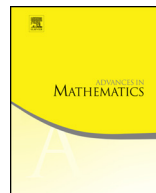




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Min-max theory for networks of constant geodesic curvature

Xin Zhou^{a,b}, Jonathan J. Zhu^{c,*}^a Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA 93106, USA^b School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA^c Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

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ABSTRACT

We prove that on a closed surface, for any $c > 0$, our min-max theory for prescribing mean curvature produces a solution given by a curve of constant geodesic curvature c which is almost embedded, except for finitely many points, at which the solution is a stationary junction with integer density. Moreover, each smooth segment has multiplicity one. The key is a classification of blowups which is new even for $c = 0$.

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1. Introduction

The min-max construction of closed geodesics dates back to Birkhoff [6], and tremendous progress has been made since then (see [11] for a nice summary). The min-max construction of closed curves (or networks) of (nonzero) constant geodesic curvature, however, has not been thoroughly investigated. In particular, it has been conjectured by

* Corresponding author.

E-mail addresses: zhou@math.ucsb.edu (X. Zhou), jjzhu@math.princeton.edu (J.J. Zhu).

Arnold [5, p. 395] and Novikov [5, Section 5] that every topological two sphere admits closed embedded curves of any prescribed constant geodesic curvature. This conjecture remains open, and we refer to [9,14,15] for more background and some partial results towards this conjecture.

The goal of this article is to show that on a closed surface, for any $c > 0$, our CMC min-max theory [19,18] (which is based on the Almgren-Pitts min-max theory for minimal hypersurfaces [3,13]) produces a solution of the same regularity as the surfaces in [19], except for finitely many points, at which the solution is a stationary junction:

Theorem 1.1. *In any closed Riemann surface (S, g) , given $c > 0$, there is a curve γ in S which is almost embedded with constant geodesic curvature c , except at finitely many stationary junctions of integer density.*

Moreover, each smooth constant geodesic curvature segment appears with multiplicity one.

See Theorem 2.1 for a precise statement.

The key is a graph theoretic argument (Section 4) to classify blowups which have a number of iterated replacements in open disks. In particular, we prove that such blowups are integer multiple of a line:

Proposition 1.2. *In any neighborhood where the c -min-max curve γ has replacements, any tangent cone of γ must be an integer multiple of a straight line.*

See Corollary 3.5 and Theorem 4.6 for details.

This classification result (for blowups) is new even for the case of geodesics, that is $c = 0$. The existence of a nontrivial geodesic network was known by Pitts [12] (based upon earlier works of Almgren [4]; note that Pitts' result also holds true in higher codimension). In fact, in [12] Pitts proved that the 1-dimensional min-max varifold is always supported in the image of its tangent varifold under the exponential map at any given point. Consequently, the min-max varifold is represented by a geodesic network. Note that Pitts's result does not preclude the tangent varifold being a bouquet of half lines (even if the min-max varifold is almost minimizing near that given point).

The existence of a geodesic network has another proof by combining Pitts [13] with Allard-Almgren [2]. Pitts [13] (based upon earlier works of Almgren [4]) proved the existence of a weak min-max solution as a nontrivial, stationary, integer rectifiable, 1-varifold in any closed manifold. The regularity theory of Allard-Almgren [2] for stationary 1-varifolds then implies that Pitts's weak solution is a geodesic networks (with constant integer multiplicity on each geodesic segment). We also refer to Calabi-Cao [7, Appendix] and Aiex [1] for other proofs of this result.

However, even on surfaces, one can not follow Pitts's regularity argument in [13] (which succeeds for hypersurfaces of dimension between 2 and 6) to prove this network regularity without Allard-Almgren [2]. The main missing ingredient for curves is that

Simons' classification for minimal stable hypercones [17] does not hold for curves. In particular, Pitts's argument [13, 7.8] cannot be extended to prove that tangent cones are linear without Simons' classification.

Nevertheless, using our new characterization for blowups, Pitts' work [13] does directly imply the geodesic network regularity of his weak solution. In fact, away from finitely many points, Pitts' weak solution has the good replacement property in small balls, so any tangent varifold satisfies the assumptions of our classification result (using an observation in [19, Lemma 5.10]), and hence is an integer multiple of a line. With this, one can proceed the same as Pitts to obtain the desired regularity.

In this paper, we carry out the process described above in the setting for $c > 0$, using the theory we developed in [19, 18].

In Section 2, we introduce the problem and state the main result. In Section 3, we collect necessary results in our previous CMC min-max theory and prove the main theorem. In Section 4, we prove the key ingredient on classifying blowups.

Remark 1.3. In [10], the authors have built upon our results, improving the regularity to show that the networks produced are either smooth or $C^{1,1}$ curves. In particular, they proved that if the (unique) junction is not a smooth point, then the tangent cone consists of two lines intersecting transversally.

2. Min-max construction for weighted length functional

In this part, we will briefly introduce the setup for the min-max construction of constant geodesic networks. We refer to [19] for more details.

Let (S, g) be a closed 2-dimensional surface with a Riemannian metric g . Fix a positive number $c > 0$. Given any Caccioppoli set $\Omega \subset S$, we define the c -weighted length functional or c -length as

$$\mathcal{L}^c(\Omega) = \text{Length}(\partial\Omega) - c \text{Area}(\Omega), \quad (2.1)$$

where Length and Area are calculated with respect to the metric g .

1-parameter families of Caccioppoli sets $\{\Omega_t\}_{t \in [0,1]}$ is said to be a *sweepout*, if

- $\Omega_0 = \emptyset$, $\Omega_1 = \Sigma$;
- the boundaries $\{\partial\Omega_t\}$ are continuous in t with respect to the flat topology.

We can then define the min-max value of $\mathcal{L}^c$ as

$$\mathbf{L}^c = \inf \left\{ \max_{t \in [0,1]} \mathcal{L}^c(\partial\Omega_t) : \{\Omega_t\}_{t \in [0,1]} \text{ is a sweepout} \right\}. \quad (2.2)$$

In this paper we will prove that

Theorem 2.1. *There exists a nontrivial 1-varifold V , finitely many points $\{p_i\}_{i=1}^n \subset S$, and a Caccioppoli set Ω , such that*

- (1) V is induced by $\partial\Omega$ (of multiplicity 1);
- (2) away from $\{p_i\}_{i=1}^n$, the boundary $\gamma = \partial\Omega$ is an almost embedded curve of constant geodesic curvature c ;
- (3) at each p_i , the density of V is an integer, and any tangent cone is a stationary geodesic network in $\mathbb{R}^2$, smooth away from 0.

Here ‘almost embedded’ means that γ is a smooth immersion, and near each self-intersection point γ decomposes to two connected embedded components which touch but do not cross.

Remark 2.2. In fact, by refining Pitts’s combinatorial argument [13, 4.10] with the observation of Colding-De Lellis (the remark after [8, Proposition 3.3]), one can show that the set $\{p_i\}_{i=1}^n$ consists of only one point. See also Remark 3.3.

3. Results from [19] and proof of Theorem 2.1

In [19,18] the authors established an existence theory, which in this setting yields that there is a certain 1-varifold V associated with $\mathbf{L}^c$. This V satisfies a list of useful properties that we will summarize in the following. In particular, the theory in [19, 18] works in any closed Riemannian manifold (M^{n+1}, g) (using the corresponding n -dimensional c -weighted area functional), and when $3 \leq n+1 \leq 7$, we proved that V is induced by the boundary of some Caccioppoli set Ω_0 , whose boundary $\Sigma_0 = \partial\Omega$ is an almost embedded closed hypersurface of constant mean curvature c .

However, since the classification of stable minimal hypercones by Simons [17] does not hold in dimension $n = 1$, we cannot directly obtain similar regularity results for V when $n = 1$. Instead, we will exploit some stronger properties of V that were obtained in [19] to achieve some partial regularity. In fact, we will use certain good replacement properties in small disks instead of just in small annuli.

Note that we used a discrete setup in [19,18] following the classical work of Almgren-Pitts [3,13]. We will not dip into these sophisticated notations, as we can start directly with the outcomes in [19].

Before summarizing what we proved in [19], we need to introduce the notion of c -replacements. A 1-varifold V is said to have c -bounded first variation, if for any smooth vector field X on S ,

$$\left| \int \operatorname{div}_S X(x) dV(x, S) \right| \leq c \cdot \int_S |X(x)| d\|V\|(x).$$

Definition 3.1. Given a 1-varifold V with c -bounded first variation and an open set U in S , V^* is said to be a c -replacement of V in U if

- (1) V coincides with V^* outside the closure $\overline{U}$, i.e., $V \llcorner Gr_1(S \setminus \overline{U}) = V^* \llcorner Gr_1(S \setminus \overline{U})$ ¹;
- (2) $\|V\|(S) - c \cdot \text{Area}(U) \leq \|V^*\|(S) \leq \|V\|(S) + c \cdot \text{Area}(U)$;
- (3) V^* , when restricted to U , is induced by the boundary of some open subset $\Omega^* \cap U$ (here Ω^* is an Caccioppoli set), that is, $V^* \llcorner Gr_1(U) = [\partial\Omega^* \cap U]$, such that $\partial\Omega^* \cap U$ is an almost embedded curve of constant geodesic curvature c ;
- (4) V^* has c -bounded first variation.

We proved in [19] that V has certain good replacement properties:

Theorem 3.2. [Theorem 5.6, Proposition 5.8, Lemma 5.9 in [19]] Given $c > 0$, let $\mathbf{L}^c$ be defined as (2.2), then there exists a 1-varifold V in (S, g) , such that

- (1) $\mathbf{L}^c > 0$ and hence V is nontrivial;
- (2) V has c -bounded first variation;
- (3) for any $p \in S$, V has a c -replacement V^* in any small enough annulus centered at p ; hence by a covering argument, there exists a finite set $\mathcal{P} = \{p_i\}_{i=1}^n$, so that for any $p \in S \setminus \mathcal{P}$, there exists a neighborhood $U \subset S \setminus \mathcal{P}$ of p , such that V has a c -replacement V^* in U ;
- (4) in any neighborhood U where V has a c -replacement, V^* also has a c -replacement V^{**} in U ; and this procedure of taking c -replacements can be iterated as many times as one wants.

Remark 3.3. In [19, Theorem 5.6, Proposition 5.8, Lemma 5.9], we proved that V is c -almost minimizing in any small annulus and hence has a c -replacement. As mentioned earlier, by a remark of Colding-De Lellis after [8, Proposition 3.3], one can prove that V is c -almost minimizing in any small open neighborhood, except about one point. (In particular one can reduce the set $\mathcal{P}$ to consist of a single point.)

To gain regularity of V^* in U , we used curvature estimates for stable hypersurfaces of constant mean curvature in [19, Theorem 2.6], but this is trivially true in dimension $n = 1$ for curves of constant geodesic curvature.

As a key step to obtain our main regularity results in [19], we analyzed the blowups of V by proving their good replacement properties. Together with the above remark, it follows that:

Proposition 3.4. Let V be as in Theorem 3.2. Given any $p \in S \setminus \{p_i\}_{i=1}^n$, and a tangent varifold $C \in \text{TanVar}(V, p)$ of V at p , then C satisfies,

¹ Here $Gr_1(U)$ is the Grassmannian bundle of 1-lines over U .

- (1) C is a stationary 1-varifold in $\mathbb{R}^2$;
- (2) given any open set $U \subset \mathbb{R}^2$, C has a 0-replacement C^* ;
- (3) C^* has 0-replacement in any open set $W \subset \mathbb{R}^2$, and this procedure of taking 0-replacements can be iterated as many times as one wants;
- (4) C is integral, that is, C is an integer multiplicity rectifiable varifold.

Proof. Properties (1)-(3) follow directly from [19, Lemma 5.10]. Note that in [19, Lemma 5.10], we only presented the proof that the tangent varifold C has replacements in any small annuli, as this was sufficient for the regularity argument in that setting. In fact, the same proof also shows that the tangent varifold C (at any point $p \notin \{p_i\}_{i=1}^n$) has replacements in any open set.

In particular, the good replacement property of C follows from that of V . To be more precise, denote $C = \lim_{i \rightarrow \infty} (\eta_{p, r_i})_{\#} V$, where $r_i > 0$, $\lim_{i \rightarrow \infty} r_i = 0$, and $\eta_{p, r_i} : x \rightarrow \frac{x-p}{r_i}$ is the rescaling map of S (by embedding S into some Euclidean space $\mathbb{R}^N$). As in [19, Lemma 5.10], to construct a replacement of C in any open set $U \subset \mathbb{R}^2$, we can first take the c -replacement V_i of V in $\eta_{p, r_i}^{-1}(U)$ (which will be a proper subset of the set where V has c -replacement for i large enough). Then the limit of rescaling $C^* = \lim_{i \rightarrow \infty} (\eta_{p, r_i})_{\#} V_i$ will be the replacement of C in U by the arguments in [19, Lemma 5.10].

Property (4) follows (1)-(3), and we provide details as follows for completeness. Using the existence of 0-replacements and the same arguments as [13, 3.13],² one can show that C has strictly positive density everywhere, and hence is rectifiable; (see [16, Theorem 42.4]). Together with (1), we know that C is rectifiable cone by [16, Theorem 19.3]. Therefore C , as a rectifiable cone, is a unit of rays starting from the origin with positive constant multiplicity along each ray. Finally we argue that the multiplicities must be integers. Indeed, given each $p \in \text{support}(V) \cap S^1$, we can take a 0-replacement C^* over the ball $B(p, \frac{1}{2})$. By the definition of 0-replacement, C^* is induced by an integer multiple of line segments in $B(p, \frac{1}{2})$, and also C^* and C have the same length in $B(p, \frac{1}{2})$. Therefore C must also have integer multiplicity inside $B(p, \frac{1}{2})$. This concludes the proof. $\square$

As a direct corollary of Theorem 4.6 in Section 4, we have,

Corollary 3.5. *Any C in Proposition 3.4 is an integer multiple of a line passing the origin.*

Now we are ready to sketch the proof of Theorem 2.1. Using Corollary 3.5 in place of [19, Proposition 5.11], the regularity of V away from $\{p_i\}_{i=1}^n$ follows from that of [19, Theorem 6.1] with minor modifications. The structure of tangent cones of V at $\{p_i\}_{i=1}^n$ follows from a classical argument of characterizing tangent cones of min-max varifold by Almgren-Pitts [13, 3.13]. We will mainly focus on the differences with the proof of [19, Theorem 6.1].

² Indeed, [13, 3.13] assumes the “almost minimizing” property, but the proof only uses the existence of 0-replacements, which is a corollary of “almost minimizing” property.

Proof of Theorem 2.1. We will prove parts (1)(2)(3) in three steps.

Step 1: We first focus on a neighborhood of a point $p \in \text{spt}\|V\| \setminus \mathcal{P}$, where the set $\mathcal{P} = \{p_i\}_{i=1}^n$ is given in Theorem 3.2. Take a small enough radius $r > 0$, such that V has c -replacements in the geodesic ball $B_r(p) \subset S$. Fix any $0 < s < r$, and take a c -replacement V^* in the annulus $A_{s,r}(p) = B_r(p) \setminus \overline{B_s(p)}$. By the definition of c -replacement, $V^* \llcorner A_{s,r}(p)$ is induced by the boundary of some Caccioppoli set Ω^* , and is an almost embedded curve, denoted by γ_1 , of constant geodesic curvature c .

Take a radius $s < t < r$, such that the sphere $\partial B_t(p)$ intersects γ_1 transversally,³ and intersects along the regular (non-touching) set of γ_1 .⁴ Now take a c -replacement V^{**} of V^* in $B_t(p)$ (usually called the second replacement). Again $V^{**} \llcorner B_t(p)$ is given by an almost embedded curve γ_2 of constant geodesic curvature c . Using Corollary 3.5 in place of [19, Proposition 5.11], we can follow the same procedure as in [19, Theorem 6.1, Steps 1 and 2] to show that $\gamma_1 = \gamma_2$ in the overlapping region $A_{s,t}(p)$, and hence they form an almost embedded curve γ in $B_r(p)$.

The next step is to use c -replacements in annuli $A_{\tau,t}(p)$, where $0 < \tau < s$. Let V_τ^{**} be the c -replacement of V^* in $A_{\tau,t}(p)$, which is induced by an almost embedded curve γ_τ . By the same reasoning, we have $\gamma_\tau = \gamma_1$ in $A_{s,t}(p)$, and hence by ODE uniqueness theory, $\gamma_\tau = \gamma \cap A_{\tau,t}(p)$.

Then by the moving sphere argument [19, Theorem 6.1, Step 5], we can show that V is induced by γ inside $B_s(p)$. This finishes the proof of the regularity of V away from $\{p_i\}_{i=1}^n$ (part (2) in Theorem 2.1).

Step 2: By the same argument as [18, Proposition 7.3], V is induced by the boundary of some Caccioppoli set Ω , and

$$\mathcal{L}^c(\Omega) = \mathbf{L}^c.$$

Remark 3.6. Note that in [19], the main goal was to produce a nontrivial CMC hypersurface with prescribed mean curvature, so we did not investigate the relation between the min-max limit varifold V and Caccioppoli sets. This part was delayed until [18, Proposition 7.3], where we proved that V is induced by the boundary of some Caccioppoli set Ω , and the $\mathcal{L}^c$ value of Ω is the assigned c -min-max value, i.e. $\mathcal{L}^c(\Omega) = \mathbf{L}^c$.

This completes part (1) of Theorem 2.1.

Step 3: Finally we prove the structure of tangent cones $\text{TanVar}(V, p_i)$ at each p_i , i.e. part (3) in Theorem 2.1. Given a tangent cone C at p_i , we know that C is stationary and integer rectifiable, since V has c -bounded first variation and is integer rectifiable. Now by smooth convergence, since V consists of constant curvature curves, C must be a geodesic network with constant integer multiplicity in each segment. Since C is a cone, $\text{spt}\|C\|$ must be a finite union of half lines coming out of the origin. The only thing left

³ The existence of such t follows from Sard's Theorem.

⁴ The touching set of γ_1 is a discrete set.

to prove is to show that the sum of all integer multiplicities must be an even number, hence the density of C at the origin - which is the same as that of V at p_i - is an integer.

Write

$$C = \lim_{j \rightarrow \infty} (\tau_{r_j, p_i})_{\#} V, \text{ as varifolds.}$$

Here $\{r_j\}$ is a sequence of positive numbers converging to 0, and $\tau_{r_j, p_i}(x) = \frac{x - p_i}{r_j}$ are the rescaling maps.⁵

Note that $V = \partial\Omega$, and consider the limit

$$\Omega' = \lim_{j \rightarrow \infty} (\tau_{r_j, p_i})_{\#} \Omega, \text{ as Caccioppoli sets.}$$

By the weak convergence, the $\text{spt } \|\partial\Omega'\| \subset \text{spt } \|C\|$, and it is easy to see that away from the origin the multiplicity of C minus the multiplicity of $\partial\Omega'$ (which is identical to 1) must be an even number. On the other hand, one can see that there must be even numbers of half lines in $\text{spt } \|\partial\Omega'\|$ (to form the boundary of a set). Summing all ingredients together, we have proven that the number of half lines of C (counting multiplicity) is even. $\square$

4. Combinatorial argument

In this part, we change gear to study geodesic networks arising in Proposition 3.4. Our main goal is to prove Theorem 4.6.

We define a stationary network V (embedded) in $\mathbb{R}^2$ to be a network whose edges vw are straight line segments with positive integer weight (multiplicity) m_{vw} , and which satisfies at each vertex v of V the stationarity condition

$$\sum_{vw \in V} m_{vw} \vec{T}_{vw} = 0. \quad (4.1)$$

Here $\vec{T}_{vw}$ is the outward unit tangent from v along the edge vw .

By a slight abuse of notation we henceforth consider stationary networks V consisting of N vertices lying on the unit circle $S^1 \subset \mathbb{R}^2$, each with an exterior radial edge to infinity; and E edges interior to the circle. In what follows let $\mathring{V}$ be the interior graph of V (consisting of those edges inside the circle); any graph theoretic concepts (degree, neighborhood, etc.) are with respect to $\mathring{V}$.

We say that such a stationary network V is admissible if it satisfies:

⁵ Here we can isometrically embed (S, g) into some Euclidean space $\mathbb{R}^L$, and the calculation $\frac{x - p_i}{r_j}$ is done in $\mathbb{R}^L$.

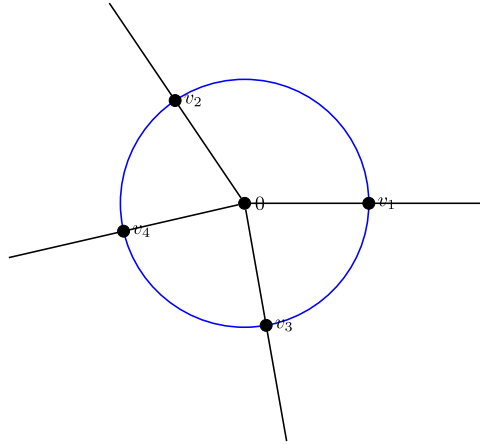


Fig. 1. A stationary junction with vertex at the origin, and intersection points v_i with the unit circle marked; models a tangent cone C .

- (1) At each vertex v , we have

$$m_v \vec{T}_v + \sum_{w \in N_v} m_{vw} \vec{T}_{vw} = 0,$$

where N_v is the set of vertices adjacent to v . Note that, treating v, w as points in $S^1 \subset \mathbb{R}^2$, we have $\vec{T}_{vw} = \frac{w-v}{|w-v|}$, and the exterior edges have unit tangent $\vec{T}_v = v$ and weight m_v . (This is just a restatement of stationarity, clarifying the notation for the exterior edges.)

- (2) There are no crossings between interior edges.

We say that V is a replaceable network if it additionally satisfies the replacement property:

- (3) At each vertex v in V , there is a replacement V'_v ; that is, there exists an admissible network V'_v whose vertex set is $\{v\} \cup \{\vec{T}_{vw} | w \in N_v\} \subset S^1$, and whose corresponding exterior edges have multiplicity m_v , resp. m_{vw} .

Finally, we say that V is a *good network* if it is a replaceable network, each replacement V' of V is also replaceable, and so forth, so that V has arbitrarily many iterated replacements. In fact we will only use four iterated replacements - two to rule out $N = 3$, another to rule out $N = 4$ and the fourth to rule out $N \geq 5$.

The goal of this section will be to show that the only good network is the straight line network. Before proving this, however, we briefly discuss the correspondence between the networks considered here and the blowups in Section 3. See Figs. 1, 2, 3 and 4 for examples of the replacement process.

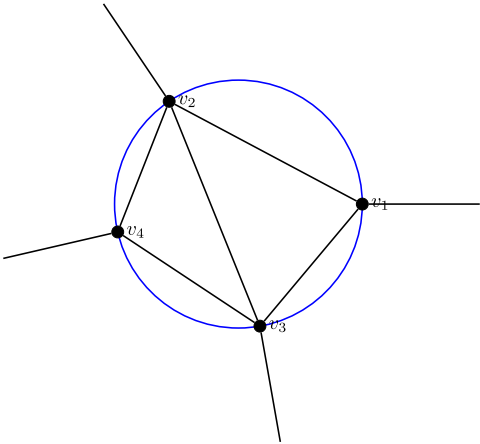


Fig. 2. An admissible network, models (for instance) a replacement C^* in the unit ball.

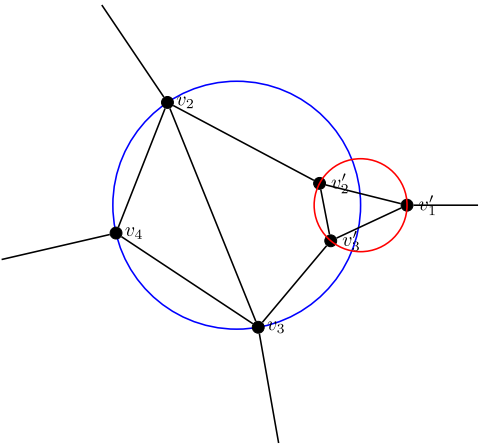


Fig. 3. An iterated replacement C^{**} inside a small (red) circle about v_1 .

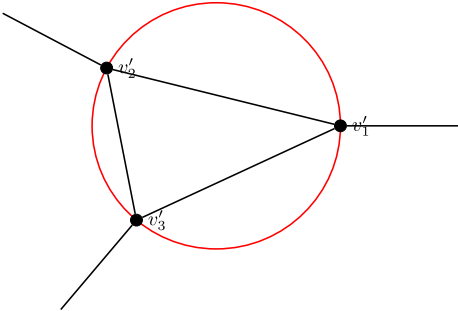


Fig. 4. The resulting (admissible) replacement network about v_1 ; a local model for C^{**} near v_1 .

4.1. Relation to min-max tangent cones

By Proposition 3.4, the tangent cone C to the min-max varifold is an integral stationary 1-varifold in $\mathbb{R}^2$. So (for instance by [2]) it is, up to translation, a network consisting of straight rays meeting at the origin, with integer multiplicities satisfying the stationarity condition (4.1).

Any iterated replacements applied to C are then also stationary, and hence consist of straight-line networks satisfying the stationarity condition at each vertex. Taking a replacement C^* in B_1 gives a network which must be regular - that is, consist of straight-line segments without crossings - except possibly on the gluing interface ∂B_1 . So in particular, admissible networks model the first replacement C^* .

The replacement networks defined above are the result of considering iterated replacements of C^* in small balls about each vertex, and then cutting and rescaling so that the replacement ball coincides with B_1 . In this way, the replacement networks are the local models for iterated 0-replacements of C^* (in the sense of Proposition 3.4), and to prove Corollary 3.5 it indeed suffices to show that any good network is a straight line.

4.2. Bounding the number of interior edges

Lemma 4.1. *Let $f(N)$ be the maximum number of straight line segments that can be drawn between N distinct points on the unit circle, which do not connect adjacent vertices, and do not have any crossings. Then $f(N) = \max(N - 3, 0)$.*

Consequently, the total number of interior edges in an admissible network is bounded by

$$E \leq F(N) := \begin{cases} 2N - 3 & , N \geq 3 \\ \max(N - 1, 0) & , N \leq 2 \end{cases}.$$

Proof. Any such edge divides the remaining vertices into a set of k vertices and a set of l vertices, $k, l \geq 1$. Then we have the recursive formula

$$f(N) = \max\{1 + f(k + 2) + f(l + 2) \mid k + l = N - 2, k, l \geq 1\}.$$

It is clear that $f(1) = f(2) = f(3) = 0$. A straightforward induction then shows that $f(N) = N - 3$ for all $N \geq 3$. $\square$

Note that if V is an admissible network and any vertex v has degree 1 in $\mathring{V}$, then the interior edge must be the diameter through v . Since no interior edges may cross, this implies that at most two vertices can have degree 1 (v and its antipode). Indeed, we have

Lemma 4.2. *Let V be an admissible network and a vertex v of (interior) degree 1. Then the number of interior edges is bounded by*

$$E \leq F_1(N) = \begin{cases} 2N - 5 & , N \geq 4 \\ 1 & , N \leq 3 \end{cases}.$$

Proof. As above, the interior edge from v must be a diameter of the circle and its antipode w must be a vertex in V . The diameter vw splits the remaining vertices into two sets of k and l vertices, where $k + l = N - 2$ and without loss of generality $0 \leq k \leq l$. If w also has degree 1, then $E \leq 1 + F(k) + F(l)$.

Otherwise, w has a second incident edge with positive weight, so to satisfy stationarity it must be connected by a third edge to the other side of vw . In particular we must have $k, l \geq 1$, and $E \leq 1 + F(k + 1) + F(l + 1)$.

Thus we have three cases: $k = 0$, in which case w must have degree 1 and

$$E \leq 1 + F(N - 2);$$

$k = 1$, in which case $N \geq 4$ and

$$E \leq 1 + F(2) + F(N - 2) = 2 + F(N - 2);$$

finally $2 \leq k \leq l$ in which case

$$E \leq 1 + F(k + 1) + F(l + 1) = 2N - 5.$$

The result follows by the cases for $F(N - 2)$. $\square$

4.3. The case $N = 3$

Let V be an admissible network with vertices $v_j = e^{i\theta_j}$.

Set $\alpha_{jk} = \theta_k - \theta_i$. Note that

$$\vec{T}_{jk} = \frac{e^{i\theta_k} - e^{i\theta_j}}{|e^{i\theta_k} - e^{i\theta_j}|} = e^{i\theta_j} \frac{e^{i\alpha_{jk}} - 1}{|e^{i\alpha_{jk}} - 1|}.$$

Also note that for $\theta \in [0, 2\pi]$, we have

$$\begin{aligned} e^{i\theta} - 1 &= 2 \sin \frac{\theta}{2} i e^{i\theta/2} = 2 \sin \frac{\theta}{2} e^{i\frac{\theta+\pi}{2}}, \\ e^{-i\theta} - 1 &= -2 \sin \frac{\theta}{2} i e^{-i\theta/2}. \end{aligned}$$

Then stationarity at each vertex v_j gives (after dividing through by $e^{i\theta_j}$ respectively)

$$m_1 + m_{12} i e^{i\alpha_{12}/2} + m_{13} i e^{i\alpha_{13}/2} = 0, \quad (4.2)$$

$$m_2 - m_{12} i e^{-i\alpha_{12}/2} + m_{23} i e^{i\alpha_{23}/2} = 0, \quad (4.3)$$

$$m_3 - m_{13}ie^{-i\alpha_{13}/2} - m_{23}ie^{-i\alpha_{23}/2} = 0. \quad (4.4)$$

Note that all vertices must have degree 2 (that is, $m_{jk} > 0$). (Otherwise, exactly one vertex has degree 1, but then $2e = \sum \deg(v) = 5$ which is impossible.)

For each vertex v_j , it is geometrically clear that the other two vertices cannot lie on the same side of the diameter through v_j , or else it would be impossible to satisfy the stationarity. Therefore $\alpha_{12} \in (0, \pi)$, $\alpha_{23} \in (0, \pi)$, $\alpha_{13} \in (\pi, 2\pi)$. (In particular $\alpha_{12} \neq \pi$, since then the only way to satisfy stationarity at v_1 would be $m_{13} = 0$, which cannot happen; and similarly $\alpha_{23}, \alpha_{13} \neq \pi$.)

Set $s_{jk} = \sin \frac{\alpha_{jk}}{2}$ and $c_{jk} = \cos \frac{\alpha_{jk}}{2}$. Note $s_{jk}, c_{12}, c_{23}, -c_{13} \in (0, 1)$.

We may rewrite the real part of the system above as

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} + \begin{pmatrix} -s_{12} & -s_{13} & 0 \\ -s_{12} & 0 & -s_{23} \\ 0 & -s_{13} & -s_{23} \end{pmatrix} \begin{pmatrix} m_{12} \\ m_{13} \\ m_{23} \end{pmatrix} = 0 \quad (4.5)$$

and the imaginary part as

$$\begin{pmatrix} c_{12} & c_{13} & 0 \\ -c_{12} & 0 & c_{23} \\ 0 & -c_{13} & -c_{23} \end{pmatrix} \begin{pmatrix} m_{12} \\ m_{13} \\ m_{23} \end{pmatrix} = 0. \quad (4.6)$$

Since the c_{ij} are nonzero, the matrix $C = \begin{pmatrix} c_{12} & c_{13} & 0 \\ -c_{12} & 0 & c_{23} \\ 0 & -c_{13} & -c_{23} \end{pmatrix}$ has rank 2, nullity 1 and one can verify that the kernel is spanned by $\begin{pmatrix} c_{13}c_{23} \\ -c_{12}c_{23} \\ c_{12}c_{13} \end{pmatrix}$.

Lemma 4.3. *Let V be an admissible network with $N = 3$. Then $e^{i\alpha_{jk}}$ are rational points on the unit circle.*

Proof. By the characterization of the kernel, we have $\begin{pmatrix} m_{12} \\ m_{13} \\ m_{23} \end{pmatrix} = \beta \begin{pmatrix} c_{13}c_{23} \\ -c_{12}c_{23} \\ c_{12}c_{13} \end{pmatrix}$ for some $\beta \neq 0$. The plugging into (4.5) we have

$$\begin{aligned} -\frac{1}{\beta} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} &= \begin{pmatrix} -s_{12} & -s_{13} & 0 \\ -s_{12} & 0 & -s_{23} \\ 0 & -s_{13} & -s_{23} \end{pmatrix} \begin{pmatrix} c_{13}c_{23} \\ -c_{12}c_{23} \\ c_{12}c_{13} \end{pmatrix} \\ &= \begin{pmatrix} c_{23}s_{23} \\ -c_{13}s_{13} \\ c_{12}s_{12} \end{pmatrix}, \end{aligned} \quad (4.7)$$

where in the last line we have used the trigonometric addition formulae. Considering the quotients $\frac{m_1 m_{23}}{m_{12} m_{13}}$ and so forth, it follows that each $\tan \frac{\alpha_{jk}}{2}$ is rational and hence $e^{i\alpha_{jk}}$ is a rational point. $\square$

Proposition 4.4. *There is no good network V with $N = 3$.*

Proof. Suppose V is a good network with $N = 3$. First take a replacement V'_1 of V at v_1 . Using that $e^{i\theta} - 1 = 2 \sin \frac{\theta}{2} e^{i \frac{\theta+\pi}{2}}$ for $\theta \in [0, 2\pi]$, the replacement network V'_1 should have vertices $v'_1 = v_1$, $v'_2 = \vec{T}_{12}$, $v'_3 = \vec{T}_{13}$ (up to a coordinate rotation); the corresponding angle differences are $\alpha_{12}^{(1)} = \frac{\alpha_{12} + \pi}{2}$, $\alpha_{13}^{(1)} = \frac{\alpha_{13} + \pi}{2}$, $\alpha_{23}^{(1)} = \frac{\alpha_{23}}{2}$. Now consider the iterated replacement V''_{11} at v'_1 ; then the angle differences will become $\alpha_{12}^{(11)} = \frac{\alpha_{12} + 3\pi}{4}$, $\alpha_{13}^{(11)} = \frac{\alpha_{13} + 3\pi}{4}$, $\alpha_{23}^{(11)} = \frac{\alpha_{23}}{4}$.

Apply the same twice iterated replacement at v_3 ; then in particular the (non-reflex) opposite angle difference halves twice, so the resulting network V''_{33} will have $\alpha_{12}^{(33)} = \frac{\alpha_{12}}{4}$. By Lemma 4.3 we must have $e^{i\alpha} \in \mathbb{Q}(i)$ for each of these α . Since $\mathbb{Q}(i)$ is a field this implies that $\exp(i(\alpha_{12}^{(11)} - \alpha_{12}^{(33)})) = e^{3i\pi/4} = \frac{1}{\sqrt{2}}(-1 + i)$ is a rational point, which is absurd. $\square$

4.4. General N

Recall that $E, \deg, \dots$ denote the edges, degree, etc. of the interior graph $\mathring{V}$.

Proposition 4.5. *There is no good network V with $N = 4$.*

Proof. Suppose V is a good network with $N = 4$. By the previous proposition, there are no good networks with $N = 3$, so by taking a replacement we see that each vertex v in $\mathring{V}$ cannot have degree equal to 2.

If there is a vertex with degree 1, then by Lemma 4.2, we have that $E \leq F_1(4) = 3$. But since there are at most two vertices with degree 1, and the remaining vertices must have degree at least 3, we have

$$E = \frac{1}{2} \sum \deg(v) \geq \frac{1}{2}(3(N - 2) + 2) = 4,$$

which is a contradiction.

On the other hand, if there are no vertices with degree 1, then they all have degree at least 3, so $\sum \deg(v) \geq 3N = 12$, but this contradicts the earlier bound

$$\sum \deg v = 2E \leq 2F(4) = 10. \quad \square$$

Theorem 4.6. *The only good network V is the straight-line network which has $N = 2$, diametrically opposite vertices and equal multiplicities.*

Proof. It is clear that there is no admissible network with $N = 1$, and the only admissible network with $N = 2$ is the straight line configuration.

Since we have proven there are no good networks with $N = 3, 4$, by taking replacements we have ruled out any vertex v in $\mathring{V}$ having degree 2 or 3, and we may assume $N \geq 5$.

If there is a vertex with degree 1, then by Lemma 4.2, we have that $E \leq F_1(N) = 2N - 5$. But since there are at most two vertices with degree 1, and the remaining vertices must have degree at least 4, we have

$$E = \frac{1}{2} \sum \deg(v) \geq \frac{1}{2}(4(N-2) + 2) = 2N - 3,$$

which is a contradiction.

On the other hand, if there are no vertices with degree 1, then all vertices must have degree at least 4 and so $\sum \deg(v) \geq 4N$. But

$$\sum \deg(v) = 2E \leq 2F(N) = 4N - 6,$$

which is a contradiction. $\square$

Remark 4.7. The stationarity of V at each vertex automatically implies that:

(1) V is stationary at infinity, that is,

$$\sum_v m_v v = 0.$$

(2) The mass of the interior graph $\mathring{V}$ is the same as the graph which extends the exterior rays into the origin, that is,

$$\sum_v m_v = \sum_{vw \in \mathring{V}} m_{vw} |v - w|.$$

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