}}^+(x_0) - Q_{\text{TP}}^-(x_0).$$

The proof of Proposition 4.2 follows by a standard variational argument that only uses the almost-minimality of u; for more details, we refer to [1] (see also [20]). Proposition 4.3 follows by the optimality of the blow-up limits and the Weiss' monotonicity formula (Lemma 2.4). We will need the following definition.

Definition 4.4 (Global solutions)**.** *Let $u : \mathbb{R}^2 \to \mathbb{R}$, $u \in H^1_{loc}(\mathbb{R}^2)$ be given.*

(OP) *We say that u is a global solution of the* **one-phase** *Bernoulli problem, if: $u \geq 0$ and, for every ball $B := B_R(x_0) \subset \mathbb{R}^2$, we have*

$$\int_B |\nabla u|^2\,dx + \Lambda|\{u>0\} \cap B| \leq \int_B |\nabla v|^2\,dx + \Lambda|\{v>0\} \cap B|, \tag{4.5}$$
$$\text{for every } v \in H^1(B) \text{ such that } u - v \in H^1_0(B).$$

(OP-c) *We say that u is a global solution of the* **one-phase constrained** *Bernoulli problem in the half-plane H, if $u \geq 0$ on H, $u = 0$ on $\mathbb{R}^2 \setminus H$ and (4.5) holds, for every ball $B := B_R(x_0) \subset \mathbb{R}^2$ and every $v \in H^1(B)$ such that $u - v \in H^1_0(B)$ and $\{v > 0\} \subset H$.*

(TP) *We say that u is a global solution of the* **two-phase** *Bernoulli problem if, for every ball $B := B_R(x_0) \subset \mathbb{R}^2$, we have*

$$\int_B \Big(|\nabla u|^2 + \Lambda_1 \mathbb{1}_{\{u>0\}} + \Lambda_2 \mathbb{1}_{\{u<0\}}\Big)\,dx \leq \int_B \Big(|\nabla v|^2 + \Lambda_1 \mathbb{1}_{\{v>0\}} + \Lambda_2 \mathbb{1}_{\{v<0\}}\Big)\,dx, \tag{4.6}$$

for every $v \in H^1(B)$ such that $u - v \in H^1_0(B)$.

Lemma 4.5 (Optimality of the blow-up limits)**.** *Let $u \in H^1(B_2)$ be as in Theorem 1.1 or Theorem 1.5 and let $u_n := u_{r_n,x_0}$ be a blow-up sequence converging to the blow-up limit $u_0 \in H^1_{loc}(\mathbb{R}^2)$. Then, we have:*

(OP) *If u is as in Theorem 1.1 and $x_0 \in \partial\Omega_u \cap B_1^+$, then $u_0 \circ A_{x_0}^{1/2}$ is a global solution of the one-phase problem with $\Lambda = Q_{\text{OP}}(x_0)$.*

(OP-c) *If u is as in Theorem 1.1 and $x_0 \in \partial\Omega_u \cap \partial H \cap B_1$, then, up to a rotation, $u_0 \circ A_{x_0}^{1/2}$ is a global solution of the constrained one-phase problem with $\Lambda = Q_{\text{OP}}(x_0)$.*

(TP) *If u is as in Theorem 1.5 and $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, then $u_0 \circ A_{x_0}^{1/2}$ is a global solution of the two-phase problem with $\Lambda_1 = Q_{\text{TP}}^+(x_0)$ and $\Lambda_2 = Q_{\text{TP}}^-(x_0)$.*

Recall that the function $\bar{u} = u \circ F_{x_0}$, where F_{x_0} is as in (3.1), is an almost-minimizer of the functional $J_{\text{OP}}^{x_0}$ (Lemma 3.2). We then refer to Lemma 4.6 in [20] applied to $\bar{u}$ for the proof of Lemma 4.5. It is also worth mentioning that the strong convergence of the blow-up sequences and the optimality of the blow-up limits are equivalent.

Lemma 4.6 (Homogeneity of the blow-up limits)**.** *Let $u \in H^1(B_2)$ be as in Theorem 1.1 or Theorem 1.5. Let $x_0 \in B_1 \cap \partial\Omega_u$ and let u_{x_0,r_n} be a blow-up sequence converging to a blow-up limit u_0. Then, u_0 is one-homogeneous.*

Proof. Assume that $x_0 = 0$ and set $\bar{u} = u \circ F_{x_0}$. Then

$$u_{x_0,r} = \bar{u}_r \circ A_{x_0}^{-1/2}, \qquad \text{where} \qquad \bar{u}_r(x) := \frac{\bar{u}(rx)}{r}.$$

We first notice that by Lemma 3.2, Lemma 2.4 and the Lipschitz continuity of u, we get that the limit $\Theta_\square := \lim_{r\to 0} W_\square(\bar{u}_r)$, $\square = \text{OP}, \text{TP}$, exists and is finite. Now the strong convergence of $\bar{u}_{r_n}$ to $\bar{u}_0 := u_0 \circ A_{x_0}^{1/2}$ (Proposition 4.2) implies that, for every $s > 0$, we have

$$\Theta_\square := \lim_{r\to 0} W_\square(\bar{u}_r) = \lim_{n\to\infty} W_\square(\bar{u}_{r_n}) = \lim_{n\to\infty} W_\square(\bar{u}_{r_n s}) = \lim_{n\to\infty} W_\square((\bar{u}_{r_n})_s) = W_\square((\bar{u}_0)_s).$$

In particular, $s \mapsto W_\square(\bar{u}_0, s)$ is constant. Now, since $\bar{u}_0$ is a global solution (Lemma 4.5), (2.5) and (2.10) imply that $E((\bar{u}_0)_s) = 0$, for every $s > 0$. Thus we have $x \cdot \nabla\bar{u}_0 = \bar{u}_0$ in $\mathbb{R}^2$, which implies that $\bar{u}_0$ (and thus, u_0) is one-homogeneous. $\square$

Proof of Proposition 4.3. We now notice that $\bar{u}_0 = u_0 \circ A_{x_0}^{1/2} : B_1 \to \mathbb{R}$ is one-homogeneous and harmonic on the cone $B_1 \cap \{\bar{u}_0 \neq 0\}$. Thus, the trace of $\bar{u}_0$ on the sphere satisfies the equation

$$-\Delta_{\mathbb{S}}\bar{u}_0 = (d-1)\bar{u}_0 \quad \text{on} \quad \mathbb{S}^{d-1} \cap \{\bar{u}_0 \neq 0\},$$

where in dimension two the spherical Laplacian $\Delta_{\mathbb{S}}$ is simply the second derivative and $d-1 = 1$. Thus, $\bar{u}_0$ is of the form $\bar{u}_0(\theta) = \sin(\theta + \theta_0)$, $\theta \in \mathbb{S}^2$, for some constant θ_0. This implies that $\{\bar{u}_0 \neq 0\}$ is a union of intervals of length π. In the one-phase case, since u is non-degenerate (see Remark 3.4), this implies that $\bar{u}_0$ is of the form (4.2), for some constant $\mu(x_0)$. Now, an internal variation argument (see [1]) implies that $\mu(x_0) = Q_{\text{OP}}^{1/2}(x_0)$, if $x_0 \in H \cap B_1^+$, and $\mu(x_0) \geq Q_{\text{OP}}^{1/2}(x_0)$, if $x_0 \in \partial H \cap B_1$. The two-phase case follows again by an internal variation argument (see [2]). $\square$

Finally, we prove a uniqueness result for the one- and two-phase (Theorem 1.1 and Theorem 1.5) blow-up limits. This is the only result of this section that cannot be immediately extended to higher dimension. This is due to the fact that the epiperimetric inequality (Theorem 2.2 and Theorem 2.3) is known (for the moment) only in dimension two.

Proposition 4.7 (Uniqueness of the blow-up and rate of convergence of the blow-up sequences)**.** *Let $u : B_2 \to \mathbb{R}$ be as in Theorem 1.1 or Theorem 1.5. There are constants $C > 0$, $\gamma > 0$ and $r_0 > 0$ such that the following claims do hold.*

(OP) *If u is as in Theorem 1.1, then for every $x_0 \in \partial\Omega_u \cap B_1$, there is a unique blow-up $u_{x_0} : \mathbb{R}^2 \to \mathbb{R}$ (of the form (4.2) or (4.3)) such that*

$$\|u_{x_0,r} - u_{x_0}\|_{L^\infty(B_1)} \leq Cr^\gamma \quad \textit{for every} \quad r \in (0, r_0). \tag{4.7}$$

(TP) *If u is as in Theorem 1.5, then for every $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, there is a unique blow-up $u_{x_0} : \mathbb{R}^2 \to \mathbb{R}$ (of the form (4.4)) such that*

$$\|u_{x_0,r} - u_{x_0}\|_{L^\infty(B_1)} \leq Cr^\gamma \quad \textit{for every} \quad r \in (0, r_0). \tag{4.8}$$

Proof. Let u be as in **(OP)** and $x_0 \in \partial\Omega_u \cap B_1$. We set $\bar u = u \circ F_{x_0}$ and $\bar u_r(x) := \frac{\bar u(rx)}{r}$, and we notice that $\bar u_r = u_{x_0,r} \circ A_{x_0}^{1/2}$. By Lemma 3.2 and Lemma 2.4, $r \mapsto W_{\text{OP}}(\bar u_r) + Cr^\delta$ is monotone. On the other hand, the homogeneity of the blow-up limits, imply that

$$\Theta_{\text{OP}} := \lim_{r\to 0} W_{\text{OP}}(\bar u_r) = \frac{\pi}{2} Q_{\text{OP}}(x_0).$$

Thus, by the epiperimetric inequality (Theorem 2.2), Lemma 3.2 and Lemma 2.5, we have that there exists a one-homogeneous function $\bar u_0$ such that, for $r > 0$ small enough,

$$\|\bar u_r - \bar u_0\|_{L^2(\partial B_1)} \le C r^{\gamma_0/2},$$

where γ_0 is the constant from Lemma 2.5. Integrating in r, we get that

$$\|\bar u_r - \bar u_0\|_{L^2(B_1)} \le C r^{\gamma_0/2}.$$

Now, since $\bar u_r = u_{x_0,r} \circ A_{x_0}^{1/2}$ and $A_{x_0}^{1/2}$ is invertible, we get

$$\|u_{x_0,r} - u_{x_0}\|_{L^2(B_1)} \le C r^{\gamma_0/2},$$

where $u_{x_0} = \bar u_0 \circ A_{x_0}^{1/2}$. Finally, we notice that the Lipschitz continuity of u implies that there is an universal bound on $\|\nabla u_{x_0,r}\|_{L^\infty(B_1)}$ and $\|\nabla u_{x_0}\|_{L^\infty(B_1)}$. Thus, we get (4.7) with $\gamma = \gamma_0/4$. The proof of **(TP)** is analogous. □

Remark 4.8. We notice that the above result does not hold at the one-phase points $x_0 \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$ of the solutions u of the two-phase problem (Theorem 1.5). This is due to the fact that the positive part u_+ is not a solution of the one-phase problem in the balls $B_r(x_0)$ that have non-empty intersection with the negative phase Ω_u^-. In fact, the blow-up limit u_{x_0} (of u at x_0) is still unique, but the decay estimate (4.7) holds only for $r < \frac12 \mathrm{dist}(x_0, \Omega_u^-)$.

5. Regularity of the one-phase free boundaries. Proof of Theorem 1.1

Let $u \in H^1(B_2)$, $u \ge 0$, be as in Theorem 1.1. By Proposition 4.7 we have that, for every $x_0 \in \partial\Omega_u \cap B_1$, there is a unique blow-up limit of u at x_0. We denote it by

$$u_{x_0}(x) = \mu(x_0)\max\{0, \nu_{x_0}\cdot x\},$$

where ν_{x_0} is of the form $A_{x_0}^{1/2}[\nu]$, for some $\nu \in \partial B_1$; and $\mu(x_0)$ satisfies the inequalities

$$Q_{\text{OP}}(x_0) \le \mu^2(x_0) \le M_{\text{A}} L^2,$$

where L is the Lipschitz constant of u. We also notice that

$$\mu(x_0) = Q_{\text{OP}}^{1/2}(x_0) \quad \text{whenever} \quad x_0 \in \partial\Omega_u \cap B_1^+.$$

Moreover, for every point $x_0 \in \partial\Omega_u \cap B_1$, we define the half-plane

$$H_{x_0} := \{x \in \mathbb{R}^2 \,:\, x\cdot\nu_{x_0} > 0\}.$$

We first prove the following:

Lemma 5.1. *Let u be as in Theorem 1.1. There are constants $C > 0$, $\gamma > 0$ and $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u \cap B_1$, we have*

$$\Omega_{x_0,r} \cap B_1 \supset \{x \in B_1 \,:\, x\cdot\nu_{x_0} > Cr^\gamma\} \quad \textit{and} \quad \Omega_{x_0,r} \cap \{x \in B_1 \,:\, x\cdot\nu_{x_0} < -Cr^\gamma\} = \emptyset, \qquad (5.1)$$

for every $r \in (0, r_0)$, where $\Omega_{x_0,r} := \{u_{x_0,r} > 0\}$.

Proof. The first part of (5.1) follows by the uniform convergence of the blow-up sequence $u_{x_0,r}$ (Proposition 4.7, equation (4.7)) and the form of the blow-up limit u_{x_0}. The second part of (5.1) follows again by (4.7), the fact that $u_{x_0} \equiv 0$ on $B_1 \setminus H_{x_0}$ and by the non-degeneracy of u, which can be written as

$$\text{If} \quad u_{x_0,r}(y_0) > 0\,, \quad \text{then} \quad \|u_{x_0,r}\|_{L^\infty(B_s(y_0))} \ge Cs\,, \quad \text{for every} \quad s \in (0,1),$$

for some $C > 0$. □

Lemma 5.2. *Let u be as in Theorem 1.1. There are constants $R, \alpha \in (0,1)$ and $C > 0$ such that, for every $x_0, y_0 \in \partial\Omega_u \cap B_R$, we have*

$$|\nu_{x_0} - \nu_{y_0}| \le C|x_0 - y_0|^\alpha \qquad \text{and} \qquad |\mu(x_0) - \mu(y_0)| \le C|x_0 - y_0|^\alpha. \tag{5.2}$$

Proof. Let $\gamma \in (0,1)$ be the exponent from Proposition 4.7 and let $\alpha := \frac{\gamma}{1+\gamma}$. Let $x_0, y_0 \in B_R \cap \partial\Omega_u$, where we choose R such that $(2R)^{1-\alpha} \le r_0$, where r_0 is the constant from Proposition 4.7. We set $r := |x_0 - y_0|^{1-\alpha}$. Recall that u is Lipschitz continuous and set $L = \|\nabla u\|_{L^\infty}$. Then, for every $x \in B_1$, we have

$$|u_{x_0,r}(x) - u_{y_0,r}(x)| = \frac{1}{r}|u(x_0 + rx) - u(y_0 + rx)| \le L\frac{|x_0 - y_0|}{r} = L|x_0 - y_0|^\alpha.$$

and then, by an integration on B_1, we get

$$\|u_{x_0,r} - u_{y_0,r}\|_{L^2(B_1)} \le |B_1|^{1/2} L|x_0 - y_0|^\alpha.$$

On the other hand, by the choice of R, we have that $r \le r_0$; applying Proposition 4.7, we get

$$\|u_{x_0,r} - u_{x_0}\|_{L^2(B_1)} \le Cr^\gamma \qquad \text{and} \qquad \|u_{y_0,r} - u_{y_0}\|_{L^2(B_1)} \le Cr^\gamma.$$

Thus, by the triangular inequality and the fact that $r^\gamma = |x_0 - y_0|^\alpha$, we obtain

$$\|u_{x_0} - u_{y_0}\|_{L^2(B_1)} \le \big(|B_1|^{1/2} L + 2C\big)|x_0 - y_0|^\alpha. \tag{5.3}$$

The conclusion now follows by a general argument. Indeed, for any pair of vectors $v_1, v_2 \in \mathbb{R}^2$, we have

$$\begin{aligned}
|v_1 - v_2| &= \left(\frac{2}{\pi}\int_{B_1} |v_1 \cdot x - v_2 \cdot x|^2\, dx\right)^{1/2} \\
&\le \left(\int_{B_1} |(v_1 \cdot x)_+ - (v_2 \cdot x)_+|^2\, dx\right)^{1/2} + \left(\int_{B_1} |(v_1 \cdot x)_- - (v_2 \cdot x)_-|^2\, dx\right)^{1/2} \\
&= 2\left(\int_{B_1} |(v_1 \cdot x)_+ - (v_2 \cdot x)_+|^2\, dx\right)^{1/2}.
\end{aligned} \tag{5.4}$$

Applying the above estimate to $v_1 = \mu(x_0)\nu_{x_0}$ and $v_2 = \mu(y_0)\nu_{y_0}$, and using (5.3), we get (5.2). □

Proof of Theorem 1.1. We first claim that, for every $\varepsilon > 0$, there exists $\rho > 0$ such that, for $x_0 \in \partial\Omega_u \cap B_\rho$ we have

$$u > 0 \ \text{ on } \ C^+(x_0, \varepsilon) \cap B_\rho(x_0) \qquad \text{and} \qquad u = 0 \ \text{ on } \ C^-(x_0, \varepsilon) \cap B_\rho(x_0), \tag{5.5}$$

where

$$C^\pm(x_0, \varepsilon) := \left\{x \in \mathbb{R}^2 \setminus \{0\} : \pm\nu_{x_0} \cdot (x - x_0) \ge \varepsilon|x - x_0|\right\}.$$

Indeed, the flatness estimate (5.1) implies (5.5) by taking ρ such that $C\rho^\gamma \le \varepsilon$, where C and γ are the constants from Lemma 5.1.

We now fix $x_0 \in B_1 \cap \partial\Omega_u$. Without loss of generality we can suppose that $x_0 = 0$ and $H_{x_0} = \{(s,t) \in \mathbb{R}^2 \,:\, t > 0\}$. Now, let $\varepsilon \in (0,1)$ and $\rho > 0$ as in (5.5) and set $\delta = \rho\sqrt{1 - \varepsilon^2}$. By (5.5) we have for every $s \in (-\delta, \delta)$

- the set $\mathcal{S}_+^s := \{t \in (-\delta, \delta) \,:\, u(s,t) > 0\}$ contains the interval $(\rho\varepsilon, \delta)$;
- the set $\mathcal{S}_0^s := \{t \in (-\delta, \delta) \,:\, u(s,t) = 0\}$ contains the interval $(-\delta, -\rho\varepsilon)$.

This implies that the function

$$g(s) := \max\{t \in \mathbb{R} : u(s,t) > 0\}$$

is well defined and such that

$$SQ_\delta \cap \Omega_u = \{(s,t) \in SQ_\delta \,:\, g(s) < t\} \qquad \text{and} \qquad SQ_\delta \setminus \Omega_u = \{(s,t) \in SQ_\delta \,:\, g(s) \ge t\},$$

where $SQ_\delta = (-\delta, \delta) \times (-\delta, \delta)$. Now, the flatness condition (5.1) implies that g is differentiable on $(-\delta, \delta)$. Furthermore, since ν is Hölder continuous, we deduce that g is a function of class $C^{1,\alpha}$. This concludes the proof. □

6. Regularity of the two-phase free boundaries. Proof of Theorem 1.5

Let u be as in Theorem 1.5. Then, by Proposition 4.7, at every point $x_0 \in \partial\Omega_u \cap B_1$ there is a unique blow-up limit u_{x_0} given by

$$\begin{aligned} u_{x_0}(x) &= \mu_+(x_0)\max\{0, x\cdot\nu_{x_0}\}, \quad \text{if } x_0 \in \Gamma_+ := (\partial\Omega_u^+ \setminus \partial\Omega_u^-) \cap B_1;\\ u_{x_0}(x) &= \mu_-(x_0)\min\{0, x\cdot\nu_{x_0}\}, \quad \text{if } x_0 \in \Gamma_- := (\partial\Omega_u^- \setminus \partial\Omega_u^+) \cap B_1;\\ u_{x_0}(x) &= \mu_+(x_0)\max\{0, x\cdot\nu_{x_0}\} + \mu_-(x_0)\min\{0, x\cdot\nu_{x_0}\}, \quad \text{if } x_0 \in \Gamma_{\text{TP}} := \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1, \end{aligned}$$

where $\nu_{x_0} \in \mathbb{R}^2$ is of the form $A_{x_0}^{-1/2}[\nu]$, for some $\nu \in \partial B_1$, and $\mu_+(x_0)$ and $\mu_-(x_0)$ are positive and such that $Q^{\pm}_{\text{TP}}(x_0) \le \mu_\pm^2(x_0) \le M_{\text{A}} L^2$, where $L = \|\nabla u\|_{L^\infty(B_2)}$ is the Lipschitz constant of u, and

$$\begin{aligned} \mu_\pm^2(x_0) &= Q^{\pm}_{\text{TP}}(x_0), \quad \text{if } x_0 \in \Gamma_\pm\\ \mu_+^2(x_0) - \mu_-^2(x_0) &= Q^{+}_{\text{TP}}(x_0) - Q^{-}_{\text{TP}}(x_0), \quad \text{if } x_0 \in \Gamma_{\text{TP}}. \end{aligned}$$

Notice that Corollary 1.3 already implies that the one-phase free boundaries Γ_+ and Γ_- are $C^{1,\alpha}$ regular. Thus, it remains to prove that $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are smooth in a neighborhood of Γ_{TP}.

Lemma 6.1 (Flatness of the free boundary at the two-phase points)**.** *Let u be as in Theorem 1.5. There are constants $C > 0$, $\gamma > 0$ and $r_0 > 0$ such that, for every $x_0 \in \partial\Gamma_{\text{TP}}$, we have*

$$\Omega^+_{x_0,r} \cap B_1 \supset \{x \in B_1 \,:\, x\cdot\nu_{x_0} > Cr^\gamma\} \quad \textit{and} \quad \Omega^-_{x_0,r} \cap B_1 \supset \{x \in B_1 \,:\, x\cdot\nu_{x_0} < -Cr^\gamma\}, \tag{6.1}$$

for every $r \in (0, r_0)$, where $\Omega^+_{x_0,r} := \{u_{x_0,r} > 0\}$ and $\Omega^-_{x_0,r} := \{u_{x_0,r} < 0\}$.

Proof. Both the inclusions of (6.5) follow by the uniform convergence of $u_{x_0,r}$ (Proposition 4.7, equation (4.8)) to the blow-up limit u_{x_0}. □

Lemma 6.2. *Let u be as in Theorem 1.5. There are constants $R, \alpha \in (0,1)$ and $C > 0$ such that, for every $x_0, y_0 \in \partial\Gamma_{\text{TP}} \cap B_R$, we have*

$$|\nu_{x_0} - \nu_{y_0}| \le C|x_0 - y_0|^\alpha \qquad \textit{and} \qquad |\mu_\pm(x_0) - \mu_\pm(y_0)| \le C|x_0 - y_0|^\alpha. \tag{6.2}$$

Proof. The proof follows step by step the one of Lemma 5.2. □

Reasoning as in the one-phase case, and using Lemma 6.1 and Lemma 6.2, one can prove that the two-phase free boundary Γ_{TP} is *contained* in a $C^{1,\alpha}$ curve. Unfortunately, this result by itself is not sufficient to deduce that $\partial\Omega_u^\pm$ are smooth. We now prove that the function u_+ (resp. u_-) is a solution of the one-phase free boundary problem

$$-\text{div}(A\nabla u_+) = f_+ \quad \text{in} \quad \Omega_u^+, \qquad |A_{x_0}^{1/2}\nabla u_+|(x_0) = \mu_+(x_0) \quad \text{for every} \quad x_0 \in \partial\Omega_u^+ \tag{6.3}$$

where the boundary equation is understood in a classical sense. This is an immediate consequence of the following lemma which states that u_+ is differentiable in Ω_u^+ up to the boundary.

Lemma 6.3 (Differentiability at points of the free boundary)**.** *Let u be as in Theorem 1.5. We consider two cases.*
(OP). *For every $x_0 \in (\partial\Omega_u^+ \setminus \partial\Omega_u^-) \cap B_1$, u_+ is differentiable at x_0 and there is $r(x_0) > 0$ such that*

$$\big|u_+(x) - \mu_+(x_0)(x - x_0)\cdot\nu_{x_0}\big| \le C|x - x_0|^{1+\gamma} \qquad \textit{for every} \qquad x \in B_{r(x_0)}(x_0) \cap \Omega_u^+.$$

(TP). *There exists a universal constant $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, the function u_+ is differentiable in $B_{r_0}(x_0) \cap \Omega_u^+$ and*

$$\big|u_+(x) - \mu_+(x_0)(x - x_0)\cdot\nu_{x_0}\big| \le C|x - x_0|^{1+\gamma} \qquad \textit{for every} \qquad x \in B_{r_0}(x_0) \cap \Omega_u^+. \tag{6.4}$$

In particular, for every $x_0 \in \partial\Omega_u^+ \cap B_1$, we have $\nabla u_+(x_0) = \mu_+(x_0)\nu_{x_0}$.

Proof. The two cases are analogous. We will prove **(TP)**. By Proposition 4.7, for every $r < r_0$, we have

$$\|\max\{0, u_{x_0,r}\} - \max\{0, u_{x_0}\}\|_{L^\infty(B_1)} \le \|u_{x_0,r} - u_{x_0}\|_{L^\infty(B_1)} \le Cr^\gamma.$$

Thus, using the flatness of the free boundary (Lemma 6.1), we get for every $x \in B_1 \cap \{u_{x_0,r} > 0\}$

$$\begin{aligned}\big|\max\{0, u_{x_0,r}(x)\} - \mu_+(x_0)x\cdot\nu_{x_0}\big| \le \big|\max\{0, u_{x_0,r}(x)\} - \max\{0, u_{x_0}(x)\}\big| \\ + \mu_+(x_0)\big|\min\{0, x\cdot\nu_{x_0}\}\big| \le Cr^\gamma.\end{aligned}$$

Now, taking $r = |x - x_0|$ and rescaling the above inequality, we obtain (6.4) □

We notice that at the two-phase free boundary point the estimate (6.4) holds in a ball whose radius does not depend on the point. Moreover, on the two-phase free boundary the gradient has a universal modulus of continuity (see Lemma 6.2). This concludes the proof of Proposition 1.8.

We next show that μ_+ is continuous on $\partial\Omega_u^+$.

Lemma 6.4. *The function $\mu_+ : \partial\Omega_u^+ \to \mathbb{R}$ is continuous.*

Proof. We start noticing that:

- on the set $\partial\Omega_u^+ \setminus \partial\Omega_u^-$, we have $\mu_+ = Q_+^{1/2}$, where we set $Q_+ := Q_{\text{TP}}^+$.
- for every $y_1, y_2 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, we have $|\mu_+(y_1) - \mu_+(y_2)| \le C|y_1 - y_2|^\alpha$.

Thus, it is sufficient to prove that if $(x_n)_{n\ge1}$ is a sequence of one-phase points, $x_n \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$, converging to a two-phase point $y_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, then $\mu_+(y_0) = Q_+^{1/2}(y_0)$. Up to a linear change of coordinates we may suppose that $A_{y_0} = Id$.

Denote by y_n the projection of x_n on the closed set $\partial\Omega_u^+ \cap \partial\Omega_u^-$ and let $r_n := |x_n - y_n|$. Since u is Lipschitz continuous, up to a subsequence, $u_n := u^+_{x_n,r_n}$ converges locally uniformly to some function u_∞. The absence of two-phase points in $B_{r_n}(x_n)$ implies that u_n is a solution of

$$-\mathrm{div}(A_n\nabla u_n) = r_n f_n \quad \text{in} \quad \{u_n > 0\}\cap B_1\,, \qquad |\nabla u_n| = q_n \quad \text{on} \quad \partial\{u_n > 0\}\cap B_1\,,$$

where $A_n(x) := A(x_n + r_n x)$, $f_n(x) := f_+(x_n + r_n x)$ and $q_n(x) = Q_+^{1/2}(x_n + r_n x)|\nu_{x_n+r_nx}|$, where we recall that $\nu_{x_n+r_nx}$ is of the form $A^{1/2}_{x_n+r_nx}[\tilde\nu]$, for some $\tilde\nu \in \partial B_1$. Passing to the limit as $n\to\infty$, we obtain that u_∞ is a viscosity solution to

$$-\Delta u_\infty = 0 \quad \text{in} \quad \{u_\infty > 0\}\cap B_1\,, \qquad |\nabla u_\infty| = Q_+^{1/2}(y_0)|\nu_{y_0}| \quad \text{on} \quad \partial\{u_\infty > 0\}\cap B_1\,.$$

On the other hand, for every $\xi \in B_1$, we have

$$u_{x_n,r_n}(\xi) = u_{y_n,r_n}(\xi + \xi_n)\,, \quad \text{where} \quad \xi_n := \frac{x_n - y_n}{r_n} \in \partial B_1\,,$$

and, up to a subsequence, we can assume that ξ_n converges to some $\xi_\infty \in \partial B_1$. Since $y_n \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, Lemma 6.3 implies that, for every $x \in B_{2r_n}(y_n) \cap \{u > 0\}$, we have

$$|u(x) - \mu_+(y_n)\max\{0, (x - y_n)\cdot\nu_{y_n}\}| \le C|x - y_n|^{1+\gamma} \le Cr_n^{1+\gamma}.$$

After rescaling, this gives

$$\big|u_{y_n,r_n}(\xi + \xi_n) - \mu_+(y_n)\max\{0, (\xi + \xi_n)\cdot\nu_{y_n}\}\big| \le Cr_n^\gamma \quad \text{for every} \quad \xi \in B_1 \cap \{u_{x_n,r_n} > 0\}.$$

Moreover, by the continuity of μ_+ on $\partial\Omega_u^+ \cap \partial\Omega_u^-$, we have that, for every $\xi \in B_1$,

$$\lim_{n\to\infty}\big|\mu_+(y_n)\max\{0, (\xi + \xi_n)\cdot\nu_{y_n}\} - \mu_+(y_0)\max\{0, (\xi + \xi_\infty)\cdot\nu_{y_0}\}\big| = 0.$$

Therefore, it follows that $u_{x_n,r_n}(\xi) = u_{y_n,r_n}(\xi + \xi_n)$ converges to

$$u_\infty(\xi) = \mu_+(y_0)\max\big\{0, (\xi + \xi_\infty)\cdot\nu_{y_0}\big\} \quad \text{for every} \quad \xi \in B_1.$$

Next we claim that $\xi_\infty\cdot\nu_{y_0} = 0$. Indeed, if $\xi_\infty\cdot\nu_{x_0} > 0$, then $u_\infty(0) > 0$ which is in contradiction with the uniform convergence of u_n; on the other hand, if $\xi_\infty\cdot e_{x_0} < 0$, then $u_\infty \equiv 0$ in a neighborhood of zero, which is in contradiction with the non-degeneracy of u_n. Thus, we get

$$u_\infty(\xi) = \mu_+(y_0)\max\big\{0, \xi\cdot\nu_{y_0}\big\} \quad \text{for every} \quad \xi \in B_1.$$

Now since $|\nabla u_\infty| = \mu_+(y_0)$, we get that $\mu_+(y_0) = Q_+^{1/2}(y_0)$. $\square$

In the next lemma we establish the Hölder continuity of μ_+.

Lemma 6.5. *The function $\mu_+ : \partial\Omega_u^+ \to \mathbb{R}$ is Hölder continuous.*

Proof. It is sufficient to show that μ_+ is Hölder continuous in a neighborhood of every two-phase point. Let $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$. Without loss of generality, we can assume that $x_0 = 0$ and $\nu_{x_0} = (0,1)$. For every $x = (s,t) \in \mathbb{R}^2$, we denote by $SQ_\delta(x)$ the square $(s-\delta, s+\delta)\times(t-\delta, t+\delta)$. By Lemma 6.1, for every $\varepsilon > 0$, there exists $\delta_0 > 0$ such that the following flatness condition holds:

For every two-phase point $x = (x_1, x_2) \in \partial\Omega_u^+ \cap \partial\Omega_u^-$ in the strip $(-\delta_0, \delta_0) \times (-\varepsilon\delta_0, \varepsilon\delta_0)$ and every $\delta \le \delta_0$ we have:

$$\begin{cases} (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 - \varepsilon\delta) \subset \Omega_u^- \cap SQ_\delta(x), \\ (x_1 - \delta, x_1 + \delta) \times (x_2 + \varepsilon\delta, x_2 + \delta) \subset \Omega_u^+ \cap SQ_\delta(x). \end{cases} \tag{6.5}$$

Notice that the flatness condition (6.5) implies that for every two-phase point $x = (x_1, x_2) \in \partial\Omega_u^+ \cap \partial\,\Omega_u^-$ and every $y = (y_1, y_2) \in \partial\Omega_u^+$, both lying in the strip $(-\delta_0, \delta_0) \times (-\varepsilon\delta_0, \varepsilon\delta_0)$, we have

$$|x_1 - y_1| \le |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \le \sqrt{1 + \varepsilon^2}\,|x_1 - y_1|. \tag{6.6}$$

Next, for every $t \in (-\delta_0, \delta_0)$ we define the vertical sections

$$\mathcal{S}^t := \{(x_1, x_2) \in SQ_{\delta_0}(0) \,:\, x_1 = t\}, \qquad \mathcal{S}^t_+ := \Omega_u^+ \cap \mathcal{S}^t, \qquad \mathcal{S}^t_- := \Omega_u^- \cap \mathcal{S}^t.$$

Let $\mathcal{U}_{\text{TP}}$ be the set of points $t \in (-\delta_0, \delta_0)$ such that there is a two-phase point $x \in SQ_{\delta_0}(0)$ lying on the section $\mathcal{S}^t$. It is immediate to check that $\mathcal{U}_{\text{TP}}$ is a closed subset of $(-\delta_0, \delta_0)$ and that, due to the flatness condition (6.5), for every $t \in \mathcal{U}_{\text{TP}}$, there is at most one two-phase point on the section $\mathcal{S}^t$. We will denote this point by x^t.

Let now $x, y \in \partial\Omega_u^+ \cap SQ_{\delta_0}(0)$. We have three possibilities:

(i) $x \in \partial\Omega_u^+ \cap \partial\Omega_u^-$ and $y \in \partial\Omega_u^+ \cap \partial\Omega_u^-$;
(ii) $x \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$ and $y \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$;
(iii) $x \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$ and $y \in \partial\Omega_u^+ \cap \partial\Omega_u^-$.

In each of this cases we will show that

$$|\mu_+(x) - \mu_+(y)| \le C|x - y|^\alpha. \tag{6.7}$$

We set $Q_+ := Q^+_{\text{TP}}$. In the case (i), (6.7) follows directly by Lemma 6.2. In the case (ii), we have that $\mu_+(x) = Q_+^{1/2}(x)$ and $\mu_+(y) = Q_+^{1/2}(y)$, so (6.7) follows by the Hölder continuity of Q_+. It remains to prove (6.7) in the case (iii). Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and, without loss of generality, suppose that $x_1 < y_1$. Let the open interval (a, t) be the connected component of $(-\delta_0, \delta_0) \setminus \mathcal{U}_{\text{TP}}$ containing x_1. Then, we have that $t \in \mathcal{U}_{\text{TP}}$ and $t \le y_1$. Let $x^t = (t, x_2^t)$ be the two-phase point lying in the section $\mathcal{S}^t$. Then, by construction of x^t, there exists (at least) one point $x_s \in \mathcal{S}^s \cap \Gamma^+$ for every $s \in (x_1, t)$. Moreover, since x^t is a two-phase point, by the flatness condition (6.5) we have that $u(t,s) > 0$ for every $s > x_2^t$ and $u(t,s) < 0$ for every $s < x_2^t$. Therefore, the sequence of one-phase point x_s converges as $s \to t$ to x^t and so, $\mu_+(x^t) = Q_+^{1/2}(x^t)$. Thus, using (i), the Hölder continuity of Q_+ and (6.6), we have

$$\begin{aligned} |\mu_+(x) - \mu_+(y)| &\le |\mu_+(x) - \mu_+(x^t)| + |\mu_+(x^t) - \mu_+(y)| \\ &= |Q_+^{1/2}(x) - Q_+^{1/2}(x^t)| + |\mu_+(x^t) - \mu_+(y)| \\ &\le C|x - x^t|^\alpha + C|x^t - y|^\alpha \\ &\le C\Big(\sqrt{1+\varepsilon^2}\Big)^\alpha \big((t - x_1)^\alpha + (y_1 - t)^\alpha\big) \\ &\le 2C\Big(\sqrt{1+\varepsilon^2}\Big)^\alpha (y_1 - x_1)^\alpha \le 2C\Big(\sqrt{1+\varepsilon^2}\Big)^\alpha |x - y|^\alpha, \end{aligned}$$

which concludes the proof. $\square$

Theorem 1.5 is now a consequence of (6.3), the Lemma 6.4 and a general result (Theorem A.1) on the regularity of the one-phase flat free boundaries, which is due to De Silva (see [14]). In the appendix we state Theorem A.1 in its full generality, for viscosity solutions of the problem (6.3), but in our case the function u_+ is a classical solution, differentiable everywhere on $\overline{\Omega}_u^+$.

7. Proof of Theorem 1.11

7.1. **Preliminary results.** In this subsection, we briefly recall the known results on the problem (1.10). The existence of a solution of (1.10) in the class of the almost-open subsets of $\mathcal{D}$ can be proved by a general variational argument (we refer to [8] and to the book [5] for more details). In the context of open sets, the existence of an optimal n-uple was proved in [3].

From now on, $(\Omega_1, \dots, \Omega_n)$ will be a solution of (1.10) and $u_i : \mathbb{R}^d \to \mathbb{R}$, for $i = 1, \dots, n$, will denote the first normalized eigenfunction of the Dirichlet Laplacian on Ω_i, that is,

$$-\Delta u_i = \lambda_1(\Omega_i) u_i \quad \text{in} \quad \Omega_i\,, \qquad u_i = 0 \quad \text{on} \quad \mathbb{R}^2 \setminus \Omega_i\,, \qquad \int_{\Omega_i} u_i^2\,dx = 1,$$

where, for every $i = 1, \dots, n$,

$$\lambda_1(\Omega_i) = \min_{u \in H_0^1(\Omega_i)\setminus\{0\}} \frac{\int_{\Omega_i} |\nabla u|^2\,dx}{\int_{\Omega_i} u^2\,dx} = \frac{\int_{\Omega_i} |\nabla u_i|^2\,dx}{\int_{\Omega_i} u_i^2\,dx},$$

where $H_0^1(\Omega_i) = \{u \in H^1(\mathbb{R}^d) \,:\, u = 0 \text{ on } \mathbb{R}^2 \setminus \Omega_i\}$. In particular, $u_i \geq 0$ on $\mathbb{R}^2$ and $\Omega_i = \{u_i > 0\}$.

Lipschitz continuity. The functions $u_i : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous on $\mathbb{R}^2$, that is, there is a universal constant $L > 0$ such that $\|\nabla u_i\|_{L^\infty(\mathbb{R}^2)} \leq L$, for every $i = 1, \dots, n$. We refer to [8] for the general case and to [3] for a simplified version in dimension two.

Non-degeneracy. There is a constant $C_0 > 0$ such that, for every $i = 1, \dots, n$, we have

$$\int_{\partial B_r} u_i\,d\mathcal{H}^1 \geq C_0 r^2 \qquad \text{for every} \qquad x_0 \in \partial\Omega_i \quad \text{and} \quad r \in (0,1).$$

Again, we refer to [8] and [3].

Absence of triple points. For every $1 \leq i < j < k \leq n$, we have that $\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k = \emptyset$ (see [8] and [23], and also [3] for a more direct proof in dimension two).

Absence of two-phase points on the boundary of the box. For every $1 \leq i < j \leq n$, we have that $\partial\Omega_i \cap \partial\Omega_j \cap \partial\mathcal{D} = \emptyset$ (see [3]).

As a consequence of the above properties, we have that, for every $i \in \{1, \dots, n\}$, the boundary $\partial\Omega_i$ can be decomposed as follows:

$$\partial\Omega_i = \bigcup_{k \neq i} (\partial\Omega_i \cap \partial\Omega_k) \cup (\partial\Omega_i \cap \partial\mathcal{D}) \cup \Gamma_{\text{OP}}(\Omega_i),$$

where $\Gamma_{\text{OP}}(\Omega_i)$ is the one-phase free boundary of Ω_i, determined by:

$$x_0 \in \Gamma_{\text{OP}}(\Omega_i) \Leftrightarrow \text{there exists } r > 0 \text{ such that } B_r(x_0) \cap \Big((\mathbb{R}^2 \setminus \mathcal{D}) \cup \bigcup_{k \neq i} \Omega_k\Big) = \emptyset.$$

We notice that already using the the regularity result of Briançon and Lamboley [4], the one-phase free boundary (lying inside the open set $\mathcal{D}$) is locally a $C^{1,\alpha}$ curve. Thus, in order to prove Theorem 1.11, it will be sufficient to show that $\partial\Omega_i$ is $C^{1,\alpha}$ in a neighborhood of the points of $\partial\Omega_i \cap \partial\mathcal{D}$ (Subsection 7.2) and $\partial\Omega_i \cap \partial\Omega_k$ (Subsection 7.3).

7.2. **One-phase points at the boundary of the box.** Let $1 \le i \le n$ and $x_0 \in \partial D \cap \partial \Omega_i$. Then, there is a neighborhood $\mathcal{U}$ of x_0 such that $\mathcal{U} \cap \Omega_j = \emptyset$, for every $j \neq i$. For the sake of simplicity, in this subsection, we will set

$$\Omega = \Omega_i\,, \quad u = u_i\,, \quad x_0 = 0 \quad \text{and} \quad D = \Omega_i \cup (\mathcal{D} \cap \mathcal{U}).$$

It is well known that the eigenvalues of the Dirichlet Laplacian have the following variational characterization:

$$\lambda_1(\Omega) = \int_\Omega |\nabla u|^2\,dx = \min\left\{\int_\Omega |\nabla v|^2\,dx : v \in H_0^1(\Omega),\ \int_\Omega v^2\,dx = 1\right\}.$$

Moreover, $\{u > 0\} = \Omega$ and u is a solution of the following minimization problem:

$$\min\left\{\int_D |\nabla v|^2\,dx + \Lambda|\{v > 0\}| : v \in H_0^1(D),\ \int_D v^2\,dx = 1\right\}. \tag{7.1}$$

We will show that the solution u of (7.1) is an almost-minimizer of the one-phase functional J_{OP}. A result in the same spirit was proved in a more general case in [20, Proposition 2.1].

Lemma 7.1 (Almost-minimality of the eigenfunction)**.** *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz continuous function, $L = \|\nabla u\|_{L^\infty}$ be the Lipschitz constant of u and $\lambda_1(\Omega_u) = \int_D |\nabla u|^2\,dx$. If u is a solution of the minimization problem (7.1), then there exists $r_0 > 0$ such that u satisfies the following almost-minimality condition:*

For every $r \in (0, r_0)$ and $x_0 \in \partial\Omega_u$,

$$\int_{B_r(x_0)} |\nabla u|^2\,dx + \Lambda|\Omega_u \cap B_r(x_0)| \le \big(1 + C_1 r^{d+2}\big)\int_{B_r(x_0)} |\nabla v|^2\,dx + \Lambda|\Omega_v \cap B_r(x_0)| + C_2 r^{d+2},$$

for every $v \in H_0^1(D)$ such that $u = v$ on $D \setminus B_r(x_0)$, where $C_1 = 2L^2$ and $C_2 = \lambda_1(\Omega_u)2L^2$.

Proof. Let $x_0 \in \partial\Omega_u$, $r > 0$ and $v \in H_0^1(D)$ be such that $u = v$ on $D \setminus B_r(x_0)$. Then, define the renormalization $w = \|v\|_{L^2}^{-1} v \in H_0^1(D)$ and notice that we have

$$\begin{aligned}\int_D |\nabla w|^2\,dx &= \Big(\int_D v^2\,dx\Big)^{-1}\int_D |\nabla v|^2\,dx \le \Big(1 - \int_{B_r(x_0)} u^2\,dx\Big)^{-1}\int_D |\nabla v|^2\,dx\\ &\le \frac{1}{1 - L^2 r^{d+2}}\int_D |\nabla v|^2\,dx \le \big(1 + 2L^2 r^{d+2}\big)\int_D |\nabla v|^2\,dx,\end{aligned}$$

where for the last inequality, we choose r_0 such that $2L^2 r_0^{d+2} \le 1$ and we use the inequality $\dfrac{1}{1-X} \le 1 + 2X$, for every $X \le 1/2$, with $X = L^2 r^{d+2}$. Now use w as a test function in (7.1) to get that

$$\int_D |\nabla u|^2\,dx + \Lambda|\{u > 0\}| \le \big(1 + 2L^2 r^{d+2}\big)\int_D |\nabla v|^2\,dx + \Lambda|\{v > 0\}|, \tag{7.2}$$

from which the claim easily follows since $\int_{D\setminus B_r(x_0)} |\nabla v|^2\,dx = \int_{D\setminus B_r(x_0)} |\nabla u|^2\,dx \le \lambda_1(\Omega_u)$. $\square$

We now notice that the C^2 regularity of $\partial\mathcal{D}$ implies that there is a constant $\delta > 0$ and a function $g : (-\delta, \delta) \to \mathbb{R}$ such that

$$\mathcal{D} \cap SQ_\delta = \big\{(x_1, x_2) \in SQ_\delta\ :\ g(x_1) < x_2\big\},$$

where $SQ_\delta = (-\delta, \delta) \times (-\delta, \delta)$. Moreover, up to a rotation of the plane, we can assume that $g'(0) = 0$. Let $\psi : SQ_\delta \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the function that straightens out the boundary of $\mathcal{D}$ and let $\phi = \psi^{-1} : \psi(SQ_\delta) \subset \mathbb{R}^2 \to \mathbb{R}^2$ be its inverse:

$$\psi(x_1, x_2) = (x_1, x_2 - g(x_1)), \quad \phi(x_1, x_2) = (x_1, x_2 + g(x_1)).$$

We define the matrix-valued function $A = (a_{ij})_{ij} : SQ_\delta \to M^2(\mathbb{R})$ by

$$A_x := \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} = \begin{pmatrix} 1 & -g'(x_1) \\ -g'(x_1) & 1+(g'(x_1))^2 \end{pmatrix} \quad \text{for every} \quad x = (x_1, x_2) \in SQ_\delta.$$

We recall that $H = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. By an elementary change of coordinates, we obtain the following result.

Lemma 7.2. *Let u and A be as above. There exist constants $C_1, C_2 > 0$ and $r_0 > 0$ such that $B_{2r_0} \subset \psi(SQ_\delta)$ and the function $\tilde{u} := u \circ \phi$ satisfies the following almost-minimality condition:*

- *For every $x_0 \in \partial\Omega_{\tilde{u}} \cap B_{r_0}$ and $r \in (0, r_0)$ we have*

$$\int_{B_r(x_0)} a_{ij}(x)\,\frac{\partial \tilde{u}}{\partial x_i}\,\frac{\partial \tilde{u}}{\partial x_j}\,dx + \Lambda|\Omega_{\tilde{u}} \cap B_r(x_0)|$$
$$\leq (1 + C_1 r^{d+2}) \int_{B_r(x_0)} a_{ij}(x)\,\frac{\partial \tilde{v}}{\partial x_i}\,\frac{\partial \tilde{v}}{\partial x_j}\,dx + \Lambda|\Omega_{\tilde{v}} \cap B_r(x_0)| + C_2 r^{d+2},$$

for every $\tilde{v} \in H^1(B_{2r_0})$ such that $\tilde{u} = \tilde{v}$ on $B_{2r_0} \setminus B_r(x_0)$ and $\Omega_{\tilde{v}} \subset H$.

Proof. Let $x_0 \in B_{r_0}$, $r \in (0, r_0)$ and $\tilde{v}$ such that $\tilde{u} = \tilde{v}$ on $B_{2r_0} \setminus B_r(x_0)$. Then, use $v \in H^1_0(D)$ defined by $v = \tilde{v} \circ \psi$ in $\psi^{-1}(B_{2r_0})$ and $v = u$ otherwise, as a test function in Lemma 7.1 to get

$$\int_{B_{c_\phi r}(y_0)} |\nabla u|^2\,dx + \Lambda|\Omega_u \cap B_{c_\phi r}(y_0)| \leq (1 + C_1 r^{d+2}) \int_{B_{c_\phi r}(y_0)} |\nabla v|^2\,dx + \Lambda|\Omega_v \cap B_{c_\phi r}(y_0)| + C r^{d+2},$$

where c_ϕ is a positive constant depending only on ϕ such that $\phi(B_r(x_0)) \subset B_{c_\phi r}(y_0)$ and $y_0 = \phi(x_0)$. Now, with a change of coordinates and noticing that $u = v$ on $\phi(B_r(x_0))$ we have

$$\begin{aligned}
\int_{B_r(x_0)} a_{ij}(x)\,\frac{\partial \tilde{u}}{\partial x_i}\,\frac{\partial \tilde{u}}{\partial x_j}\,dx + \Lambda|\Omega_{\tilde{u}} \cap B_r(x_0)| &= \int_{\phi(B_r(x_0))} |\nabla u|^2\,dx + \Lambda|\Omega_u \cap \phi(B_r(x_0))| \\
&\leq (1 + C_1 r^{d+2}) \int_{\phi(B_r(x_0))} |\nabla v|^2\,dx + \Lambda|\Omega_v \cap \phi(B_r(x_0))| + C_2 r^{d+2} \\
&= (1 + C_1 r^{d+2}) \int_{B_r(x_0)} a_{ij}(x)\,\frac{\partial \tilde{v}}{\partial x_i}\,\frac{\partial \tilde{v}}{\partial x_j}\,dx + \Lambda|\Omega_{\tilde{v}} \cap B_r(x_0)| + C_2 r^{d+2},
\end{aligned}$$

where $C_2 = \lambda_1(\Omega_u)C_1 + C$. This concludes the proof. □

Proof of Theorem 1.11 (the one-phase boundary points). We are now in position to conclude the regularity of the free boundary $\partial\Omega_i$ in a neighborhood of any one-phase boundary point $x_0 \in \partial\Omega_i \cap \partial\mathcal{D}$. Indeed, we may assume that $x_0 = 0$ and that $\partial\mathcal{D}$ is the graph of a function g. Reasoning as above, we have that $\tilde{u}_i(x_1, x_2) = u_i(x_1, x_2 + g(x_1))$ satisfies the almost-minimality condition from Lemma 7.2 in a neighborhood of the origin. On the other hand, it is immediate to check that $\tilde{u}_i$ is still Lipschitz continuous. Thus, we can apply Theorem 1.1 obtaining that, in a neighborhood of zero, $\partial\Omega_i$ is the graph of a $C^{1,\alpha}$ function. □

7.3. **Two-phase points.** Let Ω_i and Ω_j be two different sets from the optimal n-uple $(\Omega_1, \dots, \Omega_n)$, solution of (1.10). Let u_i and u_j be the first normalized eigenfunctions, respectively on Ω_i and Ω_j. Finally, let $x_0 \in \partial\Omega_i \cap \partial\Omega_j$. We know that there is a neighborhood $\mathcal{U} \subset \mathcal{D}$ of x_0 such that $\mathcal{U} \cap \Omega_k = \emptyset$, for every $k \notin \{i, j\}$. Setting $D := \Omega_i \cup \Omega_j \cup \mathcal{U}$, we get that the function $u := u_i - u_j$ is the solution of the two-phase problem

$$\min\Big\{ \int_D |\nabla v|^2\,dx + q_i|\Omega_v^+| + q_j|\Omega_v^-| \ : \ v \in H^1_0(D),\ \int_D v_+^2\,dx = \int_D v_-^2\,dx = 1 \Big\}. \tag{7.3}$$

We next show that the solutions of (7.3) satisfy a almost-minimality condition.

Lemma 7.3. *Let $D \subset \mathbb{R}^d$, $u \in H^1_0(D)$ be a Lipschitz continuous function on $\mathbb{R}^d$ and L its Lipschitz constant. Suppose that u is a solution of the minimization problem* (7.3)*. Then, there is some $r_0 > 0$ such that u satisfies the following almost-minimality condition:*

For every $r \in (0, r_0)$ and $x_0 \in \partial\Omega_u$,

$$\int_{B_r(x_0)} |\nabla u|^2\,dx + q_i|\Omega_u^+ \cap B_r(x_0)| + q_j|\Omega_u^- \cap B_r(x_0)|$$
$$\leq \big(1 + C_1 r^{d+2}\big) \int_{B_r(x_0)} |\nabla v|^2\,dx + q_i|\Omega_v^+ \cap B_r(x_0)| + q_j|\Omega_v^- \cap B_r(x_0)| + C_2 r^{d+2},$$

for every $v \in H_0^1(D)$ such that $u - v \in H_0^1(B_r(x_0))$, where $C_1 = 2L^2$ and $C_2 = C_1 \int_D |\nabla u|^2\,dx$.

Proof. Follows precisely as in Lemma 7.1. □

We are now in position to complete the proof of Theorem 1.11.

Proof of Theorem 1.11 (the two-phase free boundary). We only need to notice that in a neighborhood of any two-phase point $x_0 \cap \partial\Omega_i \cap \partial\Omega_j \cap \mathcal{D}$, Lemma 7.3 implies that u is a almost-minimizer of J_{TP}, where the matrix A is the identity, $Q_+ = q_i$ and $Q_- = q_j$. Thus, it is sufficient to apply Theorem 1.5. □

Appendix A. The flat one-phase free boundaries are $C^{1,\alpha}$

In this section we discuss a regularity theorem for viscosity solutions of the one-phase problem (without constraint). We fix the real-valued function $f : B_2 \to \mathbb{R}$ and the matrix-valued $A : B_2 \to Sym_d^+$ to be as follows:

- $f : B_2 \to \mathbb{R}$ is bounded and continuous;
- $A : B_2 \to Sym_d^+$ is coercive, bounded and has C^1-regular coefficients.

Before we state the result, we recall that a Lipschitz continuous nonnegative function $u : \mathbb{R}^d \supset B_1 \to \mathbb{R}$ is a viscosity solution to

$$-\mathrm{div}(A\nabla u) = f \quad \text{in} \quad \Omega_u \cap B_1, \qquad \big|A^{1/2}[\nabla u]\big| = g \quad \text{on} \quad \partial\Omega_u \cap B_1, \tag{A.1}$$

if the first equation holds in the open set Ω_u and if, for every $x_0 \in \partial\Omega_u$ and every $\varphi \in C^\infty(\mathbb{R}^d)$ touching $u \circ F_{x_0}$ from above (below) at zero, we have that $|\nabla\varphi|(0) \geq g(x_0)$ (resp. $|\nabla\varphi|(0) \leq g(x_0)$). Recall that *touching from above (below)* means that $\varphi(0) = 0$ and $\varphi \geq u \circ F_{x_0}$ (resp. $\varphi \leq u \circ F_{x_0}$) in $\Omega_u \cap B_1$. Moreover, we suppose that g is Hölder continuous and that there are constants $\eta_g > 0$, $C_g > 0$ and $\delta_g > 0$ such that

$$\begin{cases} |g(x) - g(y)| \leq C_g|x-y|^{\delta_g} \quad \text{for every} \quad x, y \in \partial\Omega_u \cap B_1, \\ \eta_g \leq g(x) \quad \text{for every} \quad x \in \partial\Omega_u \cap B_1. \end{cases} \tag{A.2}$$

The following result follows immediately from the results proved in [14].

Theorem A.1 (Flat free boundaries are $C^{1,\alpha}$). *Suppose that $u : B_1 \to \mathbb{R}$ is a viscosity solution of* (A.1) *and that $g : \partial\Omega_u \to \mathbb{R}$ satisfies* (A.2)*. Then, there exist $\varepsilon > 0$ and $\rho > 0$ such that if $x_0 \in \partial\Omega_u \cap B_1$ and u is such that*

$$g(x_0)\max\{0, x\cdot\nu - \varepsilon\rho\} \leq u \circ F_{x_0}(x) \leq g(x_0)\max\{0, x\cdot\nu + \varepsilon\rho\} \quad \textit{for every} \quad x \in B_\rho,$$

then $\partial\Omega_u$ is $C^{1,\alpha}$ in $B_{\rho/2}(x_0)$.

Remark A.2. Notice that since in dimension two all the blow-up limits of u_+ (given by Theorem 1.5) are half-plane solutions (Proposition 4.3), we have that the flatness assumption of the above Theorem is satisfied at every point of the free boundary $\partial\Omega_u^+$. We also notice that, in our case, we have $g = \mu_+$, which is Hölder continuous by Lemma 6.4.

Definition A.3 (Flatness). *Let $u : B_1 \to \mathbb{R}$ be continuous, $u \geq 0$ and $u \in H^1(B_1)$. We say that u is (ε, ν)-flat, if there are a matrix-valued $A = (a_{ij})_{ij} : B_1 \to Sym_d^+$ with Hölder continuous* *coefficients, and a continuous* $f : B_1 \to \mathbb{R}$ *such that:*

$$-\sum_{i,j} a_{ij}\,\partial_{ij} u = f \quad in \quad \Omega_u \cap B_1\,; \tag{A.3}$$

$$\|f\|_{L^\infty(B_1)} \le \varepsilon \quad and \quad \|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \le \varepsilon^2 \quad for\ every \quad 1 \le i,j \le d\,; \tag{A.4}$$

$$1 - \varepsilon^2 \le |\nabla u| \le 1 + \varepsilon^2 \quad on \quad \partial\Omega_u \cap B_1\,; \tag{A.5}$$

$$\max\{0, x\cdot\nu - \varepsilon\} \le u(x) \le \max\{0, x\cdot\nu + \varepsilon\} \quad for\ every \quad x \in B_1. \tag{A.6}$$

Remark A.4. The condition (A.5) is intended in a viscosity sense, that is, for any $\varphi \in C^\infty(B_1)$, we have:

- if $\varphi(x_0) = u(x_0)$ for some $x_0 \in \partial\Omega_u \cap B_1$ and $\varphi^+ \ge u$ in $\Omega_u \cap B_1$, then $|\nabla\varphi(x_0)| \ge 1 - \varepsilon^2$;
- if $\varphi(x_0) = u(x_0)$ for some $x_0 \in \partial\Omega_u \cap B_1$ and $\varphi \le u$ in $\Omega_u \cap B_1$, then $|\nabla\varphi(x_0)| \le 1 + \varepsilon^2$.

In order to prove Theorem A.1 one has to show that the flatness improves at lower scales, that is, if u is (ε,ν)-flat, then a rescaling u_r of u is $(\varepsilon/2, \nu')$-flat for some ν', which is close to ν. Of course, the essential (and hardest) part of the proof is to show the improvement of the geometric flatness (A.6). This was proved by De Silva in [14, Lemma 4.1].

Lemma A.5 (Improvement of the geometric flatness)**.** *There are universal constants $C > 0$, $r_0 > 0$ and $\varepsilon_0 > 0$ such that if u is ε-flat in the direction ν, for some $\varepsilon \in (0, \varepsilon_0)$ and $\nu \in \partial B_1$, then, for every $r \in (0, r_0)$ there is some $\nu' \in \partial B_1$ such that $|\nu - \nu'| \le C\varepsilon^2$ and*

$$\max\Big\{0, x\cdot\nu' - \frac{\varepsilon}{2}\Big\} \le u_r(x) \le \max\Big\{0, x\cdot\nu' + \frac{\varepsilon}{2}\Big\} \quad for\ every \quad x \in B_1,$$

where $u_r : B_1 \to \mathbb{R}$ is the one-homogeneous rescaling $u_r(x) = \frac{u(rx)}{r}$.

Proof of Theorem A.1. We will first prove that the flatness condition (A.3)-(A.6) improves at smaller scales. We fix $x_0 \in \partial\Omega_u \cap B_1$ and we consider the function $\tilde u = \frac{1}{g(x_0)} u \circ F_{x_0}$ (recall that $F_{x_0}(x) = x_0 + A_{x_0}^{1/2}[x]$). Let ε and r_0 be the constants from Lemma A.5. We will prove that there is $r_1 \le r_0$ such that: if $\tilde u$ is (ε,ν)-flat, then for every $r \le r_1$, $\tilde u_r$ is $(\varepsilon/2, \nu')$-flat, for ν' given again by Lemma A.5. It is sufficient that the conditions (A.3), (A.4) and (A.5) are satisfied for $\tilde u_r$ with the flatness parameter $\varepsilon/2$. We notice that $\tilde u$ is a viscosity solution of

$$-\mathrm{div}(\tilde A\nabla\tilde u) = \tilde f \quad \text{in} \quad \Omega_{\tilde u}, \qquad \big|\tilde A^{1/2}[\nabla\tilde u]\big| = \tilde g \quad \text{on} \quad \partial\Omega_{\tilde u}, \tag{A.7}$$

where $\tilde A_x = A_{x_0}^{-1/2} A_{F_{x_0}(x)} A_{x_0}^{-1/2}$, $\tilde f = \dfrac{1}{g(x_0)} f \circ F_{x_0}$, $\tilde g = \dfrac{1}{g(x_0)} g \circ F_{x_0}$ and $\tilde A_x^{1/2} = A_{F_{x_0}(x)}^{1/2} \circ A_{x_0}^{-1/2}$. Notice that $0 \in \partial\Omega_{\tilde u}$ and set $\tilde u_r(x) := \dfrac{\tilde u(rx)}{r}$. Thus, for small enough $r > 0$, $\tilde u_r$ is a viscosity solution of

$$-\mathrm{div}(\tilde A_r\nabla\tilde u_r) = \tilde f_r \quad \text{in} \quad \Omega_{\tilde u} \cap B_1, \qquad \big|\tilde A_r^{1/2}[\nabla\tilde u_r]\big| = \tilde g_r \quad \text{on} \quad \partial\Omega_{\tilde u} \cap B_1, \tag{A.8}$$

where $\tilde A_r(x) := \tilde A(rx)$, $\tilde f_r(x) = r\tilde f(rx)$, $\tilde g_r(x) = \tilde g(rx)$ and $\tilde A_r^{1/2}(x) = A_{F_{x_0}(rx)}^{1/2} \circ A_{x_0}^{-1/2}$.

We set $\tilde a_{ij}^r(x)$ to be the coefficients of $\tilde A_r(x)$ and $\tilde b^r$ to be the vector with coefficients $\tilde b_i^r = \sum_j \partial_j \tilde a_{ij}^r(x)$. Then, (A.8) can be equivalently written as

$$-\sum_{i,j} \tilde a_{ij}^r\,\partial_{ij}\tilde u_r = \tilde b^r \cdot \nabla\tilde u_r + \tilde f_r \quad \text{in} \quad \Omega_{\tilde u} \cap B_1, \qquad \big|\tilde A_r^{1/2}[\nabla\tilde u_r]\big| = \tilde g_r \quad \text{on} \quad \partial\Omega_{\tilde u} \cap B_1, \tag{A.9}$$

Now, if $\tilde u$ is (ε,ν)-flat, then the Hölder continuity of the coefficients a_{ij} and the boundedness of f imply that (A.4) holds with $\varepsilon/2$ and $\tilde u_r$, for any $r \le r_1$, where $r_1 \le r_0$, is a universal constant depending on the Hölder norm of a_{ij}. Now, in order to get (A.5) for $\varepsilon/2$ and $\tilde u_r$, we suppose that $\varphi \in C^\infty(B_1)$ touches $\tilde u_r$ from below at a point $y_0 \in B_1 \cap \partial\{\tilde u_r > 0\}$. Thus, we have that

$$\Big|A_{F_{x_0}(ry_0)}^{1/2} \circ A_{x_0}^{-1/2}[\nabla\varphi(y_0)]\Big| \le \frac{g(F_{x_0}(ry_0))}{g(x_0)},$$

and so, if $\|\cdot\| = \|\cdot\|_{\mathcal{L}(\mathbb{R}^d)}$ stands for the space of $d \times d$ matrices, we have

$$|\nabla\varphi(y_0)| \le \left\| A_{x_0}^{1/2} \circ A_{F_{x_0}(ry_0)}^{-1/2} \right\| \frac{g(F_{x_0}(ry_0))}{g(F_{x_0}(0))}.$$

Now, by the Hölder continuity (and the uniform boundedness from below) of g, we can choose r_1 such that

$$\frac{g(F_{x_0}(ry_0))}{g(F_{x_0}(0))} \le 1 + \frac{\varepsilon^2}{10}.$$

On the other hand, there are universal constants C and $\delta > 0$, depending only on the Hölder exponent δ_A and the norm C_A, of the matrix-valued function A, such that

$$\left\| A_{x_0}^{1/2} \circ A_{F_{x_0}(ry_0)}^{-1/2} - Id \right\| \le \left\| A_{x_0}^{1/2} - A_{F_{x_0}(ry_0)}^{1/2} \right\| \cdot \left\| A_{F_{x_0}(ry_0)}^{-1/2} \right\| \le C|ry_0|^\delta \le Cr_1^\delta.$$

Choosing r_1 such that $Cr_1^\delta \le \frac{\varepsilon^2}{10}$ and using the triangular inequality, we get

$$|\nabla\varphi(y_0)| \le \left\| A_{x_0}^{1/2} \circ A_{F_{x_0}(ry_0)}^{-1/2} \right\| \frac{g(F_{x_0}(ry_0))}{g(F_{x_0}(0))} \le \left(1 + \frac{\varepsilon^2}{10}\right)^2 \le 1 + (\varepsilon/2)^2,$$

which completes the proof of the improvement of flatness for $\tilde{u}$, the case when φ touches from above being analogous. Now, the claim follows by a standard argument, similar to the one we used in Section 5. □

Acknowledgments. The first author has been partially supported by the NSF grant DMS 1810645. The second and third author have been partially supported by Agence Nationale de la Recherche (ANR) by the projects GeoSpec (LabEx PERSYVAL-Lab, ANR-11-LABX-0025-01) and CoMeDiC (ANR-15-CE40-0006).

References

[1] H. W. Alt, and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1081), 105–144.

[2] H. W. Alt, L. A. Caffarelli, A. Friedman. Variational problems with two phases and their free boundaries. *Trans. Amer. Math. Soc.* **282** (2) (1984), 431–461.

[3] B. Bogosel, B. Velichkov. Multiphase Optimization Problems for Eigenvalues: Qualitative Properties and Numerical Results. *SIAM J. Numer. Anal.* **54** (1) (2015), 210–241.

[4] T. Briançon, J. Lamboley. Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26** (4) (2009), 1149–1163.

[5] D. Bucur, G. Buttazzo. Variational Methods in Shape Optimization Problems. *Progress in Nonlinear Differential Equations*, **65**, Birkhäuser Verlag, Basel, 2005.

[6] D. Bucur, G. Buttazzo, B. Velichkov. Spectral optimization problems with internal constraint. *Ann. I. H. Poincare* **30** (3) (2013), 477–495.

[7] D. Bucur, D. Mazzoleni, A. Pratelli, B. Velichkov. Lipschitz regularity of the eigenfunctions on optimal domains. *Arch. Rat. Mech. Anal.* **216** (1) (2015), 117-151.

[8] D. Bucur, B. Velichkov. Multiphase shape optimization problems. *SIAM J. Control Optim.* **52** (6) (2014), 3556–3591.

[9] G. Buttazzo, G. Dal Maso. An existence result for a class of shape optimization problems. *Arch. Rat. Mech. Anal.* **122** (1993), 183–195.

[10] L. Caffarelli, H. Shahgholian, K. Yeressian. A minimization problem with free boundary related to a cooperative system. *Duke Math. J.* **167** (10) (2018), 1825–1882.

[11] H. Chang-Lara, O. Savin. Boundary regularity for the free boundary in the one-phase problem. *Preprint ArXiv* (2017).

[12] G. David, M. Engelstein, T. Toro. Free boundary regularity for almost-minimizers. *Preprint ArXiv* (2017).

[13] G. David, T. Toro. Regularity for almost minimizers with free boundary. *Calc. Var. PDE* **54** (1) (2015), 455–524.

[14] D. De Silva. Free boundary regularity from a problem with right hand side. *Interfaces and Free Boundaries* **13** (2) (2011), 223–238.

[15] D. De Silva, O. Savin. *Almost minimizers of the one-phase free boundary problem.* Preprint ArXiv (2019).

[16] M. Giaquinta, E. Giusti. Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form. *J. Reine Angew. Math.* **351** (1984), 55–65.

[17] A. Henrot, M. Pierre. *Variation et Optimisation de Formes. Une Analyse Géométrique.* Mathématiques & Applications **48**, Springer-Verlag, Berlin (2005).

[18] D. Kriventsov, F. Lin. Regularity for Shape Optimizers: The Nondegenerate Case. *Comm. Pure Appl. Math.* **71** (8) (2018), 1525–2596.

[19] D. Kriventsov, F. Lin. Regularity for Shape Optimizers: The Degenerate Case. *Preprint Arxiv* (2018).

[20] D. Mazzoleni, S. Terracini, B. Velichkov. Regularity of the optimal sets for some spectral functionals. *Geom. Funct. Anal.* **27** (2) (2017), 373–426.

[21] O. Queiroz, L. Tavares. Almost minimizers for semilinear free boundary problems with variable coefficients. *Mathematische Nachrichten* **291** (10) (2018), 1486–1501.

[22] L. Spolaor, B. Velichkov. An epiperimetric inequality for the regularity of some free boundary problems: the 2-dimensional case. *Comm. Pure Appl. Math.* **72** (2) (2018), 375–421.

[23] B. Velichkov. A note on the monotonicity formula of Caffarelli-Jerison-Kenig. *Rend. Lincei Mat. Appl.* **25** (2) (2014), 165–189.

[24] G. S. Weiss. Partial regularity for a minimum problem with free boundary. *J. Geom. Anal.* **9** (2) (1999), 317–326.

Luca Spolaor:
Massachusetts Institute of Technology (MIT),
77 Massachusetts Avenue, Cambridge MA 02139, USA
Email address: `lspolaor@mit.edu`

Baptiste Trey:
Laboratoire Jean Kuntzmann (LJK), Université Grenoble Alpes
Bâtiment IMAG, 700 Avenue Centrale, 38401 Saint-Martin-d'Hères
Email address: `baptiste.trey@univ-grenoble-alpes.fr`

Bozhidar Velichkov:
Laboratoire Jean Kuntzmann (LJK), Université Grenoble Alpes
Bâtiment IMAG, 700 Avenue Centrale, 38401 Saint-Martin-d'Hères
Email address: `bozhidar.velichkov@univ-grenoble-alpes.fr`