

## HYDRODYNAMIC LIMIT OF GRANULAR GASES TO PRESSURELESS EULER IN DIMENSION 1

BY

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**Abstract.** We investigate the behavior of granular gases in the limit of small Knudsen number, that is, very frequent collisions. We deal with the strongly inelastic case in one dimension of space and velocity. We are able to prove the convergence toward the pressureless Euler system. The proof relies on dispersive relations at the kinetic level, which leads to the so-called Oleinik property at the limit.

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**1. Introduction.** The granular gases equation is a Boltzmann-like kinetic equation describing a rarefied gas composed of macroscopic particles interacting via energy-dissipative binary collisions (pollen flow in a fluid or planetary rings, for example). More precisely, the phase space distribution  $f^\varepsilon(t, x, v)$  solves the equation

$$\begin{cases} \frac{\partial f^\varepsilon}{\partial t} + v \frac{\partial f^\varepsilon}{\partial x} = \frac{1}{\varepsilon} \mathcal{Q}_\alpha(f^\varepsilon, f^\varepsilon), \\ f^\varepsilon(0, x, v) = f_0^\varepsilon(x, v), \end{cases} \quad (1.1)$$

where  $f_0^\varepsilon$  is a given non-negative distribution,  $t \geq 0$ ,  $v \in \mathbb{R}$  and  $x \in \mathbb{R}$ . The collision operator  $\mathcal{Q}_\alpha$  is the so-called *granular gases* operator (sometimes known as the inelastic Boltzmann operator), describing an energy-dissipative microscopic collision dynamics, which we will present in the following section. The parameter  $\varepsilon > 0$  is the scaled *Knudsen* number, that is, the ratio between the mean free path of particles before a collision and the length scale of observation.

As  $\varepsilon \rightarrow 0$ , the frequency of collisions increases to infinity. The *particle distribution function*  $f^\varepsilon$  then formally converges towards a Dirac mass centered on the mean velocity,

$$\rho(x) \delta_0(v - u(x)), \quad \forall (x, v) \in \mathbb{R} \times \mathbb{R}. \quad (1.2)$$

This is due to the energy dissipation which ensures that all particles occupying the same position in space necessarily have the same velocity.

The form (1.2) of  $f^\varepsilon$  is usually called monokinetic and greatly reduces the complexity of (1.1): The solution is completely described by its local hydrodynamic fields, namely its *mass*  $\rho \geq 0$  and its *velocity*  $u \in \mathbb{R}$ .

Before the limit  $\varepsilon \rightarrow 0$ , the same *macroscopic* quantities can be obtained from the distribution function  $f^\varepsilon$  by computing its first moments in velocity:

$$\rho^\varepsilon(t, x) = \int_{\mathbb{R}} f^\varepsilon(t, x, v) dv, \quad \rho^\varepsilon(t, x) u^\varepsilon(t, x) = \int_{\mathbb{R}} f^\varepsilon(t, x, v) v dv. \quad (1.3)$$

However those quantities cannot be solved independently as they do not satisfy a closed system for  $\varepsilon > 0$ . Instead one has by integrating (1.1) (see the properties of the collision operator just below)

$$\begin{aligned} \partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) &= 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \partial_x(E^\varepsilon) &= 0, \end{aligned}$$

where  $E^\varepsilon = \int_{\mathbb{R}} f^\varepsilon(t, x, v) |v|^2 dv$  and cannot be expressed directly in terms of  $\rho^\varepsilon$  and  $u^\varepsilon$ . But at the limit  $\varepsilon \rightarrow 0$ , if (1.2) holds, then one has that  $E = \rho u^2$  and  $\rho$ ,  $u$  now satisfy the pressureless Euler dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0. \end{cases} \quad (1.4)$$

This system of equation is mostly known as a model for the formation of large scale structures in the universe (*e.g.* aggregates of galaxies) [29].

The purpose of this article is to justify rigorously this limit of (1.1) to (1.4).

Such hydrodynamic limits for collisional models have been famously investigated for *elastic collisions* (preserving the kinetic energy) such as the Boltzmann equation. They are connected to the rigorous derivation of Fluid Mechanics models (such as incompressible Navier-Stokes or Euler); this long-standing conjecture formulated by Hilbert was finally solved in [17, 18, 28].

The inelasticity (loss of kinetic energy for each collision) leads however to a very distinct behavior and requires different techniques. In fact even classical formal techniques such as Hilbert or Chapman-Enskog expansions (see *e.g.* [14] for a mathematical introduction in the elastic case) are not applicable. The limit system for instance is very singular (see the corresponding subsection below), to the point that well posedness for (1.4) is only known in dimension 1. This is the main reason why our study is limited to this one-dimensional case.

We continue this introduction by explaining more precisely the collision operator. We then present the current theory for the limit system (1.4) before giving the main result of the article.

1.1. *The collision operator.* Let  $\alpha \in [0, 1]$  be the restitution coefficient of the microscopic collision process, that is, the ratio of kinetic energy dissipated during a collision, in the direction of impact. This quantity can depend on the magnitude of the relative velocity before collision  $|v - v_*|$  (see the book [12] for a long discussion of this topic).

If  $\alpha = 1$ , no energy is dissipated, and the collision is *elastic*. If  $\alpha \in (0, 1)$ , the collision is said to be *inelastic*. We define a strong form of the *collision operator*  $\mathcal{Q}_\alpha$  by

$$\begin{aligned} \mathcal{Q}_\alpha(f, g)(v) &= \int_{\mathbb{R}} |v - v_*| \left( \frac{f' g'_*}{\alpha^2} - f g_* \right) dv_* \\ &= \mathcal{Q}_\alpha^+(f, g)(v) - f(v) L(g)(v), \end{aligned} \quad (1.5)$$

where we have used the usual shorthand notation  $f' := f(v')$ ,  $f'_* := f(v'_*)$ ,  $f := f(v)$ ,  $f_* := f(v_*)$ . In (1.5),  $v'$  and  $v'_*$  are the pre-collisional velocities of two particles of given velocities  $v$  and  $v_*$ , defined by

$$\begin{cases} v' = \frac{1}{2}(v + v_*) + \frac{\alpha}{2}(v - v_*), \\ v'_* = \frac{1}{2}(v + v_*) - \frac{\alpha}{2}(v - v_*). \end{cases} \quad (1.6)$$

The operator  $\mathcal{Q}_\alpha^+(f, g)(v)$  is usually known as the *gain* term because it can be understood as the number of particles of velocity  $v$  created by collisions of particles of pre-collisional velocities  $v'$  and  $v'_*$ , whereas  $f(v)L(g)(v)$  is the *loss* term, modeling the loss of particles of pre-collisional velocities  $v'$ .

We can also give a weak form of the collision operator, which is compatible with sticky collisions. Let us reparametrize the post-collisional velocities  $v'$  and  $v'_*$  as

$$\begin{cases} v' = v - \frac{1-\alpha}{2}(v - v_*), \\ v'_* = v_* + \frac{1-\alpha}{2}(v - v_*). \end{cases}$$

Then we have the weak representation, for any smooth test function  $\psi$ ,

$$\int_{\mathbb{R}} Q_\alpha(f, g) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} |v - v_*| f_* g (\psi' + \psi'_* - \psi - \psi_*) dv dv_*. \quad (1.7)$$

Thanks to this expression, we can compute the macroscopic properties of the collision operator  $Q_\alpha$ . Indeed, we have the microscopic conservation of impulsion and dissipation of kinetic energy:

$$\begin{aligned} v' + v'_* &= v + v_*, \\ (v')^2 + (v'_*)^2 - v^2 - v_*^2 &= -\frac{1 - \alpha^2}{2} (v - v_*)^2 \leq 0. \end{aligned}$$

Then if we integrate the collision operator against  $\varphi(v) = (1, v, v^2)$ , we obtain the preservation of mass and momentum and the dissipation of kinetic energy:

$$\int_{\mathbb{R}} Q_\alpha(f, f)(v) \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \\ -(1 - \alpha^2)D(f, f) \end{pmatrix}, \quad (1.8)$$

where  $D(f, f) \geq 0$  is the *energy dissipation* functional given by

$$D(f, f) := \int_{\mathbb{R} \times \mathbb{R}} f f_* |v - v_*|^3 dv dv_* \geq 0. \quad (1.9)$$

The conservation of mass implies an *a priori* bound for  $f$  in  $L^\infty(0, T; L^1(\mathbb{R} \times \mathbb{R}))$ . Moreover, these macroscopic properties of the collision operator, together with the conservation of positiveness, imply that the equilibrium profiles of  $Q_\alpha$  are trivial Dirac masses (see *e.g.* the review paper [32] of Villani).

Finally, we can give a precise estimate of the energy dissipation functional. Indeed, applying Jensen's inequality to the convex function  $v \mapsto |v|^3$  and to the measure  $f(v_*) dv_*$ , we get

$$\int_{\mathbb{R}} f(v_*) |v - v_*|^3 dv_* \geq \left| v \int_{\mathbb{R}} f(v_*) dv_* - \int_{\mathbb{R}} v_* f(v_*) dv_* \right| = |\rho(v - u)|^3.$$

Using Hölder inequality, we find that the energy dissipation is such that

$$\begin{aligned} D(f, f) &\geq \rho^3 \int_{\mathbb{R}} f(v) |v - u|^3 dv \\ &\geq \rho^{5/2} \left( \int_{\mathbb{R}} f(v) |v - u|^2 dv \right)^{\frac{3}{2}}. \end{aligned} \quad (1.10)$$

REMARK 1. Let us define the *temperature* of a particle distribution function  $f$  by

$$\theta(t, x) := \int_{\mathbb{R}} |v - u|^2 f(t, x, v) dv.$$

Multiplying equation (1.1) by  $|v - u|^2$  and integrating with respect to the velocity and space variables yield, thanks to (1.10), the so-called *Haff's Law* [19]

$$\int_{\mathbb{R}} \theta^\varepsilon(t, x) dx \lesssim \frac{1 - \alpha}{\varepsilon} \frac{1}{(1 + t)^2}. \quad (1.11)$$

This asymptotic behavior of the macroscopic temperature is characteristic of granular gases and has been proved to be optimal in the space homogeneous case for a constant

restitution coefficient by Mischler and Mouhot in [24]. These results have then been extended to a more general class of collision kernel and restitution coefficients by Alonso and Lods in [2, 3] and by the second author in [27]. Nevertheless, in all these works, additional constraints on the smoothness of the initial data (a somehow nonphysical  $L^p$  bound for  $p > 1$ ) are required for the results to hold.

The existence in the general  $\mathbb{R}_x^3 \times \mathbb{R}_v^3$  setting for a large class of velocity-dependent restitution coefficient but close to vacuum was obtained in [1]. The stability in  $L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$  under the same assumptions was derived for instance in [33]. Finally the existence and convergence to equilibrium in  $\mathbb{T}_x^3 \times \mathbb{R}_v^3$  for a diffusively heated, weakly inhomogeneous granular gas were proved in [31].

As one can imagine, the theory in the dimension 1 case (as concerns us here) is much simpler. The existence of solutions for the granular gases equation (1.1) in one dimension of physical space and velocity, with a constant restitution coefficient, was proved in [6] for compact initial data. The velocity-dependent restitution coefficient case, for small data, was then proven in [4]. More precisely, one has:

**THEOREM 1.1** (From [4]). Let us assume that there exists  $\gamma \in (0, 1)$  such that

$$\alpha = \alpha(|v - v_*|) = \frac{1}{1 + |v - v_*|^\gamma}.$$

Then, for  $0 \leq f^0 \in L^\infty(\mathbb{R}_x \times \mathbb{R}_v)$  with small total mass, there exists a unique mild, bounded solution in  $L^\infty(\mathbb{R}_x \times \mathbb{R}_v)$  of (1.1).

The main argument is reminiscent of work due to Bony in [8] concerning discrete velocity approximation of the Boltzmann equation in dimension 1.

Finally, the problem of the hydrodynamic limit was only tackled formally and in the quasi-elastic setting  $\alpha \rightarrow 1$ . The first results for this case can be found in [5] for the one-dimensional case. The review paper [30] summarizes most of the known formal results for the general case.

**1.2. Pressureless Euler: The sticky particles dynamics.** The pressureless system (1.4) is rather delicate. It can (and will in general) exhibit shocks as the velocity  $u$  formally solves the Burgers equation where  $\rho > 0$ . The implied lack of regularity on  $u$  leads to concentrations on the density  $\rho$  which is only a non-negative measure in general.

System (1.4) is hence in general ill-posed as classical solutions cannot exist for large times and weak solutions are not unique. It is however possible to recover a well posed theory by imposing a semi-Lipschitz condition on  $u$ . This theory was introduced in [9] and later extended in [10] and [20] (see also [16] and [15]). We cite below the main result of [20], where  $M^1(\mathbb{R})$  denotes the space of Radon measures on  $\mathbb{R}$  and  $L^2(\rho)$  for  $\rho \geq 0$  in  $M^1(\mathbb{R})$  denotes the space of functions which are square integrable against  $\rho$ .

**THEOREM 1.2** (From [20]). For any  $\rho^0 \geq 0$  in  $M^1(\mathbb{R})$  and any  $u^0 \in L^2(\rho^0)$ , there exist  $\rho \in L^\infty(\mathbb{R}_+, M^1(\mathbb{R}))$  and  $u \in L^\infty(\mathbb{R}_+, L^2(\rho))$  solutions to system (1.4) in the sense of distribution and satisfying the semi-Lipschitz Oleinik-type bound

$$u(t, x) - u(t, y) \leq \frac{x - y}{t}, \quad \text{for a.e. } x > y. \quad (1.12)$$

Moreover the solution is unique if  $u^0$  is semi-Lipschitz or if the kinetic energy is continuous at  $t = 0$ :

$$\int_{\mathbb{R}} \rho(t, dx) |u(t, x)|^2 \longrightarrow \int_{\mathbb{R}} \rho^0(dx) |u^0(x)|^2, \quad \text{as } t \rightarrow 0.$$

The proof of Theorem 1.2 is quite delicate, relying on duality solutions. For this reason, we only explain the rationale behind the bound (1.12), which can be seen very simply from the discrete sticky particles dynamics. We refer in particular to [11] for the limit of this sticky particles dynamics as  $N \rightarrow \infty$ .

Consider  $N$  particles on the real line. We describe the  $i^{\text{th}}$  particle at time  $t > 0$  by its position  $x_i(t)$  and its velocity  $v_i(t)$ . Since we are dealing with one-dimensional dynamics, we can always assume the particles to be initially ordered:

$$x_1^{in} < x_2^{in} < \dots < x_N^{in}.$$

The dynamics is characterized by the following properties:

- (i) The particle  $i$  moves with velocity  $v_i(t)$ :  $\frac{d}{dt}x_i(t) = v_i(t)$ .
- (ii) The velocity of the  $i^{\text{th}}$  particle is constant as long as it does not collide with another particle:  $v_i(t)$  is constant as long as  $x_i(t) \neq x_j(t)$  for all  $j \neq i$ .
- (iii) The velocity jumps when a collision occurs: if at time  $t_0$  there exists  $j \in \{1, \dots, N\}$  such that  $x_j(t_0) = x_i(t_0)$  and  $x_j(t) \neq x_i(t)$  for any  $t < t_0$ , then all the particles with the same position take as new velocity the average of all the velocities

$$v_i(t_0+) = \frac{1}{|j| x_j(t_0) = x_i(t_0)} \sum_{j|x_j(t_0)=x_i(t_0)} v_j(t_0-).$$

Note in particular that particles having the same position at a given time will then move together at the same velocity. Hence, only a finite number of collisions can occur as the particles aggregate.

This property also leads to the Oleinik regularity. Consider any two particles  $i$  and  $j$  with  $x_i(t) > x_j(t)$ . Because they occupy different positions, they have never collided, and hence  $x_i(s) > x_j(s)$  for any  $s \leq t$ . If neither had undergone any collision, then  $x_i(0) = x_i(t) - v_i(t)t > x_j(0) = x_j(t) - v_j(t)t$  or

$$\frac{(v_i - v_j)_+}{(x_i - x_j)_+} < \frac{1}{t}, \quad (1.13)$$

where  $x_+ := \max(x, 0)$ . It is straightforward to check that (1.13) still holds if particles  $i$  and  $j$  had some collisions between time 0 and  $t$ .

As one can see, this bound is a purely dispersive estimate based on free transport and the exact equivalent of the traditional Oleinik regularization for Scalar Conservation Laws; see [25]. It obviously leads to the semi-Lipschitz bound (1.12) as  $N \rightarrow \infty$ .

We conclude this subsection with the following remark, which foresees our main method.

REMARK 2. Define the empirical measure of the distribution of particles

$$f_N(t, x, v) := \sum_{i=1}^N \delta_0(x - x_i(t)) \delta_0(v - v_i(t)). \quad (1.14)$$

The empirical measure is a solution to the following kinetic equation:

$$\partial_t f_N + v \partial_x f_N = -\partial_{vv} m_N, \quad (1.15)$$

for some non-negative measure  $m_N$ . This equation embeds the fundamental properties of the dynamics: conservation of mass and momentum, and dissipation of kinetic energy. It is in several respects a sort of kinetic formulation, rather similar to the ones introduced for some conservation laws [22, 23]; see also [26].

The kinetic formulation (1.15) has to be coupled with a constraint on  $f_n$  (just like for Scalar Conservation Laws). Unsurprisingly this constraint is that  $f_N$  has to be monokinetic:

$$f_N = \rho_N(t, x) \delta(v - u_N(t, x)).$$

1.3. *Main result.* We are now ready to state the main result of this article.

THEOREM 1.3. Consider a sequence of weak solutions  $f_\varepsilon(t, x, v) \in L^\infty([0, T], L^p(\mathbb{R}^2))$  for some  $p > 2$  and with total mass 1 to the granular gases (1.1) such that all initial  $v$ -moments are uniformly bounded in  $\varepsilon$ ,

$$\sup_\varepsilon \int_{\mathbb{R}^2} |v|^k f_\varepsilon^0(x, v) dx dv < \infty; \quad (1.16)$$

some moment in  $x$  is uniformly bounded, for instance

$$\sup_\varepsilon \int_{\mathbb{R}^2} |x|^2 f_\varepsilon^0(x, v) dv < \infty; \quad (1.17)$$

and  $f_\varepsilon^0$  is, uniformly in  $\varepsilon$ , in some  $L^p$  for  $p > 1$ ,

$$\sup_\varepsilon \int_{\mathbb{R}^2} (f_\varepsilon^0(x, v))^p dx dv < \infty. \quad (1.18)$$

Then any weak-\* limit  $f$  of  $f_\varepsilon$  is monokinetic,  $f = \rho(t, x) \delta(v - u(t, x))$  for a.e.  $t$ , where  $\rho, u$  constitute a solution in the sense of distributions to the pressureless system (1.4), while  $u$  has the Oleinik property for any  $t > 0$ ,

$$u(t, x) - u(t, y) \leq \frac{x - y}{t}, \quad \text{for } \rho \text{ a.e. } x \geq y.$$

REMARK 3. It is possible to replace the  $L^p$  condition on  $f_\varepsilon^0$  by assuming that  $f_\varepsilon^0$  is well prepared in the sense that  $f_\varepsilon^0 \rightarrow \rho^0 \delta(v - u^0(x))$  for some  $u^0$  Lipschitz with the convergence in an appropriate sense (made precise in Remark 4 after Theorem 3.1). In that case one knows in addition that the limit is the unique “sticky particles” solution to the pressureless system (1.4) as obtained in [9, 20].

The basic idea of the proof of Theorem 1.3 is to use the kinetic description (1.15) to compare the granular gases dynamics to the pressureless gas system. The main difficulty is to show that  $f_\varepsilon$  becomes monokinetic at the limit. This is intimately connected to the Oleinik property (1.12), just as this property is critical to pass to the limit from the discrete sticky particles dynamics.

Unfortunately it is not possible to obtain (1.12) directly. Contrary to the sticky particles dynamics, this bound cannot hold for any finite  $\varepsilon$  (or for any distribution that is not monokinetic). This is the reason why it is very delicate to obtain the pressureless gas system from kinetic equations (no matter how natural it may seem). Indeed we are only aware of one other such example in [21].

One of the main contributions of this article is a complete reworking of the Oleinik estimate, still based on dispersive properties of the free transport operator  $v \partial_x$  but compatible with kinetic distributions that are not monokinetic.

The next section is devoted to the introduction and properties of the corresponding new functionals. This will allow us to prove a more general version of Theorem 1.3 in the last section.

## 2. A new dissipative functional for kinetic equations.

2.1. *Basic definitions.* The heart of our proof relies on new dissipative properties of kinetic equations which are

- Contracting in velocity;
- Close to monokinetic.

Mathematically speaking, consider  $f \in L^\infty([0, T], M^1(\mathbb{R}^2))$  a solution to

$$\partial_t f + v \partial_x f = -\partial_{vv} m, \quad m \in M^1([0, T] \times \mathbb{R}^2), \quad m \geq 0. \quad (2.1)$$

We also need a notion of trace for  $f$  and more precisely that

$$\Lambda_{f,k}(t) = \limsup_{\delta \rightarrow 0} \frac{1}{\delta} \int_{x \in \mathbb{R}} \int_{x < y < x + \delta} \int_{v, w \in \mathbb{R}^2} (v - w)_+^k f(t, x, v) f(t, y, w) dv dw dy dx \in L^1([0, T]). \quad (2.2)$$

This system is now dissipative and will yield as a dissipation rate a control on the following nonlinear functional for any  $\eta, \mu > 0, k \geq 1$ :

$$\mathcal{L}_{\eta, \mu, k}(f)(t) := \int \frac{(v - w)_+^{k+2}}{(x - y + \eta)^k} \chi_\mu(x - y) f(t, x, v) f(t, y, w) dv dw dx dy, \quad (2.3)$$

where the function  $\chi_\mu$  is a smooth, non-centered approximation of the Heaviside function, as in Figure 1. In particular  $\chi_\mu$  is non-increasing in  $\mu$  and

$$0 \leq \chi_\mu(x) \leq \mathbb{I}_{x > 0}, \quad 0 \leq \chi'_\mu(x) \leq \frac{2}{\mu} \mathbb{I}_{0 < x < \mu u}. \quad (2.4)$$

This functional is somehow similar to the one described by Bony in [8] and is used by Cercignani in [13] and by Biryuk, Craig and Panferov in [7].

To make notation consistent, we define when  $k = 0$ ,

$$\mathcal{L}_{\eta, \mu, 0}(f)(t) := - \int (v - w)_+^2 \log(x - y + \eta) \chi_\mu(x - y) f(t, x, v) f(t, y, w) dv dw dx dy. \quad (2.5)$$

We also define, from the monotonicity of  $\chi_\mu$ ,

$$\mathcal{L}_{\eta, 0+, k}(f)(t) := \sup_{\mu \rightarrow 0} \mathcal{L}_{\eta, \mu, k}(f)(t) = \limsup_{\mu \rightarrow 0} \mathcal{L}_{\eta, \mu, k}(f)(t). \quad (2.6)$$

Observe that for  $\mu = 0$ ,  $\mathcal{L}_{\eta, 0, k}(f)(t)$  may not be well defined and may in fact depend on the way the Heavyside function  $\mathcal{I}(x - y)$  is approximated. This is the reason

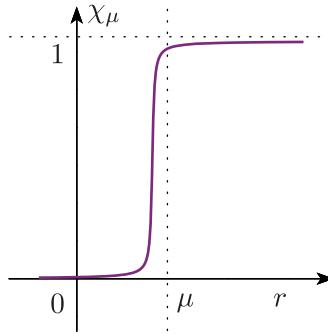


FIG. 1. Smoothed, non-centered approximation of the Heaviside function  $\chi_\mu$ .

for the precise definition above of  $\mathcal{L}_{\eta,0+,k}(f)(t)$ . Furthermore from the trace property (2.2), whatever the definition of  $\mathcal{L}_{\eta,0,k}(f)(t)$ , one would have that  $\int_0^T \mathcal{L}_{\eta,0,k}(f)(t) dt \leq \int_0^T \mathcal{L}_{\eta,0+,k}(f)(t) dt + 2 \int_0^T \Lambda_{f,k}(t) dt$  as explained below.

EXAMPLE. It is possible to prove that  $\mathcal{L}_{\eta,\mu,k}$  is bounded for the sticky particle dynamics. Indeed, let  $(x_i(t), v_i(t))_{1 \leq i \leq N}$  for  $N \in \mathbb{N}$  be the solution to the sticky particles system (iii) and  $f_N$  be the associated empirical measure given by (1.14). We already observed in Remark 2 that  $f_N$  solves (2.1); moreover it has the trace property (2.2) with  $\Lambda_{f,k} = 0$ .

In that simple example, it is possible to bound  $\mathcal{L}_{\eta,\mu,k}$  directly by using (1.13), so that

$$0 \leq \mathcal{L}_{\eta,\mu,k}(f_N)(t) \leq |\max v_i|^2 \left( \sup_t \sup_{i \neq j} \frac{(v_i - v_j)_+}{(x_i - x_j)_+} \right)^k \leq C^k,$$

independently of  $\eta$  and  $\mu$ .

Let us start with some basic properties of  $\mathcal{L}_{\eta,\mu,k}(f_N)(t)$ .

LEMMA 2.1. Assume that  $f \in L^\infty([0, T], M^1(\mathbb{R}^2))$  solves (2.1) and has bounded moments in  $v$  for some  $k \geq 0$ ,

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^{k+3} f(t, x, v) dx dv < \infty. \quad (2.7)$$

Then for any  $\eta, \mu > 0$ ,  $\mathcal{L}_{\eta,\mu,k}(f)(t)$  is  $BV$  in  $t$ ; in particular  $\mathcal{L}_{\eta,\mu,k}(f)(t)$  is continuous at a.e.  $t$  and has a left and right trace at every  $t$ . Furthermore for any  $s, t$ ,

$$\int_s^t \mathcal{L}_{\eta,\mu,k}(f)(r) dr \longrightarrow \int_s^t \mathcal{L}_{\eta,0+,k}(f)(r) dr, \quad \text{as } \mu \rightarrow 0. \quad (2.8)$$

The functional  $\int_s^t \mathcal{L}_{\eta,\mu,k}(f)(r) dr$  is also continuous in  $f$  and  $\int_s^t \mathcal{L}_{\eta,0+,k}(f)(t) dt$  is lower semi-continuous in the following sense: If  $f_n$  is a sequence of solutions to (2.1) with right-hand sides  $m_n \geq 0$  s.t.

$$\sup_n \sup_{t \in [0, T]} \int_{\mathbb{R}^2} (|x|^2 + |v|^{k+3}) f_n(t, x, v) dx dv < \infty,$$

and  $f_n \rightarrow f$  in  $w-*L^\infty([0, T], M^1(\mathbb{R}^2))$ , then

$$\int_s^t \mathcal{L}_{\eta, \mu, k}(f)(r) dr = \lim \int_s^t \mathcal{L}_{\eta, \mu, k}(f_n)(r) dr, \quad \int_s^t \mathcal{L}_{\eta, 0+, k}(f)(r) dr = \liminf \int_s^t \mathcal{L}_{\eta, 0+, k}(f_n)(r) dr.$$

*Proof.* First of all the  $\mathcal{L}_{\eta, \mu, k}(f)$  are bounded by moments of  $f$ :

$$\begin{aligned} \mathcal{L}_{\eta, \mu, k}(f)(t) &\leq \eta^{-k} \int_{\mathbb{R}^4} (|v|^{k+2} + |w|^{k+2}) f(t, x, v) f(t, y, w) dx dy dv dw \\ &= \eta^{-k} \left( \int_{\mathbb{R}^2} |v|^{k+2} f(t, x, v) dx dv \right)^2. \end{aligned}$$

By its definition  $\mathcal{L}_{\eta, \mu, k}(f)(t)$  converges pointwise in  $t$  to  $\mathcal{L}_{\eta, 0+, k}(f)(t)$ . Thus the previous bound implies by dominated convergence that for any  $s, t$ ,

$$\int_s^t \mathcal{L}_{\eta, \mu, k}(f)(r) dr \longrightarrow \int_s^t \mathcal{L}_{\eta, 0+, k}(f)(r) dr,$$

as  $\mu \rightarrow 0$ .

Next denoting  $f' = f(t, y, w)$ , from the equation (2.1) on  $f$ , since every term in  $\mathcal{L}_{\eta, \mu, k-1}(f)(t)$  is smooth, one has that

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_{\eta, \mu, k}(f)(t) &= \int [f' \partial_t f + f \partial_t f'] \frac{(v-w)_+^{k+2}}{(x-y+\eta)^k} \chi_\mu(x-y) dv dw dx dy \\ &= \int \left[ f' \left( -v \partial_x f - \partial_{vv} m \right) + f \left( -w \partial_y f' - \partial_{ww} m' \right) \right] \\ &\quad \times \frac{(v-w)_+^{k+2}}{(x-y+\eta)^k} \chi_\mu(x-y) dv dw dx dy. \end{aligned}$$

Integrating by parts the free transport terms of the last relation, with respect to  $x$  and  $y$ , we obtain that for  $k \geq 1$ ,

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_{\eta, \mu, k}(f)(t) &= \int \left\{ (v-w)_+^{k+3} f f' \left[ -k \frac{\chi_\mu(x-y)}{(x-y+\eta)^{k+1}} + \frac{\chi'_\mu(x-y)}{(x-y+\eta)^k} \right] \right. \\ &\quad \left. - (k+1)(k+2) [f' m + f m'] \frac{(v-w)_+^k}{(x-y+\eta)^k} \chi_\mu(x-y) \right\} dv dw dx dy. \end{aligned}$$

Recalling that  $m \geq 0$ , this leads to

$$\frac{d}{dt} \mathcal{L}_{\eta, \mu, k}(f)(t) \leq \int (v-w)_+^{k+3} f f' \left[ -k \frac{\chi_\mu(x-y)}{(x-y+\eta)^{k+1}} + \frac{\chi'_\mu(x-y)}{(x-y+\eta)^k} \right] dx dy dv dw, \quad (2.9)$$

and hence by (2.4)

$$\frac{d}{dt} \mathcal{L}_{\eta, \mu, k}(f)(t) \leq \frac{4}{\mu \eta^k} \int |v|^{k+3} f dx dv,$$

which is bounded by (2.7).

On the other hand, if  $k = 0$  by the definition of  $\mathcal{L}_{\eta,\mu,k}(f)$  and with similar calculations,

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_{\eta,\mu,0}(f)(t) &= - \int_{x \leq 1+y+\eta} \left\{ (v-w)_+^3 f f' \left[ \frac{\chi_\mu(x-y)}{(x-y+\eta)} + \chi'_\mu(x-y) \log(x-y+\eta) \right] \right. \\ &\quad \left. + 2 [f' m + f m'] \mathbb{I}_{v-w \geq 0} \log(x-y+\eta) \chi_\mu(x-y) \right\} dv dw dx dy. \end{aligned}$$

Note that since  $\log x \leq 0$  for  $x \leq 1$  and  $m \geq 0$ , one has similarly in this case

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_{\eta,\mu,0}(f)(t) &\leq - \int_{x \leq 1+y+\eta} (v-w)_+^3 f f' \left[ \frac{\chi_\mu(x-y)}{(x-y+\eta)} + \chi'_\mu(x-y) \log(x-y+\eta) \right] dx dy dv dw, \end{aligned} \quad (2.10)$$

leading by (2.4) to

$$\frac{d}{dt} \mathcal{L}_{\eta,\mu,0}(f)(t) \leq \frac{4 |\log \eta|}{\mu} \int |v|^3 f dx dv.$$

In all cases,  $\mathcal{L}_{\eta,\mu,k}(f)(t)$  is hence semi-Lipschitz and thus  $BV$ .

Consider now any sequence  $f_n$  of solutions to (2.1). Observe that

$$\sup_n \int_0^T \int_{\mathbb{R}^2} m_n(dt, dx, dv) = \sup_n \int_{\mathbb{R}^2} |v|^2 (f_n^0(dx, dv) - f_n(T, dx, dv)) < \infty.$$

Therefore by (2.1),  $\partial_t f_n$  is bounded in  $M_{loc}^1([0, T] \times \mathbb{R}^2) + L^\infty(W_x^{-1,1} L_x^1)$ . That implies that  $f_n$  is compact in  $L^2([0, T])$  with values in some weak space.

On the other hand, the function  $(v-w)_+^{k+2} (x-y+\eta)^{-k} \chi_\mu(x-y)$  is smooth ( $C^\infty$ ) for any  $\eta, \mu > 0$ . The uniform control on the moments of  $f_n$  then implies that

$$I_n(t, x, v) = \int_{\mathbb{R}^2} (v-w)_+^{k+2} (x-y+\eta)^{-k} \chi_\mu(x-y) f_n(t, dy, dw)$$

is compact in  $L^2([0, T], C_{x,v}^1)$ . Therefore we can easily pass to the limit in

$$\int_s^t \int_{\mathbb{R}^2} f_n(r, dx, dv) I_n(r, x, v) dr = \int_s^t \mathcal{L}_{\eta,\mu,k}(f_n)(r) dr.$$

This obviously cannot work for  $L_{\eta,0+,k}(f_n)(t)$ . However as  $L_{\eta,\mu,k}(f_n)(t)$  is increasing in  $\mu$ , and by (2.8)

$$\int_s^t \mathcal{L}_{\eta,0+,k}(f)(r) dr = \sup_{\mu > 0} \int_s^t \mathcal{L}_{\eta,\mu,k}(f)(r) dr.$$

The supremum of any family of continuous functions is automatically lower semi-continuous, thus finishing the proof.  $\square$

**2.2. Dissipation properties.** Our main goal is to use the dispersive properties of the free transport to bound  $\mathcal{L}_{\eta,0+,k}(f)$  in terms of  $\mathcal{L}_{\eta,0+,k-1}(f)$ .

**THEOREM 2.1.** Assume that  $f \in L^\infty([0, T], M^1(\mathbb{R}^2))$  solves (2.1), satisfies (2.2) and has bounded moments in  $v$  for some  $k \geq 0$ :

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^{k+2} f(t, x, v) dx dv < \infty. \quad (2.11)$$

Then for any  $\mu, \eta > 0$ ,  $0 \leq s \leq t$  if  $k \geq 2$ ,

$$k \int_s^t \mathcal{L}_{\eta,0+,k}(f)(r) dr + \mathcal{L}_{\eta,0+,k-1}(f)(t-) \leq \mathcal{L}_{\eta,0+,k-1}(f)(s+) + \frac{2}{\eta^{k-1}} \int_s^t \Lambda_{f,k+2}(r) dr, \quad (2.12)$$

and if  $k = 1$ ,

$$\int_s^t \mathcal{L}_{\eta,0+,1}(f)(r) dr + \mathcal{L}_{\eta,0+,0}(f)(t-) \leq \mathcal{L}_{\eta,0+,0}(f)(s+) + 2 |\log \eta| \int_s^t \Lambda_{f,2}(r) dr. \quad (2.13)$$

*Proof.* The proof is straightforward after Lemma 2.7. We begin by working with  $\mathcal{L}_{\eta,\mu,k}(f)$  for  $\mu > 0$ . Differentiating in time, one again obtains (2.9), that is,

$$\frac{d}{dt} \mathcal{L}_{\eta,\mu,k-1}(f)(t) \leq -k \mathcal{L}_{\eta,\mu,k}(f)(t) + \int (v-w)_+^{k+2} f f' \frac{\chi'_\mu(x-y)}{(x-y+\eta)^{k-1}} dx dy dv dw,$$

for  $k \geq 2$ , and if  $k-1=0$  by (2.10), then

$$\frac{d}{dt} \mathcal{L}_{\eta,\mu,0}(f)(t) \leq -\mathcal{L}_{\eta,\mu,1}(f)(t) - \int (v-w)_+^3 f f' \chi'_\mu(x-y) \log(x-y+\eta)_- dx dy dv dw.$$

We now use the property (2.4) to bound for  $k \geq 2$ ,

$$\begin{aligned} & \int (v-w)_+^{k+2} f f' \frac{\chi'_\mu(x-y)}{(x-y+\eta)^{k-1}} dx dy dv dw \\ & \leq \frac{2}{\mu \eta^{k-1}} \int_{x < y < x+\mu} (v-w)_+^{k+2} f f' dx dy dv dw. \end{aligned}$$

Therefore integrating in time between  $s$  and  $t$  the inequality above, one has that

$$\begin{aligned} \mathcal{L}_{\eta,\mu,k-1}(f)(t-) - \mathcal{L}_{\eta,\mu,k-1}(f)(s+) & \leq -k \int_s^t \mathcal{L}_{\eta,\mu,k}(f)(r) dr \\ & + \frac{2}{\mu \eta^{k-1}} \int_s^t \int_{x < y < x+\mu} (v-w)_+^{k+2} f f' dx dy dv dw dr. \end{aligned}$$

Take the limit  $\mu \rightarrow 0$  and observe that by its definition

$$\limsup_{\mu \rightarrow 0} \frac{1}{\mu} \int_s^t \int_{x < y < x+\mu} (v-w)_+^{k+2} f f' dx dy dv dw dr \leq \int_s^t \Lambda_{f,k+2}(r) dr.$$

The passage to the limit in  $\mathcal{L}_{\eta,\mu,k-1}(f)$  and  $\int_s^t \mathcal{L}_{\eta,\mu,k}(f)(r) dr$  is provided by Lemma 2.1, which concludes the proof in that case. The case  $k=1$  is handled similarly.  $\square$

**2.3. The connection with monokinetic solutions.** It turns out that the functionals  $\mathcal{L}_{\eta,\mu,k}(f)$  can control the concentration in velocity of a solution to (2.1). Roughly speaking it is not possible to have a bound on  $\mathcal{L}_{\eta,\mu,k}(f)$  uniform in  $\eta$  and  $\mu$  if  $f$  is not monokinetic. This is due to the fact that  $(x-y)^k$  is not integrable if  $k \geq 1$ , and thus the only way to keep the integral bounded is to have  $(v-w)_+$  small if  $x$  is close to  $y$ .

This is formalized in the following.

**PROPOSITION 2.1.** Assume that  $f \in L^\infty([0, T], M^1(\mathbb{R}^2))$  solves (2.1) and has bounded  $v$ -moments for some  $k \geq 0$ ,

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^{k+2} f(t, x, v) dx dv < \infty.$$

Assume moreover that

$$\sup_{\mu, \eta} \int_0^T \mathcal{L}_{\eta, \mu, k}(f)(r) dr < \infty.$$

Then  $f$  is monokinetic for a.e.  $t$ : There exist  $\rho \in L^\infty([0, T], M^1(\mathbb{R}))$ ,  $u \in L^\infty([0, T], L^{k+2}(\rho))$  s.t. for a.e.  $t$ ,

$$f(t, x, v) = \rho(t, x) \delta(v - u(t, x)).$$

*Proof.* First of all notice that it is always possible to define  $\rho$  and  $u$  by

$$\rho(t, x) = \int_{\mathbb{R}} f(t, x, dv), \quad \rho u(t, x) = \int_{\mathbb{R}} v f(t, x, dv).$$

One necessarily has that  $u \in L^\infty([0, T], L^{k+2}(\rho))$  because

$$u(t, x) = \int_{\mathbb{R}} v \frac{f(t, x, dv)}{\rho(t, x)},$$

and by Jensen's inequality

$$\int_{\mathbb{R}} |u(t, x)|^{k+2} \rho(t, dx) \leq \int_{\mathbb{R}^2} |v|^{k+2} f(t, x, dv).$$

Furthermore by (2.1),  $f$  is  $BV$  in time with value in a weak space in  $x$  and  $v$  (as in the proof of Theorem 2.1), and using the moments this proves that  $\rho$  and  $\rho u$  are also  $BV$  in time.

The Radon-Nikodym theorem implies that it is possible to decompose  $f$  according to  $\rho$ ,

$$f(t, x, v) = \rho(t, x) M(t, x, v),$$

and the goal is thus to prove that  $M$  is concentrated on a Dirac mass. We proceed in two steps by considering the atomic and non-atomic parts of  $\rho$ . We write accordingly

$$\rho(t, x) = \sum_{n=1}^{\infty} \rho_n(t) \delta(x - x_n(t)) + \tilde{\rho}(t, x),$$

where  $\tilde{\rho}$  does not contain any Dirac mass.

STEP 1 (Control of the non-atomic part). This part does not require any further use of (2.1). Start by remarking that by Jensen's inequality again

$$\int_{\mathbb{R}^2} \frac{(u(x) - u(y))_+^{k+2}}{(x - y + \eta)^k} \chi_{\mu}(x - y) \rho(t, dx) \rho(t, dy) \leq \mathcal{L}_{\eta, \mu, k}(f)(t).$$

Instead of replacing both  $v$  and  $w$  in  $\mathcal{L}_{\eta, \mu, k}(f)(t)$ , it is also possible to use Jensen's inequality to replace only  $v$  for instance. Thus one has as well

$$\int_{\mathbb{R}^3} \frac{(v - u(y))_+^{k+2}}{(x - y + \eta)^k} \chi_{\mu}(x - y) f(t, dx, dv) \rho(t, dy) \leq \mathcal{L}_{\eta, \mu, k}(f)(t).$$

Now  $(a+b)^k \leq 2^k (a^k + b^k)$  and combining the two previous inequalities, one obtains

$$\begin{aligned}
& \int_{\mathbb{R}^3} \frac{(v - u(x))_+^{k+2}}{(x - y + \eta)^k} \chi_\mu(x - y) f(t, dx, dv) \rho(t, dy) \\
&= \int_{\mathbb{R}^3} \frac{(v - u(y) + u(y) - u(x))_+^{k+2}}{(x - y + \eta)^k} \chi_\mu(x - y) f(t, dx, dv) \rho(t, dy) \\
&\leq 2^k \int_{\mathbb{R}^3} \frac{(v - u(y))_+^{k+2}}{(x - y + \eta)^k} \chi_\mu(x - y) f(t, dx, dv) \rho(t, dy) \\
&\quad + 2^k \int_{\mathbb{R}^2} \frac{(u(x) - u(y))_+^{k+2}}{(x - y + \eta)^k} \chi_\mu(x - y) \rho(t, dx) \rho(t, dy) \\
&\leq 2^{k+1} \mathcal{L}_{\eta, \mu, k}(f)(t).
\end{aligned}$$

In the left-hand side, only  $\rho(t, dy)$  depends on  $y$ , and this leads us to define

$$\alpha_{\mu, \eta} = \int_{\mathbb{R}} \frac{\chi_\mu(x)}{(x + \eta)^k} dx, \quad K_{\mu, \eta} = \frac{\alpha_{\mu, \eta}^{-1} \chi_\mu(x)}{(x + \eta)^k}, \quad \alpha_\eta = \lim_{\mu \rightarrow 0} \alpha_{\mu, \eta}, \quad K_\eta = \lim_{\mu \rightarrow 0} K_{\mu, \eta} = \frac{\alpha_\eta^{-1} \mathbb{I}_{x > 0}}{(x + \eta)^k}.$$

The previous inequality can be written as

$$\int_{\mathbb{R}^2} (v - u(x))_+^{k+2} K_{\mu, \eta} \star \rho(t, x) f(t, dx, dv) \leq \alpha_{\mu, \eta}^{-1} 2^{k+1} \mathcal{L}_{\eta, \mu, k}(f)(t).$$

One has that  $\lim_{\mu \rightarrow 0} K_{\mu, \eta} \star \rho = \sup_{\mu} K_{\mu, \eta} \star \rho = K_\eta \star \rho$ . Note that  $K_\eta$  is not continuous and in particular it is defined with  $\mathbb{I}_{x > 0}$  and not  $\mathbb{I}_{x \geq 0}$ . This makes a difference if  $\rho$  contains Dirac masses, and as we will see it is the reason why additional calculations are required for the atomic part.

In the meantime integrating in time, taking the supremum in  $\mu$  and using the decomposition of  $f$ , one obtains that

$$\int_0^T \int_{\mathbb{R}^2} (v - u(t, x))_+^{k+2} K_\eta \star \rho(t, x) \rho(t, dx) M(t, x, dv) dt \leq \int_0^T \alpha_\eta^{-1} 2^{k+1} \mathcal{L}_{\eta, 0+, k}(f)(t) dt \rightarrow 0,$$

as  $\eta \rightarrow 0$ , since  $x^{-k}$  is not integrable for  $k \geq 1$  and thus  $\alpha_\eta \rightarrow +\infty$ . Therefore for  $\rho(t, dx) dt$  almost every point  $t$  and  $x$  s.t.

$$\liminf_{\eta \rightarrow 0} K_\eta \star \rho(t, x) > 0, \quad (2.14)$$

one must have that the support of  $M(t, x, .)$  in  $v$  is included in  $(-\infty, u(t, x)]$ . However by their definition, one has that for  $\rho(t, dx) dt$  almost every point  $t$  and  $x$ ,

$$\int_{\mathbb{R}} M(t, x, dv) = u(t, x).$$

Thus at such points  $t$  and  $x$  s.t. (2.14) holds, one must have that  $M(t, x, v) = \delta(v - u(t, x))$ , which is our goal.

In this argument, we treated differently  $x$  and  $y$  in  $\mathcal{L}_{\eta, 0+, k}(f)(t)$ . We can make the symmetric argument, deducing that for  $\rho(t, dy) dt$  almost every point  $t$  and  $y$  s.t.

$$\liminf_{\eta \rightarrow 0} \int_{\mathbb{R}} K_\eta(x - y) \rho(t, dx) > 0,$$

the support of  $M(t, y, .)$  in  $w$  is included in  $[u(t, x), +\infty)$ , and again one must have that  $M(t, y, w) = \delta(w - u(t, x))$ .

Combining those two arguments, we deduce that  $M(t, x, v) = \delta(v - u(t, x))$  for  $\rho(t, dx) dt$  almost every point  $t$  and  $x$  s.t.

$$\liminf_{\eta \rightarrow 0} \int_{\mathbb{R}} (K_\eta(x - y) + K_\eta(y - x)) \rho(t, dy) = \liminf_{\eta \rightarrow 0} \int_{y \neq x} \frac{\rho(t, dy)}{(|x - y| + \eta)^k} > 0.$$

We emphasize that  $\rho(t, dy)$  is only integrated on  $y \neq x$  so that a Dirac mass at  $x$  in  $\rho$  does not contribute to the previous integral. Finally

$$\int_{y \neq x} \frac{\rho(t, dy)}{(|x - y| + \eta)^k} \geq 2\eta^{-1} \int_{B(x, \eta), y \neq x} \rho(t, dy),$$

yielding

$$M(t, x, v) = \delta(v - u(t, x)) \quad \text{for } \rho dt \text{ a.e. } t, x \text{ s.t. } \liminf_{\eta \rightarrow 0} \eta^{-1} \int_{B(x, \eta), y \neq x} \rho(t, dy) > 0. \quad (2.15)$$

To conclude this step, use the classical Besicovitch derivation theorem, which implies that for  $dt \tilde{\rho}$  a.e.  $t, x$ ,

$$\liminf_{\eta \rightarrow 0} \frac{1}{2\eta} \int_{B(x, \eta), y \neq x} \tilde{\rho}(t, dy) = \liminf_{\eta \rightarrow 0} \frac{1}{2\eta} \int_{B(x, \eta)} \tilde{\rho}(t, dy) = \lim_{\eta \rightarrow 0} \frac{1}{2\eta} \int_{B(x, \eta)} \tilde{\rho}(t, dy) > 0,$$

as  $\tilde{\rho}$  does not have any Dirac mass.

This means that for  $dt \tilde{\rho}$  a.e.  $t, x$ ,  $M(t, x, v) = \delta(v - u(t, x))$  and

$$f(t, x, v) = \tilde{\rho}(t, x) \delta(v - u(t, x)) + \sum_{n=1}^{\infty} \rho_n(t) \delta(x - x_n(t)) M(t, x_n, v).$$

STEP 2 (Control of the atomic part). As noticed the previous step does not control the atomic part of  $f$ . Given that  $f$  is  $BV$  in time, by contradiction if  $f$  is not monokinetic at a.e.  $t$ , then there exist  $t_0, x_0, \rho_0 > 0$  and  $M_0(v) \neq \delta(v - u(t_0, x_0))$  s.t.

$$f(t_0+, x, v) = g + \rho_0 \delta(x - x_0) M_0(v), \quad g \geq 0, \quad \int_{\mathbb{R}} M_0(dv) = 1.$$

The main idea then is to use (2.1) to show that in that case the atom at  $x_0$  has to split at  $t > t_+$ . The corresponding pieces will now necessarily interact in  $\mathcal{L}_{\eta, 0+, k}(f)(t)$ , not being at the same point, and this will lead to a contradiction.

Since  $M_0$  is not a Dirac mass, it is possible to find two smooth non-negative functions  $\varphi_1$  and  $\varphi_2$ , supported on distinct intervals  $I_1$  and  $I_2$  s.t.

$$\inf \{(v - w)_+ : v \in I_1, w \in I_2\} \geq C_* > 0 \quad (2.16)$$

and

$$\inf_{i=1, 2} \int_{\mathbb{R}} M_0(dv) \varphi_i(v) \geq \frac{1}{3}.$$

Denote these intervals as  $I_i := [\underline{v}_i, \bar{v}_i]$ , and calculate using (2.1) for  $t > t_0$ :

$$\begin{aligned} \frac{d}{dt} \int_{x_0 + \underline{v}_i(t-t_0)}^{x_0 + \bar{v}_i(t-t_0)} \int_{\mathbb{R}} \varphi_i(v) f(t, dx, dv) &= \int_{\mathbb{R}} \bar{v}_i \varphi_i f(t, x_0 + \bar{v}_i(t-t_0), dv) \\ &- \int_{\mathbb{R}} \underline{v}_i \varphi_i f(t, x_0 + \underline{v}_i(t-t_0), dv) - \int_{x_0 + \underline{v}_i(t-t_0)}^{x_0 + \bar{v}_i(t-t_0)} \int_{\mathbb{R}} \varphi_i(v) v \cdot \partial_x f(t, dx, dv) \\ &- \int_{x_0 + \underline{v}_j t}^{x_0 + \bar{v}_j t} \int_v \varphi_i(v) \partial_{vv} \mu(t, dx, dv). \end{aligned}$$

Integrating by parts the term in  $v \partial_x f$ , we find that

$$\begin{aligned} \frac{d}{dt} \int_{x_0 + \underline{v}_i(t-t_0)}^{x_0 + \bar{v}_i(t-t_0)} \int_{\mathbb{R}} \varphi_i(v) f(t, dx, dv) &= \int_{\mathbb{R}} (\bar{v}_i - v) \varphi_i f(t, x_0 + \bar{v}_i(t-t_0), dv) \\ &\int_{\mathbb{R}} (v - \underline{v}_i) \varphi_i f(t, x_0 + \underline{v}_i(t-t_0), dv) - \int_{x_0 + \underline{v}_j t}^{x_0 + \bar{v}_j t} \int_v \varphi_i(v) \partial_{vv} m(t, dx, dv). \end{aligned}$$

Since  $\varphi_i$  is supported on the interval  $I_i$ , we have there that  $\bar{v}_i - v \geq 0$  and  $v - \underline{v}_i \geq 0$ , so integrating between  $t_0$  and  $t$ ,

$$\int_{x_0 + \underline{v}_i(t-t_0)}^{x_0 + \bar{v}_i(t-t_0)} \int_{\mathbb{R}} \varphi_i(v) f(t, dx, dv) \geq \rho_0 \int_{\mathbb{R}} \varphi_i(v) M_0(dv) - \int_{t_0+}^t \int_{\mathbb{R}^2} |\partial_{vv} \varphi_i| m(dt, dx, dv),$$

and hence for some constant  $C > 0$ ,

$$\int_{x_0 + \underline{v}_i(t-t_0)}^{x_0 + \bar{v}_i(t-t_0)} \int_{\mathbb{R}} \varphi_i(v) f(t, dx, dv) \geq \frac{\rho_0}{3} - C \int_{t_0+}^t \int_{\mathbb{R}^2} m(dt, dx, dv).$$

The measure  $m$  has finite total mass as it can be checked by integrating (2.1) against  $|v|^2$ :

$$\int_0^T \int_{\mathbb{R}^2} m(dt, dx, dv) \leq \int_{\mathbb{R}^2} |v|^2 f^0(dx, dv) < \infty.$$

In particular this implies that

$$\int_{t_0+}^t \int_{\mathbb{R}^2} m(dt, dx, dv) \rightarrow 0, \quad \text{as } t \rightarrow t_0,$$

and that there exists a critical time  $t_c > t_0$  s.t.

$$\int_{t_0+}^{t_c} \int_{\mathbb{R}^2} m(dt, dx, dv) \leq \frac{\rho_0}{6C}.$$

Consequently for any  $t_0 < t < t_c$ ,

$$\inf_{i=1, 2} \int_{x_0 + \underline{v}_i(t-t_0)}^{x_0 + \bar{v}_i(t-t_0)} \int_{\mathbb{R}} \varphi_i(v) f(t, dx, dv) \geq \frac{\rho_0}{6}. \quad (2.17)$$

Inserting this decomposition in  $\mathcal{L}_{\eta, \mu, k}(f)(t)$ , we obtains

$$\begin{aligned} & \int_{t_0}^{t_c} \mathcal{L}_{\eta, 0+, k}(f)(t) dt \\ & \geq \int_{t_0}^{t_c} \int_{x_0 + \underline{v}_1(t-t_0)}^{x_0 + \overline{v}_1} \int_{x_0 + \underline{v}_2(t-t_0)}^{x_0 + \overline{v}_2} \int_{\mathbb{R}^2} \frac{(v-w)_+^{k+2}}{(x-y+\eta)^k} \chi_\mu(x-y) \varphi_1(v) \varphi_2(w) f(t, dx, dv) f(t, dy, dw) dt \\ & \geq \int_{t_0}^{t_c} \int_{x_0 + \underline{v}_1(t-t_0)}^{x_0 + \overline{v}_1} \int_{x_0 + \underline{v}_2(t-t_0)}^{x_0 + \overline{v}_2} \int_{\mathbb{R}^2} \frac{C_*^{k+2}}{(x-y+\eta)^k} \chi_\mu(x-y) \varphi_1(v) \varphi_2(w) f(t, dx, dv) f(t, dy, dw) dt, \end{aligned}$$

by (2.16) since  $\varphi_i$  is supported in  $I_i$ .

If  $x \in [x_0 + \underline{v}_1(t-t_0), x_0 + \overline{v}_1(t-t_0)]$  and  $y \in [x_0 + \underline{v}_2(t-t_0), x_0 + \overline{v}_2(t-t_0)]$ , then by (2.16)

$$x - y \geq (\underline{v}_1 - \overline{v}_2)(t-t_0) \geq C_*(t-t_0).$$

Therefore

$$\begin{aligned} & \int_{t_0}^{t_c} \mathcal{L}_{\eta, 0+, k}(f)(t) dt \\ & \geq \int_{t_0 + \mu/C_*}^{t_c} \int_{x_0 + \underline{v}_1(t-t_0)}^{x_0 + \overline{v}_1} \int_{x_0 + \underline{v}_2(t-t_0)}^{x_0 + \overline{v}_2} \int_{\mathbb{R}^2} \frac{C_*^{k+2}}{(C_*(t-t_0) + \eta)^k} \varphi_1(v) \varphi_2(w) f(t, dx, dv) f(t, dy, dw) dt \\ & \geq \int_{t_0 + \mu/C_*}^{t_c} \frac{C_*^{k+2}}{(C_*(t-t_0) + \eta)^k} \frac{\rho_0^2}{36}, \end{aligned}$$

by (2.17). Finally if  $k > 1$  this implies that

$$\int_{t_0}^{t_c} \mathcal{L}_{\eta, 0+, k}(f)(t) dt \geq \frac{\rho_0^2}{36} \frac{C_*^{k+2}}{k(\mu + \eta)^{k-1}},$$

and if  $k = 1$ , then

$$\int_{t_0}^{t_c} \mathcal{L}_{\eta, 0+, k}(f)(t) dt \geq -\frac{\rho_0^2}{36} C_*^3 \log(\mu + \eta).$$

In both cases, one obtains that

$$\sup_{\eta, \mu} \int_0^T \mathcal{L}_{\eta, 0+, k}(f)(t) dt = \infty,$$

which is a contradiction.  $\square$

### 3. Hydrodynamic limit: Proof of Theorem 1.3.

3.1. *A general hydrodynamic limit.* We prove here a more general version of Theorem 1.3 which can apply to many different systems.

**THEOREM 3.1.** Assume that one has a sequence  $f_\varepsilon \in L^\infty([0, T], M^1(\mathbb{R}^2))$  of solutions to (2.1) with mass 1 for a corresponding sequence of non-negative measures  $m_\varepsilon$ . Assume that all  $v$ -moments of  $f_\varepsilon$  are bounded uniformly in  $\varepsilon$ : For any  $k$ ,

$$\sup_\varepsilon \sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^k f_\varepsilon(t, dx, dv) < \infty,$$

together with one moment in  $x$ , for instance

$$\sup_{\varepsilon} \sup_{t \in [0, T]} \int_{\mathbb{R}^2} |x|^2 f_{\varepsilon}(t, dx, dv) < \infty.$$

Assume moreover that  $f_{\varepsilon}$  satisfies the condition (2.2) with

$$\int_0^T \Lambda_{f_{\varepsilon}, k}(t) dt \longrightarrow 0, \quad \text{as } \varepsilon \rightarrow 0 \text{ for any fixed } k, \quad (3.1)$$

with finally that

$$\sup_{\varepsilon} \sup_{\eta, \mu} \mathcal{L}_{\eta, \mu, 0}(f_{\varepsilon})(t = 0) < \infty. \quad (3.2)$$

Then any weak-\* limit  $f$  of  $f_{\varepsilon}$  solves the sticky particles dynamics in the sense that  $\rho = \int_{\mathbb{R}} f(t, x, dv)$  and  $j = \int_{\mathbb{R}} v f(t, x, dv) = \rho u$  constitute a distributional solution to the pressureless system (1.4), while  $u$  has the Oleinik property for any  $t > 0$ ,

$$u(t, x) - u(t, y) \leq \frac{x - y}{t}, \quad \text{for } \rho \text{ a.e. } x \geq y. \quad (3.3)$$

**REMARK 4.** As already mentioned in the introduction, it is known from [9, 20] that there exists a unique solution  $(\rho, u)$  to the pressureless Euler equations (1.4) (called the *entropy* solution) under the so-called *Oleinik condition* (3.3) for any  $t > 0$  and if the measure  $\rho u^2$  weakly converges to  $\rho_{in} u_{in}^2$  as  $t$  goes to 0. Therefore once  $f$  is known in Theorem 3.1 at some time  $t_0$ , it is necessarily unique after that time  $t_0$ . The only problem for uniqueness can occur at  $t = 0$ . This can be remedied if the initial data is well prepared, for example

$$f^0(x, v) = \rho^0(x) \delta(v - u^0(x)), \quad u^0 \text{ Lipschitz.} \quad (3.4)$$

*Proof.* We divide the proof into distinct steps: First passing to the limit in  $f_{\varepsilon}$  and its moments. Then proving that  $f$  is monokinetic, which implies that  $\rho, j$  solve the pressureless system (1.4) and finally obtaining the *Oleinik condition* (3.3).

STEP 1 (Extracting limits). First of all, since the total mass is 1 at any  $t$ , the sequence  $f_{\varepsilon}$  is uniformly bounded in  $L^{\infty}([0, T], M^1(\mathbb{R}^2))$ . It is possible to extract a subsequence, still denoted  $f_{\varepsilon}$  for simplicity, that converges to some  $f$  in the appropriate weak-\* topology: For any  $\phi \in L^1([0, T], C_c(\mathbb{R}^2))$ ,

$$\int_0^T \int_{\mathbb{R}^2} \Phi(t, x, v) f_{\varepsilon}(t, dx, dv) dt \longrightarrow \int_0^T \int_{\mathbb{R}^2} \Phi(t, x, v) f(t, dx, dv) dt.$$

Since moments up to order at least 3 of  $f_{\varepsilon}$  are uniformly bounded in  $\varepsilon$ , it is also possible to pass to the limit in moments of  $f_{\varepsilon}$  and

$$\begin{aligned} \rho_{\varepsilon} &= \int_{\mathbb{R}} f_{\varepsilon}(t, x, dv) \rightarrow \rho = \int_{\mathbb{R}} f(t, x, dv), & j_{\varepsilon} &= \int_{\mathbb{R}} v f_{\varepsilon}(t, x, dv) \rightarrow j = \int_{\mathbb{R}} v f(t, x, dv), \\ E_{\varepsilon} &= \int_{\mathbb{R}} v^2 f_{\varepsilon}(t, x, dv) \rightarrow E = \int_{\mathbb{R}} v^2 f(t, x, dv), \end{aligned}$$

in the weak-\* topology of  $L^{\infty}([0, T], M^1(\mathbb{R}))$ .

Multiplying (2.1) by  $|v|^2$  one finds that

$$\sup_{\varepsilon} \int_0^T \int_{\mathbb{R}^2} m_{\varepsilon}(dt, dx, dv) \leq \sup_{\varepsilon} \int_{\mathbb{R}} v^2 f_{\varepsilon}(t=0, x, dv) < \infty.$$

Therefore one may further extract a converging subsequence  $m_{\varepsilon} \rightarrow m \geq 0$  in the weak-\* topology of  $M^1([0, T] \times (\mathbb{R}))$ .

This proves that  $f$  and  $m$  still solve (2.1). From the bounded moments of  $f$ , one may integrate this system against 1 first and  $v^2$  second to find the system

$$\begin{aligned} \partial_t \rho + \partial_x j &= 0, \\ \partial_t j + \partial_x E &= 0. \end{aligned} \tag{3.5}$$

STEP 2 ( $f$  is monokinetic). We now apply Theorem 2.1 to  $f_{\varepsilon}$  for  $k=1$  and find from (2.12) that

$$\int_0^T \mathcal{L}_{\eta,0+,1}(f_{\varepsilon})(t) dt \leq \mathcal{L}_{\eta,0+,0}(f_{\varepsilon})(t=0) + 2 |\log \eta| \int_0^T \Lambda_{f_{\varepsilon},2}(t) dt.$$

This means in particular that for any  $\mu > 0$ ,

$$\int_0^T \mathcal{L}_{\eta,\mu,1}(f_{\varepsilon})(t) dt \leq \mathcal{L}_{\eta,0+,0}(f_{\varepsilon})(t=0) + 2 |\log \eta| \int_0^T \Lambda_{f_{\varepsilon},2}(t) dt.$$

We use Lemma 2.1 on the sequence  $f_{\varepsilon}$  to obtain that for any  $\mu > 0$  and  $\eta > 0$ ,

$$\int_0^T \mathcal{L}_{\eta,\mu,1}(f_{\varepsilon})(t) dt \rightarrow \int_0^T \mathcal{L}_{\eta,\mu,1}(f)(t) dt.$$

By the assumptions of Theorem 3.1 we also have that  $\int_0^T \Lambda_{f_{\varepsilon},2}(t) dt \rightarrow 0$  and that  $C := \sup_{\varepsilon, \eta} \mathcal{L}_{\eta,0+,0}(f_{\varepsilon})(t=0) < \infty$ . Thus

$$\int_0^T \mathcal{L}_{\eta,\mu,1}(f)(t) dt \leq C,$$

and in particular

$$\sup_{\mu, \eta} \int_0^T \mathcal{L}_{\eta,\mu,1}(f)(t) dt < \infty.$$

We may now apply Proposition 2.1, which implies that  $f$  is monokinetic, that is,  $f = \rho(t, x) \delta(v - u(t, x))$ , while  $u$  satisfies that for any  $k$ ,

$$\sup_{[0, T]} \int_{\mathbb{R}} |u(t, x)|^k \rho(t, dx) < \infty.$$

Therefore one automatically has that  $j = \rho u$  and  $E = \rho |u|^2$ . From system (3.5),  $\rho$  and  $\rho u$  solve the pressureless gas dynamics (1.4).

STEP 3 (The Oleinik condition). We only have to show that  $u$  is semi-Lipschitz in the sense of (3.3). Since all moments of  $f_{\varepsilon}$  are bounded, we may apply Theorem 2.1 to  $f_{\varepsilon}$  for any  $k$ , for which we repeat the conclusion

$$\mathcal{L}_{\eta,0+,k-1}(f_{\varepsilon})(t) + k \int_s^t \mathcal{L}_{\eta,0+,k}(f_{\varepsilon})(r) dr \leq \mathcal{L}_{\eta,0+,k-1}(f_{\varepsilon})(s) + \frac{2}{\eta^{k-1}} \int_s^t \Lambda_{f_{\varepsilon},k+2}(r) dr. \tag{3.6}$$

Observe that by a simple Hölder inequality

$$\begin{aligned} \mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t) &= \int_{\mathbb{R}^4} \mathbb{I}_{x>y} \frac{(v-w)_+^{k-1}}{(x-y+\eta)^{k-1}} (v-w)_+^2 f_\varepsilon f'_\varepsilon \\ &\leq \left( \int_{\mathbb{R}^4} \mathbb{I}_{x>y} \frac{(v-w)_+^k}{(x-y+\eta)^k} (v-w)_+^2 f_\varepsilon f'_\varepsilon \right)^{(k-1)/k} \left( \int_{\mathbb{R}^4} (v-w)_+^2 f_\varepsilon f'_\varepsilon \right)^{1/k}. \end{aligned}$$

Therefore since  $\int |v|^2 f_\varepsilon(dx, dv)$  is uniformly bounded in  $\varepsilon$ , for some uniform constant  $C$  one has that

$$\mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t) \leq C^{1/k} (\mathcal{L}_{\eta,0+,k}(f_\varepsilon)(t))^{(k-1)/k},$$

which from the inequality (3.6) leads to, for a.e.  $s < t$ ,

$$\begin{aligned} \mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t) + k C^{-\frac{1}{k-1}} \int_s^t (\mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(r))^{\frac{k}{k-1}} dr \\ \leq \mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(s) + \frac{2}{\eta^{k-1}} \int_s^t \Lambda_{f_\varepsilon,k+2}(r) dr. \end{aligned}$$

This is now a closed inequality on  $\mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t)$ . In order to derive a bound in a simple manner, assume momentarily that  $\Lambda_{f_\varepsilon,k+2}$  is  $L^\infty$  in time, or more precisely approximate it by such a bounded function. Then the inequality would imply that  $\mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)$  is Lipschitz and could be rewritten in the more direct form

$$\frac{d}{dt} \mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t) \leq -k C^{-\frac{1}{k-1}} (\mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(r))^{\frac{k}{k-1}} + \frac{2}{\eta^{k-1}} \Lambda_{f_\varepsilon,k+2}(t).$$

We introduce the intermediary quantity  $M(t) = t^{k-1} \mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t)$ , which now satisfies

$$\frac{dM}{dt} \leq (k-1) \frac{M - C^{-\frac{1}{k-1}} M^{1+\frac{1}{k-1}}}{t} + \frac{2 t^{k-1}}{\eta^{k-1}} \Lambda_{f_\varepsilon,k+2}(t).$$

At a given point  $t$ , either  $M \leq C$  or

$$\frac{dM}{dt} \leq \frac{2 t^{k-1}}{\eta^{k-1}} \Lambda_{f_\varepsilon,k+2}(t).$$

Therefore obviously

$$M(t) \leq C + \frac{2 T^{k-1}}{\eta^{k-1}} \int_0^T \Lambda_{f_\varepsilon,k+2}(r) dr.$$

This final bound now depends only on the  $L^1$  norm of  $\Lambda_{f_\varepsilon,k+2}$  (and thus is independent of the chosen approximation of  $\Lambda_{f_\varepsilon,k+2}$ ), leading to the inequality

$$t^{k-1} \mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t) \leq C + \frac{2 T^{k-1}}{\eta^{k-1}} \int_0^T \Lambda_{f_\varepsilon,k+2}(r) dr.$$

Integrating this inequality between 0 and  $T$  and recalling that  $\mathcal{L}_{\eta,\mu,k-1}(f_\varepsilon)(t) \leq \mathcal{L}_{\eta,0+,k-1}(f_\varepsilon)(t)$ , one obtains that for any  $\mu > 0$  and  $\eta > 0$ ,

$$\int_0^T r^{k-1} \mathcal{L}_{\eta,\mu,k-1}(f_\varepsilon)(r) dr \leq C T + \frac{2 T^k}{\eta^{k-1}} \int_0^T \Lambda_{f_\varepsilon,k+2}(r) dr.$$

Because of  $r^{k-1}$  it is now possible to pass to the limit as  $\varepsilon \rightarrow 0$  by Lemma 2.1. Recall that from the assumption of Theorem 3.1,  $\int_0^T \Lambda_{f_\varepsilon, k+2}(r) dr \rightarrow 0$  to obtain

$$\int_0^T r^{k-1} \mathcal{L}_{\eta, \mu, k-1}(f)(r) dr \leq CT.$$

Take the supremum in  $\mu$  to find from Lemma 2.1 that

$$\int_0^T r^{k-1} \mathcal{L}_{\eta, 0+, k-1}(f)(r) dr \leq CT,$$

or recalling the definition of  $\mathcal{L}_{\eta, 0+, k-1}(f)$  and the fact that  $f$  is monokinetic,

$$\int_0^T \int_{\mathbb{R}^2} \mathbb{I}_{x > y} (u(t, x) - u(t, y))_+^2 \left( t \frac{(u(t, x) - u(t, y))_+}{(x - y + \eta)} \right)^{k-1} \rho(t, dx) \rho(t, dy) dt \leq CT.$$

For a fixed  $\eta$ , take the limit  $k \rightarrow \infty$  in this inequality. The only possibility for the left-hand side to remain bounded is that on the support of  $\mathbb{I}_{x > y} \rho(t, x) \rho(t, y)$ , one has that

$$t \frac{(u(t, x) - u(t, y))_+}{(x - y + \eta)} \leq 1.$$

This is uniform in  $\eta$ , and thus passing finally to the limit  $\eta \rightarrow 0$ , one recovers the Oleinik bound (3.3).  $\square$

**3.2. Proof of Theorem 1.3.** Let us start by checking that  $f_\varepsilon$  is a solution to (2.1). Given that  $f_\varepsilon$  solves (1.1), this is equivalent to showing that for any  $\alpha$  and any  $f$  the collision kernel  $\mathcal{Q}_\alpha(f, f)$  can be represented as  $-\partial_{vv} m$  for some non-negative measure  $m$ .

Thus we have to show that

$$\int_{\mathbb{R}} \mathcal{Q}_\alpha(f, f) dv = 0, \quad \int_{\mathbb{R}} v \mathcal{Q}_\alpha(f, f) dv = 0,$$

which is just the conservation of mass and momentum, and that for any  $\psi(v)$  with  $\partial_{vv}\psi \geq 0$ , that is,  $\psi$  convex,

$$\int_{\mathbb{R}} \psi(v) \mathcal{Q}_\alpha(f, f) dv \leq 0.$$

This is a consequence of the weak formulation of the operator (1.7), which reads as we recall that for any smooth test function  $\psi$ ,

$$\int_{\mathbb{R}} \psi(v) \mathcal{Q}_\alpha(f, f) dv = \frac{1}{2} \int_{\mathbb{R}^2} |v - v_*| f(v_*) f(v) (\psi(v') + \psi(v'_*) - \psi(v_*) - \psi(v)) dv dv_*. \quad (3.7)$$

Now rewriting  $v'$  and  $v'_*$ ,

$$\begin{aligned} \psi(v') + \psi(v'_*) - \psi(v_*) - \psi(v) &= \psi\left(\frac{1+\alpha}{2} v + \frac{1-\alpha}{2} v_*\right) + \psi\left(\frac{1-\alpha}{2} v + \frac{1+\alpha}{2} v_*\right) - \psi(v_*) - \psi(v) \\ &\leq 0, \end{aligned}$$

if  $\psi$  convex for  $\alpha < 1$ .

This implies that propagating moments is easy:

$$\frac{d}{dt} \int_{\mathbb{R}^2} |v|^k f_\varepsilon dx dv = \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |v|^k Q_\alpha(f_\varepsilon, f_\varepsilon) dx dv \leq 0.$$

This immediately proves that

$$\sup_{\varepsilon} \sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^k f_{\varepsilon}(t, x, v) dx dv \leq \sup_{\varepsilon} \int_{\mathbb{R}^2} |v|^k f_{\varepsilon}^0(x, v) dx dv < \infty.$$

Next note that

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 f_{\varepsilon}(t, x, v) dx dv = 2 \int_{\mathbb{R}^2} x \cdot v f_{\varepsilon}(t, x, v) dx dv \leq \int_{\mathbb{R}^2} (|x|^2 + |v|^2) f_{\varepsilon}(t, x, v) dx dv,$$

so that

$$\sup_{\varepsilon} \sup_{t \in [0, T]} \int_{\mathbb{R}^2} |x|^2 f_{\varepsilon}(t, x, v) dx dv \leq e^T \sup_{\varepsilon} \int_{\mathbb{R}^2} (|x|^2 + |v|^2) f_{\varepsilon}^0(x, v) dx dv < \infty.$$

In addition the dissipation term from the  $v$ -moments actually leads to a control on  $\Lambda_{f_{\varepsilon}, k}(t)$ . Since we assumed that  $f_{\varepsilon} \in L^{\infty}([0, T], L^p(\mathbb{R}^2))$  for  $p > 2$  and every moment of  $f_{\varepsilon}$  is bounded then for any fixed  $v$ , we find that

$$\int_{\mathbb{R}} (v - w)_+^k f_{\varepsilon}(t, x, w) dw$$

is bounded in  $L^2([0, T] \times \mathbb{R})$ , and by standard approximation by convolution

$$\int_{\mathbb{R}^2} \frac{\mathbb{I}_{x < y < \delta}}{\delta} (v - w)_+^k f_{\varepsilon}(t, y, w) dy dw - \int_{\mathbb{R}} (v - w)_+^k f_{\varepsilon}(t, x, w) dw \longrightarrow 0,$$

in  $L^2([0, T] \times \mathbb{R})$  as  $\delta \rightarrow 0$ . Of course this convergence only holds for a fixed  $\varepsilon$  (and is not in principle uniform in  $\varepsilon$ ). But for a fixed  $\varepsilon$ , it now directly implies that for a.e.  $t$ ,

$$\Lambda_{f_{\varepsilon}, k} = \int_{\mathbb{R}^3} (v - w)_+^k f_{\varepsilon}(t, x, v) f_{\varepsilon}(t, x, w) dx dv dw.$$

As suggested in the introduction for the energy,  $k = 2$ , this term is then controlled by the dissipation of the moment of order  $k$ . More precisely if  $\psi(v) = |v|^k$ , then for some  $C_k > 0$ ,

$$\psi(v') + \psi(v'_*) - \psi(v_*) - \psi(v) \geq \frac{|v - v_*|^k}{C_k}.$$

Therefore

$$\int_0^T \Lambda_{f_{\varepsilon}, k} \leq C_k \varepsilon \int_0^T \int_{\mathbb{R}^2} \mathcal{Q}_{\alpha}(f_{\varepsilon}, f_{\varepsilon}) dx dv dt \leq C_k \varepsilon \int_{\mathbb{R}^2} |v|^k f_{\varepsilon}^0(x, v) dx dv \longrightarrow 0,$$

as  $\varepsilon \rightarrow 0$ .

The last assumption of Theorem 3.1 to check is a bound  $\mathcal{L}_{\eta, \mu, 0}(f_{\varepsilon})(t = 0)$  uniformly in  $\varepsilon$ ,  $\eta$ ,  $\mu$ . This follows from the uniform  $L^p$  bound on  $f_{\varepsilon}^0$  through a straightforward Hölder

estimate to compensate for the log singularity. Denote  $q = \frac{1+p}{2}$  and  $q^*$  s.t.  $1/q^* = 1 - 1/q$ ; then

$$\begin{aligned} \mathcal{L}_{\eta, \mu, 0}(f_\varepsilon)(t=0) &= - \int_{\mathbb{R}^4} \chi_\mu(x-y) |v-w|_+^2 \log(x-y+\eta) f_\varepsilon^0(x, v) f_\varepsilon^0(y, w) dx dy dv dw \\ &\leq 2 \int_{\mathbb{R}^4} \mathbb{I}_{|x-y|\leq 2} |\log|x-y|| |v|^2 f_\varepsilon^0(x, v) f_\varepsilon^0(y, w) dx dy dv dw \\ &\leq 2 \int_{\mathbb{R}^2} |v|^2 f_\varepsilon^0(x, v) \left( \int_{x-2}^{x+2} \int_{\mathbb{R}} \frac{|\log|x-y||^{q^*}}{1+|w|^2} dy dw \right)^{1/q^*} \\ &\quad \times \left( \int_{\mathbb{R}^2} (1+|w|^2) |f_\varepsilon^0(y, w)|^q dy dw \right)^{1/q} dx dv \\ &\leq C \int_{\mathbb{R}^2} |v|^2 f_\varepsilon^0(x, v) dx dv \left( \int_{\mathbb{R}^2} (1+|w|^2) |f_\varepsilon^0(y, w)|^q dy dw \right)^{1/q}, \end{aligned}$$

since  $|\log x|^l$  is integrable at 0 for any  $l > 0$ . Finally by Cauchy-Schwartz

$$\int_{\mathbb{R}^2} (1+|w|^2) |f_\varepsilon^0(y, w)|^q dy dw \leq \left( \int_{\mathbb{R}^2} (1+|w|^2)^2 f_\varepsilon^0(y, w) dy dw \right)^{1/2} \left( \int_{\mathbb{R}^2} |f_\varepsilon^0(y, w)|^p dy dw \right)^{1/2},$$

which gives

$$\mathcal{L}_{\eta, \mu, 0}(f_\varepsilon)(t=0) \leq C \|f_\varepsilon^0\|_{L^p}^{p/2} \left( 1 + \int_{\mathbb{R}^2} |v|^4 f_\varepsilon^0(x, v) dx dv \right)^{3/2}$$

and the uniform bound.

Since the sequence  $f_\varepsilon$  satisfies all the assumptions of Theorem 3.1, its conclusions apply, thus proving Theorem 1.3.

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