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Verification of stationary action trajectories via optimal control

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***Abstract*— A new optimal control based representation for stationary action trajectories is constructed by exploiting connections between semiconvexity, semiconcavity, and stationarity. This new representation is used to verify a known two-point boundary value problem characterization of stationary action.**

I. Introduction

The principle of stationary action, or *action principle*, is a fundamental variational postulate that underpins conservation laws in modern physics [5], [6], [8], [10]. A corollary of this principle states that *any trajectory of a conservative system must render the corresponding action stationary in the calculus of variation sense*, in which the *action* is the time integral of the corresponding Lagrangian.

When dynamical evolution is restricted to sufficiently short time horizons, the action involved is typically a convex function of the generalized velocity trajectory, at least where the generalized position space is finite dimensional, see for example [10], [4]. Consequently, on such short time horizons, stationary action is achieved as *least action*, and the trajectories involved can be characterized using tools from classical optimal control. In particular, for a specific conservation law, an optimal control problem can be formulated with respect to a cost function defined as the sum of the integrated Lagrangian and a terminal cost, with the latter used to capture terminal data. Dynamic programming may then be applied to characterize optimal trajectories, which necessarily correspond to trajectories of the underlying conservative system, subject to the imposed boundary conditions.

Recent efforts by the authors have successfully exploited this connection between least action and optimal control on short time horizons to develop a variety of fundamental solutions for conservative systems, including for the gravitational N-body problem [10]. However, on longer time horizons, or for systems evolving in infinite dimensions, this connection breaks down, typically due to a loss of convexity of the action. Indeed, the value functions associated with optimal control problems posed on these longer time horizons are typically afflicted by finite escape phenomena. As a minimum cannot be achieved in these cases, stationarity must explicitly be considered [3], [4], [11], [9].

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In this paper, connections between stationarity and *stationary control* [10], [4], [11], [12], [16] are summarized and further explored, with a view to expanding the applicability of generalized optimal control tools to the evolution of conservative systems over longer time horizons. In Section II, the aforementioned connections between least action and optimal control are reviewed and formalized, and the indicated short horizon constraint elucidated. Subsequently, Section III briefly summarizes the relaxation of optimal control to stationary control that is required to deal with longer horizons, and provides the expected two-point boundary value problem (TPBVP) characterization of the stationary trajectory (i.e. along which the action is stationary). Finally, Section IV exploits connections between semiconvexity, semiconcavity, and stationarity in order to formulate two auxiliary optimal control problems that can be used to characterize the *staticizing* velocity input that yields upon integration the aforementioned stationary trajectory. The obtained characterization is used to verify the TPBVP formulation of Section III.

Throughout, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ denote the real, integer, and natural numbers respectively, with extended reals defined as $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{\pm\infty\}$. The space of continuous mappings between Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ is denoted by $C(\mathscr{X};\mathscr{Y})$. The set of bounded linear operators between the same spaces is denoted by $\mathcal{L}(\mathscr{X};\mathscr{Y})$, or $\mathcal{L}(\mathscr{X})$ if $\mathscr{X}$ and $\mathscr{Y}$ coincide. A function $f \in C(\mathscr{X};\mathscr{Y})$ is Fréchet differentiable at $x \in \mathscr{X}$, with derivative $Df(x) \in \mathcal{L}(\mathscr{X};\mathscr{Y})$, if $0 = \lim_{\|h\|_{\mathscr{X}} \to 0} \|df_x(h)\|_{\mathscr{Y}}$, with $df_x : \mathscr{X} \to \mathscr{X}$ defined by

$$df_x(h) \doteq \begin{cases} 0, & \|h\|_{\mathscr{X}} = 0, \\ \frac{f(x+h) - f(x) - Df(x)\, h}{\|h\|_{\mathscr{X}}}, & \|h\|_{\mathscr{X}} > 0. \end{cases}$$

By definition, the map $h \mapsto df_x(h)$ is continuous at 0.

II. Least action and optimal control

For a conservative system with generalized position evolving in a real Hilbert space $\mathscr{X}$, the action is formalized as a function defined with respect to a coercive inertia operator $\mathcal{M} \in \mathcal{L}(\mathscr{X})$, a potential field $V : \mathscr{X} \to \mathbb{R}$, and a convex terminal cost $\Psi : \mathscr{X} \to \mathbb{R}$ that is included to encode terminal data [10], [4]. Given $t, T \in \mathbb{R}$, $t < T$, and $\mathscr{U}[t,T] \doteq \mathscr{L}_2([t,T];\mathscr{X})$, it is explicitly defined by $J_t[\Psi] : \mathscr{X} \times \mathscr{U}[t,T] \to \mathbb{R}$, with

$$J_t[\Psi](x,u) \doteq \int_t^T \tfrac{1}{2} \langle u_s, \mathcal{M}\, u_s \rangle - V(\xi_s)\, ds + \Psi(\xi_T), \quad (1)$$

for all $x \in \mathscr{X}$, $u \in \mathscr{U}[t,T]$, in which $s \mapsto \xi_s$ is the generalized position trajectory

$$\xi_s \doteq x + \int_t^s u_\sigma \, d\sigma, \quad s \in [t,T], \quad (2)$$

defined with respect to a corresponding generalized velocity trajectory $s \mapsto u_s$ for $s \in [t, T]$. For convenience, in addition to coercivity of $\mathcal{M}$, it is assumed throughout that V, Ψ are three times continuously Fréchet differentiable with uniformly bounded Hessian as per [3], i.e.

$$\begin{aligned} &V, \Psi \in C^3(\mathscr{X}; \mathbb{R}), \; m \doteq \inf_{h \in \mathscr{X}} \left\{ \langle h, \mathcal{M}\, h \rangle / \|h\|^2 \right\} > 0, \quad (3) \\ &\kappa \doteq 2 \sup_{x \in \mathscr{X}} \max \left\{ \|\nabla^2 V(x)\|_{\mathcal{L}(\mathscr{X})}, \|\nabla^2 \Psi(x)\|_{\mathcal{L}(\mathscr{X})} \right\} < \infty. \end{aligned}$$

For sufficiently short time horizons $T - t > 0$, and in the company of (3), the action $J_t[\Psi](x, \cdot) : \mathscr{U}[t, T] \to \mathbb{R}$, $x \in \mathscr{X}$, can be shown to be strictly convex and coercive for finite dimensional $\mathscr{X}$ [10], [4], see Theorem 2.1 below. In that case, an optimal control problem can be formulated to describe stationary action as *least* action. The value function involved is defined by $\overline{W}_t : \mathscr{X} \to \overline{\mathbb{R}}$, with

$$\overline{W}_t(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t[\Psi](x, u) \tag{4}$$

for all $x \in \mathscr{X}$. The Hamiltonian $H : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ involved is subsequently defined by

$$\begin{aligned} H(x, p) &\doteq \tfrac{1}{2} \langle p, \mathcal{M}^{-1} p \rangle + V(x) \\ &= \sup_{u \in \mathscr{X}} \{ -\langle p, u \rangle - \tfrac{1}{2} \langle u, \mathcal{M}\, u \rangle \} + V(x) \end{aligned} \tag{5}$$

for all $x, p \in \mathscr{X}$, in which the (second) equality follows by completion of squares.

The following is a consequence of [3, Theorem 3.6, Assertion 2] and Pontryagin's minimum principle [15], [1].

Theorem 2.1: Given m, κ as per (3) and $t_0 \in \mathbb{R}_{<T}$ satisfying $\max(T - t_0, 1)\,(T - t_0) < \frac{m}{\kappa}$, the following properties concerning the optimal control problem (4) hold:

(i) Given any $t \in [t_0, T)$, $x \in \mathscr{X}$, the action (cost) $J_t[\Psi](x, \cdot) : \mathscr{U}[t, T] \to \mathbb{R}$ is strictly convex and coercive, and there exists a unique optimal input $\bar{u}^* \in \mathscr{U}[t, T]$ such that $\overline{W}_t(x) = J_t(x, \bar{u}^*) \in \mathbb{R}$;

(ii) There exists a classical solution of the two-point boundary value problem (TPBVP)

$$\begin{cases} \dot{\bar{x}}_s = -\nabla_p H(\bar{x}_s, \bar{p}_s) = -\mathcal{M}^{-1} \bar{p}_s, & \bar{x}_t = x, \\ \dot{\bar{p}}_s = \nabla_x H(\bar{x}_s, \bar{p}_s) = \nabla V(\bar{x}_s), & \bar{p}_T = \nabla \Psi(\bar{x}_T), \end{cases} \tag{6}$$

for all $s \in [t, T]$, in which $\nabla_x H$ and $\nabla_p H$ denote Riesz representations of the Fréchet derivatives of the Hamiltonian (5), and $\bar{u}^*$ of *(i)* satisfies

$$\bar{u}_s^* = -\mathcal{M}^{-1} \bar{p}_s, \quad s \in [t, T]. \tag{7}$$

Remark 2.2: Classical solutions for the characteristic system (6) in the statement of Theorem 2.1 *(ii)* can be asserted via a global Lipschitz property that follows from (3).

Remark 2.3: As any solution of TPBVP (6) is a classical solution, differentiating the first equation in (6) yields

$$\ddot{\bar{x}}_s = -\mathcal{M}^{-1} \dot{\bar{p}}_s = -\mathcal{M}^{-1} \nabla V(\bar{x}_s), \quad s \in (t, T),$$

which is a generalized form of Newton's second law. Moreover, applying (5), (6), and the chain rule,

$$\begin{aligned} &\tfrac{d}{ds} H(\bar{x}_s, \bar{p}_s) = \langle \nabla_x H(\bar{x}_s, \bar{p}_s), \dot{\bar{x}}_s \rangle + \langle \nabla_p H(\bar{x}_s, \bar{p}_s), \dot{\bar{p}}_s \rangle \\ &= \langle \nabla V(\bar{x}_s), -\mathcal{M}^{-1} \bar{p}_s \rangle + \langle \mathcal{M}^{-1} \bar{p}_s, \nabla V(\bar{x}_s) \rangle = 0, \end{aligned}$$

for all $s \in (t, T)$. Consequently, the Hamiltonian is the conserved quantity, along the characteristic flow, as expected by the minimum principle underlying (7). □

Theorem 2.1 demonstrates that solutions of the TPBVP (6) describe those trajectories that render the action stationary in the statement of the action principle. As expected, it is also possible to characterize these trajectories via a verification theorem involving a solution of the corresponding HJB PDE

$$\begin{cases} 0 = -\dfrac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)), \\ W_T(x) = \Psi(x), \end{cases} \tag{8}$$

for all $t \in [t_0, T]$, $x \in \mathscr{X}$. The proof of the following is standard [1].

Theorem 2.4: Under the conditions of Theorem 2.1, suppose there exists $(t, x) \mapsto W_t(x) \in C^1((t_0, T) \times \mathscr{X}; \mathbb{R})$ such that (8) holds, with $(\frac{\partial}{\partial t} W_t(x), \nabla W_t(x)) \in \mathbb{R} \times \mathscr{X}$ denoting its Fréchet derivative at $(t, x) \in (t_0, T) \times \mathscr{X}$. Then, $W_t(x) \leqslant J_t[\Psi](x, u)$ for all $u \in \mathscr{U}[t, T]$. Furthermore, if there exists a solution $s \mapsto \bar{x}_s^*$, $s \in (t, T)$, of (2) satisfying

$$\bar{x}_s^* = x + \int_t^s \bar{u}_\sigma^* \, d\sigma, \quad \bar{u}_\sigma^* = -\mathcal{M}^{-1} \nabla W_\sigma(\bar{x}_\sigma^*), \tag{9}$$

such that $\bar{x}_s^* \in \mathscr{X}$ for all $s \in (t, T)$, then $W_t(x) = J_t[\Psi](x, \bar{u}_t^*) = \overline{W}_t(x)$ for all $x \in \mathscr{X}$.

III. Stationary Action and Stationary Control

Theorem 2.1 guarantees that stationarity of the action (1) is achieved at a minimum, provided that the maximal time horizon $T - t_0$ is sufficiently short. For longer horizons, Theorem 2.1 is no longer applicable, typically due to a loss of convexity of (1). This is manifested in the optimal control problem (4) as finite escape phenomena exhibited by the value function $t \mapsto W_t$ as $T - t > 0$ increases.

As the connection between stationary (least) action and optimal control breaks down for longer time horizons, stationarity of the action is instead formalized by replacing the *inf* operation in (4) with a *stat* operation [11], [12]. This *stat* operation, along with the corresponding *argstat* operation, can be defined for Fréchet differentiable functions $F \in C^1(\mathscr{W}; \mathbb{R})$ on any real Hilbert space $\mathscr{W}$ by

$$\begin{aligned} &\operatorname*{stat}_{w \in \mathscr{W}} F(w) \doteq \left\{ F(\overline{w}) \;\middle|\; \overline{w} \in \operatorname*{arg\,stat}_{w \in \mathscr{W}} F(w) \right\}, \\ &\operatorname*{arg\,stat}_{w \in \mathscr{W}} F(w) \doteq \{ \overline{w} \in \mathscr{W} \mid \nabla F(\overline{w}) = 0 \}, \end{aligned} \tag{10}$$

in which $\nabla F : \mathscr{W} \to \mathcal{L}(\mathscr{W}; \mathbb{R})$ denotes the Riesz representation of the derivative. As the action $J_t[\Psi](x, \cdot) : \mathscr{U}[t, T] \to \mathbb{R}$ is continuously Fréchet differentiable, see [3, Theorem 3.6], and $\mathscr{U}[t, T]$ is a real Hilbert space, it is possible to select $\mathscr{W} \doteq \mathscr{U}[t, T]$ and $F \doteq J_t[\Psi](x, \cdot)$ in (10).

The ensuing *stationary control problem* is defined for *any* time horizon $T - t > 0$, $t, T \in \mathbb{R}$, by a (possibly set-valued) *stat* value function $\widetilde{W}_t : \mathscr{X} \to \overline{\mathbb{R}}$, with

$$\widetilde{W}_t(x) \doteq \operatorname*{stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x, u), \tag{11}$$

for all $x \in \mathscr{X}$. Given a specific $x \in \mathscr{X}$, trajectories that render the action $J_t[\Psi](x, \cdot)$ stationary as per the action principle can be characterized as follows [3, Theorem 3.9].

Theorem 3.1: Suppose (3) holds. Given $t, T \in \mathbb{R}$, $t \leqslant T$, $x \in \mathscr{X}$, an input $\bar{u} \in \mathscr{U}[t, T]$ is *staticizing*, i.e.

$$\bar{u} \in \operatorname*{arg\,stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x, u) \tag{12}$$

if and only if there exists a classical solution of the TPBVP

$$\begin{cases} \dot{\bar{x}}_s = -\mathcal{M}^{-1} \bar{p}_s, & \bar{x}_t = x, \\ \dot{\bar{p}}_s = \nabla V(\bar{x}_s), & \bar{p}_T = \nabla \Psi(\bar{x}_T), \end{cases} \tag{13}$$

for all $s \in [t, T]$. Furthermore, $\bar{u} \in \mathscr{U}[t, T]$ satisfies

$$\bar{u}_s = -\mathcal{M}^{-1} \bar{p}_s, \quad s \in [t, T]. \tag{14}$$

Proof: [Sketch] The argument used is a minor generalization of [3, Theorem 3.9], involving the replacement of a scalar inertia with the coercive operator $\mathcal{M} \in \mathcal{L}(\mathscr{X})$. ■

By inspection, TPBVPs (6), (13) are identical except for the time horizon on which solutions are sought. For short time horizons, as required in (4), (6), the input $\bar{u} \in \mathscr{U}[t, T]$ defined by (7) is a minimizer for the action (1). Theorems 2.1 and 2.4 provide a means for synthesizing this via solution of HJB PDE (8) and the application of Theorem 2.4.

For longer horizons, as allowed in (11), (13), the input $\bar{u} \in \mathscr{U}[t, T]$ defined by (12) need only render the action (1) stationary. As Theorems 2.1 and 2.4 are unavailable on these longer horizons, it is not possible to construct $\bar{u}$ via HJB PDE (8). However, verification of the stationary control is possible using an alternative approach that again appeals to a pair of optimal control and corresponding HJB PDEs, via semiconvex and semiconcave duality.

IV. Verification of the Staticizing Control

The aim is to develop a verification argument for the staticizing input (12), applicable to longer time horizons. Crucial to this development is a new characterization of the argstat operation (10) using semiconvex and semiconcave duality. This characterization is applicable for any real Hilbert space $\mathscr{W}$, although its application here will be restricted to the case $\mathscr{W} \doteq \mathscr{U}[t, T]$, given $t, T \in \mathbb{R}$, $t < T$. Unlike [11], this development will make use of a pair of optimal control problems, rather than a single stationary control problem.

A. Duality based characterization of argstat

Some preliminary definitions are required. A function $\psi : \mathscr{W} \to \overline{\mathbb{R}}$ is convex if its epigraph $\{(w, \alpha) \in \mathscr{W} \times \mathbb{R} \,|\, \psi(w) \leqslant \alpha\}$ is convex [13]. It is lower closed if $\psi = \mathrm{cl}^- \psi$, in which $\mathrm{cl}^- \psi$ is the lower closure of ψ, defined with respect to the corresponding lower semicontinuous envelope $\operatorname{lsc} \psi$ by

$$\mathrm{cl}^- \psi(w) \doteq \begin{cases} \operatorname{lsc} \psi(w), & \operatorname{lsc} \psi(w) > -\infty \ \forall \ w \in \mathscr{W}, \\ -\infty & \text{otherwise}, \end{cases}$$

for all $w \in \mathscr{W}$. A function ψ is concave if $-\psi$ is convex, and upper closed if $-\psi$ is lower closed, see [13, pp.15-17]. Recall that a proper lsc function is convex if and only if it is the pointwise supremum of its affine support functions, see for example [14, Theorem 8.13, p.309].

Semiconvexity and semiconcavity, and subsequent relaxed notions of duality, are defined with respect to a bivariate quadratic support or basis function $\varphi : \mathscr{W} \times \mathscr{W} \to \mathbb{R}$ that has a fixed coercive Hessian $\mathcal{C} \in \mathcal{L}(\mathscr{W})$. Explicitly,

$$\varphi(v, w) \doteq -\tfrac{1}{2} \langle v - w, \, \mathcal{C}\,(v - w) \rangle \tag{15}$$

for all $v, w \in \mathscr{W}$. Using (15), the spaces $\mathscr{S}_\varphi^+$ and $\mathscr{S}_\varphi^-$ of (uniformly) semiconvex and semiconcave functions are defined respectively by

$$\begin{aligned} \mathscr{S}_\varphi^+ &\doteq \left\{ \psi : \mathscr{W} \to \overline{\mathbb{R}} \;\middle|\; \begin{array}{c} v \mapsto \psi(v) - \varphi(v, 0) \\ \text{convex, lower closed} \end{array} \right\}, \\ \mathscr{S}_\varphi^- &\doteq \left\{ \phi : \mathscr{W} \to \overline{\mathbb{R}} \;\middle|\; -\phi \in \mathscr{S}_\varphi^+ \right\}. \end{aligned} \tag{16}$$

These spaces are in duality, via either the semiconvex transform $\mathcal{D}_\varphi^+$, see for example [7], [2], or the analogously defined semiconcave transform $\mathcal{D}_\varphi^-$, i.e.

$$\mathscr{S}_\varphi^+ \underset{\mathcal{D}_\varphi^- \,\equiv\, [\mathcal{D}_\varphi^+]^{-1}}{\overset{\mathcal{D}_\varphi^+ \,\equiv\, [\mathcal{D}_\varphi^-]^{-1}}{\rightleftarrows}} \mathscr{S}_\varphi^-$$

The semiconvex transform and its inverse are given by

$$\begin{aligned} (\mathcal{D}_\varphi^+ \, \psi)(w) &\doteq -\sup_{v \in \mathscr{W}} \{\varphi(v, w) - \psi(v)\}, & \psi \in \mathscr{S}_\varphi^+, \\ ([\mathcal{D}_\varphi^+]^{-1} \, \phi)(v) &= \sup_{w \in \mathscr{W}} \{\varphi(v, w) + \phi(w)\}, & \phi \in \mathscr{S}_\varphi^-, \end{aligned} \tag{17}$$

for all $v, w \in \mathscr{W}$, while for the semiconcave transform,

$$\begin{aligned} \mathcal{D}_\varphi^- \, \phi &\doteq -\mathcal{D}_\varphi^+[-\phi] = [\mathcal{D}_\varphi^+]^{-1} \phi, & \phi \in \mathscr{S}_\varphi^-, \\ [\mathcal{D}_\varphi^-]^{-1} \, \psi &= -[\mathcal{D}_\varphi^+]^{-1} [-\psi] = \mathcal{D}_\varphi^+ \, \psi, & \psi \in \mathscr{S}_\varphi^+, \end{aligned} \tag{18}$$

in which the symmetry of φ with respect to its arguments is used to obtain the right-hand equivalences in (18).

A new characterization of the argstat operation (10), using the spaces of semiconvex and semiconcave functions (16) and their respective transforms (17), (18), is as follows.

Theorem 4.1: Suppose $F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-$. Then,

$$(\mathcal{D}_\varphi^+ F)(w) \leqslant F(w) \leqslant (\mathcal{D}_\varphi^- F)(w) \tag{19}$$

for all $w \in \mathscr{W}$, and

$$\overline{w} \in \operatorname*{arg\,stat}_{w \in \mathscr{W}} F(w) \iff (\mathcal{D}_\varphi^+ F)(\overline{w}) = (\mathcal{D}_\varphi^- F)(\overline{w}). \tag{20}$$

Proof: Fix $F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-$. With $w \in \mathscr{W}$, recalling (17), (18), and noting the symmetry of φ of (15), define

$$\begin{aligned} a(w) &\doteq (\mathcal{D}_\varphi^+ F)(w) = \inf_{v \in \mathscr{W}} \{F(v) - \varphi(v, w)\}, \\ b(w) &\doteq (\mathcal{D}_\varphi^- F)(w) = \sup_{v \in \mathscr{W}} \{F(v) + \varphi(v, w)\}. \end{aligned} \tag{21}$$

As $w \in \mathscr{W}$ is suboptimal in both right-hand sides in (21), and $\varphi(w, w) = 0$, by inspection,

$$a(w) \leqslant F(w) \leqslant b(w). \tag{22}$$

That is, (19) holds.

[Necessity]. Suppose the right-hand statement in (20) holds. That is, recalling (21), there exists $\overline{w} \in \mathscr{W}$ such that

$$a(\overline{w}) = b(\overline{w}). \tag{23}$$

Together, (22), (23), yield

$$a(\overline{w}) = F(\overline{w}) = b(\overline{w}). \tag{24}$$

Fix any $h \in \mathscr{W}$. As $\overline{w} + h \in \mathscr{W}$ is also suboptimal in the definitions of a and b, (21), (24) imply that

$$\begin{aligned} F(\overline{w}) &= a(\overline{w}) \leqslant F(\overline{w}+h) - \varphi(\overline{w}+h, \overline{w}), \\ F(\overline{w}) &= b(\overline{w}) \geqslant F(\overline{w}+h) + \varphi(\overline{w}+h, \overline{w}). \end{aligned}$$

These inequalities and (15) together yield

$$\begin{aligned} |F(\overline{w}+h) - F(\overline{w})| &\leqslant -\varphi(\overline{w}+h, \overline{w}) = \tfrac{1}{2}\langle h, \mathcal{C}\, h\rangle \\ &\leqslant \|\mathcal{C}\|_{\mathcal{L}(\mathscr{W})}\, |h|^2. \end{aligned}$$

As $h \in \mathscr{W}$ is arbitrary, it follows that F is Fréchet differentiable at $\overline{w}$, with derivative and its Riesz representation given by $DF(\overline{w}) = 0 \in \mathcal{L}(\mathscr{W})$ and $\nabla F(\overline{w}) = 0 \in \mathscr{W}$ respectively. Hence, recalling (10), $\overline{w} \in \arg\mathrm{stat}_{w \in \mathscr{W}} F(w)$, as required.

[Sufficiency]. Suppose the left-hand statement in (20) holds, i.e. there exists a $\overline{w} \in \arg\mathrm{stat}_{w\in\mathscr{W}} F(w)$. By definition (10), $\nabla F(\overline{w}) = 0$. Note further by (15) that $\nabla_v \varphi(v, \overline{w})|_{v=\overline{w}} = 0$, so that

$$\nabla_v [F(v) - \varphi(v, \overline{w})]|_{v=\overline{w}} = 0. \tag{25}$$

Recall that $v \mapsto F(v) - \varphi(v, 0)$ is convex, as $F \in \mathscr{S}_\varphi^+$. As $\varphi(v, 0) - \varphi(v, \overline{w})$ is affine, the map $v \mapsto F(v) - \varphi(v, \overline{w})$ must also be convex, while simultaneously satisfying (25). Hence, it has a global minimum at $v = \overline{w}$, so that

$$a(\overline{w}) = \inf_{v \in \mathscr{W}} \{F(v) - \varphi(v, \overline{w})\} = F(\overline{w}) - \varphi(\overline{w}, \overline{w}) = F(\overline{w}),$$

by (21). Similarly, as $F \in \mathscr{S}_\varphi^-$, the map $v \mapsto F(v) + \varphi(v, \overline{w})$ is concave, simultaneously satisfying (25). Hence, it has a global maximum at $v = \overline{w}$, so that

$$b(\overline{w}) = \sup_{v \in \mathscr{W}} \{F(v) + \varphi(v, \overline{w})\} = F(\overline{w}) + \varphi(\overline{w}, \overline{w}) = F(\overline{w}),$$

by (21). Hence, combining these two conclusions yields

$$(\mathcal{D}_\varphi^+ F)(\overline{w}) = a(\overline{w}) = F(\overline{w}) = b(\overline{w}) = (\mathcal{D}_\varphi^- F)(\overline{w}),$$

which completes the proof. ■

The following lemma is useful in the subsequent application of Theorem 4.1.

Lemma 4.2: With $F \in C^2(\mathscr{W}; \mathbb{R})$, suppose that the first Fréchet derivative of the Riesz representation of its first Fréchet derivative $D\nabla F : \mathscr{W} \to \mathcal{L}(\mathscr{W})$ is uniformly bounded, i.e. $\bar{c} \doteq \sup_{w \in \mathscr{W}} \|D\nabla F(w)\|_{\mathcal{L}(\mathscr{W})} < \infty$. Then, for any $\mathcal{C} \in \mathcal{L}(\mathscr{W})$, $\epsilon \geqslant 0$, satisfying $\langle h, \mathcal{C}\, h\rangle \geqslant (\bar{c} + \epsilon)\, |h|^2$ for all $h \in \mathscr{W}$, the support φ defined by (15) is such that *(i)* $w \mapsto F(w) - \varphi(w, 0)$ is (strictly) convex and $w \mapsto F(w) + \varphi(w, 0)$ is (strictly) concave for $(\epsilon > 0)$ $\epsilon \geqslant 0$; and *(ii)* for any $\epsilon \geqslant 0$,

$$F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-. \tag{26}$$

Proof: Fix $F \in C^2(\mathscr{W}; \mathbb{R})$ and $\bar{c} < \infty$ as per the lemma statement. Fix $\epsilon \in \mathbb{R}_{\geqslant 0}$. Select any coercive $\mathcal{C} \in \mathcal{L}(\mathscr{W})$ such that $\langle h, \mathcal{C}\, h\rangle \geqslant (\bar{c} + \epsilon)\, |h|^2$ for all $h \in \mathscr{W}$, e,g, $\mathcal{C} \doteq c\,\mathcal{I}$, $c \geqslant \bar{c} + \epsilon$. Using this $\mathcal{C}$, define φ as per (15).

(i) Fix any $w, h \in \mathscr{W}$. As $F \in C^2(\mathscr{W}; \mathbb{R})$, its first Fréchet derivative at w satisfies $DF(w)\, h = \langle \nabla F(w), h\rangle$, in which $\nabla F(w) \in \mathscr{W}$ is the corresponding Riesz representation. Moreover,

$$D\,\nabla F(w) \in \mathcal{L}(\mathscr{W}), \quad D^2 F(w)\, h\, h = \langle h,\, D\,\nabla F(w)\, h\rangle,$$

see for example [3, Appendix]. Hence, for $\mu \doteq \pm 1$,

$$\begin{aligned} D^2[\,\mu\, F(w) - \varphi(w, 0)]\, h\, h &= \mu\, \langle h,\, D\,\nabla F(w)\, h\rangle + \langle h,\, \mathcal{C}\, h\rangle \\ &\geqslant -\bar{c}\, |h|^2 + \langle h,\, \mathcal{C}\, h\rangle \geqslant \epsilon\, |h|^2. \end{aligned}$$

As $w, h \in \mathscr{W}$ are arbitrary, it follows that $w \mapsto \pm F(w) - \varphi(w, 0)$ is (strictly) convex, as $(\epsilon > 0)$ $\epsilon \geqslant 0$.

(ii) The maps $w \mapsto \pm F(w) - \varphi(w, 0)$ are continuous by definition, and hence lower closed. Hence, applying *(i)* and (16), $F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-$. ■

B. Application to stationary control

Given fixed $t_0, T \in \mathbb{R}$ with $t_0 < T$ via (3), the intention is to apply Theorem 4.1 to the stationary control problem (11), (12) for any $t \in [t_0, T)$, with $\mathscr{W} \doteq \mathscr{U}[t, T]$ and $F \doteq J_t[\Psi](x, \cdot)$. In order to explicitly define the quadratic support function $\varphi : \mathscr{U}[t, T] \times \mathscr{U}[t, T] \to \mathbb{R}$ as per (15), let

$$\begin{aligned} \mathcal{C} &\doteq c\,\mathcal{I} \in \bigcup_{t \in [t_0, T)} \mathcal{L}(\mathscr{U}[t, T]), \quad c \in \mathbb{R},\ c \geqslant c_{t_0}, \qquad (27) \\ c_{t_0} &\doteq 1 + \|\mathcal{M}\|_{\mathcal{L}(\mathscr{X})} + \kappa\, \max(T - t_0, 1)\, (T - t_0) < \infty. \end{aligned}$$

Lemma 4.3: Suppose (3) holds. Given any $t \in [t_0, T)$, $x \in \mathscr{X}$, and support φ as per (15), (27), the following properties concerning the action (1) hold: *(i)* $u \mapsto J_t[\Psi](x, u) - \varphi(u, 0)$ is strictly convex and $u \mapsto J_t[\Psi](x, u) + \varphi(u, 0)$ is strictly concave; and *(ii)* $J_t[\Psi](x, \cdot)$ is simultaneously semiconvex and semiconcave, i.e.

$$J_t[\Psi](x, \cdot) \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-. \tag{28}$$

Proof: Fix any $t \in [t_0, T)$, $x \in \mathscr{X}$, and $u, h, \tilde{h} \in \mathscr{U}[t, T]$. Recall by [3, Theorem 3.6] that the action (1) is three times continuously Fréchet differentiable, i.e. $J_t[\Psi](x, \cdot) \in C^3(\mathscr{U}[t, T]; \mathbb{R})$. *(i)* From (1), the Fréchet derivative of the Riesz representation of the first Fréchet derivative of $J_t[\Psi](x, \cdot)$, i.e. $D_u \nabla_u J_t[\Psi](x, u) \in \mathcal{L}(\mathscr{U}[t, T])$, satisfies

$$\begin{aligned} &\langle \tilde{h},\, D_u \nabla_u J_t[\Psi](x, u)\, h\rangle \leqslant \|\mathcal{M}\|_{\mathcal{L}(\mathscr{X})}\, |\tilde{h}|\, |h| \\ &\quad + \kappa\, \max(T - t, 1) \int_t^T |\tilde{h}_r|\, dr \int_t^T |h_\rho|\, d\rho \\ &\leqslant \left[\|\mathcal{M}\|_{\mathcal{L}(\mathscr{X})} + \kappa\, \max(T - t, 1)\, (T - t)\right] |\tilde{h}|\, |h|. \end{aligned}$$

As $h, \tilde{h} \in \mathscr{U}[t, T]$ are arbitrary, it follows immediately by definition (27) of c_{t_0} that $c_{t_0} \geqslant \bar{c}_{t_0} + \epsilon$, with $\epsilon \doteq 1$ and

$$\bar{c}_{t_0} \doteq \|D_u \nabla_u J_t[\Psi](x, u)\|_{\mathcal{L}(\mathscr{U}[t, T])}.$$

Hence, by definitions (15), (27) of φ, $\mathcal{C}$, Lemma 4.2 *(i)* implies that assertion *(i)* holds. Assertion *(ii)*, i.e. (28), is subsequently immediate by Lemma 4.2 *(ii)* ■

Lemma 4.4: Suppose (3) holds. Given any $t \in [t_0, T)$, $x \in \mathscr{X}$, and $u \in \mathscr{U}[t, T]$,

$$(\mathcal{D}_\varphi^+ J_t[\Psi](x, \cdot))(u) \leqslant (\mathcal{D}_\varphi^- J_t[\Psi](x, \cdot))(u). \qquad (29)$$

Moreover, the argstat condition (12) holds, i.e. $\bar{u} \in \operatorname{arg\,stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x, u)$, if and only if

$$(\mathcal{D}_\varphi^+ J_t[\Psi](x, \cdot))(\bar{u}) = (\mathcal{D}_\varphi^- J_t[\Psi](x, \cdot))(\bar{u}), \qquad (30)$$

in which $\mathcal{D}_\varphi^\pm$ and φ are as per (18) and (15), (27).

Proof: Fix any $t \in [t_0, T)$, $x \in \mathscr{X}$, and $u \in \mathscr{U}[t, T]$. Observe by Lemma 4.3 that (28) holds. Hence, applying Theorem 4.1 with $\mathscr{W} \doteq \mathscr{U}[t, T]$, $F \doteq J_t[\Psi](x, \cdot)$, and φ defined via (15), (27), yields inequality (29) via (19), and the stated equivalence between (12) and (30) via (20). ∎

C. Auxiliary optimal control problems

In order to apply (30), it is useful to first rewrite both sides in a more familiar form. In particular, given $t \in [t_0, T)$ and $x \in \mathscr{X}$, and recalling the definitions (18) and (1) of $\mathcal{D}_\varphi^\pm$ and $J_t[\Psi](x, \cdot)$, observe that

$$\begin{aligned} (\mathcal{D}_\varphi^+ J_t[\Psi](x, \cdot))(\bar{u}) &= \inf_{u \in \mathscr{U}[t,T]} \{J_t[\Psi](x, u) - \varphi(u, \bar{u})\}, \\ (\mathcal{D}_\varphi^- J_t[\Psi](x, \cdot))(\bar{u}) &= \sup_{u \in \mathscr{U}[t,T]} \{J_t[\Psi](x, u) + \varphi(u, \bar{u})\}. \end{aligned} \qquad (31)$$

That is, the two sides of (30) define a pair of auxiliary optimal control problems, parameterized by $\bar{u} \in \mathscr{U}[t, T]$. In view of (31), given any $v \in \mathscr{U}[t_0, T]$, explicitly define the auxiliary cost functions

$$J_t^v[\Psi],\ \hat{J}_t^v[\Psi] : \mathscr{X} \times \mathscr{U}[t, T] \to \mathbb{R}$$

via (1) and (15), (27) by

$$\begin{aligned} J_t^v[\Psi](x, u) &\doteq J_t[\Psi](x, u) - \varphi(u, v) \\ &= \int_t^T \tfrac{1}{2}\langle u_s, \mathcal{E}\, u_s\rangle - \langle u_s, \mathcal{C}\, v_s\rangle + \tfrac{1}{2}\langle v_s, \mathcal{C}\, v_s\rangle - V(\xi_s)\, ds \\ &\qquad + \Psi(\xi_T), \end{aligned} \qquad (32)$$

$$\begin{aligned} \hat{J}_t^v[\Psi](x, u) &\doteq J_t[\Psi](x, u) + \varphi(u, v) \\ &= \int_t^T -\tfrac{1}{2}\langle u_s, \hat{\mathcal{E}}\, u_s\rangle + \langle u_s, \mathcal{C}\, v_s\rangle - \tfrac{1}{2}\langle v_s, \mathcal{C}\, v_s\rangle - V(\xi_s)\, ds \\ &\qquad + \Psi(\xi_T), \end{aligned} \qquad (33)$$

for all $t \in [t_0, T]$, $x \in \mathscr{X}$, and $u \in \mathscr{U}[t, T]$, in which

$$\mathcal{E}, \hat{\mathcal{E}} \in \mathcal{L}(\mathscr{X}), \quad \mathcal{E} \doteq \mathcal{C} + \mathcal{M}, \quad \hat{\mathcal{E}} \doteq \mathcal{C} - \mathcal{M}, \qquad (34)$$

are coercive by (27). The aforementioned auxiliary optimal control problems are defined via their respective value functions $W_t^v, \widehat{W}_t^v : \mathscr{X} \times \mathscr{U}[t, T] \to \mathbb{R}$, with

$$W_t^v(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t^v[\Psi](x, u), \qquad (35)$$

$$\widehat{W}_t^v(x) \doteq \sup_{u \in \mathscr{U}[t,T]} \hat{J}_t^v[\Psi](x, u), \qquad (36)$$

for all $x \in \mathscr{X}$, $v \in \mathscr{U}[t, T]$. Analogously to the short horizon case of (4), (8), and Theorem 2.1, the relevant Hamiltonians are defined with respect to (5) by

$$H^v(t, x, p) \doteq H(x, p) - \tfrac{1}{2}\langle p + \mathcal{M}\, v_t,\, \mathcal{G}(p + \mathcal{M}\, v_t)\rangle, \qquad (37)$$

$$\hat{H}^v(t, x, p) \doteq H(x, p) - \tfrac{1}{2}\langle p + \mathcal{M}\, v_t,\, \hat{\mathcal{G}}(p + \mathcal{M}\, v_t)\rangle, \qquad (38)$$

for all $v \in \mathscr{U}[t, T]$, $t \in [t_0, T]$, $x, p \in \mathscr{X}$, in which

$$\mathcal{G}, \hat{\mathcal{G}} \in \mathcal{L}(\mathscr{X}), \ \mathcal{G} \doteq \mathcal{M}^{-1} - \mathcal{E}^{-1}, \ \hat{\mathcal{G}} \doteq \mathcal{M}^{-1} + \hat{\mathcal{E}}^{-1}, \qquad (39)$$

are coercive, by coercivity of $\mathcal{E}$, $\hat{\mathcal{E}}$ of (34). By (5), (39), the maps $p \mapsto H^v(t, x, p)$ and $p \mapsto \hat{H}^v(t, x, p)$ are respectively convex and concave. Properties of these auxiliary optimal control problems follow analogously to Theorem 2.1, while being applicable to longer time horizons.

Theorem 4.5: Suppose (3) holds. Given arbitrary $v \in C([t_0, T]; \mathscr{X})$, the following properties of (35) hold:

(i) Given $t \in [t_0, T]$, $x \in \mathscr{X}$, there exists a unique optimal input $u_v^* \in \mathscr{U}[t, T]$ such that $W_t^v(x) = J_t^v[\Psi](x, u_v^*) \in \mathbb{R}$; and

(ii) There exists a classical solution of the TPBVP

$$\begin{cases} \dot{x}_s = v_s - \mathcal{E}^{-1}(\mathcal{M}\, v_s + p_s), & x_t = x, \\ \dot{p}_s = \nabla V(x_s), & p_T = \nabla\Psi(x_T), \end{cases} \qquad (40)$$

for all $s \in [t, T]$, such that u_v^* of *(i)* satisfies

$$[u_v^*]_s = v_s - \mathcal{E}^{-1}(\mathcal{M}\, v_s + p_s), \quad s \in [t, T]. \qquad (41)$$

Proof: *(i):* Fix $t_0, T \in \mathbb{R}$ with $t_0 < T$ via (3), and let $\mathcal{C} \in \mathcal{L}(\mathscr{X})$ be as per (27). Fix $t \in [t_0, T]$. Observe by Lemma 4.3 that $J_t^v[\Psi](x, \cdot) : \mathscr{U}[t, T] \to \mathbb{R}$ is strictly convex and coercive. Hence, there exists a unique optimal control $u_v^* \in \mathscr{U}[t, T]$ that is the minimizer of $J_t^v[\Psi](x, \cdot)$, i.e. $W_t^v(x) = J_t^v[\Psi](x, u_v^*) \in \mathbb{R}$. *(ii):* The characteristic system (40) follows by inspection of (37). Existence of a solution to (40) follows by Pontryagin's minimum principle and *(i)*. ∎

Theorem 4.6: Suppose (3) holds. Given arbitrary $v \in C([t_0, T]; \mathscr{X})$, the following properties concerning the value function (36) hold:

(i) Given $t \in [t_0, T]$, $x \in \mathscr{X}$, there exists a unique optimal input $\hat{u}_v^* \in \mathscr{U}[t, T]$ such that $\widehat{W}_t^v(x) = \hat{J}_t^v[\Psi](x, \hat{u}_v^*) \in \mathbb{R}$; and

(ii) There exists a classical solution of the TPBVP

$$\begin{cases} \dot{\hat{x}}_s = v_s + \hat{\mathcal{E}}^{-1}(\mathcal{M}\, v_s + \hat{p}_s), & \hat{x}_t = x, \\ \dot{\hat{p}}_s = \nabla V(\hat{x}_s), & \hat{p}_T = \nabla\Psi(\hat{x}_T), \end{cases} \qquad (42)$$

for all $s \in [t, T]$, such that $\hat{u}_v^*$ of *(i)* satisfies

$$[\hat{u}_v^*]_s = v_s + \hat{\mathcal{E}}^{-1}(\mathcal{M}\, v_s + \hat{p}_s), \quad s \in [t, T]. \qquad (43)$$

Proof: *(i):* Fix $t_0, T \in \mathbb{R}$ with $t_0 < T$ via (3), and let $\mathcal{C} \in \mathcal{L}(\mathscr{X})$ be as per (27). Fix $t \in [t_0, T]$. Observe by Lemma 4.3 that $-\hat{J}_t^v[\Psi](x, \cdot) : \mathscr{U}[t, T] \to \mathbb{R}$ is strictly convex and coercive. Hence, there exists a unique optimal control $\hat{u}_v^* \in \mathscr{U}[t, T]$ that is the maximizer of $\hat{J}_t^v[\Psi](x, \cdot)$, i.e. $\widehat{W}_t^v(x) = J_t^v[\Psi](x, \hat{u}_v^*) \in \mathbb{R}$. The remaining assertion *(ii)* follows analogously as per Theorem 4.5. ∎

Verification theorems analogous to Theorem 2.4 follow, with the HJB PDEs corresponding to (35), (36) given by

$$\begin{cases} 0 = -\dfrac{\partial W_t}{\partial t}(x) + \mathcal{H}^v(t, x, \nabla_x W_t(x)), \\ W_T(x) = \Psi(x), \end{cases} \tag{44}$$

for all $t \in [t_0, T]$, $x \in \mathscr{X}$, in which $\mathcal{H}^v \in \{H^v, \widehat{H}^v\}$, and H^v, $\widehat{H}^v$ are as per (37), (38). Their proofs are standard [1].

Theorem 4.7: Under the conditions of Theorem 4.5, with $v \in C([t,T]; \mathscr{X})$ fixed, suppose there exists $(t,x) \mapsto W_t(x) \in C^1((t_0,T) \times \mathscr{X}; \mathbb{R})$ such that (44) holds with $\mathcal{H}^v \doteq H^v$, and $(\frac{\partial}{\partial t} W_t(x), \nabla W_t(x)) \in \mathbb{R} \times \mathscr{X}$ denoting the Fréchet derivative at $(t,x) \in (t_0,T) \times \mathscr{X}$. Then, $W_t^v(x) \leqslant J_t^v[\Psi](x,u)$ for all $u \in \mathscr{U}[t,T]$. Furthermore, if there exists a mild solution $s \mapsto (x_v^*)_s$, $s \in (t,T)$ of (2) satisfying

$$\begin{aligned} [x_v^*]_s &= x + \int_t^s [u_v^*]_\sigma \, d\sigma, \\ [u_v^*]_\sigma &= v_\sigma - \mathcal{E}^{-1}(\mathcal{M}\, v_\sigma + \nabla W_t^v([x_v^*]_\sigma)), \end{aligned} \tag{45}$$

such that $[x_v^*]_s \in \mathscr{X}$ for all $s \in (t,T)$, then $W_t^v(x) = J_t^v[\Psi](x, u_v^*)$ for all $x \in \mathscr{X}$.

Theorem 4.8: Under the conditions of Theorem 4.6, with $v \in C([t,T]; \mathscr{X})$ fixed, suppose there exists $(t,x) \mapsto W_t(x) \in C^1((t_0,T) \times \mathscr{X}; \mathbb{R})$ such that (44) holds with $\mathcal{H}^v \doteq \widehat{H}^v$, and $(\frac{\partial}{\partial t} W_t(x), \nabla W_t(x)) \in \mathbb{R} \times \mathscr{X}$ denoting the Fréchet derivative at $(t,x) \in (t_0,T) \times \mathscr{X}$. Then, $W_t^v(x) \geqslant \widehat{J}_t^v[\Psi](x,u)$ for all $u \in \mathscr{U}[t,T]$. Furthermore, if there exists a mild solution $s \mapsto (x_v^*)_s$, $s \in (t,T)$ of (2) satisfying

$$\begin{aligned} [\hat{x}_v^*]_s &= x + \int_t^s [\hat{u}_v^*]_\sigma \, d\sigma, \\ [\hat{u}_v^*]_\sigma &= v_\sigma + \widehat{\mathcal{E}}^{-1}(\mathcal{M}\, v_\sigma + \nabla W_t^v([\hat{x}_v^*]_\sigma)), \end{aligned} \tag{46}$$

such that $[\hat{x}_v^*]_s \in \mathscr{X}$ for all $s \in (t,T)$, then $W_t^v(x) = \widehat{J}_t^v[\Psi](x, \hat{u}_v^*)$ for all $x \in \mathscr{X}$.

Theorem 4.9: Suppose (3) holds. Then, $W_t^v(x) \leqslant \widehat{W}_t^v(x)$ for all $t \in [t_0, T)$, $x \in \mathscr{X}$, $v \in \mathscr{U}[t,T]$. Moreover, the argstat condition (12) holds, i.e. $\bar{u} \in \operatorname{arg\,stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x,u)$, if and only if $W_t^{\bar{u}}(x) = \widehat{W}_t^{\bar{u}}(x)$.

Proof: The hypothesis is a restatement of Lemma 4.4, via (31), (32), (33), (35), (36). The proof is immediate. ■

D. Verification of Theorem 3.1 via optimal control

Theorems 4.5, 4.6, and 4.9 may be applied to directly verify the long time horizon $\operatorname{arg\,stat}$ characterization provided by Theorem 3.1. In particular, by application of Theorems 4.5 and 4.6, it is evident that given $t \in [t_0, T]$, $x \in \mathscr{X}$, and $v \in \mathscr{U}[t,T]$ defined via the TPBVP

$$\begin{cases} \dot{x}_s = v_s \doteq -\mathcal{M}^{-1} p_s, & x_t = x, \\ \dot{p}_s = \nabla V(x_s), & p_T = \nabla \Psi(x_T) \end{cases}$$

for $s \in [t, T]$, that

$$J_t^v[\Psi](x, u_v^*) = W_t^v(x) = \widehat{W}_t^v(x) = \widehat{J}_t^v(x, \hat{u}_v^*),$$

and $u_v^* = v = \hat{u}_v^*$. Hence, Theorem 4.9 immediately yields that $v \in \operatorname{arg\,stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x,u)$, as per Theorem 3.1.

V. Conclusions

The stationary action principle is a fundamental physical postulate that underpins the temporal evolution of dynamical systems that obey conservation laws. Where this evolution involves a finite dimensional generalized position space, and is over a sufficiently short time horizon, this action principle can be encapsulated within an optimal control problem, and tools from classical optimal control can be brought to bear in the computation of system trajectories. However, on longer time horizons, this encapsulation is known to break down, typically due to a loss of convexity of the integrated Lagrangian. In this paper, a new characterization of the stationary action principle is developed that exploits connections between stationarity, semiconvexity, semiconcavity, and optimal control. In particular, it is shown that the stationary action principle can be characterized by a pair of related optimal control problems that are well-defined on arbitrarily long finite horizons.

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