2020 American Control Conference
Denver, CO, USA, July 1-3, 2020

A min-plus fundamental solution semigroup for a class of approximate infinite dimensional optimal control problems

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Abstract— **By exploiting min-plus linearity, semiconcavity, and semigroup properties of dynamic programming, a fundamental solution semigroup for a class of approximate finite horizon linear infinite dimensional optimal control problems is constructed. Elements of this fundamental solution semigroup are parameterized by the time horizon, and can be used to approximate the solution of the corresponding finite horizon optimal control problem for any terminal cost. They can also be composed to compute approximations on longer horizons. The value function approximation provided takes the form of a min-plus convolution of a kernel with the terminal cost. A general construction for this kernel is provided, along with a spectral representation for a restricted class of sub-problems.**

I. Introduction

Infinite dimensional linear quadratic regulator problems describe a class of optimal control problem whose formulation utilizes a standard quadratic cost functional constrained by linear infinite dimensional dynamics described by partial differential equations. This class of optimal control problem has been well-studied in the literature, and includes substantial theory, tools, and representations for their solution, see for example [3], [2].

In the finite dimensional setting, the study of corresponding less specialized nonlinear regulator problems has motivated the development of idempotent methods for their solution. These idempotent methods exploit min-plus or max-plus linearity [9], [10], [1], semiconcavity or semiconvexity, and semigroup properties of dynamic programming to facilitate construction of elements of an idempotent fundamental solution semigroup that can be used to propagate the associated value function to longer time horizons. These developments have yielded min- and max-plus eigenvector methods [10], [1], [4] that provide sparse solution approximations, and even curse-of-dimensionality free methods for efficiently computing solutions for specific problem classes in higher dimensions [11]. Moreover, specific tailoring to finite dimensional linear quadratic regulator problems has yielded new fundamental solutions for the standard differential Riccati equations that arise [12], [8].

In seeking to translate these finite dimensional advances to the infinite dimensional setting, recent efforts have yielded *dual space* solution approaches for particular classes of infinite dimensional optimal control problems, and new fundamental solutions for corresponding operator differential Riccati equations, see [5]. More recent efforts have focussed on *primal space* solution approaches, yielding fundamental solutions for classes of related wave equations [6], [7].

In the current work by the authors, a min-plus primal space fundamental solution is developed for a class of approximate infinite dimensional linear quadratic regulator problems that admits a possibly non-quadratic terminal cost. After setting out the problem class in Section II, the fundamental solution semigroup concept and its construction is detailed in Section III. This concept is further specialized via a spectral representation in Section IV, to facilitate future computations.

Throughout, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ denote the real, integer, and natural numbers respectively, with extended reals defined as $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{\pm\infty\}$. The space of continuous mappings between Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ is denoted by $C(\mathscr{X};\mathscr{Y})$. The set of bounded linear operators between the same spaces is denoted by $\mathcal{L}(\mathscr{X};\mathscr{Y})$, or $\mathcal{L}(\mathscr{X})$ if $\mathscr{X}$ and $\mathscr{Y}$ coincide. The corresponding space of self-adjoint bounded linear operators is denoted by $\Sigma(\mathscr{X})$. The space of strongly (C_0) continuous mappings from an interval $I \subset \mathbb{R}$ to $\mathscr{X}$ is denoted by $C_0(I;\mathscr{X})$, while those with continuous Fréchet derivatives are denoted by $C^1(I;\mathscr{X})$. The complete min-plus algebra $(\overline{\mathbb{R}}, \oplus, \otimes)$ is a commutative semi-field over $\overline{\mathbb{R}}$, equipped with addition and multiplication operations $a \oplus b \doteq \min(a,b)$ and $a \otimes b \doteq a + b$ for all $a, b \in \overline{\mathbb{R}}$.

II. Optimal control problem

Let $\mathscr{U}$, $\mathscr{X}$ denote a pair of infinite dimensional Hilbert spaces. Attention is restricted to a standard finite horizon infinite dimensional linear optimal control problem [3], [2] defined via the value and cost functions

$$W_t(x) \doteq (\mathcal{S}_t\,\Psi)(x) \doteq \inf_{u \in \mathscr{U}[0,t]} J_t[\Psi](x,u), \tag{1}$$

$$J_t[\Psi](x,u) \doteq \int_0^t \tfrac{1}{2}\langle x_s, \mathcal{C}\, x_s\rangle + \tfrac{1}{2}\|u_s\|^2\, ds + \Psi(x_t), \tag{2}$$

for all $t \in \mathbb{R}_{\geqslant 0}$, $u \in \mathscr{U}[0,t] \doteq \mathscr{L}_2([0,t];\mathscr{U})$, $x \in \mathscr{X}$, in which $\mathcal{C} \in \Sigma(\mathscr{X})$ is non-negative and $\Psi : \mathscr{X} \to \overline{\mathbb{R}}$ denotes a possibly extended real valued terminal cost. In (2), $s \mapsto x_s \in C([0,t];\mathscr{X})$ denotes the unique mild solution of the abstract Cauchy problem

$$\dot{x}_s = \mathcal{A}\, x_s + \mathcal{B}\, u_s\,, \quad x_0 = x \in \mathsf{dom}\,(\mathcal{A}) \subset \mathscr{X}, \tag{3}$$

for all $s \in [0,t]$, with $\mathcal{A}$ densely defined in $\mathscr{X}$ and generating a C_0-semigroup, and $\mathcal{B} \in \mathcal{L}(\mathscr{U};\mathscr{X})$. The optimal control problem (2) can be approximated by a corresponding

This research was partially supported by AFOSR grants FA2386-16-1-4066 and FA9550-18-1-0268

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978-1-5386-8266-1/$31.00 ©2020 AACC

problem in which unbounded $\mathcal{A}$ is replaced by its (bounded) Yosida approximation [14],

$$\mathcal{A}^\mu \doteq \mathcal{A}\,(\mathcal{I} - \mu^2\,\mathcal{A})^{-1} \in \mathcal{L}(\mathscr{X}), \quad \mu \in \mathbb{R}_{>0}\,. \tag{4}$$

The analogous value and cost functions are defined as per (1), (2), with

$$W_t^\mu(x) \doteq (\mathcal{S}_t^\mu\,\Psi)(x) \doteq \inf_{u \in \mathscr{U}[0,t]} J_t^\mu[\Psi](x,u), \tag{5}$$

$$J_t^\mu[\Psi](x,u) = \int_0^t \tfrac{1}{2}\langle \xi_s^\mu,\, \mathcal{C}\,\xi_s^\mu\rangle + \tfrac{1}{2}\,\|u_s\|^2\,ds + \Psi(\xi_t^\mu), \tag{6}$$

in which $s \mapsto \xi_s^\mu \in C([0,t];\mathscr{X})$ is the unique classical solution of the Cauchy problem approximating (3) via

$$\dot{\xi}_s = \mathcal{A}^\mu\,\xi_s + \mathcal{B}\,u_s\,, \quad \xi_0 = x \in \mathscr{X}, \tag{7}$$

for all $s \in [0,t]$. A standard verification theorem follows for the optimal control problem (5), and its proof is omitted.

Theorem 2.1 (Verification): Given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$, suppose there exists a functional $(s,x) \mapsto V_s(x) \in C([0,t] \times \mathscr{X};\mathbb{R}) \cap C^1((0,t) \times \mathscr{X};\mathbb{R})$ such that

$$0 = -\frac{\partial V_s}{\partial s}(x) + H(x, \nabla_x V_s(x)), \; V_0(x) = \Psi(x), \tag{8}$$

for all $s \in (0,t)$, $x \in \mathscr{X}$, where $\nabla_x V_s(x) \in \mathscr{X}$ denotes the Riesz representation of the Fréchet derivative of $x \mapsto V_s(x)$, and $H : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ is the Hamiltonian

$$H(x,p) \doteq \tfrac{1}{2}\langle x,\, \mathcal{C}\,x\rangle + \langle p,\, \mathcal{A}^\mu\,x\rangle - \tfrac{1}{2}\langle p,\, \mathcal{B}\,\mathcal{B}'\,p\rangle, \tag{9}$$

for all $x, p \in \mathscr{X}$. Then, $V_t(x) \leqslant J_t^\mu(x,u)$ for all $x \in \mathscr{X}$, $u \in \mathscr{U}[0,t]$. Furthermore, if there exists a mild solution $s \mapsto \xi_s^*$ of the Cauchy problem

$$\begin{aligned} &\dot{\xi}_s^* = \mathcal{A}^\mu\,\xi_s^* + \mathcal{B}\,u_s^*\,, && u_s^* = k_s(\xi_s^*)\,, \\ &k_s(y) \doteq -\mathcal{B}'\,\nabla_x V_{t-s}(y)\,, && s \in [0,t], \; y \in \mathscr{X}\,, \end{aligned} \tag{10}$$

with $\xi_s \in \mathscr{X}$ for all $s \in [0,t]$, then $V_s(x) = J_s^\mu(x, u_s^*) = W_s^\mu(s)$ for all $s \in [0,t]$.

III. Min-plus fundamental solution semigroup

A. Min-plus convolution representation

The dynamic programming evolution operators $\mathcal{S}_t$, $\mathcal{S}_t^\mu$ defined in (1), (5) are *min-plus linear* operators. In particular, given any $\Psi, \Phi : \mathscr{X} \to \overline{\mathbb{R}}$, $c \in \overline{\mathbb{R}}$, $\mu \in \mathbb{R}_{>0}$, and $t \in \mathbb{R}_{\geqslant 0}$, inspection of (2), (6) reveals that $J_t^\mu[\Psi \oplus (c \otimes \Phi)](x,u) = J_t^\mu[\Psi](x,u) \oplus (c \otimes J_t^\mu[\Phi](x,u))$, for all $x \in \mathscr{X}$, $u \in \mathscr{U}[0,t]$. That is, the map $\Psi \to J_t^\mu[\Psi](x,u)$ is min-plus linear, which implies by (5) and an interchange of $\inf$ and $\min$ that

$$\mathcal{S}_t^\mu\,[\Psi \oplus (c \otimes \Phi)] = \mathcal{S}_t^\mu\,[\Psi] \oplus (c \otimes \mathcal{S}_t^\mu[\Phi])\,. \tag{11}$$

Moreover, the corresponding identity operator $\mathcal{I}^\oplus$ acting on terminal costs is given by the min-plus convolution

$$\begin{aligned} (\mathcal{I}^\oplus\,\Psi)(x) &\doteq \inf_{y \in \mathscr{X}} \{\delta(x,y) + \Psi(y)\}\,, \\ \delta(x,y) &\doteq \left\{ \begin{array}{cl} 0 & \|x-y\| = 0, \\ +\infty & \text{otherwise}, \end{array} \right. \end{aligned} \tag{12}$$

for all $\Psi : \mathscr{X} \to \overline{\mathbb{R}}$, $x, y \in \mathscr{X}$. The interplay of the min-plus linearity property (11) with this min-plus convolution representation of the identity yields a corresponding min-plus convolution representation for the value function (5), see also [9], [13].

Theorem 3.1: Given $\mu \in \mathbb{R}_{>0}$ and $\Psi : \mathscr{X} \to \overline{\mathbb{R}}$, the value function (5) has the min-plus convolution representation

$$W_t^\mu(x) = (\mathcal{G}_t^\mu\,\Psi)(x) \doteq \inf_{y \in \mathscr{X}} \{G_t^\mu(x,y) + \Psi(y)\} \tag{13}$$

for all $t \in \mathbb{R}_{\geqslant 0}$, $x \in \mathscr{X}$, in which the min-plus convolution kernel $G_t^\mu : \mathscr{X} \times \mathscr{X} \to \overline{\mathbb{R}}$ of the defined min-plus linear operator $\mathcal{G}_t^\mu$ is the bivariate functional given by

$$G_t^\mu(x,y) \doteq (\mathcal{S}_t^\mu[\delta(\cdot,y)])(x), \tag{14}$$

for all $x, y \in \mathscr{X}$.

Proof: Let $\Psi : \mathscr{X} \to \overline{\mathbb{R}}$, and fix $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, $x, y \in \mathscr{X}$. Together, (5), (11), (12), and a swap of inf order yields $W_t^\mu(x) = (\mathcal{S}_t^\mu\,\mathcal{I}^\oplus\,\Psi)(x) = \mathcal{S}_t^\mu[\inf_{y \in \mathscr{X}}\{\delta(\cdot,y) + \Psi(y)\}](x) = \inf_{y \in \mathscr{X}}\{(\mathcal{S}_t^\mu[\delta(\cdot,y)])(x) + \Psi(y)\}$, which yields (13) via (14). ∎

Dynamic programming and the min-plus convolution representation for the value function provided by Theorem 3.1 defines a semigroup of min-plus convolution kernels $\{G_t^\mu\}_{t \in \mathbb{R}_{\geqslant 0}}$ and a corresponding semigroup of min-plus convolution operators $\{\mathcal{G}_t^\mu\}_{t \in \mathbb{R}_{\geqslant 0}}$. The latter is referred to as a *min-plus primal space fundamental solution semigroup* [13], [6], [4], [7]. It describes a fundamental solution insofar as the value function (5) for any terminal cost Ψ can be represented using (13).

Observe by inspection of (14) that the kernel G_t^μ is the value of an optimal control problem with fixed initial and final states. Note in particular that $G_t^\mu(x,y) = +\infty$ if no open-loop control $u \in \mathscr{U}[0,t]$ exists such that $x_0 = x$ and $x_t = y$. A homotopy argument can be used to construct this kernel, via a limit representation of (14), see for example [13]. However, a potentially simpler approach is to exploit semiconcavity properties of the optimal control problem (5), analogously to the semiconvexity argument employed in [7, Theorem 3.3]. This is the approach adopted here.

B. Kernel construction via semiconcave duality

In order to introduce suitable notions of uniform semiconvexity and semiconcavity, corresponding semiconvex and semiconcave transforms, and semiconcave duality, it is useful to define the bi-quadratic basis functional $\varphi : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ with respect to a coercive operator $\mathcal{M} \in \Sigma(\mathscr{X})$ by

$$\varphi(x,y) \doteq \tfrac{1}{2}\,\langle x - y,\, \mathcal{M}\,(x-y)\rangle, \tag{15}$$

for all $x, y \in \mathscr{X}$. The spaces of uniformly semiconvex and semiconcave functions $\mathscr{S}_\varphi^+$ and $\mathscr{S}_\varphi^-$ are subsequently defined with respect to (15) by

$$\begin{aligned} \mathscr{S}_\varphi^+ &\doteq \left\{ \Psi : \mathscr{X} \to \overline{\mathbb{R}} \;\middle|\; \begin{array}{l} x \mapsto \Psi(x) + \varphi(x,0) \\ \text{convex, lower closed} \end{array} \right\}, \\ \mathscr{S}_\varphi^- &\doteq \left\{ \Phi : \mathscr{X} \to \overline{\mathbb{R}} \;\middle|\; -\Phi \in \mathscr{S}_\varphi^+ \right\}. \end{aligned} \tag{16}$$

These spaces are in duality, see [10], [4], via either the semiconvex transform $\mathcal{D}_\varphi^+$ or the semiconcave transform $\mathcal{D}_\varphi^-$. The semiconvex transform and its inverse are defined by

$$(\mathcal{D}_\varphi^+ \Psi)(z) \doteq -\sup_{x\in\mathscr{X}}\{-\varphi(x,z) - \Psi(x)\}\,, \quad \Psi \in \mathscr{S}_\varphi^+\,,$$
$$([\mathcal{D}_\varphi^+]^{-1}\,\Phi)(x) \doteq \sup_{z\in\mathscr{X}}\{-\varphi(x,z) + \Phi(z)\}\,, \qquad \Phi \in \mathscr{S}_\varphi^-\,,$$

for all $x, z \in \mathscr{X}$, while for the semiconcave transform,

$$\begin{aligned} \mathcal{D}_\varphi^-\,\Phi &\doteq -\mathcal{D}_\varphi^+[-\Phi] = [\mathcal{D}_\varphi^+]^{-1}\Phi\,, \qquad \Phi \in \mathscr{S}_\varphi^-\,, \\ [\mathcal{D}_\varphi^-]^{-1}\,\Psi &\doteq -([\mathcal{D}_\varphi^+]^{-1}[-\Psi] = \mathcal{D}_\varphi^+\,\Psi\,, \quad \Psi \in \mathscr{S}_\varphi^+\,, \end{aligned} \tag{17}$$

in which the symmetry of φ implies the right-hand equivalences. In representing the kernel (14) via the semiconcave transform, a parameterized auxiliary optimal control problem is considered. In particular, given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, $y \in \mathscr{X}$, and basis (15), define the auxiliary value function $S_t^\mu(\cdot, y) : \mathscr{X} \to \overline{\mathbb{R}}$ of interest by

$$S_t^\mu(x,y) \doteq [\mathcal{S}_t^\mu\,\varphi(\cdot,y)](x)\,, \tag{18}$$

for all $x \in \mathscr{X}$. An explicit representation for this auxiliary value function is provided via an application of the verification Theorem 2.1.

Theorem 3.2: Given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, the auxiliary value function (18) has the explicit quadratic form

$$S_t^\mu(x,y) = \tfrac{1}{2}\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \mathcal{X}_t^\mu & \mathcal{Y}_t^\mu \\ (\mathcal{Y}_t^\mu)' & \mathcal{Z}_t^\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \tag{19}$$

for all $x, y \in \mathscr{X}$, in which $\mathcal{X}_t^\mu, \mathcal{Z}_t^\mu \in \Sigma(\mathscr{X})$, $\mathcal{Y}_t^\mu \in \mathcal{L}(\mathscr{X})$ are evaluations of the respective classical solutions

$$s \mapsto \mathcal{X}_s^\mu, \mathcal{Z}_s^\mu \in C([0,t]; \Sigma(\mathscr{X})) \cap C^1((0,t); \Sigma(\mathscr{X}))\,,$$
$$s \mapsto \mathcal{Y}_s^\mu \in C([0,t]; \mathcal{L}(\mathscr{X})) \cap C^1((0,t); \mathcal{L}(\mathscr{X}))\,,$$

for $s \in (0,t)$, of the operator-valued Cauchy problems

$$\begin{aligned} \dot{\mathcal{X}}_s^\mu &= (\mathcal{A}^\mu)'\,\mathcal{X}_s^\mu + \mathcal{X}_s^\mu\,\mathcal{A}^\mu - \mathcal{X}_s^\mu\,\mathcal{B}\mathcal{B}'\,\mathcal{X}_s^\mu + \mathcal{C}\,, & \mathcal{X}_0^\mu &= \mathcal{M}\,, \\ \dot{\mathcal{Y}}_s^\mu &= (\mathcal{A}^\mu - \mathcal{B}\mathcal{B}'\,\mathcal{X}_s^\mu)'\,\mathcal{Y}_s^\mu\,, & \mathcal{Y}_0^\mu &= -\mathcal{M}\,, \\ \dot{\mathcal{Z}}_s^\mu &= -(\mathcal{Y}_s^\mu)'\,\mathcal{B}\mathcal{B}'\,\mathcal{Y}_s^\mu\,. & \mathcal{Z}_0^\mu &= \mathcal{M}\,. \end{aligned} \tag{20}$$

Proof: Fix $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, and $x, y \in \mathscr{X}$. With $\mathcal{A}^\mu \in \mathcal{L}(\mathscr{X})$ by (4) and the Hille-Yosida Theorem [14, Theorem 3.1, p.8], the existence of a unique classical solution $s \mapsto \mathcal{X}_s^\mu \in C([0,t]; \Sigma(\mathscr{X})) \cap C^1((0,t); \Sigma(\mathscr{X}))$ of the first Cauchy problem in (20) follows by (for example) [2, Proposition 3.2, p.399]. Existence of unique classical solutions for the remaining two Cauchy problems in (20) follow similarly. Let $\widehat{S}_t^\mu$ denote the explicit quadratic form (19). As $\mathcal{X}_s^\mu, \mathcal{Z}_s^\mu \in C^1((0,t); \Sigma(\mathscr{X}))$, $\mathcal{Y}_s^\mu \in C^1((0,1); \mathcal{L}(\mathscr{X}))$, it follows that $s \mapsto \widehat{S}_s^\mu(x,y)$ of (19) is Fréchet differentiable. Similarly, $x \mapsto \widehat{S}_s^\mu(x,y)$ is Fréchet differentiable by inspection. Hence, by substitution, $(s,x) \mapsto \widehat{S}_s^\mu(x,y)$ is a classical solution of HJB PDE (8), and the stated assertion regarding (18) and (19) follows by Theorem 2.1 and (20). ■

Theorem 3.3: Given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, the min-plus primal space fundamental solution kernel G_t^μ of (14) satisfies

$$G_t^\mu(x,\cdot) \in \mathscr{S}_\varphi^+\,, \quad G_t^\mu(x,y) = [\mathcal{D}_\varphi^-\,S_t^\mu(x,\cdot)](y)\,, \tag{21}$$

for all $x, y \in \mathscr{X}$, in which $S_t^\mu(x,\cdot) \in \mathscr{S}_\varphi^-$ is the auxiliary value function (18), (19).

Proof: Fix $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, and $x, y \in \mathscr{X}$. By inspection of (15), (19),

$$\begin{aligned} -\,S_t^\mu(0,y) + \varphi(y,0) &= \tfrac{1}{2}\langle y,\, (\mathcal{M} - \mathcal{Z}_t^\mu)\,y\rangle \\ &= \tfrac{1}{2}\left\langle y,\, \left(\int_0^t (\mathcal{Y}_s^\mu)'\,\mathcal{B}\mathcal{B}'\mathcal{Y}_s^\mu\,ds\right) y\right\rangle, \end{aligned}$$

which is convex in y. As $S_t^\mu(0,y) - S_t^\mu(x,y)$ is linear and hence convex in y, note that $-S_t^\mu(x,y) + \varphi(y,0) = [-S_t^\mu(0,y) + \varphi(y,0)] + [S_t^\mu(0,y) - S_t^\mu(x,y)]$ is also convex in y. Note further that this functional is continuous, and hence lower closed [15]. Hence, $-S_t^\mu(x,\cdot) \in \mathscr{S}_\varphi^+$, so that $S_t^\mu(x,\cdot) \in \mathscr{S}_\varphi^- = \mathrm{dom}\,(\mathcal{D}_\varphi^-)$, see (16). Hence, the right-hand side of the right-hand equality in (21) is well-defined, and $G_t^\mu(x,\cdot) \in \mathscr{S}_\varphi^+$. Moreover, recalling Theorem 3.1,

$$\begin{aligned} S_t^\mu(x,y) &= (\mathcal{S}_t^\mu\varphi(\cdot,y))(x) = \inf_{z\in\mathscr{X}}\{G_t^\mu(x,z) + \varphi(z,y)\} \\ &= \inf_{z\in\mathscr{X}}\{\varphi(y,z) + G_t^\mu(x,z)\} = ([\mathcal{D}_\varphi^-]^{-1}\,G_t^\mu(x,\cdot))(y)\,, \end{aligned}$$

in which $[\mathcal{D}_\varphi^-]^{-1}$ is the inverse semiconcave transform, see (17). Hence, taking the semiconcave transform of both sides, with respect to y, yields (21). ■

Theorems 3.2 and 3.3 may now be combined via semiconcave duality to yield an explicit representation of the kernel G_t^μ of any element of the min-plus primal space fundamental solution semigroup $\{\mathcal{G}_t^\mu\}_{t\in\mathbb{R}_{\geqslant 0}}$ for the approximate optimal control problem (5).

Theorem 3.4: Given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, and $x, y \in \mathscr{X}$, the kernel $G_t^\mu(x,y)$ of (14), (21) satisfies $G_t^\mu(x,y) > -\infty$, and if finite, has the explicit quadratic representation

$$G_t^\mu(x,y) = \tfrac{1}{2}\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \mathcal{P}_t^\mu & \mathcal{Q}_t^\mu \\ (\mathcal{Q}_t^\mu)' & \mathcal{R}_t^\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \tag{22}$$

for all $x, y \in \mathscr{X}$, with operators $\mathcal{P}_t^\mu, \mathcal{R}_t^\mu \in \Sigma(\mathscr{X})$, $\mathcal{Q}_t^\mu \in \mathcal{L}(\mathscr{X})$ satisfying

$$\begin{aligned} \mathcal{P}_t^\mu &\doteq \mathcal{X}_t^\mu + \mathcal{Y}_t^\mu\,(\mathcal{M} - \mathcal{Z}_t^\mu)^\sharp\,(\mathcal{Y}_t^\mu)'\,, \\ \mathcal{Q}_t^\mu &\doteq \mathcal{Y}_t^\mu\,(\mathcal{M} - \mathcal{Z}_t^\mu)^\sharp\,\mathcal{M}\,, \\ \mathcal{R}_t^\mu &\doteq -\mathcal{M} + \mathcal{M}\,(\mathcal{M} - \mathcal{Z}_t^\mu)^\sharp\,\mathcal{M}\,, \end{aligned} \tag{23}$$

in which $\mathcal{X}_t^\mu, \mathcal{Z}_t^\mu \in \Sigma(\mathscr{X})$, $\mathcal{Y}_t^\mu \in \mathcal{L}(\mathscr{X})$ are as per (19), and $(\cdot)^\sharp$ denotes the Moore-Penrose inverse. Alternatively, if $\mathcal{M} - \mathcal{Z}_t^\mu$ appearing in (23) is coercive, then G_t^μ is always finite, and (22), (23) hold with $(\cdot)^\sharp = (\cdot)^{-1}$.

Proof: Fix $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geqslant 0}$, and $x, y \in \mathscr{X}$. Applying Theorems 3.2 and 3.3, and in particular (19), (21) via (17),

$$\begin{aligned} G_t^\mu(x,y) &= [\mathcal{D}_\varphi^-\,S_t^\mu(x,\cdot)](y) = ([\mathcal{D}_\varphi^+]^{-1}\,S_t^\mu(x,\cdot))(y) \\ &= -\tfrac{1}{2}\langle y,\, \mathcal{M}\,y\rangle + \tfrac{1}{2}\langle x,\, \mathcal{X}_t^\mu\,x\rangle + \sup_{z\in\mathscr{X}} \pi_{x,y}(z)\,, \end{aligned} \tag{24}$$

in which $\pi_{x,y} : \mathscr{X} \to \overline{\mathbb{R}}$ is defined by $\pi_{x,y}(z) \doteq \langle z,\, (\mathcal{Y}_t^\mu)'\,x + \mathcal{M}\,y\rangle - \tfrac{1}{2}\langle z,\, (\mathcal{M} - \mathcal{Z}_t^\mu)\,z\rangle$. As $\pi_{x,y}(0) = 0$, and $z = 0$ is suboptimal in (24), $G_t^\mu(x,y) \geqslant -\tfrac{1}{2}\langle y,\, \mathcal{M}\,y\rangle + \tfrac{1}{2}\langle x,\, \mathcal{X}_t^\mu\,x\rangle > -\infty$, as per the first assertion. Suppose further that $G_t^\mu(x,y) < \infty$. Applying [5, Lemma E.2] yields existence of the Moore-Penrose inverse $(\mathcal{M} - \mathcal{Z}_t^\mu)^\sharp$, and

$\sup_{z\in\mathscr{X}} \pi_{x,y}(z) = \frac{1}{2}\langle(\mathcal{Y}_t^\mu)' x + \mathcal{M} y,\, (\mathcal{M} - \mathcal{Z}_t^\mu)^\sharp [(\mathcal{Y}_t^\mu)' x + \mathcal{M} y]\rangle$. Hence, (22), (23) follow by (24). Coercivity of $\mathcal{M} - \mathcal{Z}_t^\mu$ implies existence of the corresponding (bounded) inverse, so that the final assertion is immediate. ∎

IV. Spectral representation

A. Restricted problem class

In illustrating Theorems 3.1, 3.2, 3.3, and 3.4, a convenient restricted problem class is considered, with $\mathscr{U} = \mathscr{X}$ and

$$\mathcal{A} \doteq -\Lambda\,, \; \mathcal{B} = \mathcal{C} = \mathcal{I}, \tag{25}$$

in which $\Lambda : \mathsf{dom}\,(\Lambda) \subset \mathscr{X} \to \mathscr{X}$ is a densely defined, unbounded, positive, and self-adjoint operator with a compact inverse, and $\mathcal{I} \in \mathcal{L}(\mathscr{X})$ is the identity operator on $\mathscr{X}$. The compactness restriction on Λ implies via the spectral theorem [3, Theorem A.4.25, p.619] that it has the representation $\Lambda\, x = \sum_{n=1}^\infty \lambda_n \langle x,\, \varphi_n\rangle \varphi_n$ for all $x \in \mathscr{X}$, in which $\{\lambda_n^{-1}\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ denote the sets of eigenvalues and corresponding orthonormal eigenvectors of Λ^{-1}. Note in particular that $\{\lambda_n^{-1}\}_{n\in\mathbb{N}}$ is a positive, strictly decreasing sequence, with $\lim_{n\to\infty} \lambda_n^{-1} = 0$. This representation is inherited by $\mathcal{A}$ and $\mathcal{A}^\mu$, with

$$\mathcal{A}\, x = \sum_{n=1}^\infty -\lambda_n \langle x,\, \varphi_n\rangle \varphi_n\,, \quad \mathcal{A}^\mu\, x = \sum_{n=1}^\infty -\lambda_n^\mu \langle x,\, \varphi_n\rangle \varphi_n\,, \tag{26}$$

for all $x \in \mathscr{X}$, and (for convenience)

$$\lambda_n^\mu \doteq \frac{\lambda_n}{1 + \mu^2\, \lambda_n}\,, \quad \omega_n^\mu \doteq \sqrt{1 + (\lambda_n^\mu)^2}\,, \quad n \in \mathbb{N}\,. \tag{27}$$

Note by inspection that $\lambda_1^\mu \leqslant \lambda_n^\mu < \frac{1}{\mu^2}$ and $\omega_n^\mu < \sqrt{1 + \frac{1}{\mu^4}}$, for all $n \in \mathbb{N}$. The first bound further implies that $\|\mathcal{A}^\mu\| \leqslant \frac{1}{\mu^2}$. In (15), the operator $\mathcal{M} = \mathcal{M}^\mu \in \Sigma(\mathscr{X})$ is selected with the same spectral representation (26), with eigenvalues $\{m_n^\mu\}_{n\in\mathbb{N}} \subset \mathbb{R}_{>0}$ restricted to satisfy a priori fixed bounds $\underline{m}^\mu$, $\overline{m}^\mu$, with

$$m_n^\mu \in [\underline{m}^\mu,\, \overline{m}^\mu] \subset \mathbb{R}_{>0}\,, \quad n \in \mathbb{N}\,, \tag{28}$$
$$\text{with} \; \begin{cases} \underline{m}^\mu \doteq 2 \max_{n\in\mathbb{N}} g(\lambda_n^\mu) = 2\, g(\lambda_1^\mu) > 0\,, \\ \underline{m}^\mu \leqslant \overline{m}^\mu < \infty\,, \end{cases}$$

defined with respect to the strictly positive and strictly decreasing map $\lambda \mapsto g(\lambda) \doteq \sqrt{1 + \lambda^2} - \lambda > 0$, $\lambda \in \mathbb{R}_{\geqslant 0}$.

It is emphasized that $\mathcal{M} = \mathcal{M}^\mu$ of (15), (28) depends on the approximation parameter $\mu \in \mathbb{R}_{>0}$. A similar dependence arises in a related max-plus / semiconvex setting [6, (43)].

B. Kernel

In order to compute the kernel G_t^μ via (22), (23), i.e. Theorems 3.2 and 3.4, solutions of the operator DREs (20) are required. To this end, it is convenient to define candidate solutions $s \mapsto \widehat{\mathcal{X}}_s^\mu, \widehat{\mathcal{Y}}_s^\mu, \widehat{\mathcal{Z}}_s^\mu$ of the form (26), with respective eigenvalues given by

$$[\hat{x}_s^\mu]_n \doteq -\lambda_n^\mu + \omega_n^\mu \coth(\omega_n^\mu\, s + \theta_n^\mu)\,, \tag{29}$$
$$[\hat{y}_s^\mu]_n \doteq -m_n^\mu \sinh\theta_n^\mu \operatorname{csch}(\omega_n^\mu\, s + \theta_n^\mu)\,,$$
$$[\hat{z}_s^\mu]_n \doteq m_n^\mu - \frac{(m_n^\mu)^2}{\omega_n^\mu} \sinh^2\theta_n^\mu\, [\coth\theta_n^\mu - \coth(\omega_n^\mu\, s + \theta_n^\mu)],$$

in which $\{m_n^\mu\}_{n\in\mathbb{N}} \subset \mathbb{R}_{>0}$ denote the eigenvalues of $\mathcal{M} = \mathcal{M}^\mu \in \Sigma(\mathscr{X})$ as per (15), (28), and

$$\theta_n^\mu \doteq \coth^{-1}\left(\frac{m_n^\mu + \lambda_n^\mu}{\omega_n^\mu}\right)\,, \quad n \in \mathbb{N}. \tag{30}$$

Some straightforward algebraic manipulations yield boundedness of these sequences, and corresponding sequences of time derivatives of (29).

Lemma 4.1: Given $\mu \in \mathbb{R}_{>0}$, and $\underline{m}^\mu$, $\overline{m}^\mu$ as per (28), the sequence $\{\theta_n^\mu\}_{n\in\mathbb{N}}$ of (30) is well-defined and satisfies the uniform bounds

$$\theta_n^\mu \in (\underline{\theta}^\mu,\, \overline{\theta}^\mu) \subset \mathbb{R}_{>0}\,, \quad n \in \mathbb{N}\,, \tag{31}$$
$$\text{with} \; \begin{cases} \underline{\theta}^\mu \doteq \coth^{-1}\left(\dfrac{\overline{m}^\mu + \lambda_\infty^\mu}{\omega_1^\mu}\right) > 0\,, \\ \overline{\theta}^\mu \doteq \coth^{-1}\left(\dfrac{\underline{m}^\mu}{2\,\omega_\infty^\mu} + 1\right) > \underline{\theta}^\mu\,. \end{cases}$$

Furthermore, given $t \in \mathbb{R}_{>0}$, and $\mathcal{M} = \mathcal{M}^\mu$ of (15), (28), the eigenvalue sequences $\{[\hat{x}_s^\mu]_n\}_{n\in\mathbb{N}}$, $\{[\hat{y}_s^\mu]_n\}_{n\in\mathbb{N}}$, $\{[\hat{z}_s^\mu]_n\}_{n\in\mathbb{N}}$ of (29) are well-defined and bounded uniformly in $s \in [0, t]$, $n \in \mathbb{N}$. Moreover, the sequences of first and second derivatives with respect to $s \in (0, t)$ are likewise bounded uniformly, and in particular the first derivatives satisfy

$$\begin{aligned} \tfrac{d}{ds}[\hat{x}_s^\mu]_n &= (\omega_n^\mu)^2\, [1 - \coth^2(\omega_n^\mu\, s + \theta_n^\mu)] \\ &= 1 - 2\,\lambda_n^\mu\, [\hat{x}_s^\mu]_n - [\hat{x}_s^\mu]_n^2\,, \\ \tfrac{d}{ds}[\hat{y}_s^\mu]_n &= m_n^\mu\, \omega_n^\mu \sinh\theta_n^\mu \coth(\omega_n^\mu\, s + \theta_n^\mu) \times \\ &\qquad \operatorname{csch}(\omega_n^\mu\, s + \theta_n^\mu) \\ &= -[\hat{x}_s^\mu]_n\, [\hat{y}_s^\mu] - \lambda_n^\mu\, [\hat{y}_s^\mu]_n\,, \\ \tfrac{d}{ds}[\hat{z}_s^\mu]_n &= -(m_n^\mu)^2 \sinh^2\theta_n^\mu \operatorname{csch}^2(\omega_n^\mu\, s + \theta_n^\mu) \\ &= -[\hat{y}_s^\mu]_n^2\,, \end{aligned} \tag{32}$$

for all $s \in (0, t)$, $n \in \mathbb{N}$.

Lemma 4.1 implies that the candidate solutions defined by (29) are indeed solutions of the Cauchy problems (20).

Lemma 4.2: Given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$, the operator valued maps $s \mapsto \widehat{\mathcal{X}}_s^\mu, \widehat{\mathcal{Y}}_s^\mu, \widehat{\mathcal{Z}}_s^\mu$ of (29) are Fréchet differentiable and solve the Cauchy problems (20), for all $s \in [0, t]$.

Proof: Fix $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$. Note by Lemma 4.1 that the sequences $\{[\hat{x}_s^\mu]_n\}_{n\in\mathbb{N}}$, $\{\frac{d}{ds}[\hat{x}_s^\mu]_n\}_{n\in\mathbb{N}}$, $\{\frac{d^2}{ds^2}[\hat{x}_s^\mu]_n\}_{n\in\mathbb{N}}$ are bounded uniformly in $s \in (0, t)$, $n \in \mathbb{N}$. Fix $s \in (0, t)$, $n \in \mathbb{N}$. By the mean value theorem, given $h \in (-s, t - s)$, there exists $\sigma \in (0, t)$ depending implicitly on $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$, $s \in (0, t)$, $n \in \mathbb{N}$, and h such that

$$[\hat{x}_{s+h}^\mu]_n - [\hat{x}_s^\mu]_n - h\, \tfrac{d}{ds}[\hat{x}_s^\mu]_n = \tfrac{1}{2}\, h^2\, \tfrac{d^2}{ds^2}[\hat{x}_\sigma^\mu]_n\,.$$

The aforementioned uniform boundedness implies existence of $K_1 \in \mathbb{R}_{\geqslant 0}$ such that

$$\begin{aligned} &\sup_{s\in[0,t]} \sup_{n\in\mathbb{N}} \left|[\hat{x}_{s+h}^\mu]_n - [\hat{x}_s^\mu]_n - h\, \tfrac{d}{ds}[\hat{x}_s^\mu]_n\right| \\ &\leqslant \tfrac{1}{2}\, h^2 \sup_{\sigma\in[0,t]} \sup_{n\in\mathbb{N}} \left|\tfrac{d^2}{ds^2}[\hat{x}_\sigma^\mu]_n\right| = h^2\, K_1 < \infty\,, \end{aligned}$$

Hence, $s \mapsto \widehat{\mathcal{X}}_s^\mu$ is Fréchet differentiable, with derivative

$$\dot{\widehat{\mathcal{X}}}_s^\mu\, x \doteq (\tfrac{d}{ds}\widehat{\mathcal{X}}_s^\mu)\, x \doteq \sum_{n=1}^\infty \tfrac{d}{ds}[\hat{x}_s^\mu]_n \langle x,\, \varphi_n\rangle \varphi_n\,, \tag{33}$$

for all $s \in [0,t]$ and $x \in \mathscr{X}$. The operator-valued maps $s \mapsto \widehat{\mathcal{Y}}_s^\mu, \widehat{\mathcal{Z}}_s^\mu$ are likewise Fréchet differentiable, and their corresponding derivatives have eigenvalues given by the sequences $\{\frac{d}{ds}[\hat{y}_s^\mu]_n\}_{n\in\mathbb{N}}$ and $\{\frac{d}{ds}[\hat{z}_s^\mu]_n\}_{n\in\mathbb{N}}$ in (32). Moreover, recalling (29), (30), (32), it follows that

$$\begin{aligned}
\dot{\widehat{\mathcal{X}}}_s^\mu &= \mathcal{I} + (\mathcal{A}^\mu)'\, \widehat{\mathcal{X}}_s^\mu + \widehat{\mathcal{X}}_s^\mu\, \mathcal{A}^\mu - \widehat{\mathcal{X}}_s^\mu\, \widehat{\mathcal{X}}_s^\mu\,, & \widehat{\mathcal{X}}_0^\mu &= \mathcal{M}^\mu\,,\\
\dot{\widehat{\mathcal{Y}}}_s^\mu &= (\mathcal{A}^\mu - \widehat{\mathcal{X}}_s^\mu)'\, \widehat{\mathcal{Y}}_s^\mu\,, & \widehat{\mathcal{Y}}_0^\mu &= -\mathcal{M}^\mu\,,\\
\dot{\widehat{\mathcal{Z}}}_s^\mu &= -(\widehat{\mathcal{Y}}_s^\mu)'\, \widehat{\mathcal{Y}}_s^\mu\,, & \widehat{\mathcal{Z}}_0^\mu &= \mathcal{M}^\mu\,.
\end{aligned}$$

That is, the operator-valued functions $s \mapsto \widehat{\mathcal{X}}_s^\mu, \widehat{\mathcal{Y}}_s^\mu, \widehat{\mathcal{Z}}_s^\mu$ satisfy (20), given (25), as required. ∎

Theorem 3.2 and Lemma 4.2 subsequently provide the explicit representation (19) for the auxiliary value function S_t^μ of (18) that is the foundation for the application of Theorems 3.3 and 3.4. In particular, (19) holds with

$$\mathcal{X}_s^\mu \doteq \widehat{\mathcal{X}}_s^\mu\,, \quad \mathcal{Y}_s^\mu \doteq \widehat{\mathcal{Y}}_s^\mu\,, \quad \mathcal{Z}_s^\mu \doteq \widehat{\mathcal{Z}}_s^\mu\,, \tag{34}$$

for all $s \in [0,t]$, see (29). With a view to applying Theorem 3.4 in particular, observe by (28), (29), (34) that

$$\begin{aligned}
(\mathcal{M}^\mu - \mathcal{Z}_s^\mu)\, x &= \sum_{n=1}^\infty [\delta_s^\mu]_n\, \langle x,\, \varphi_n\rangle\, \varphi_n\,, \qquad (35)\\
[\delta_s^\mu]_n &\doteq \frac{(m_n^\mu)^2}{\omega_n^\mu} \sinh^2\theta_n^\mu\, [\coth\theta_n^\mu - \coth(\omega_n^\mu\, s + \theta_n^\mu)]\,,
\end{aligned}$$

for all $x \in \mathscr{X}$, $s \in [0,t]$, $n \in \mathbb{N}$. As sinh is increasing and coth is decreasing, by restricting $s \in [\tau, t]$ for some a priori fixed $\tau \in (0,t]$, note further that

$$\begin{aligned}
[\delta_s^\mu]_n &\geqslant \frac{(\underline{m}^\mu)^2}{\omega_\infty^\mu} \sinh^2\theta_n^\mu\, [\coth\theta_n^\mu - \coth(\omega_n^\mu\, s + \theta_n^\mu)]\\
&\geqslant \frac{(\underline{m}^\mu)^2}{\omega_\infty^\mu} \sinh^2\underline{\theta}^\mu\, [\coth\theta_n^\mu - \coth(\omega_1^\mu\, \tau + \theta_n^\mu)]\\
&\geqslant \underline{\delta}^\mu \doteq \frac{(\underline{m}^\mu)^2}{\omega_\infty^\mu} \sinh^2\underline{\theta}^\mu\, [\coth\overline{\theta}^\mu - \coth(\omega_1^\mu\, \tau + \overline{\theta}^\mu)] > 0\,,
\end{aligned}$$

for all $s \in [\tau,t]$, $n \in \mathbb{N}$. Hence, $\mathcal{M}^\mu - \mathcal{Z}_s^\mu$ is coercive, uniformly in $s \in [\tau,t]$, with coercivity constant $\underline{\delta}^\mu > 0$.

Lemma 4.3: Given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$, and $\mathcal{M}^\mu \in \mathcal{L}(\mathscr{X})$ as per (15), (28), $\mathcal{M} - \mathcal{Z}_s^\mu \in \mathcal{L}(\mathscr{X})$ is boundedly invertible for all $s \in (0,t]$, with

$$(\mathcal{M}^\mu - \mathcal{Z}_s^\mu)^{-1} x = \sum_{n=1}^\infty \frac{1}{[\delta_s^\mu]_n}\, \langle x,\, \varphi_n\rangle\, \varphi_n\,, \tag{36}$$

for all $x \in \mathscr{X}$, in which $[\delta_s^\mu]_n$ is as per (35).

Proof: With $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$, and any $s \in (0,t]$ fixed, the assertion follows by coercivity of $\mathcal{M}^\mu - \mathcal{Z}_s^\mu$, see for example [3, Example A.4.2, p.609]. ∎

The bounded inverse $(\mathcal{M}^\mu - \mathcal{Z}_s^\mu)^{-1} \in \mathcal{L}(\mathscr{X})$ provided by Lemma 4.3 replaces the Moore-Penrose pseudo inverse $(\mathcal{M}^\mu - \mathcal{Z}_s^\mu)^\dagger$ in (23). This inverse also has the corresponding spectral representation (26), (29), which implies that $\mathcal{P}_s^\mu$, $\mathcal{Q}_s^\mu$, $\mathcal{R}_s^\mu$ in (22), (23) also do, for $s \in [\tau, t]$.

Theorem 4.4: Given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$, the kernel G_t^μ of (14) is finite, and has the explicit form (22), with $\mathcal{P}_t^\mu, \mathcal{R}_t^\mu \in \Sigma(\mathscr{X})$, $\mathcal{Q}_t^\mu \in \mathcal{L}(\mathscr{X})$ of (22), (23) taking the spectral form (26), with their respective eigenvalues given by

$$\begin{aligned}
[p_t^\mu]_n = -\lambda_n^\mu + \omega_n^\mu \coth(\omega_n^\mu\, t), \quad [q_t^\mu]_n &= -\omega_n^\mu \operatorname{csch}(\omega_n^\mu\, t),\\
[r_t^\mu]_n = \lambda_n^\mu + \omega_n^\mu \coth(\omega_n^\mu\, t), & \qquad (37)
\end{aligned}$$

for all $n \in \mathbb{N}$.

Proof: Fix $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$. By Theorem 3.4 and Lemma 4.3, G_t^μ is finite, and the operators $\mathcal{P}_t^\mu, \mathcal{R}_t^\mu \in \Sigma(\mathscr{X})$, $\mathcal{Q}_t^\mu \in \mathcal{L}(\mathscr{X})$ take the same spectral form (26), by (23) and (29), (34), (36). In particular, the eigenvalues of $\mathcal{P}_t^\mu$ satisfy

$$\begin{aligned}
[p_t^\mu]_n &= [\hat{x}_t^\mu]_n + \frac{[\hat{y}_t^\mu]_n^2}{m_n^\mu - [\hat{z}_t^\mu]_n}\\
&= -\lambda_n^\mu + \omega_n^\mu \coth(\omega_n^\mu\, t + \theta_n^\mu)\\
&\quad + \frac{(m_n^\mu)^2 \sinh^2\theta_n^\mu \operatorname{csch}^2(\omega_n^\mu\, t + \theta_n^\mu)}{\frac{(m_n^\mu)^2}{\omega_n^\mu} \sinh^2\theta_n^\mu\, [\coth\theta_n^\mu - \coth(\omega_n^\mu\, t + \theta_n^\mu)]}\\
&= -\lambda_n^\mu + \omega_n^\mu \left[\frac{1 - \tanh(\omega_n^\mu\, t + \theta_n^\mu)\tanh\theta_n^\mu}{\tanh(\omega_n^\mu\, t + \theta_n^\mu) - \tanh\theta_n^\mu}\right]\\
&= -\lambda_n^\mu + \omega_n^\mu \coth(\omega_n^\mu\, t)\,,
\end{aligned}$$

for all $n \in \mathbb{N}$, via sum-of-angles. The eigenvalues for $\mathcal{Q}_t^\mu$, $\mathcal{R}_t^\mu$ follow similarly, and the details are omitted. ∎

C. Optimal control

A feedback characterization of the optimal control can be obtained via Theorems 2.1, 3.1, and 3.4, under certain circumstances. With this in mind, attention is further restricted in (25) to the case where, given $\mu \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$,

$$\begin{gathered}
\Psi \in C^1(\mathscr{X};\mathbb{R}), \quad \operatorname{dom} W_{t-s}^\mu \neq \emptyset, \qquad (38)\\
\mathscr{Y}_s^\mu(x) \doteq \operatorname{int}\operatorname{dom} G_{t-s}^\mu(x,\cdot) \neq \emptyset,\\
G_{t-s}^\mu(x,\cdot) + \Psi \text{ is strictly convex on } \mathscr{Y}_s^\mu(x),\\
\forall\ x \in \operatorname{dom} W_{t-s}^\mu,\ \ s \in [0,t).
\end{gathered}$$

Given $x \in \mathscr{X}$, $s \in [0,t)$, (38) and Theorem 3.4 imply that $y \mapsto G_{t-s}^\mu(x,y)$ takes the explicit quadratic form (22) and is Fréchet differentiable at every $y \in \mathscr{Y}_s^\mu(x)$. Moreover, as Ψ is Fréchet differentiable and $y \mapsto G_{t-s}^\mu(x,y) + \Psi(y)$ is strictly convex, $y_s^\mu(x) \in \mathscr{Y}_s^\mu(x)$ exists, with $\infty > W_{t-s}^\mu(x) = \min_{y\in\mathscr{Y}_s^\mu(x)}\{G_{t-s}^\mu(x,y) + \Psi(y)\} = G_{t-s}^\mu(x, y_s^\mu(x)) + \Psi(y_s^\mu(x))$, and

$$0 = \nabla_y G_{t-s}^\mu(x, y_s(x)) + \nabla_y \Psi(y_s(x)). \tag{39}$$

If $x \mapsto y_s^\mu(x)$ is itself Fréchet differentiable, note further that $D_x W_{t-s}^\mu(x) = D_x G_{t-s}^\mu(x, y_s(x)) + [D_y G_{t-s}^\mu(x, y_s(x)) + D_y \Psi(y_s(x))]\, D_x y_s(x) = D_x G_{t-s}^\mu(x, y_s(x))$, by (39), or in terms of the Riesz representation,

$$\nabla_x W_{t-s}^\mu(x) = \nabla_x G_{t-s}^\mu(x, y_s(x))\,. \tag{40}$$

In the specific case of a quadratic terminal cost

$$\Psi(x) \doteq \tfrac{1}{2}\,\langle x - \bar{x},\, \mathcal{T}(x - \bar{x})\rangle, \quad x \in \mathscr{X}, \tag{41}$$

given coercive $\mathcal{T} \in \Sigma(\mathscr{X})$ of spectral form (26), and fixed $\bar{x} \in \mathscr{X}$, note that (38) holds for any $\mu > 0$, $t > 0$. In particular, $\mathscr{Y}_s^\mu(x) = \mathscr{X}$, $\mathcal{R}_{t-s}^\mu + \mathcal{T}$ is boundedly invertible, and $y_s^\mu(x) = (\mathcal{R}_{t-s}^\mu + \mathcal{T})^{-1}\,[\mathcal{T}\,\bar{x} - (\mathcal{Q}_{t-s}^\mu)'\, x]$, for all

$x \in \mathscr{X}$, $s \in [0,t)$, by (39). As $x \mapsto y_s^\mu(x)$ is Fréchet differentiable by inspection, (40) applies, with

$$\begin{aligned}
\nabla_x W_{t-s}^\mu(x) &= \nabla_x G_{t-s}^\mu(x, y_s^\mu(x)) = \mathcal{P}_{t-s}^\mu\, x + \mathcal{Q}_{t-s}^\mu\, y_s^\mu(x) \\
&= [\mathcal{P}_{t-s}^\mu - \mathcal{Q}_{t-s}^\mu\, (\mathcal{R}_{t-s}^\mu + \mathcal{T})^{-1}\, (\mathcal{Q}_{t-s}^\mu)']\, x \\
&\quad + \mathcal{Q}_{t-s}^\mu\, (\mathcal{R}_{t-s}^\mu + \mathcal{K})^{-1}\, \mathcal{T}\, \bar{x} \doteq \mathcal{K}_{t-s}^\mu\, x + \mathcal{L}_{t-s}^\mu\, \bar{x}. \quad (42)
\end{aligned}$$

Consequently, according to Theorem 2.1, the optimal feedback control has the feedback characterization

$$\begin{aligned}
\dot{\xi}_s^\mu &= \mathcal{A}^\mu\, \xi_s^\mu + u_s^\mu, \qquad \xi_0^\mu = x, \quad (43) \\
u_s^\mu &= -\nabla_x W_{t-s}^\mu(\xi_s^\mu) = -\mathcal{K}_{t-s}^\mu\, \xi_s^\mu - \mathcal{L}_{t-s}^\mu\, \bar{x},
\end{aligned}$$

for all $s \in [0,t)$. Letting $\{t_n\}_{n\in\mathbb{N}}$ denote the eigenvalues of $\mathcal{T} \in \Sigma(\mathscr{X})$, (42) and the spectral form (26) imply that $\mathcal{K}_{t-s}$, $\mathcal{L}_{t-s}$ are of the same form, with eigenvalues given by

$$[k_{t-s}^\mu]_n \doteq [p_{t-s}^\mu]_n - \frac{[q_{t-s}^\mu]_n^2}{[r_{t-s}^\mu]_n + t_n}, \; [l_{t-s}^\mu]_n \doteq \frac{[q_{t-s}^\mu]_n\, t_n}{[r_{t-s}^\mu]_n + t_n}.$$

for all $s \in [0,t)$, $n \in \mathbb{N}$.

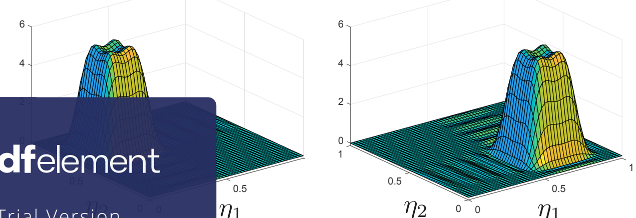
V. Example

The min-plus primal space fundamental solution semigroup $\{\mathcal{G}_t^\mu\}_{t\in\mathbb{R}_{\geq 0}}$ of (13) underlies propagation of the approximate value function (5) to longer horizons $t \in \mathbb{R}_{>0}$ for any terminal cost, see Theorem 3.1. The kernel G_t^μ involved can be characterized via semiconcave duality, see Theorems 3.2, 3.3, 3.4. In the restricted setting (25), explicit spectral representations for the operators involved yield a state feedback characterization for the optimal control, see Theorem 4.4 and (42), (43).

As an illustration of these developments, a heat equation in considered, and the feedback characterization (42), (43) of the optimal control developed is simulated in feedback with it. The problem data of (1), (2), (3), and (5), (6), (7) is specified via (25), (41), with

$$\begin{aligned}
&\Lambda \doteq -\partial_1^2 - \partial_2^2, \; \mathrm{dom}\,(\Lambda) \subset \mathscr{X} \doteq \mathscr{L}_2(\Omega;\mathbb{R}), \; \Omega \doteq [0,1]^2, \\
&\mathrm{dom}\,(\Lambda) \doteq \left\{ x \in \mathscr{X} \;\middle|\; \begin{array}{c} x,\, \partial_1 x,\, \partial_2 x \text{ abs. cts.} \\ x|_{\partial\Omega} = 0,\, \partial_1^2 x,\, \partial_2^2 x \in \mathscr{X} \end{array} \right\}, \\
&\qquad t \doteq 0.01, \; \mu \doteq 0.01, \; \mathcal{T} \doteq 10^3\, \mathcal{I}. \quad (44)
\end{aligned}$$

Figure 1 illustrates an initial and desired final temperature distributions, defining $x, \bar{x} \in \mathscr{X}$ in (7), (41). Figure 2 illustrates the approximate controlled temperature distribution $\xi_s^\mu \in \mathscr{X}$ for $s \in [0,t]$, demonstrating evolution from $\xi_0^\mu = x$ towards $\xi_t^\mu = \bar{x}$.



(a) $x_0^\mu = x \in \mathscr{X}$. (b) $\bar{x} \in \mathscr{X}$.

Fig. 1: Initial and final temperature distributions.

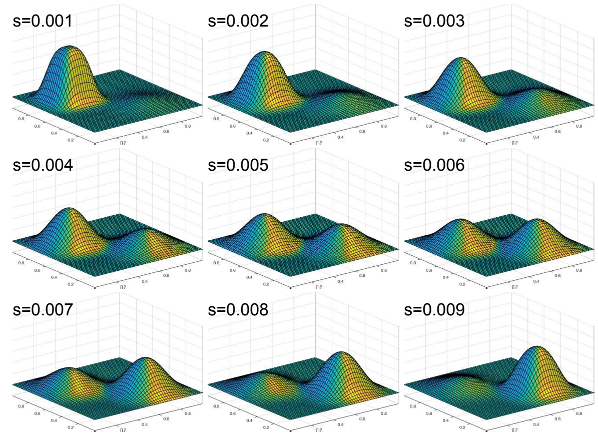


Fig. 2: Controlled temperature distribution $s \mapsto x_s \in \mathscr{X}$.

VI. Conclusions

A min-plus primal space fundamental solution is developed for a class of approximate infinite dimensional optimal control problems. For a sub-class of problems, an explicit spectral representation for this fundamental solution is obtained and subsequently applied to yield a state feedback characterization of the optimal control involved.

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