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On optimal steering of a non-Markovian Gaussian process

Daniele Alpago, Yongxin Chen, Tryphon Georgiou and Michele Pavon

***Abstract*— At present, the problem to steer general non-Markovian processes between specified end-point marginal distributions with minimum energy remains unsolved. Herein, we consider the special case of a non-Markovian process $y(t)$ which assumes a finite-dimensional stochastic realization with a Markov *state process* that is fully observable. In this setting, and over a finite time horizon $[0,T]$, we determine an optimal (least) finite-energy control law that steers the stochastic system to a final distribution that is compatible with a specified distribution for the terminal output process $y(T)$; the solution is given in closed-form. This work provides a key step towards the important problem to steer a stochastic system based on partial observations of the state (i.e., an output process) corrupted by noise.**

I. Introduction

Throughout we will be considering a controlled evolution of the n-dimensional Gauss-Markov process $\{x(t) \mid 0 \leq t \leq T\}$ that obeys the linear stochastic differential equation

$$dx^u = A(t)x^u(t)dt + B(t)u(t)dt + B(t)dw(t), \qquad \text{(1a)}$$
$$x^u(0) = \xi \ \text{ a.s.}$$

Here, as it is customary, w is an m-dimensional standard Wiener process and ξ is an n-dimensional Gaussian random vector which is independent of w. For simplicity we suppose that ξ has zero mean, and that it has density

$$\rho_0(x) = (2\pi)^{-n/2} \det(\Sigma_0^x)^{-1/2} \exp\left\{-\frac{1}{2}x'\left(\Sigma_0^x\right)^{-1}x\right\}. \qquad \text{(1b)}$$

We also assume that $A(\cdot)$ and $B(\cdot)$ are continuous matrix functions taking values in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$, respectively, and the pair (A,B) is controllable. The assumption that control and noise enter through the same channel (same $B(\cdot)$ matrix) may be seen as restrictive; though it should be noted that such problems have been developed in the literature under the even stronger assumption that B is the identity matrix for both, control and stochastic excitation, see the survey [31]. The case of different channels for the control and stochastic excitation, which is indeed more interesting from an applications' standpoint, has been addressed in [11] for Markovian Gauss processes. It is considerably more involved. In this first paper we focus on the case of identical channels, and only briefly discuss the general cases in Remark 7. We note however, that this special case of identical channels is not devoid of practical interest. Indeed, it is applicable to models of RLC networks with Nyquist-Johnson noisy resistors, see Section V for further motivation. It is also used in robotics where $dw(t)$ captures the noise in implementing a control strategy u. That is, the real input to the system is $udt + dw$ instead of udt due to disturbances in real implementations; see [44] and references therein for more discussions.

D. Alpago is with the Dipartimento di Ingegneria dell'Informazione, Università di Padova, 35131 Padova, Italy; daniele.alpago@phd.unipd.it
Y. Chen is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332;yongchen@gatech.edu
T.T. Georgiou is with the Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697; tryphon@uci.edu
M. Pavon is with the Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, 35121 Padova, Italy; pavon@math.unipd.it
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In recent years, there has been considerable interest in the problem of *minimum-energy steering* of the (Gaussian) distribution of $x(t)$ to a target distribution $\mathcal{N}(0,\Sigma_T^x)$ at time $t = T$, [10], [11], [26], [24], [2], as well as in the infinite-horizon case where the goal is to achieve with minimum power a specified stationary state. The latter generalizes the classical work on covariance control of Skelton et al. [27], [25]. Motivation for such problems is multifarious: they represent a natural relaxation of classical LQR steering problems and have applications in quality control and industrial manufacturing, vehicle path planning [36], statistical physics as in *cooling* and the control of nano-to-meter scale resonators, atomic force microscopy and so forth, see e.g., [21], [12].

Historically, the origin of this stochastic steering problem stems from a *Gedankenexperiment* formulated by Schrödinger in the early thirties [41], [42], seeking the most likely flow of particle distribution between observed end-point marginals. Schrödinger's problem was later recognized to be a problem in the theory of large deviations (which was unavailable at that time), cf. [22]. Indeed, thanks to Sanov's theorem [40], Schrödinger's problem amounts to seeking a probability distribution on particle trajectories having maximum entropy among those in agreement with the specified end-points marginal distributions [23], [4], [28], [22], [46]. Then, in the late eighties and early nineties, based on the work of Jamison, Föllmer, Nagasawa, Wakolbinger, Fleming, Holland, Mitter and others, a clear connection was made with stochastic control [18], [19], [37]. The distribution on paths, corresponding to the uncontrolled evolution, plays the role of the "prior" measure in the maximum entropy problem which generalizes Schrödinger's original one. At about the same time, Blaquiere [5] studied the control of the Fokker-Planck equation and later Brockett studied the Louiville equation [8] along a similar spirit, to steer the distribution to a target one. A variation of the stochastic control problem for Markov processes stemming from another application domain has been considered in [20] where, given a random-vector ζ, the stronger condition $x^u(T) = \zeta$ almost surely, is imposed. This circle of control problems for uncertain systems has recently been linked to yet another fast

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developing topic, Optimal Mass Transport (OMT) [45], when it was realized that *Schrödinger's bridge problem* (SBP) may be viewed as a regularization of OMT [33], [34], [35], [31], [13]; interestingly, SBP naturally mitigates computational challenges of OMT and has been developed to this end from different angles [3], [15], [16], [39].

Extending the Schrödinger problem to the case of non-Markov processes is a tantalizing and natural next step. While the general case is currently open[1], in the present paper we work out the special case of steering the output of a Gauss-Markov model whose state is fully observable. More specifically, in conjunction with (1a), we consider the output process

$$y(t) = C(t)x^u(t), \tag{1c}$$

where $C(\cdot)$ is continuous and takes values in $\mathbb{R}^{p\times n}$ for $p<n$.

In general, y by itself is not a (vector) Markov process.[2] This situation arises when we consider steering only some components of the state to a prescribed terminal distribution (see Section V). Problems where only a portion of the state needs to be specified arise, for instance, in thickness control (film extrusion) [1], [2] where the remaining components of the state vector might either not be of interest or may be difficult/expensive to measure. In path planning problems for robot manipulators, the target is usually specified in the workspace (output), which may correspond to infinitely many possible configurations (state) [30]. This also serves as a motivating example for our problem formulation. In Section V, we discuss a case where it is of interest to regulate only the distribution of the momentum of a stochastic oscillator. Although the setting considered in this paper is not the most general, in that a finite-dimensional stochastic realization of the non-Markovian process is assumed to be available, the fact that the optimal control can be obtained in closed-form is still noteworthy. Indeed, our main contribution in this paper is the *closed-form* optimal control strategy for the covariance steering problem (Problem 3) for the system (1a)-(1c).

The outline of the paper is as follows. In Section II, we recall some central results from [10] in the case of a Markovian prior. In Section III, we give a precise formulation of our stochastic control problem. In Section IV, we provide a closed-form solution to our problem by finding the terminal time state covariance which can be reached with minimum energy among those complying with the assigned covariance of $y(T)$. Section V illustrates the results in a problem of steering the momentum distribution of a stochastic oscillator to a desired final one. Finally, in Section VI, we draw the conclusions and discuss some future developments.

[1] See [38] for a considerably simpler "half-bridge" problem where only the final distribution is prescribed.

[2] It might be worthwhile to recall that a Gaussian process always admits a Markovian representation/stochastic realization (1a)-(1c) where, however, the state process might be infinite-dimensional or of dimension varying with time, see [32].

II. Background

Let $\mathcal{U}(\Sigma_0^x,\Sigma_T^x)$ be the family of *adapted*[3]*, finite energy* control functions such that (1a) has a strong solution on $[0,T]$ and $x(T)$ has distribution $\mathcal{N}(0,\Sigma_T^x)$. The optimal steering problem reads

Problem 1: Determine

$$u^* := \underset{u\in\mathcal{U}(\Sigma_0^x,\Sigma_T^x)}{\operatorname{argmin}}\ J(u) := \mathbb{E}\left\{\int_0^T u(t)'u(t)\,dt\right\}.$$

In [10, Theorem 8], it was shown that, under controllability of the pair $(A(\cdot),B(\cdot))$ on the given time interval, $\mathcal{U}(\Sigma_0^x,\Sigma_T^x)$ is nonempty and the (unique) optimal control is a linear feedback of the state given by

$$u^\star(t) = -B(t)'Q(t)^{-1}x(t). \tag{2}$$

Here, $P(t)$ and $Q(t)$, taking values in the set of symmetric, $n\times n$ matrices, are the unique *nonsingular* solutions on $[0,T]$ of the system of linear matrix equations

$$\dot{P}(t) = A(t)P(t)+P(t)A(t)'+B(t)B(t)', \tag{3a}$$

$$\dot{Q}(t) = A(t)Q(t)+Q(t)A(t)'-B(t)B(t)', \tag{3b}$$

nonlinearly coupled through the boundary conditions

$$(\Sigma_0^x)^{-1} = P(0)^{-1}+Q(0)^{-1}, \tag{4a}$$

$$(\Sigma_T^x)^{-1} = P(T)^{-1}+Q(T)^{-1}. \tag{4b}$$

The solutions to these equations can actually be provided in closed form as a function of (Σ_0^x,Σ_T^x), see [10, Section III] for further details. The case in which the initial and final Gaussian distributions have prescribed non-zero means can be handled in a similar way. In fact, the control of mean and covariance can be fully decoupled and therefore can be considered separately. The mean control is a standard optimal control problem see [10, Remark 9] for the details. Therefore, throughout, we assume that the means are 0.

Let $C(0,T;\mathbb{R}^n)$ be the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$, and let P_0 and P_u be the probability measures on $C(0,T;\mathbb{R}^n)$ corresponding to the solutions of (1a) with control 0, and $u\in\mathcal{U}(\Sigma_0^x,\Sigma_T^x)$, respectively. Also let $\pi_0(x_0,x_T)$ and $\pi_u(x_0,x_T)$ be their initial-final joint densities, respectively. In [10, Section IV], a well known decomposition of the relative entropy [22] was extended to the case of degenerate diffusions to show that the Schrödinger bridge problem with marginals densities $\rho_0=\mathcal{N}(0,\Sigma_0^x)$ and $\rho_T=\mathcal{N}(0,\Sigma_T^x)$ can be reduced to the following maximum entropy problem for distributions on a finite-dimensional space:

Problem 2: Minimize over densities π_u on $\mathbb{R}^n\times\mathbb{R}^n$ the Kullback-Leibler index

$$\mathbb{D}(\pi_u\|\pi_0) := \int\int\left[\log\frac{\pi_u(x,y)}{\pi_0(x,y)}\right]\pi_u(x,y)dxdy \tag{5}$$

subject to the (linear) constraints

$$\int \pi_u(x,y)dy = \rho_0(x), \quad \int \pi_u(x,y)dx = \rho_T(y). \tag{6}$$

[3] $u(t)$ only depends on t and on $\{x^u(s); 0\le s\le t\}$ for each $t\in[0,T]$.

Let $\Sigma^u_{0,T}$ be the covariance of $\pi_u(x_0,x_T)$. Since $u \in \mathcal{U}(\Sigma^x_0,\Sigma^x_T)$, $\Sigma^u_{0,T}$ has necessarily the structure

$$\Sigma^u_{0,T} = \begin{bmatrix} \Sigma^x_0 & Y^u \\ (Y^u)' & \Sigma^x_T \end{bmatrix} \tag{7}$$

for some Y^u. Let $S_{0,T}$ instead be the covariance corresponding to $\pi_0(x_0,x_T)$. Then, it has the form

$$S_{0,T} = \begin{bmatrix} \Sigma^x_0 & \Sigma^x_0\Phi(T,0)' \\ \Phi(T,0)\Sigma^x_0 & S_T \end{bmatrix} \tag{8}$$

where

$$S_T = \Phi(T,0)\Sigma^x_0\Phi(T,0)' + \int_0^T \Phi(T,\tau)B(\tau)B(\tau)'\Phi(T,\tau)'d\tau,$$

with $\Phi(t,s)$ denoting the state-transition of $A(\cdot)$ determined by

$$\frac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s), \quad \Phi(t,t) = I.$$

It was shown in [10, Section IV] that the covariance $S_{0,T}$ is non-singular under the assumption that (A, B) is controllable. Thanks to the explicit form of relative entropy (Kullback-Leibler index) for Gaussian distributions [17], Problem 2 can be expressed in terms of covariances as follows:

$$\underset{(Y^u)\in\mathcal{Q}^x}{\operatorname{argmin}} \quad -\log\det\Sigma^u_{0,T} + \operatorname{trace}(S^{-1}_{0,T}\Sigma^u_{0,T}) \tag{9}$$

where $\Sigma^u_{0,T}$ is as in (7) and

$$\mathcal{Q}^x := \left\{Y \in \mathbb{R}^{n\times n} : \Sigma^x_T - Y'(\Sigma^x_0)^{-1}Y > 0\right\},$$

see [10, Section IV] for the details.

III. Problem formulation

We consider the output process in (1c) and assume that the state $x^u(t)$ is fully observable and that the finite-dimensional *Markovian representation* (*stochastic realization*) for y provided by (1a)-(1c) is available. Such a representation, as is well-known, constitutes the starting point of Kalman filtering and much of optimal control theory, and the construction of such a model with minimal state vector dimension has been the subject of intense study [32]. This is also our starting point.

Let us denote by $\mathcal{U}(\Sigma^x_0,\Sigma^y_T)$ the family of adapted control functions such that (1a) has a strong solution on $[0,T]$ and $y(T)$ has distribution $\mathcal{N}(0,\Sigma^y_T)$. We formulate the following *Schrödinger Bridge Problem* with non-Markov prior (1a)-(1c):

Problem 3: Determine

$$u^* := \underset{u\in\mathcal{U}(\Sigma^x_0,\Sigma^y_T)}{\operatorname{argmin}} \; J(u) := \mathbb{E}\left\{\int_0^T u(t)'u(t)\,dt\right\}.$$

Notice that on one side, at $t = 0$, the boundary constraint requires matching the covariance for the state vector (which can be relaxed) while on the other end, at $t = T$, requires matching the covariance of the output

$$\Sigma^y_T = C(T)\Sigma^x_T C(T)'. \tag{10}$$

The value of Σ^x_T is a parameter and there are in general several values for it such that (10) is satisfied[4]. Corresponding to each one of them, there is a feedback control in $\mathcal{U}(\Sigma^x_0,\Sigma^x_T)$ optimally performing the transfer of distributions according to [10]. Thus, the problem may be also viewed as that of determining the one final covariance Σ^x_T, among those compatible with Σ^y_T, whose corresponding optimal control (2) has minimum energy.

Inspired by the reduction of the classical case leading to Problem 2, we proceed in the next section to derive a closed-form solution of Problem 3.

IV. Solution to the non-Markovian steering problem

Note that the matching of the covariance of the output (10) is only required at time $t = T$. For simplicity, below, we suppress the argument and write C instead of $C(T)$. Problem 3 can be rewritten as

$$\begin{aligned} &\underset{u\in\mathcal{U}(\Sigma^x_0,X)}{\operatorname{argmin}} && \mathbb{E}\left\{\int_0^T u(t)'u(t)\,dt\right\} \\ &\text{subject to} && x(0)\sim\mathcal{N}(0,\Sigma^x_0), \quad x(T)\sim\mathcal{N}(0,X), \\ &&& CXC' = \Sigma^y_T, \end{aligned} \tag{11}$$

where $\Sigma^x_0 > 0$, $\Sigma^y_T > 0$ constitute the given data while X is a parameter. We assume that C is full row rank to avoid triviality; otherwise the problem is infeasible. We now proceed to cast Problem (11) as in (9).

Problem 4: Given Σ^x_0, Σ^y_T, and $S = S_{0,T}$ as in (8)

$$\begin{aligned} &\underset{(X,Y)\in\mathcal{S}_+\times\mathbb{R}^{n\times n}}{\operatorname{argmin}} && -\log\det\Sigma + \operatorname{trace}(S^{-1}\Sigma) \\ &\text{subject to} && \Sigma = \begin{bmatrix}\Sigma^x_0 & Y \\ Y' & X\end{bmatrix} > 0, \quad CXC' = \Sigma^y_T, \end{aligned}$$

where $\mathcal{S}_+$ denotes the cone of $n\times n$ symmetric positive definite matrices.

Let now

$$\mathcal{Q} := \left\{(X,Y) \in \mathcal{S}_+\times\mathbb{R}^{n\times n} : X - Y'(\Sigma^x_0)^{-1}Y > 0\right\},$$

and partition

$$S^{-1} = \begin{bmatrix} N & V \\ V' & P \end{bmatrix}$$

conformally to Σ. By applying the Schur complement on Σ, the optimization problem (11) can be recast as

$$\begin{aligned} &\underset{(X,Y)\in\mathcal{Q}}{\operatorname{argmin}} && -\log\det\left(X - Y'\Sigma_0^{-1}Y\right) + \operatorname{trace}(PX + 2V'Y) \\ &\text{subject to} && CXC' = \Sigma^y_T, \end{aligned} \tag{12}$$

Suppressing the argument for simplicity, and *writing* Σ_0 *instead of* Σ^x_0, we consider the Lagrangian for problem (12)

$$\begin{aligned} \mathcal{L}(X,Y,M) = &-\log\det\left(X - Y'\Sigma_0^{-1}Y\right) + \operatorname{trace}(PX) \\ &+ \operatorname{trace}\left[M(CXC' - \Sigma^y_T)\right] + 2\operatorname{trace}(V'Y), \end{aligned}$$

[4]The case where only Σ^y_0 and Σ^y_T are prescribed can be treated in a similar fashion by optimizing also with respect to Σ^x_0.

where $M = M'$ is a Lagrange multiplier. We consider next the *unconstrained* minimization problem

$$\inf_{(X,Y)\in\mathcal{Q}} \mathcal{L}(X,Y,M). \tag{13}$$

We first check the convexity of $\mathcal{L}$ with respect to (X,Y).

Proposition 1: $\mathcal{L}$ is jointly convex in (X,Y) over $\mathcal{Q}$.

Proof: We compute $\delta\mathcal{L}(X,Y,M;\delta X,\delta Y)$, the first variation of $\mathcal{L}$ in direction $(\delta X,\delta Y)$ (since $\mathcal{Q}$ is open, any pair (X,Y) is an interior point in any direction). Applying the chain rule, we get

$$\begin{aligned}
&\delta\mathcal{L}(X,Y,M;\delta X,\delta Y) = \\
&\frac{d}{d\varepsilon}\mathcal{L}(X+\varepsilon\delta X, Y+\varepsilon\delta Y, M)|_{\varepsilon=0} \\
&= -\operatorname{trace}[(X-Y'\Sigma_0^{-1}Y)^{-1}\,\delta\left(X-Y'\Sigma_0^{-1}Y;\delta X,\delta Y\right)] \\
&+ \operatorname{trace}\left[(P+C'MC)\delta X + 2V'\delta Y\right] \\
&= -\operatorname{trace}[(X-Y'\Sigma_0^{-1}Y)^{-1}(\delta X - Y'\Sigma_0^{-1}\delta Y - \delta Y'\Sigma_0^{-1}Y)] \\
&+ \operatorname{trace}\left[(P+C'MC)\delta X + 2V'\delta Y\right].
\end{aligned}$$

To check the convexity, we compute the second variation

$$\delta^2\mathcal{L}(X,Y,M;\delta X,\delta Y) = \frac{d^2}{d\varepsilon^2}\mathcal{L}(X+\varepsilon\delta X, Y+\varepsilon\delta Y, M)|_{\varepsilon=0}.$$

We have

$$\begin{aligned}
&\delta^2\mathcal{L}(X,Y,M;\delta X,\delta Y) = \\
&= \operatorname{trace}\left[\left((X-Y'\Sigma_0^{-1}Y)^{-1}(\delta X - Y'\Sigma_0^{-1}\delta Y - \delta Y'\Sigma_0^{-1}Y)\right)^2\right] \\
&+ 2\operatorname{trace}\left[(X-Y'\Sigma_0^{-1}Y)^{-1}(\delta Y'\Sigma_0^{-1}\delta Y)\right].
\end{aligned}$$

which is clearly non-negative on $\mathcal{Q}$. ■

Thus

$$\delta\mathcal{L}(X,Y,M;\delta X,\delta Y) = 0, \qquad \forall(\delta X,\delta Y)\in\mathcal{S}\times\mathbb{R}^{n\times n}, \tag{14}$$

is a *sufficient* condition for a minimum pair for $\mathcal{L}$ over $\mathcal{Q}$. Condition (14) yields the two equations

$$P + C'MC - (X-Y'\Sigma_0^{-1}Y)^{-1} = 0, \tag{15}$$

$$V + \Sigma_0^{-1}Y(X-Y'\Sigma_0^{-1}Y)^{-1} = 0. \tag{16}$$

To compute the optimal (X,Y), we use these equations in the Lagrangian and then proceed to maximize the resulting (concave) functional with respect to M. Accordingly, the last equation we need is given by

$$\delta\mathcal{L}(X,Y,M;\delta M) = 0,\ \forall\,\delta M\in\mathcal{S} \iff CXC' = \Sigma_T^y. \tag{17}$$

Let $Z := X - Y'\Sigma_0^{-1}Y$ and note that $Z = Z' > 0$. We immediately get $X = Z + Y'\Sigma_0^{-1}Y$ and, from (15)-(16),

$$(15) \iff Z^{-1} = P + C'MC, \tag{18}$$

$$(16) \iff Y = -\Sigma_0 VZ.$$

Therefore, $X = Z + ZV'\Sigma_0 VZ$ and

$$(17) \iff CZC' + CZV'\Sigma_0 VZC' = \Sigma_T^y. \tag{19}$$

At this point we only need to find Z from equations (18), (19). Since we can always find a state space transformation $\hat{x} = U^{-1}x$ such that $\hat{C} = CU = [I\ 0]$, without loss of generality, we can always assume that $C = [I\ 0]$.

Remark 5: One way to construct the transformation matrix U is as follows. Write U as $[U_1\ U_2]$ with U_1 being of the same dimension with C'. It is easy to find a matrix U_1 so that $CU_1 = I$. Clearly U_1 is of full column rank. Next, choose a matrix U_2 of compatible dimension whose columns span the orthogonal complement of the range of U_1. This is standard and leads to $CU_2 = 0$. The resulting matrix U satisfies $CU = [CU_1\ CU_2] = [I\ 0]$ and is nonsingular.

Let

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22}\end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22}\end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}.$$

Equation (19) becomes

$$Z_{11} + \begin{bmatrix} Z_{11} & Z_{12}\end{bmatrix} \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}}_{V'\Sigma_0 V > 0} \begin{bmatrix} Z_{11} \\ Z_{12}\end{bmatrix} = \Sigma_T^y, \tag{20}$$

while equation (18) can be equivalently written as

$$\begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix} = \begin{bmatrix} Z_{11}(M+P_{11}) + Z_{12}P_{12} & Z_{11}P_{12} + Z_{12}P_{22} \\ Z_{21}(M+P_{11}) + Z_{22}P_{21} & Z_{21}P_{12} + Z_{22}P_{22}\end{bmatrix} \tag{21}$$

which reduces to the system of equations

$$\begin{cases} Z_{11}P_{12} + Z_{12}P_{22} = 0 \\ Z_{21}P_{12} + Z_{22}P_{22} = I \\ Z_{12} = Z_{21}' \end{cases} \iff \begin{cases} Z_{21} = -P_{22}^{-1}P_{21}Z_{11} \\ Z_{12} = Z_{21}' \\ Z_{22} = P_{22}^{-1} - Z_{21}P_{12}P_{22}^{-1} \end{cases} \tag{22}$$

Plugging Z_{12}, Z_{21} and Z_{22} into (20), we get

$$Z_{11} + Z_{11}WZ_{11} = \Sigma_T^y, \tag{23}$$

where

$$W := \begin{bmatrix} I & -P_{12}P_{22}^{-1}\end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix} \begin{bmatrix} I \\ -P_{22}^{-1}P_{21}\end{bmatrix} > 0.$$

Equation (23) is a quadratic equation with two solutions

$$Z_{11}^{\pm} = W^{-\frac{1}{2}}\left[\pm\left(\frac{1}{4}I + W^{\frac{1}{2}}\Sigma_T^y W^{\frac{1}{2}}\right)^{\frac{1}{2}} - \frac{1}{2}I\right]W^{-\frac{1}{2}}, \tag{24}$$

where $F^{1/2}$ denotes the unique positive symmetric square root of the positive definite symmetric matrix F. Clearly, $Z = X - Y'\Sigma_0^{-1}Y > 0$ by Schur complement, which implies $Z_{11} > 0$. This singles out the solution Z_{11}^{+}. We can now recover Z from (22) and then $X = Z + ZV'\Sigma_0 VZ$ and $Y = -\Sigma_0 VZ$. Finally, from (21), one can find the multiplier M:

$$M = (Z_{11}^{+})^{-1} - P_{11} - (Z_{11}^{+})^{-1}Z_{12}P_{12}.$$

The above results can be summarized as follows.

Theorem 6: Assume a coordinate transform has been applied so that $C = [I\ 0]$. Let Z_{11}^{+} be as in (24) and Z,X,Y be derived accordingly. Then (X,Y) solves Problem 4. Furthermore, the solution to Problem 3 coincides with the solution to Problem 1 with $\Sigma_T^x = X$.

Remark 7: When the input and noise enter the system through different channels, that is, the noise term in (1a) is $B_1 dw(t)$, Problem 3 doesn't have a nice closed-form solution. Nevertheless, the problem allows for an optimization formulation

$$\begin{aligned} &\min_{K(\cdot)} \int_0^T \operatorname{trace}(K(t)\Sigma(t)K(t)')dt \\ &\dot{\Sigma}(t) = (A - BK(t))\Sigma(t) + \Sigma(t)(A - BK(t))' + B_1 B_1' \\ &\Sigma(0) = \Sigma_0^x, \quad C\Sigma(T)C' = \Sigma_T^y, \end{aligned}$$

where the controller is restricted to linear state feedback $u(t) = -K(t)x(t)$. With a standard change of variable $U(t) = K(t)\Sigma(t)$, it can be transformed into a convex optimization problem. See [11] for more details on feasibility and computation issues.

V. Example

Controlled stochastic oscillators play an important role in Atomic Force Microscopy, molecular dynamics and cooling of nano-to-meter scale resonators, see e.g. [6], [7], [29], [21], [12]. A standard model for these applications is the Ornstein-Uhlenbeck process. Consider controlling the Ornstein-Uhlenbeck model

$$\begin{aligned} dq^u(t) &= p^u(t)dt \\ dp^u(t) &= -\beta\, p^u(t)dt - Kq^u(t)dt + u(t)dt + dw(t) \end{aligned} \tag{25}$$

corresponding to a given quadratic potential $V(q) = \frac{1}{2}q'Kq$ with K symmetric, positive-definite, and $u(\cdot)$ the control force. By setting

$$x = \begin{pmatrix} q \\ p \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -K & -\beta I \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I \end{pmatrix},$$

model (25) becomes

$$\begin{aligned} dx^u(t) &= Ax^u(t)dt + Bu(t)dt + Bdw(t) \\ x^u(0) &= \xi \text{ a.s.} \end{aligned}$$

where ξ is zero-mean Gaussian with $\Sigma_0^x = I/2$, and the pair (A,B) is controllable. We consider a state dimension of $n = 2$ and we assume, to simplify the exposition, that the units are such that $K = I$ and $\beta = 1$. We would like to steer the Gaussian distribution of the momentum to a final distribution at time $T = 1$ with $\Sigma_1^p = 1/16$ minimizing the quadratic control energy under the controlled dynamics (25). In other words, we are prescribing only the final covariance matrix of $y(t) = Cx(t)$ with $C = [0\ I]$. Figure 1 shows the trajectories of the state variables in phase space (left) and the corresponding control efforts (right). Figure 2 highlights instead the trajectories of position (left) and momentum (right) with the corresponding confidence interval. In all figures, the transparent blue tube represent the "3σ" confidence interval, whose intersection with the plane $\{(\tau,q,p) \in \mathbb{R}^3 : \tau = t\}$ is given by

$$\left\{ (q,p) \in \mathbb{R}^2 \,\middle|\, \begin{bmatrix} q & p \end{bmatrix} \Sigma_t^{-1} \begin{bmatrix} q \\ p \end{bmatrix} \leq 3^2 \right\}.$$

The figures highlight the reduction of the variance of the momentum process as time increases to $T = 1$.

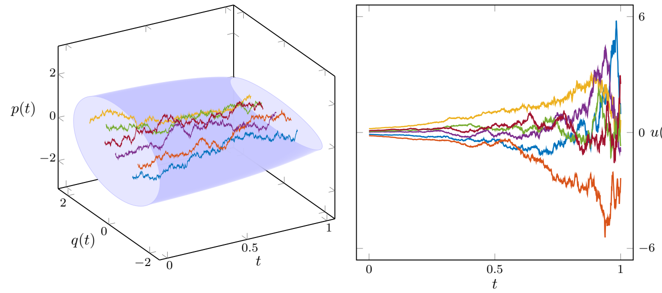

Fig. 1: Steering the momentum distribution. Trajectories in phase space (left) and relative control efforts (right).

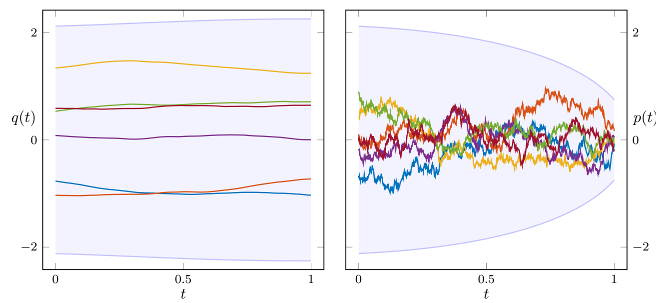

Fig. 2: Steering the momentum distribution. Position's trajectories (left) and momentum's trajectories (right).

VI. Conclusion and Outlook

In this paper, we have considered optimal steering for the distribution of the output of a linear stochastic system whose state is fully observable. The problem has been transformed into a maximum entropy problem whose solution has been provided in closed-form. Using the results of [10], we then obtained a closed-form expression for the minimum-energy control that steers the output from the initial to the final prescribed distributions. This work should be seen as a first step towards the more challenging steering problem of general Gaussian non-Markov processes. A research direction of great interest consists of the steering problem where only partial observations of the state corrupted by noise are available. To this end, we expect that the results of this paper, together with those in [9], will help accomplishing the task; we note that this latter paper considers steering the state under partial observation while allowing for different control and noise channels.

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