Backward Reachability using Integral Quadratic Constraints for Uncertain Nonlinear Systems

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***Abstract*—A method is proposed to compute robust inner-approximations to the backward reachable set of nonlinear systems and to generate a robust control law that drives trajectories starting in these inner-approximations to a target set. The method merges dissipation inequalities and integral quadratic constraints (IQCs) with both hard and soft IQC factorizations, allowing for a variety of perturbations including parametric uncertainty, unmodeled dynamics, and uncertain time delays. Computational algorithms are developed using the generalized S-procedure and sum-of-squares techniques, and illustrated on a 6-state quadrotor with actuator uncertainties.**

I. Introduction

The backward reachable set (BRS) is the set of initial conditions whose successors can be driven to a target set at the end of a finite time horizon with an admissible controller. The BRS is of vital importance for safety-critical systems, since it indicates where the trajectories should start from in order to reach the target set.

Backward reachability has been studied with several approaches. Occupation measure-based methods [1]–[3] compute BRS outer-approximations, but do not guarantee reaching the target set. In contrast, the exact BRS is computed in [4]–[6] as the sublevel set of the solution to Hamilton-Jacobi (HJ) partial differential equations. Other results provide BRS inner-approximations using relaxed HJ equations [7], [8] and Lyapunov-based methods [9].

A shortcoming of the existing reachability tools is that they rely on accurate system models. Only limited forms of uncertainty have been addressed, such as parametric uncertainty in [3], [4], [7]–[9] and both parametric uncertainty and $\mathcal{L}_2$ disturbances in our earlier work [10], [11].

In this paper, we propose a method to compute inner-approximations to the BRS that are robust to a more general class of perturbations. We model the uncertain nonlinear system as an interconnection of the nominal system G and the perturbation Δ, as in Fig. 1. The input-output relationship of Δ is described using the integral quadratic constraint (IQC) framework [12], [13], which accounts for parametric uncertainties, unmodeled dynamics, and uncertain time delays. We characterize BRS inner-approximations by sublevel sets of storage functions that satisfy a dissipation inequality that is compatible with IQCs. We then formulate an iterative convex optimization procedure to compute storage functions and associated control laws using the generalized S-procedure [14] and sum-of-squares (SOS) techniques [15].

The specific contributions of this paper are threefold. First, we propose a general framework for robust backward reachability of uncertain nonlinear systems, allowing for various types of uncertainty beyond parametric uncertainty. Second, we incorporate both hard and soft IQC factorizations in the framework. The use of dissipation inequalities typically requires IQCs that are valid over any finite time horizon, known as hard IQCs. However, many IQCs are specified in the frequency domain, which are equivalent to time-domain constraints over infinite horizons (soft IQCs). We obtain improved BRS bounds by incorporating soft IQCs by means of the finite-horizon bound derived in [13]. Third, we overcome a technical challenge that arises when the input of the perturbation Δ depends directly on the control command, as in the case of actuator uncertainty. This dependence creates a source of nonconvexity, which we circumvent by introducing auxiliary states in the control law.

In [16] we employed IQCs for *forward* reachability analysis and did not pursue control synthesis. Here we study *backward* reachability and produce a control law, while also maximizing the volume of the BRS inner-approximation.

Notation: $\mathbb{S}^n$ denotes the set of n-by-n real, symmetric matrices. $\mathbb{RL}_\infty$ is the set of rational functions with real coefficients that have no poles on the imaginary axis. $\mathbb{RH}_\infty \subset \mathbb{RL}_\infty$ contains functions that are analytic in the closed right-half of the complex plane. $\mathcal{L}_2^{n_r}$ is the space of measurable functions $r : [0,\infty) \to \mathbb{R}^{n_r}$ with $\|r\|_2^2 := \int_0^\infty r(t)^\top r(t) dt < \infty$. Associated with $\mathcal{L}_2^{n_r}$ is the extended space $\mathcal{L}_{2e}^{n_r}$ of functions whose truncation $r_T(t) := r(t)$ for $t \le T$; $r_T(t) := 0$ for $t > T$, is in $\mathcal{L}_2^{n_r}$ for all $T > 0$. Define the finite-horizon $\mathcal{L}_2$ norm as $\|r\|_{2,[0,T]} := \left(\int_0^T r(t)^\top r(t) dt\right)^{1/2}$. $\mathcal{L}_2^{n_r}[0,T]$ is the space of finite-horizon $\mathcal{L}_2$-measurable functions $r : [0,T] \to \mathbb{R}^{n_v}$ with $\|r\|_{2,[0,T]} < \infty$. The finite horizon induced $\mathcal{L}_2$ to $\mathcal{L}_2$ norm of an operator is denoted as $\|\cdot\|_{2\to 2,[0,T]}$. For $\xi \in \mathbb{R}^n$, $\mathbb{R}[\xi]$ represents the set of polynomials in ξ with real coefficients, and $\mathbb{R}^m[\xi]$ and $\mathbb{R}^{m\times p}[\xi]$ denote all vector and matrix valued polynomial functions. The subset $\Sigma[\xi] := \{\pi = \sum_{i=1}^M \pi_i^2 : \pi_1, ..., \pi_M \in \mathbb{R}[\xi]\}$ of $\mathbb{R}[\xi]$ is the set of SOS polynomials in ξ. For $\eta \in \mathbb{R}$, and continuous $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, define $\Omega(g,t,\eta) := \{x \in \mathbb{R}^n : g(t,x) \le \eta\}$, a t-dependent set. KYP denotes a mapping to the block 2-by-2 matrix: $KYP(Y,A,B,C,D,M) := \begin{bmatrix} A^\top Y + YA & YB \\ B^\top Y & 0 \end{bmatrix} + [\,C\ D\,]^\top M [\,C\ D\,]$.

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II. BACKWARD REACHABILITY WITH HARD IQCS

A. Problem Setup

Consider an uncertain nonlinear system defined on $[0, T]$:

$$\dot{x}_G(t) = f(x_G(t), w(t), d(t)) + g(x_G(t), w(t), d(t))u(t), \tag{1a}$$

$$v(t) = h(x_G(t), w(t), d(t)), \tag{1b}$$

$$w(\cdot) = \Delta(v(\cdot)), \tag{1c}$$

which is an interconnection (Fig. 1a) of the nominal system G and the perturbation Δ, denoted as $F_u(G, \Delta)$. In (1), $x_G(t) \in \mathbb{R}^{n_G}$ is the state, $u(t) \in U \subseteq \mathbb{R}^{n_u}$ is the control input, $d(t) \in \mathbb{R}^{n_d}$ is the external disturbance, and $v(t) \in \mathbb{R}^{n_v}$ and $w(t) \in \mathbb{R}^{n_w}$ are the inputs and outputs of Δ. The mappings f, g, and h define the nominal system G. The perturbation $\Delta : \mathcal{L}_{2e}^{n_v} \to \mathcal{L}_{2e}^{n_w}$ is a bounded and causal operator. Note that in (1b), v does not depend directly on u. Well-posedness of $F_u(G, \Delta)$ is defined as follows.

Definition 1. *$F_u(G, \Delta)$ is well-posed if for all $x_G(t_0) \in \mathbb{R}^{n_G}$ and $d \in \mathcal{L}_{2e}^{n_d}$ there exist unique solutions $x_G \in \mathcal{L}_{2e}^{n_G}$, $v \in \mathcal{L}_{2e}^{n_v}$, and $w \in \mathcal{L}_{2e}^{n_w}$ satisfying (1) with a causal dependence on d.*

Assumption 1. *(i) d satisfies $d \in \mathcal{L}_2^{n_d}$ with:*

$$\|d\|_{2,[0,T]} < R, \text{ for some } R > 0, \text{ and} \tag{2}$$

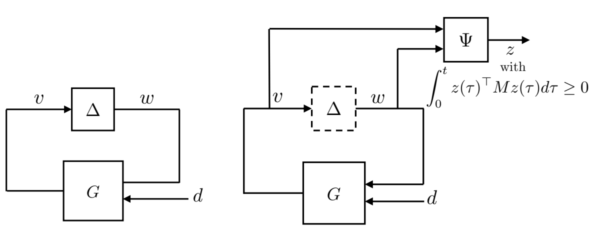
(ii) the set of control constraints is given as a polytope $U := \{u \in \mathbb{R}^{n_u} : Pu \leq b\}$, where $P \in \mathbb{R}^{n_p \times n_u}$ and $b \in \mathbb{R}^{n_p}$.

Let $x_G(t; \xi, u, d)$ define the solution to the uncertain system (1), at time t $(0 \leq t \leq T)$, from the initial condition ξ, under the control u and the disturbance d. Let $X_T \subset \mathbb{R}^{n_G}$ denote the target set for $x_G(t; \xi, u, d)$ to reach at time T.

Definition 2. *Under Assumption 1, the BRS of $F_u(G, \Delta)$ (1) is defined as $BRS(T, X_T, U, R, F_u(G, \Delta)) :=$*

$$\{\xi \in \mathbb{R}^{n_G} : \exists\ u, \text{ s.t. } u(t) \in U\ \forall\ t \in [0, T], \text{ and } x_G(T; \xi, u, d) \in X_T\ \forall\ d \text{ with } \|d\|_{2,[0,T]} < R\}.$$

Our goal is to compute an BRS inner-approximation and an associated controller.



(a) $F_u(G, \Delta)$ (b) Extended system

Fig. 1: The original uncertain system $F_u(G, \Delta)$ and the extended system of G and Ψ.

B. Integral Quadratic Constraints

The perturbation Δ can represent various types of uncertainties and nonlinearities, including parametric uncertainty, unmodeled dynamics, slope-bounded nonlinearities, and uncertain time delays [12], [13]. To characterize Δ with an integral quadratic constraint (IQC) we apply a 'virtual' filter Ψ to the input v and output w of Δ, and impose quadratic constraints on its output z. Ψ is an LTI system driven by (v, w), with zero initial condition $x_\psi(0) = 0$, and dynamics:

$$\dot{x}_\psi(t) = A_\psi x_\psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \tag{3a}$$

$$z(t) = C_\psi x_\psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t), \tag{3b}$$

where $x_\psi(t) \in \mathbb{R}^{n_\psi}$ is the state, and $z(t) \in \mathbb{R}^{n_z}$ is the output. IQCs can be defined in both the time and frequency domain. The use of time domain IQCs is required for the dissipation-type results used later in this paper. Time domain IQCs consist of hard IQCs and soft IQCs, which are quadratic constraints on z over finite and infinite horizons, respectively. In this section, we focus on the analysis with hard IQCs.

Definition 3. *Given $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v + n_w)}$ and $M \in \mathbb{S}^{n_z}$. A bounded, causal operator $\Delta : \mathcal{L}_{2e}^{n_v} \to \mathcal{L}_{2e}^{n_w}$ satisfies the hard IQC defined by (Ψ, M) if, $\forall v \in \mathcal{L}_{2e}^{n_v}$, and $w = \Delta(v)$,*

$$\int_0^t z(\tau)^\top M z(\tau) d\tau \geq 0,\ \forall\ t \in [0, T]. \tag{4}$$

where $z = \Psi \begin{bmatrix} v \\ w \end{bmatrix}$ is defined in (3b).

The notation $\Delta \in \text{HardIQC}(\Psi, M)$ indicates that Δ satisfies the hard IQC defined by (Ψ, M).

Example 1. *(a) Consider the set of LTI uncertainties with a given norm bound $\sigma > 0$: $\Delta \in \mathbb{RH}_\infty$ with $\|\Delta\|_\infty \leq \sigma$. It is proven in [17] that $\Delta \in \text{HardIQC}(\Psi, M_D)$, where $\Psi = \texttt{blkdiag}(\Psi_{11}, \Psi_{11})$ with $\Psi_{11} \in \mathbb{RH}_\infty^{n_z \times 1}$ and*

$$M_D \in \mathcal{M}_1 := \left\{ \begin{bmatrix} \sigma^2 M_{11} & 0 \\ 0 & -M_{11} \end{bmatrix} : M_{11} \succeq 0 \right\}. \tag{5}$$

A typical choice for Ψ_{11} [13] is

$$\Psi_{11}^{d,m} = \left[1, \tfrac{1}{(s+m)}, \ldots, \tfrac{1}{(s+m)^d} \right]^\top, \text{ with } m > 0, \tag{6}$$

where m and d are selected by the user.

(b) Consider the set of nonlinear, time varying, uncertainties with a given norm-bound σ: $\|\Delta\|_{2\to 2,[0,T]} \leq \sigma$. Δ satisfies the hard IQCs defined by $\Psi = I_{n_v + n_w}$ and

$$M \in \mathcal{M}_2 := \left\{ \begin{bmatrix} \sigma^2 \lambda I_{n_v} & 0 \\ 0 & -\lambda I_{n_w} \end{bmatrix} : \lambda \geq 0 \right\}. \tag{7}$$

C. Robust Backward Reachability

As illustrated in the previous examples, each type of Δ can be characterized by corresponding hard IQCs associated with a filter Ψ and a matrix M. The analysis on $F_u(G, \Delta)$ can be instead performed on the extended system shown in Fig. 1b, with an additional constraint $\Delta \in \text{HardIQC}(\Psi, M)$. The extended system is an interconnection of G and Ψ, with combined state vector $x := [x_G; x_\psi] \in \mathbb{R}^n$, $n = n_G + n_\psi$, whose dynamics can be rewritten as

$$\dot{x}(t) = F(x(t), w(t), d(t), u(t)), \tag{8a}$$

$$z(t) = H(x(t), w(t), d(t)), \tag{8b}$$

where F and H depend on the dynamics of G and Ψ. F is still affine in u. For any input $d \in \mathcal{L}_2^{n_d}$ and initial condition $x_G(t_0) \in \mathbb{R}^{n_G}$, the solutions $v \in \mathcal{L}_{2e}^{n_v}$ and $w \in \mathcal{L}_{2e}^{n_w}$ to $F_u(G, \Delta)$ (1) satisfy the IQC (4). The extended system (8) with the IQC (4) "covers" the responses of the original uncertain system $F_u(G,\Delta)$. Indeed, given any input $d \in \mathcal{L}_2^{n_d}$ and initial condition $x_G(t_0) \in \mathbb{R}^{n_G}$, the input $w \in \mathcal{L}_{2e}^{n_w}$ is implicitly constrained in the extended system so that the pair (v, w) satisfies the IQC (4). This set of (v, w) that satisfies the IQC (4) includes all input/output pairs of Δ.

We consider the memoryless, time-varying state-feedback control $u(t) = k(t, x_G(t))$, $k : \mathbb{R} \times \mathbb{R}^{n_G} \to \mathbb{R}^{n_u}$. We don't allow k to depend on x_ψ, since x_ψ is introduced by the virtual filter Ψ. The following theorem provides a BRS inner-approximation for the extended system G and Ψ, and therefore for the original uncertain system $F_u(G,\Delta)$.

Theorem 1. *Let Assumption 1 hold, and further assume (i) $F_u(G,\Delta)$ is well-posed, (ii) $\Delta \in$ HardIQC(Ψ, M), with Ψ and M given. Given $X_T \subset \mathbb{R}^{n_G}$, $P \in \mathbb{R}^{n_p \times n_u}$, $b \in \mathbb{R}^{n_p}$, $R > 0$, F, H defined in* (8)*, $T > 0$, and $\gamma \in \mathbb{R}$, if there exists a $\mathcal{C}^1$ function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, and a control law $k : \mathbb{R} \times \mathbb{R}^{n_G} \to \mathbb{R}^{n_u}$ that is continuous in t and locally Lipschitz in x_G, such that*

$$\begin{aligned} &\partial_t V(t,x) + \partial_x V(t,x) \cdot F(x,w,d,k) + z^\top M z \le d^\top d, \\ &\quad \forall\ (t,x,w,d) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_d}, \\ &\quad \text{s.t. } V(t,x) \le \gamma + R^2, \end{aligned} \tag{9}$$

$$\{x_G : V(T,x) \le \gamma + R^2\} \subseteq X_T,\ \forall\ x_\psi \in \mathbb{R}^{n_\psi}, \tag{10}$$

$$\begin{aligned} &P_i k(t,x_G) \le b_i, \forall (t,x) \in [0,T] \times \mathbb{R}^n, \\ &\quad \text{s.t. } V(t,x) \le \gamma + R^2,\ \forall\ i = 1,...,n_p, \end{aligned} \tag{11}$$

where $x = [x_G; x_\psi]$, then the intersection of $\Omega(V,0,\gamma)$ with the hyperplane $x_\psi = 0$ is an inner-approximation to $BRS(T, X_T, U, R, F_u(G,\Delta))$ under the control law k.

Proof. Since the dissipation inequality (9) only holds on the region $\Omega(V,t,\gamma+R^2)$, we first need to prove that all the state trajectories starting from $\Omega(V,0,\gamma)$ won't leave $\Omega(V,t,\gamma+R^2)$ for all $t \in [0,T]$. This is proven by contradiction. Assume there exists a time instance $T_1 \in [0,T]$, such that a trajectory starting from $x(0) \in \Omega(V,0,\gamma)$ satisfies $V(T_1, x(T_1)) > \gamma + R^2$. Define $T_2 = \inf_{V(t,x(t)) > \gamma + R^2} t$, and integrate (9) over $[0, T_2]$:

$$\begin{aligned} &V(T_2, x(T_2)) - V(0, x(0)) \\ &\quad + \int_0^{T_2} z(t)^\top M z(t) dt \le \int_0^{T_2} d(t)^\top d(t) dt. \end{aligned}$$

Apply $x(0) \in \Omega(V,0,\gamma)$ and $\Delta \in \text{HardIQC}(\Psi, M)$ to show

$$V(T_2, x(T_2)) \le \gamma + \int_0^{T_2} d(t)^\top d(t) dt.$$

Next recall that d is assumed to satisfy (2):

$$\gamma + R^2 = V(T_2, x(T_2)) < \gamma + R^2,$$

which is a contradiction. As a result, $x(0) \in \Omega(V,0,\gamma)$ implies $x(t) \in \Omega(V,t,\gamma+R^2)$ for all $t \in [0,T]$, and thus $V(T, x(T)) \le \gamma + R^2$. Combining it with (10) shows that $\Omega(V,0,\gamma)$ is an inner-approximation to the BRS of the extended system, and the intersection of $\Omega(V,0,\gamma)$ with $x_\psi = 0$ is an inner-approximation to $BRS(T, X_T, U, R, F_u(G,\Delta))$. Lastly, constraint (11) ensures the control signal derived from $u(t) = k(t, x_G(t))$ satisfies the control constraints $Pu(t) \le b\ \forall\ t \in [0,T]$. □

D. Robust Backward Reachability with SOS Programming

To find a V and a k satisfying (9)–(11), we make use of sum-of-squares (SOS) programming. To do so, we restrict the decision variables to polynomials $V \in \mathbb{R}[(t,x)]$, $k \in \mathbb{R}^{n_u}[(t, x_G)]$, and make the following assumption.

Assumption 2. *The nominal system G given in* (1) *has polynomial dynamics: $f \in \mathbb{R}^{n_G}[(x_G, w, d)]$, $g \in \mathbb{R}^{n_G \times n_u}[(x_G, w, d)]$, and $h \in \mathbb{R}^{n_v}[(x_G, w, d)]$. Therefore, F and H in* (8) *are polynomials. X_T is a semi-algebraic set: $X_T := \{x_G : p_x(x_G) \le 0\}$, where $p_x \in \mathbb{R}[x_G]$ is provided.*

In Example 1, we have seen that for each type of perturbation, any IQC defined by a properly chosen Ψ and a M drawn from the constraint set $\mathcal{M}$ is valid. Therefore, along with V and k, we also treat $M \in \mathcal{M}$ as a decision variable. Assume $\mathcal{M}$ is described by linear matrix inequalities [13]. Define $p_t := t(T-t)$, which is nonnegative for all $t \in [0,T]$. By applying the generalized S-procedure [14] to (9)–(11), and choosing the volume of $\Omega(V,0,\gamma)$ as the objective function (to be maximized), we obtain the following optimization:

$$\begin{aligned} \sup_{V,M,k,s_i}\ & \text{Volume}(\Omega(V,0,\gamma)) \\ \text{s.t. }\ & V \in \mathbb{R}[(t,x)], k \in \mathbb{R}^{n_u}[(t,x_G)], M \in \mathcal{M}, \\ & -(\partial_t V + \partial_x V \cdot F|_{u=k} + z^\top M z - d^\top d) - s_1 p_t \\ & \quad + (V - \gamma - R^2) s_2 \in \Sigma[(t,x,w,d)], && \text{(12a)} \\ & -s_3 p_x + V|_{t=T} - \gamma - R^2 \in \Sigma[x], && \text{(12b)} \\ & -(P_i k - b_i) - s_{4,i} p_t + (V - \gamma - R^2) s_{5,i} \\ & \quad \in \Sigma[(t,x)],\ \forall\ i = 1,...,n_p, && \text{(12c)} \end{aligned}$$

where $s_1, s_2 \in \Sigma[(t,x,w,d)]$, $(s_3 - \epsilon) \in \Sigma[x]$, and $s_{4,i}, s_{5,i} \in \Sigma[(t,x)]$. The positive number ϵ ensures that s_3 is uniformly bounded away from 0. The optimization (12) is nonconvex, since it is bilinear in two sets of decision variables, V and $(k, s_2, s_{5,i})$. As summarized in Algorithm 1, the nonconvex optimization (12) is handled by alternating the search over these two sets of decision variables, since holding one set fixed and optimizing over the other results in a convex problem. Since an explicit expression is not available for the volume of $\Omega(V,0,\gamma)$ for a generic V, we instead enlarge it by maximizing γ in the (k,γ)-step when V is fixed. Combining it with the constraint (13) in the V-step, which enforces $\Omega(V^{j-1}, 0, \gamma^j) \subseteq \Omega(V^j, 0, \gamma^j)$, we are able to prove that the volume of the inner-approximations will not decrease with each iteration [11, Theorem 2]. A linear state feedback for the linearization about the equilibrium point was used to compute the initial iterate V^0.

III. Extension to Actuator Uncertainty

This section considers the case where the control inputs are subject to actuator uncertainty. For example, unmodeled actuator dynamics can be modeled by a plant input u_{pert} given as the sum of the controller command u and a norm-bounded nonlinearity Δ: $u_{\text{pert}} = u + \Delta(u)$. The input v to Δ and the IQC filter output z were previously defined (Equations (1b) and (8b)) to be independent of the control command u.

Algorithm 1: Alternating direction method for hard IQCs

Input: function V^0 such that constraints (12) are feasible by proper choice of s_i, k, γ, M.
Output: k, γ, V, M.
1: **for** $j = 1 : N_{\text{iter}}$ **do**
2: **(k, γ)-step**: decision variables (s_i, k, γ, M). Maximize γ subject to (12) using $V = V^{j-1}$. This yields $(s_2^j, s_{5,i}^j, k^j)$ and optimal reward γ^j.
3: **V-step**: decision variables $(s_0, s_1, s_3, s_{4,i}, V, M)$; Maximize the feasibility subject to (12) as well as $s_0 \in \Sigma[x]$, and

$$(\gamma^j - V|_{t=0}) + (V^{j-1}|_{t=0} - \gamma^j)s_0 \in \Sigma[x], \quad (13)$$

using $(\gamma = \gamma^j, s_2 = s_2^j, s_{5,i} = s_{5,i}^j, k = k^j)$. This yields V^j.
4: **end for**

However, the inclusion of the actuator uncertainty implies that v and z must now depend on u.

This motivates the following generalization of the proposed method. Assume the entire input vector u is subject to the actuator uncertainty. The perturbation input and IQC filter output are now given by the following modifications to Equations (1b) and (8b):

$$v(t) = h(x_G(t), w(t), d(t), u(t)), \quad (14)$$
$$z(t) = H(x(t), w(t), d(t), u(t)). \quad (15)$$

A consequence of this generalization is that optimization over k is nonconvex even when V is fixed, since $z^\top M z$ in (9) depends nonlinearly on k. A remedy is to introduce auxiliary state $\tilde{x} \in \mathbb{R}^{n_u}$ for the perturbed control input u, and to design a dynamic controller of the form

$$\dot{\tilde{x}}(t) = \tilde{k}(t, x_G(t), \tilde{x}(t)), \quad (16a)$$
$$u(t) = \tilde{x}(t). \quad (16b)$$

where $\tilde{k} : \mathbb{R} \times \mathbb{R}^{n_G} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is to be determined. If we restrict the initial condition of $\tilde{x}$ to be zero: $\tilde{x}(0) = 0^{n_u}$, allow $\tilde{k}$ to depend on $\tilde{x}$, but not on x_ψ, and V to depend on the new state $\tilde{x}$: $V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}$, then the dissipation inequality becomes:

$$\begin{aligned}
&\partial_t V(t, x, \tilde{x}) + \partial_x V(t, x, \tilde{x}) \cdot F(x, w, d, \tilde{x}) \\
&\quad + \partial_{\tilde{x}} V(t, x, \tilde{x}) \cdot \tilde{k}(t, x_G, \tilde{x}) + z^\top M z \le d^\top d, \\
&\quad \forall\ (t, x, \tilde{x}, w, d) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_d}, \\
&\quad \text{s.t. } V(t, x, \tilde{x}) \le \gamma + R^2. \quad (17)
\end{aligned}$$

The term $z^\top M z$ in (17) is then nonlinear in the state variable $\tilde{x}$, rather than in the control law. The dissipation inequality is therefore bilinear in V and $\tilde{k}$, and can be solved in a way similar to Algorithm 1. Next, we provide the theorem that incorporates actuator uncertainties.

Theorem 2. *Let Assumption 1 hold, and further assume (i) $F_u(G, \Delta)$ is well-posed, (ii) $\Delta \in$ HardIQC(Ψ, M), with Ψ and M given. Given $X_T \subset \mathbb{R}^{n_G}$, $P \in \mathbb{R}^{n_p \times n_u}$, $b \in \mathbb{R}^{n_p}$, $R > 0$, F defined in* (8a)*, H defined in* (15)*, $T > 0$, and $\gamma \in \mathbb{R}$, if there exists a $\mathcal{C}^1$ function $V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}$, and control law $\tilde{k} : \mathbb{R} \times \mathbb{R}^{n_G} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$, such that* (17),

$$\begin{aligned}
&\{x_G : V(T, x, \tilde{x}) \le \gamma + R^2\} \subseteq X_T, \\
&\qquad \forall\ (x_\psi, \tilde{x}) \in \mathbb{R}^{n_\psi} \times U, \quad (18) \\
&P_i \tilde{x} \le b_i, \forall (t, x, \tilde{x}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n_u}, \\
&\qquad \textit{s.t. } V(t, x, \tilde{x}) \le \gamma + R^2, \forall\ i = 1, ..., n_p, \quad (19)
\end{aligned}$$

where $x = [x_G; x_\psi]$, then the intersection of $\Omega(V, 0, \gamma)$ with the hyperplane $(x_\psi, \tilde{x}) = 0$ is an inner-approximation to $BRS(T, X_T, U, R, F_u(G, \Delta))$ under the control (16).

The conditions of Theorem 2 can be formulated as an SOS optimization similar to (12), and is omitted. Although we assumed all control inputs are perturbed by uncertainty, the results can be extended to the case where a subset of the actuators are perturbed. This extension involves mainly notational changes and is also omitted.

IV. Backward Reachability with soft IQCs

Previously we assumed $\Delta \in$ HardIQC(Ψ, M). However, many IQCs are specified in the frequency domain [12]. Their time domain representation results in so called 'soft IQC'. The definitions for frequency domain and time domain soft IQCs are given below.

Definition 4. *Let $\Pi = \Pi^* \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be given. A bounded, causal operator $\Delta : \mathcal{L}_{2e}^{n_v} \to \mathcal{L}_{2e}^{n_w}$ satisfies the frequency domain IQC defined by the multiplier Π if, $\forall v \in \mathcal{L}_2^{n_v}$, and $w = \Delta(v)$,*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \ge 0, \quad (20)$$

where $\hat{v}$ and $\hat{w}$ are Fourier transforms of v and w.

Definition 5. *Given $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v+n_w)}$ and $M \in \mathbb{S}^{n_z}$. A bounded, causal operator $\Delta : \mathcal{L}_{2e}^{n_v} \to \mathcal{L}_{2e}^{n_w}$ satisfies the soft IQC defined by (Ψ, M) if, $\forall v \in \mathcal{L}_2^{n_v}$, and $w = \Delta(v)$,*

$$\int_0^\infty z(\tau)^\top M z(\tau) d\tau \ge 0, \ \textit{where } z = \Psi \begin{bmatrix} v \\ w \end{bmatrix}. \quad (21)$$

Let $\Delta \in$ FreqIQC(Π) and $\Delta \in$ SoftIQC(Ψ, M) indicate that Δ satisfies corresponding frequency domain and time domain soft IQCs, respectively. Note that if Δ satisfies a time domain (hard or soft) IQC defined by (Ψ, M), then $\Delta \in$ FreqIQC$(\Psi^* M \Psi)$. Conversely, any frequency domain multiplier Π can be factorized (non-uniquely) as: $\Pi = \Psi^* M \Psi$ with Ψ stable. By Parseval's theorem [18], $\Delta \in$ FreqIQC(Π) implies $\Delta \in$ SoftIQC(Ψ, M) for any such factorization. However, $\Delta \in$ FreqIQC(Π) doesn't imply $\Delta \in$ HardIQC(Ψ, M) in general. Hence, the library of IQCs specified in frequency domain can always be translated into soft IQCs, but not into hard IQCs. For example, the Popov multiplier doesn't have a hard factorization [12]. In addition, when both hard and soft factorizations exist, the latter is usually less restrictive. Therefore, it is helpful to incorporate soft IQCs in the analysis. Here, we provide one type of uncertainty and its frequency and time domain IQCs.

Example 2. *Consider the set of real constant parametric uncertainties: $w(t) = \Delta(v(t)) = \delta v(t)$, satisfying $\delta \le \sigma$. A soft IQC for δ is given by a filter $\Psi = \texttt{blkdiag}(\Psi_{11}^{d,m}, \Psi_{11}^{d,m})$, where $\Psi_{11}^{d,m}$ is defined in* (6)*, and $M_{DG} = \begin{bmatrix} \sigma^2 M_{11} & M_{12} \\ M_{12}^\top & -M_{11} \end{bmatrix}$, where decision matrices are subject to $M_{11} = M_{11}^\top$, $M_{12} = -M_{12}^\top$, and $\Psi_{11}^{d,m*} M_{11} \Psi_{11}^{d,m} \ge 0$, which can be enforced by a KYP LMI [19]. Notice that δ is a special case of the perturbation considered in Example 1 (a), and thus $\delta \in$ HardIQC(Ψ, M_D) as well. However, since M_D is a special case of M_{DG} with $M_{12} \equiv 0$, the analysis using M_{DG} can be less conservative than using M_D. A method is proposed in [20] to iteratively refine the choice of Ψ.*

Since soft IQCs hold over the infinite horizon, they cannot be incorporated in the analysis based on a finite-horizon dissipation inequality directly. To alleviate this issue, we use the following lemma which provides lower bounds for soft IQCs over all finite horizons, and thus allows for soft IQCs in the finite horizon reachability analysis. Let $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$ be a partition conformal with the dimensions of v and w.

Lemma 1. *([21]) Let $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v + n_w)}$ and $M \in \mathbb{S}^{n_z}$ be given. Define $\Pi := \Psi^* M \Psi$. If $\Pi_{22}(j\omega) < 0 \ \forall \omega$, then*

- *$D_{\psi 2}^\top M D_{\psi 2} < 0$ and there exists a $Y_{22} \in \mathbb{S}^{n_\psi}$ satisfying*

$$KYP(Y_{22}, A_\psi, B_{\psi 2}, C_\psi, D_{\psi 2}, M) < 0. \tag{22}$$

- *If $\Delta \in$ SoftIQC(Ψ, M) then for all $t \ge 0$, $v \in \mathcal{L}_2^{n_v}$, $w = \Delta(v)$, and $Y_{22} \in \mathbb{S}^{n_\psi}$ satisfying* (22),

$$\int_0^t z(\tau)^\top M z(\tau) d\tau \ge -x_\psi(t)^\top Y_{22} x_\psi(t). \tag{23}$$

Based on this lemma, the following theorem provides a BRS inner-approximation for $F_u(G, \Delta)$ with $\Delta \in \text{SoftIQC}(\Psi, M)$, also allowing for actuator uncertainties.

Theorem 3. *Let Assumption 1 hold, and further assume (i) $F_u(G, \Delta)$ is well-posed, (ii) $\Delta \in$ SoftIQC(Ψ, M), with Ψ and M given, (iii) $\Pi = \Psi^* M \Psi$ satisfying $\Pi_{22} < 0 \ \forall \omega$. Given $X_T \subset \mathbb{R}^{n_G}$, $P \in \mathbb{R}^{n_p \times n_u}$, $b \in \mathbb{R}^{n_p}$, $R > 0$, F defined in* (8a)*, H defined in* (15)*, $T > 0$, and $\gamma \in \mathbb{R}$, if there exists a $\mathcal{C}^1$ function $V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}$, a matrix $Y_{22} \in \mathbb{S}^{n_\psi}$ satisfying* (22)*, and a $\tilde{k} : \mathbb{R} \times \mathbb{R}^{n_G} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$, such that*

$$\begin{aligned}
&\partial_t V(t, x, \tilde{x}) + \partial_x V(t, x, \tilde{x}) \cdot F(x, w, d, \tilde{x}) \\
&\quad + \partial_{\tilde{x}} V(t, x, \tilde{x}) \cdot \tilde{k}(t, x_G, \tilde{x}) + z^\top M z \le d^\top d, \\
&\quad \forall\ (t, x, \tilde{x}, w, d) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_d}, \\
&\quad \textit{s.t. } \mathcal{V}(t, x, \tilde{x}) \le \gamma + R^2, && (24a) \\
&\{x_G : \mathcal{V}(T, x, \tilde{x}) \le \gamma + R^2\} \subseteq X_T, \\
&\quad \forall\ (x_\psi, \tilde{x}) \in \mathbb{R}^{n_\psi} \times U, && (24b) \\
&P_i \tilde{x} \le b_i, \forall (t, x, \tilde{x}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n_u}, \\
&\quad \textit{s.t. } \mathcal{V}(t, x, \tilde{x}) \le \gamma + R^2, \forall\ i = 1, ..., n_p, && (24c)
\end{aligned}$$

where $\mathcal{V} = V - x_\psi^\top Y_{22} x_\psi$, $x = [x_G; x_\psi]$, then the intersection of $\Omega(V, 0, \gamma)$ with the hyperplane $(x_\psi, \tilde{x}) = 0$ is an inner-approximation to $BRS(T, X_T, U, R, F_u(G, \Delta))$ under (16).

The proof is given in the extended version [22] of this paper. Similar to (12), we formulate an SOS optimization using the constraints of Theorem 3 :

$$\begin{aligned}
\sup_{V, M, Y_{22}, \tilde{k}, s_i} \quad & \text{Volume}(\Omega(V, 0, \gamma)) \\
\text{s.t.} \quad & V \in \mathbb{R}[(t, x, \tilde{x})], \tilde{k} \in \mathbb{R}^{n_u}[(t, x_G, \tilde{x})], \\
& M \in \mathcal{M} \text{ and } Y_{22} \in \mathbb{S}^{n_\psi} \text{ satisfy (22)}, \\
& -(\partial_t V + \partial_x V \cdot F|_{u=\tilde{x}} + \partial_{\tilde{x}} V \cdot \tilde{k} \\
& \quad + z^\top M z - d^\top d) + (\mathcal{V} - \gamma - R^2) s_2 \\
& \quad - s_1 p_t \in \Sigma[(t, x, \tilde{x}, w, d)], && (25a) \\
& - s_3 p_x + \mathcal{V}|_{t=T} - \gamma - R^2 \\
& \quad + \Sigma_{i=1}^{n_p} (P_i \tilde{x} - b_i) s_{6,i} \in \Sigma[(x, \tilde{x})], && (25b) \\
& -(P_i \tilde{x} - b_i) + (\mathcal{V} - \gamma - R^2) s_{5,i} \\
& \quad - s_{4,i} p_t \in \Sigma[(t, x, \tilde{x})], \forall\ i = 1, ..., n_p, && (25c)
\end{aligned}$$

where $s_1, s_2 \in \Sigma[(t, x, \tilde{x}, w, d)]$, $(s_3 - \epsilon), s_{6,i} \in \Sigma[(x, \tilde{x})]$ and $s_{4,i}, s_{5,i} \in \Sigma[(t, x, \tilde{x})]$. The optimization (25) is bilinear in (V, Y_{22}) and $(s_2, s_{5,i}, \tilde{k})$. Similar to Algorithm 1, Algorithm 2 tackles (25) by decomposing it into convex subproblems, and it also guarantees the improvement of the quality of the inner-approximation through iterations. $Y_{22}^0 = 0^{n_\psi}$ and a $M^0 \in \mathcal{M}$ can be used as initializations. It is demonstrated in [22, Section V (A)] that soft IQCs have richer knowledge of uncertainties than hard IQCs, and thus yield less conservative inner-approximations.

Algorithm 2: Alternating direction method for soft IQCs

Input: V^0, M^0 and Y_{22}^0 such that constraints (25) are feasible by proper choice of $s_i, \tilde{k}, \gamma$.
Output: $\tilde{k}$, γ, V, M, Y_{22}.

1: **for** $j = 1 : N_{\text{iter}}$ **do**
2: **(k, γ)-step**: decision variables $(s_i, \tilde{k}, \gamma)$. Maximize γ subject to (25) using $V = V^{j-1}$, $M = M^{j-1}$, and $Y_{22} = Y_{22}^{j-1}$. This yields $(s_2^j, s_{5,i}^j, \tilde{k}^j, \gamma^j)$.
3: **V-step**: $(s_0, s_1, s_3, s_{4,i}, s_{6,i}, V, M, Y_{22})$ are decision variables. Maximize the feasibility subject to (25) as well as $s_0 \in \Sigma[(x, \tilde{x})]$, and

$$(\gamma^j - V|_{t=0}) + (V^{j-1}|_{t=0} - \gamma^j) s_0 \in \Sigma[(x, \tilde{x})],$$

using $\gamma = \gamma^j, s_2 = s_2^j, s_{5,i} = s_{5,i}^j, \tilde{k} = \tilde{k}^j$. This yields V^j, M^j and Y_{22}^j.
4: **end for**

Computational complexity: A polynomial decision variable of degree $2d_{\text{poly}}$ in n_{var} variables has a $m \times m$ Gram matrix representation where $m := \binom{n_{\text{var}} + d_{\text{poly}}}{d_{\text{poly}}}$, and thus introduces $\mathcal{O}(m^2)$ decision variables due to the Gram matrix. Using higher-degree polynomial decision variables can provide a less conservative BRS estimate, but it takes longer to solve and might be intractable for high-dimensional systems.

V. Quadrotor Example

Consider the 6-state planar quadrotor dynamics [23]:

$\dot{x}_1 = x_3, \dot{x}_2 = x_4, \dot{x}_3 = u_1 K \sin(x_5), \dot{x}_4 =$

$$u_1 K \cos(x_5) - g_n, \dot{x}_5 = x_6, \dot{x}_6 = -d_0 x_5 - d_1 x_6 + n_0 u_2,$$

where $x_G = [x_1, ..., x_6]$ is the state, x_1 to x_6 are horizontal position (m), vertical position (m), horizontal velocity (m/s), vertical velocity (m/s), roll (rad), and roll velocity (rad/s), respectively. u_1 and u_2 are total thrust and desired roll angle. Control saturation limits are $u_1(t) \in [-1.5, 1.5] + g_n/K$, and $u_2(t) \in [-\pi/12, \pi/12]$. $g_n = 9.8$, $K = 0.89/1.4$, $d_0 = 70$, $d_1 = 17$, and $n_0 = 55$ are taken from [23].

The control objective of this example is to design controllers for u_1 and u_2 to maintain the trajectories of the quadrotor starting from the BRS to stay within the safe set X_t during the time horizon $[0, T]$ with $T = 2$. X_t is given as $X_t = \{x_G : x_G^\top N x_G \leq 1\}$, where $N = diag(1/1.7^2, 1/0.85^2, 1/0.8^2, 1/1^2, 1/(\pi/12)^2, 1/(\pi/2)^2)$. $\sin(x_5)$ is approximated by $(-0.166x_5^3 + x_5)$ and $\cos(x_5)$ is approximated by $(-0.498x_5^2 + 1)$, using least squares regression for $x_5 \in [-\pi/12, \pi/12]$. The validity of this bound on x_5 is guaranteed by the state constraint X_t. Assume that the control input u_2 is perturbed by an additive norm-bounded nonlinearity $\|\Delta\|_{2\to 2,[0,T]} \leq 0.2$, which introduces one auxiliary state $\tilde{x}$ to the analysis. We use the hard IQC discussed in Example 1(b) with a fixed filter Ψ and search for M over the constraint set given in (7). The SOS optimization problem was formulated using the SOS module in SOSOPT on MATLAB, and solved by the SDP solver MOSEK. The computation of BRS inner-approximations takes 1.1×10^3 and 3.6×10^4 seconds using degree-2 and degree-4 polynomial storage functions.

Fig. 2 shows the projections of the resulting inner-approximations. The one computed using degree-2 storage function is shown with the solid magenta curve, and the one computed using degree-4 storage function is shown with the red dash-dotted curve. The projections of X_t are shown with the blue solid curves.

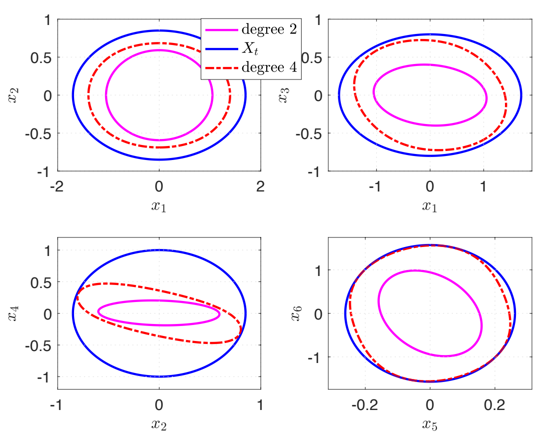


Fig. 2: BRS inner-approximations for the quadrotor

VI. Conclusions

A method is proposed to compute the BRS inner-approximations and control laws for uncertain nonlinear systems. The proposed framework merges dissipation inequalities and IQCs, with both hard and soft factorizations, allowing for a large class of perturbations. The effectiveness of the method is illustrated on a 6-state quadrotor example.

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