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A LOCAL TO GLOBAL ARGUMENT ON LOW DIMENSIONAL MANIFOLDS

SAM NARIMAN

Abstract. For an oriented manifold M whose dimension is less than 4, we use the contractibility of certain complexes associated to its submanifolds to cut M into simpler pieces in order to do local to global arguments. In particular, in these dimensions, we give a different proof of a deep theorem of Thurston in foliation theory that says the natural map between classifying spaces $\mathrm{BHomeo}^{\delta}(M) \to \mathrm{BHomeo}(M)$ induces a homology isomorphism where $\mathrm{Homeo}^{\delta}(M)$ denotes the group of homeomorphisms of M made discrete. Our proof shows that in low dimensions, Thurston's theorem can be proved without using foliation theory. Finally, we show that this technique gives a new perspective on the homotopy type of homeomorphism groups in low dimensions. In particular, we give a different proof of Hacher's theorem that the homeomorphism groups of Haken 3-manifolds with boundary are homotopically discrete without using his disjunction techniques.

1. Introduction

Often, in h-principle type theorems (e.g. Smale-Hirsch theory), it is easy to check that the statement holds for the open disks (local data) and then one wishes to glue them together to prove that the statement holds for closed compact manifolds (global statement). But there are cases where one has a local statement for a closed disk relative to the boundary. To use such local data to great effect, instead of covering the manifold by open disks, we use certain "resolutions" associated to submanifolds (see Section 2) to cut the manifold into closed disks.

1.1. **Thurston's h-principle theorem for C^0-foliated bundles.**

The main example that led us to such a local to global argument comes from foliation theory. Let $\mathrm{Homeo}(D^n, \partial D^n)$ denote the group of compactly supported homeomorphisms of the interior of the disk D^n with the compact-open topology. By the Alexander trick, we know that the group $\mathrm{Homeo}(D^n, \partial D^n)$ is contractible for all n. Let $\mathrm{Homeo}^{\delta}(D^n, \partial D^n)$ denote the same group as $\mathrm{Homeo}(D^n, \partial D^n)$ but with the discrete topology. By an infinite repetition trick due to Mather ([Mat71]), it is known that $\mathrm{BHomeo}^{\delta}(D^n, \partial D^n)$ is acyclic; i.e., its reduced homology groups vanish. Therefore, the natural map

$$\mathrm{BHomeo}^{\delta}(D^n, \partial D^n) \to \mathrm{BHomeo}(D^n, \partial D^n)$$

induced by the identity homomorphism is in particular a homology isomorphism. Thurston generalized Mather's work ([Mat73]) on foliation theory in [Thu74a] and as a corollary he obtained the following surprising result.

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Theorem 1.1 (Thurston)**.** *For a compact closed connected manifold M, the map*

$$\eta\colon \mathrm{BHomeo}^{\delta}(M) \to \mathrm{BHomeo}(M)$$

induces an isomorphism on homology.

In this paper, we give a proof of this theorem when $\dim(M) \leq 3$. Our proof is inspired by Jekel's calculation of the group homology of $\mathrm{Homeo}^{\delta}(S^1)$ in [Jek12, Theorem 4]. The first proof of Thurston's theorem in the literature in all dimensions was given by McDuff following Segal's program in foliation theory (see [McD80, Seg78]). Thurston in fact proved a more general homology h-principle theorem for foliations such that Theorem 1.1 is just its consequence for C^0-foliations.

Mather and Thurston used foliation theory to study the homotopy fiber of η. To briefly explain their point of view, let us recall the notion of Haefliger groupoid. Haefliger defined a topological groupoid Γ_q^r whose space of objects is $\mathbb{R}^q$ with the usual topology and the space of morphisms between two points is given by germs of C^r-diffeomorphisms sending x to y (see [Hae71, Section 1] for more details). The homotopy type of the classifying space of this groupoid, $\mathrm{B}\Gamma_q^r$, plays an important role in the classification of C^r-foliations (see [Thu74b] and [Thu76]). One of Thurston's deep theorems in foliation theory relates the homotopy type of $\mathrm{B}\Gamma_q^r$ to the group homology of C^r-diffeomorphism groups made discrete. For $r = 0$, he first uses Mather's theorem ([Mat71]) to show that $\mathrm{B}\Gamma_q^0$ is weakly equivalent to the classifying space of rank q microbundles, $\mathrm{BTop}(q)$, and as a consequence he deduces that the map η in Theorem 1.1 is in fact acyclic; in particular, its homotopy fiber has vanishing reduced homology groups.

In fact there are general h-prinicple theorems in all dimensions that identify the homotopy fiber of η as a subspace of certain section spaces of a bundle associated to the manifold M. Our goal in this paper is to show that in low dimensions one can directly study the maps between classifying spaces like η instead of their homotopy fibers. To do so, we provide the strategy in detail for Theorem 1.1 in low dimensions that does not use any foliation theory.

1.2. **On homotopy type of** $\mathrm{Homeo}_0(M)$**.** Using this technique, we also give a different proof of the contractibility of the identity component of homeomorphism groups in low dimensions.

Theorem 1.2 (Earle-Eells-Schatz, Hatcher)**.** *The identity components of homeomorphism groups of hyperbolic surfaces* (*see* [ES70, EE69]) *and Haken manifolds with boundary* (*see* [Hat76]) *are contractible.*

Remark 1.3. The same statement holds for diffeomorphism groups. In dimensions less than 4, it is known that for a compact manifold M, the group $\mathrm{Homeo}_0(M)$ with the compact-open topology and $\mathrm{Diff}_0(M)$ with the C^∞-topology have the same homotopy type. In $\dim(M) = 2$, it is a theorem of Hamstrom [Ham74], and in $\dim(M) = 3$ it is a deep theorem of Cerf [Cer61, Theorem 8].

Remark 1.4. Recall that $\mathrm{Homeo}(D^n, \partial D^n)$ is contractible for all n and that $\mathrm{Diff}(D^n, \partial D^n)$ is contractible for $n \leq 3$ (see [Sma59, Hat83]). One could also use this local data and the technique of this paper to re-prove Hamstrom and Cerf's theorem that $\mathrm{Homeo}_0(M)$ and $\mathrm{Diff}_0(M)$ have the same homotopy type.

Instead of working with the homeomorphism groups, we work with their classifying spaces. Considering the delooping of these topological groups has the advantage that one can apply homological techniques to the classifying spaces to extract homotopical information about homeomorphism groups.

The reason that we restrict ourselves to low dimensions is that for surfaces and 3-manifolds, there is a procedure to split up the manifold into disks. For the surfaces, this procedure is given by cutting along handles. For 3-manifolds, however, it is more subtle to cut it into simpler pieces. In that case, we use the prime decomposition theorem and Haken's hierarchy to cut the manifold into disks.

1.3. **Outline.** The paper is organized as follows: in Section 2, we discuss the main idea and give a different model for the map η which will be technically more convenient. In Section 3, we discuss the case where M is a surface and semisimplicial resolutions for the classifying spaces of homeomorphisms of surfaces. In Section 4, we will treat the case of 3-manifolds. In Section 5, we give a different proof of the contractibility of the identity component of the homeomorphism groups for certain low dimensional manifolds.

2. Resolving classifying spaces by embedded submanifolds

Let us first sketch the idea for Theorem 1.1. Let M be a smooth oriented closed manifold and let $\mathrm{Homeo}_0(M)$ denote the identity component of the topological group $\mathrm{Homeo}(M)$. Since the group $\mathrm{Homeo}(M)$ is locally path connected and in fact it is locally contractible (see [Č69]), the group of connected components $\pi_0(\mathrm{Homeo}(M))$ is a discrete group and sits in a short exact sequence

$$1 \to \mathrm{Homeo}_0(M) \to \mathrm{Homeo}(M) \to \pi_0(\mathrm{Homeo}(M)) \to 1.$$

Using the Serre spectral sequence, one could reduce Theorem 1.1 to proving that the map

$$\eta\colon \mathrm{BHomeo}_0^{\delta}(M) \to \mathrm{BHomeo}_0(M)$$

induces a homology isomorphism. To prove this version, we want to inductively reduce Theorem 1.1 to the case of a simpler manifold. Such simpler manifolds are obtained from M by cutting along its submanifolds.

Let ϕ be an embedding of a manifold into M. To cut along this embedding, we construct a semisimplicial space $A_\bullet(M,\phi)$ on which the topological group $\mathrm{Homeo}_0(M)$ acts (see [ERW19] or [RW16, Section 2] for definitions of (augmented) semisimplicial objects and their realizations).[1] Similarly we construct a semisimplicial set $A_\bullet^{\delta}(M,\phi)$ as the underlying semisimplicial set of the semisimplicial space $A_\bullet(M,\phi)$ on which the group $\mathrm{Homeo}_0^{\delta}(M)$ acts. These semisimplicial spaces are constructed so that their realizations are weakly contractible. Therefore, we obtain semisimplicial resolutions[2]

$$|A_\bullet^{\delta}(M,\phi) /\!\!/ \mathrm{Homeo}_0^{\delta}(M)| \xrightarrow{\simeq} \mathrm{BHomeo}_0^{\delta}(M),$$

$$|A_\bullet(M,\phi) /\!\!/ \mathrm{Homeo}_0^{\delta}(M)| \xrightarrow{\simeq} \mathrm{BHomeo}_0(M).$$

[1] We shall use the same notation for the realizations of semisimplicial spaces and simplicial spaces in this paper.

[2] For a topological group G acting on a topological space X, the homotopy quotient is denoted by $X /\!\!/ G$ and is given by $X \times_G \mathrm{E}G$ where $\mathrm{E}G$ is a contractible space on which G acts freely.

We then construct a zig-zag of maps from the space $A_\bullet^\delta(M,\phi) /\!\!/ \mathrm{Homeo}_0^\delta(M)$ to the space $A_\bullet(M,\phi) /\!\!/ \mathrm{Homeo}_0(M)$ which induces a commutative diagram

$$\begin{array}{ccc} H_*(|A_\bullet^\delta(M,\phi) /\!\!/ \mathrm{Homeo}_0^\delta(M)|;\mathbb{Z}) & \xrightarrow{f_*} & H_*(|A_\bullet(M,\phi) /\!\!/ \mathrm{Homeo}_0(M)|;\mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ H_*(\mathrm{BHomeo}_0^\delta(M);\mathbb{Z}) & \xrightarrow{\eta_*} & H_*(\mathrm{BHomeo}_0(M);\mathbb{Z}). \end{array}$$

Therefore, it is enough to prove that f_* is an isomorphism. As we shall see in Section 3.2, proving that f_* induces a homology isomorphism is equivalent to the statement of Theorem 1.1 for a manifold that is obtained from M by cutting it along ϕ. Then, by induction we can reduce Theorem 1.1 to the case of a disk relative to its boundary that

$$\mathrm{BHomeo}^\delta(D^n,\partial D^n) \to \mathrm{BHomeo}(D^n,\partial D^n)$$

induces a homology isomorphism ([Mat71]).

We restricted ourselves to the case of closed oriented manifold M of dimension less than 4 because we still do not know how to make a certain surgery argument in Lemma 3.31 work in dimensions higher than 3.

To prove Theorem 1.1 for closed manifolds, we need to work with manifolds with boundary. We first fix two notations to deal with homeomorphisms that are relative to the boundary.

Definition 2.1. For an oriented manifold M with (possibly nonempty) boundary, we let $\mathrm{Homeo}(M,\partial M)$ and $\mathrm{Homeo}_\partial(M)$ be respectively the group of compactly supported orientation preserving homeomorphisms of $\mathrm{int}(M)$, interior of M, and the group of orientation preserving homeomorphisms that are the identity on the boundary with the compact open topology.

It is, however, technically more convenient to work with simplicial groups to avoid subtleties of working with the topological group $\mathrm{Homeo}(M)$. We shall define the corresponding simplicial groups.

Definition 2.2. The set of the p-simplices of the simplicial group $S_\bullet(\mathrm{Homeo}_\partial(M))$, namely, the singular complex of $\mathrm{Homeo}(M)$, can be described as the commutative diagram

$$\begin{array}{ccc} \Delta^p \times M & \xrightarrow{\phi} & \Delta^p \times M \\ & {}_{pr_1}\searrow \quad \swarrow_{pr_1} & \\ & \Delta^p, & \end{array}$$

where ϕ is a homeomorphism which is the identity on $\Delta^p \times \partial M$. Similarly, one defines $S_\bullet(\mathrm{Homeo}_{0,\partial}(M))$ and $S_\bullet(\mathrm{Homeo}_0(M,\partial M))$.

Using the theorem of Milnor ([Mil57]), we know that the geometric realization $|S_\bullet(\mathrm{Homeo}_{0,\partial}(M))|$ is a topological group which is weakly equivalent to $\mathrm{Homeo}_{0,\partial}(M)$. The composite of the maps

$$\mathrm{Homeo}_{0,\partial}^\delta(M) \to |S_\bullet(\mathrm{Homeo}_{0,\partial}(M))| \xrightarrow{\mathrm{ev}} \mathrm{Homeo}_{0,\partial}(M),$$

where the first one is induced by the inclusion of the 0-simplices and the second map is induced by the evaluation map $S_\bullet(\mathrm{Homeo}_{0,\partial}(M)) \times \Delta^\bullet \to \mathrm{Homeo}_{0,\partial}(M)$, is the identity homomorphism. Therefore, the map η is factored as

$$\mathrm{BHomeo}^{\delta}_{0,\partial}(M) \to \mathrm{B}|S_\bullet(\mathrm{Homeo}_{0,\partial}(M))| \xrightarrow{\simeq} \mathrm{BHomeo}_{0,\partial}(M).$$

So we reformulate Theorem 1.1 as follows.

Theorem 2.3. *For a compact oriented smooth manifold M whose dimension is less than* 4, *the map*

$$\eta\colon \mathrm{BHomeo}^{\delta}_{0,\partial}(M) \to \mathrm{B}|S_\bullet(\mathrm{Homeo}_{0,\partial}(M))|$$

induces a homology isomorphism.

Remark 2.4. The same statement holds for $\mathrm{Homeo}_{0,\partial}(M)$. Using the pushing collar technique ([Nar17, Corollary 2.3][3]), one can show that the map

$$\mathrm{BHomeo}^{\delta}_{0}(M, \partial M) \to \mathrm{BHomeo}^{\delta}_{0,\partial}(M)$$

induces a homology isomorphism. On the other hand, since the space of collars for topological manifolds is contractible (see [Arm70]), the inclusion map $\mathrm{Homeo}_0(M, \partial M) \hookrightarrow \mathrm{Homeo}_{0,\partial}(M)$ is a weak equivalence (see [Kup15, Section 4.3] for a similar discussion). Hence, Theorem 2.3 also implies that the map

$$\eta\colon \mathrm{BHomeo}^{\delta}_{0}(M, \partial M) \to \mathrm{B}|S_\bullet(\mathrm{Homeo}_{0}(M, \partial M))|$$

induces a homology isomorphism.

We want to cut up M into disks in a "contractible space of choices" (e.g. see Proposition 3.9 and Lemma 3.31). As we shall see in Remark 3.25, the easiest case is when M is homeomorphic to a circle (see also [Jek12, Theorem 4]). For M being a surface, we define a certain space of handles to cut the surface along them. Finally, if M is a 3-manifold, we first reduce to the case of irreducible 3-manifolds and we cut it along incompressible surfaces in a "contractible space of choices". For this reason, we consider the case of 3-manifolds separately. To cut along submanifolds, we need to consider "nicely" embedded submanifolds. This is more essential in dimensions higher than 2. So let us recall the definition of locally flat embeddings.

Definition 2.5. The k simplices of the simplicial set of *locally flat embeddings* $\mathrm{Emb}^{\mathrm{lf}}_\bullet(N, M)$ is given by the commutative diagram

$$\begin{array}{ccc} \Delta^k \times N & \xrightarrow{\quad f \quad} & \Delta^k \times M \\ & {\scriptstyle pr_1}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & \Delta^k, & \end{array}$$

where f is a homeomorphism onto its image which is also locally flat. To recall the condition of being locally flat, let the codimension of the map f be p. Then for all $(t, n) \in \Delta^k \times N$ there exist open neighborhoods U and V around t and n, respectively, such that there is a map $U \times V \times \mathbb{R}^p \to \Delta^k \times M$ over Δ^k which extends $f|_{U\times V}$ and is a homeomorphism onto its image. If N has a boundary, we consider those embeddings that restrict to locally flat embeddings of the interior and locally flat embeddings of the boundary.

[3]This corollary that says certain pushing collar maps between diffeomorphism groups induce homology isomorphisms also works for homeomorphism groups.

Remark 2.6. To cut codimension 0 submanifolds with boundary and obtain a manifold, we need to have a bicollared boundary. This is guaranteed by Brown's result [Bro62, Theorem 3] that locally flat two-sided codimension 1 submanifolds are bicollared.

Remark 2.7. We can also consider the space of locally flat embeddings $Emb^{\mathrm{lf}}(N, M)$ as a subspace of embeddings of N into M with the compact-open topology. In codimension 0 as in codimension 3 and higher (see [Las76, Appendix] for the comparison between different versions of the embedding spaces), it is known that the realization of $\mathrm{Emb}^{\mathrm{lf}}_{\bullet}(N, M)$ has the same homotopy type as $Emb^{\mathrm{lf}}(N, M)$ and in fact in these cases $\mathrm{Emb}^{\mathrm{lf}}_{\bullet}(N, M)$ is equal to the singular set $S_{\bullet}(Emb^{\mathrm{lf}}(N, M))$.

3. Cutting surfaces into disks

In this section M is an oriented surface with possibly nonempty boundary.

3.1. **0-handle resolutions.** The first step is to reduce the statement of Theorem 2.3 to the case of the surfaces with nonempty boundary so that one could remove 1-handles. Hence we first want to remove disks (0-handles) from a closed surface M. To parametrize different choices of removing 0-handles, we define the following semisimplicial spaces.

Definition 3.1. We first define the semisimplicial simplicial set on which the simplicial group $S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))$ acts.

- **Topological versions:** Let $[p]$ denote the set $\{0, 1, \dots, p\}$ of $p + 1$ ordered elements.
  - Let
  $$A_p(M)_{\bullet} = \mathrm{Emb}^{\mathrm{lf}}_{\bullet}(\coprod_{[p]} D^2, M)$$
  denote the simplicial set of locally flat embeddings consisting of orientation preserving embeddings of p disjoint closed unit 2-disks into M. The collection $A_{\bullet}(M)_{\bullet}$ is a semisimplicial simplicial set where the face maps in the semisimplicial direction are given by forgetting disks. We shall write $A_p(M)$ for the realization of $A_p(M)_{\bullet}$ in the simplicial direction.
  - Let $A^{\mathbf{t}}_p(M)$ denote the space $Emb^{\mathrm{lf}}(\coprod_{[p]} D^2, M)$ equipped with the compact-open topology. By Remark 2.7, the natural map $A_p(M) \to A^{\mathbf{t}}_p(M)$ is a weak equivalence. Let $A^{\mathbf{t},\delta}_p(M)$ be the underlying set of the space $A^{\mathbf{t}}_p(M)$; in other words, we have $A^{\mathbf{t},\delta}_p(M) = A_p(M)_0$.
  - We also define an auxiliary semisimplicial simplicial set $\overline{A}_{\bullet}(M)_k$ whose set of 0-simplices in semisimplicial direction is the same as $A_0(M)_k$, but its p-simplices in the semisimplicial direction have k-simplices consisting of $(p+1)$-tuples $(\phi_0(t), \phi_1(t), \dots, \phi_p(t))$ of k-simplices of $A_0(M)_k$, where for all $t \in \Delta^k$ the centers of the embedded disks $\phi_i(t)$ are pairwise disjoint but the disks may overlap. We shall write $\overline{A}_p(M)$ for the realization of $\overline{A}_p(M)_{\bullet}$ in the simplicial direction.
- **Discrete version:** In the 0-simplices $\mathrm{Emb}^{\mathrm{lf}}_0(\coprod_{[p]} D^2, M)$ of the simplical set $A_p(M)_{\bullet}$, we say two embeddings g_1 and g_2 have the same germ if there exists an open neighborhood $U \subset D^2$ around the origin so that $g_1|_{\coprod_{[p]} U} = g_2|_{\coprod_{[p]} U}$.

– Let
$$A^{\delta}_{\bullet}(M)=\mathrm{Emb}^{\mathrm{g}}_0(\coprod_{[\bullet]}D^2,M)$$
denote the set of germs of embeddings of disjoint union of $p+1$ disks compatible with the orientation of M.
– We define an auxiliary semisimplicial set $\overline{A}^{\delta}_{\bullet}(M)$ which is given by 0-simplices in the simplicial direction of the semisimplicial simplicial set $\overline{A}_{\bullet}(M)_{\bullet}$.

The simplicial group $S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))$ acts on the simplical set $A_p(M)_{\bullet}$. To define the 0-handle resolution of $\mathrm{B}|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))|$, we need to consider the homotopy quotient of the action of $S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))$ on $A_p(M)$. To do so, we recall the two-sided bar construction. Recall that for a group G acting on a topological space X, the two-sided bar construction $B_{\bullet}(X,G,*)=X\times G^{\bullet}$ is a simplicial space with the usual face maps and degeneracies. For a discrete group (or well-pointed topological group), the realization of this bar construction is a model for the homotopy quotient $X/\!\!/G$.

Definition 3.2. We define the 0-handle resolution $X_{\bullet,n,k}(M)$ to be an augmented semisimplicial bisimplicial set
$$X_{\bullet,n,k}(M)=B_n(A_{\bullet}(M)_k,S_k(\mathrm{Homeo}_{0,\partial}(M)),*)\xrightarrow{\epsilon}B_n(*,S_k(\mathrm{Homeo}_{0,\partial}(M)),*).$$
We denote the realization of $X_{p,n,k}(M)$ in the bisimplicial directions by $X_p(M)$. The face maps of $X_p(M)$ are induced by the face maps of $A_p(M)$ in the semisimplicial direction.

Since in realizing a bisimplicial set the order of realization in each direction does not matter ([Qui73, Lemma, p. 86]), the realization of the augmentation map ϵ is given by
$$A_p(M)/\!\!/|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))|\xrightarrow{|\epsilon|}\mathrm{B}|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))|\xrightarrow{\simeq}\mathrm{BHomeo}_{0,\partial}(M).$$
The semisimplicial space $X_p(M)$ is called a semisimplicial resolution for the classifying space $\mathrm{BHomeo}_0(M,\partial M)$, because, as we shall see in Proposition 3.9, if we realize $X_p(M)$ in the semisimplicial direction we obtain a map
$$||\epsilon||\colon|X_{\bullet}(M)|\to\mathrm{B}|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))|,$$
which turns out to be a weak equivalence. The fiber of the map $||\epsilon||$ is the realization $|A_{\bullet}(M)|$ of the semisimplicial simplicial set $A_{\bullet}(M)_{\bullet}$, which means first realizing in the simplicial direction to obtain a semisimplicial space and then realizing in the semisimplicial direction. Using Proposition 2.18 in [Kup15], to prove that the map $||\epsilon||$ is a weak equivalence it is enough to show that its fiber $|A_{\bullet}(M)|$ is contractible, as we shall prove in Proposition 3.16.

Remark 3.3. A more geometric model for this homotopy quotient is to consider the simplicial set
$$\mathrm{Emb}^{\mathrm{lf}}_{\partial}(M;\mathbb{R}^{\infty})\times_{S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))}A_p(M)_{\bullet},$$
but we do not need this model for our argument.

On the other hand, the discrete group $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ acts on $A^{\delta}_{\bullet}(M)$. So we define the semisimplicial resolution for $\mathrm{BHomeo}^{\delta}_{0,\partial}(M)$ as follows.

Definition 3.4. The 0-handle resolution for $\mathrm{BHomeo}^{\delta}_{0,\partial}(M)$ is the augmented semisimplicial space

$$\theta\colon X^{\delta}_{\bullet}(M) := A^{\delta}_{\bullet}(M) /\!\!/ \mathrm{Homeo}^{\delta}_{0,\partial}(M) \to \mathrm{BHomeo}^{\delta}_{0,\partial}(M).$$

Similarly, to prove that $|\theta|\colon |X^{\delta}_{\bullet}(M)| \to \mathrm{BHomeo}^{\delta}_{0}(M)$ is a weak equivalence, it is enough to show that its fiber, the realization $|A^{\delta}_{\bullet}(M)|$, is contractible, as we shall prove in Proposition 3.9.

Note that there are natural maps

$$A_{\bullet}(M)_{\bullet} \to \overline{A}_{\bullet}(M)_{\bullet} \leftarrow \overline{A}^{\delta}_{\bullet}(M) \to A^{\delta}_{\bullet}(M),$$

where the first map is the inclusion. By scaling the disks, it is easy to see that the first map induces a weak equivalence after realization in the simplicial directions. The second map is the inclusion to the 0-simplices in the simplicial direction, and the last map is induced by taking germs of embeddings of disks at their centers.

3.1.1. *The homotopy type of $X_p(M)$ and $X^{\delta}_p(M)$.* For the semisimplical space $X_p(M)$, we have a spectral sequence

$$E^1_{p,q}(X_{\bullet}(M)) = H_q(X_p(M)) \Rightarrow H_{p+q}(|X_{\bullet}(M)|)$$

for any coefficient systems that pulls back from $|X_{\bullet}(M)|$ (see [ERW19, Section 1.4]), so we often suppress the coefficients for brevity. Similarly, we have a spectral sequence that calculates $H_{p+q}(|X^{\delta}_{\bullet}(M)|)$. In order to be able to compare these spectral sequences, we need to compare the homotopy types of $X_p(M)$ and $X^{\delta}_p(M)$. The first step is to make sure that they have the same number of connected components.

The set of the connected components for $X^{\delta}_p(M)$, which is the homotopy quotient $A^{\delta}_p(M) /\!\!/ \mathrm{Homeo}^{\delta}_{0,\partial}(M)$, is in bijection with the set of the orbits of the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ on $A^{\delta}_p(M)$. On the other hand, since the map $A_p(M) \xrightarrow{\simeq} A^{\mathbf{t}}_p(M)$ is equivariant with respect to the homomorphism $|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))| \xrightarrow{\simeq} \mathrm{Homeo}_{0,\partial}(M)$, we obtain a map

$$X_p(M) = A_p(M) /\!\!/ |S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))| \to A^{\mathbf{t}}_p(M) /\!\!/ \mathrm{Homeo}_{0,\partial}(M),$$

which is a weak equivalence by the comparison of the long exact sequences of homotopy groups for fibrations. Therefore, the set of the connected components for $X_p(M)$ is in bijection with the set of the orbits of the action of $\mathrm{Homeo}_{0,\partial}(M)$ on $A^{\mathbf{t}}_p(M)$.

Lemma 3.5. *The set of the connected components of $X_p(M)$ is in bijection with that of $X^{\delta}_p(M)$ for all p.*

Proof. From the above discussion, it is enough to show that the set of the orbits of the action of $\mathrm{Homeo}_{0,\partial}(M)$ on $A^{\mathbf{t}}_p(M)$ is in bijection with that of the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ on $A^{\delta}_p(M)$. As we shall see, these actions are transitive, but what matters and will be useful later when the action is not transitive is that the set of the orbits is determined by the action on the core of the handles. In other words, in this case, the orbits are determined by the action on the center of the disks (or rather germs of the disks at their centers).

Suppose we have two embeddings e_1 and e_2 in $A_p(M)_0$. Each embedding gives a configuration of $p+1$ disjoint unparameterized disks $e_i(M) \subset M$. Since we work with orientation preserving embeddings and homeomorphisms, if we show that we can find an element f in $\mathrm{Homeo}_{0,\partial}(M)$ that sends the unparameterized $e_1(M)$ to

$e_2(M)$, we can change f up to isotopy to send e_1 to e_2. We first arrange f to send the centers of the disks in $e_1(M)$ to the centers of the disk in $e_2(M)$. This is easy by the fact that the action of $\mathrm{Homeo}_{0,\partial}(M)$ is strongly k-transitive for all k (see [Ban97, Lemma 2.1.10]), which means that the action of $\mathrm{Homeo}_{0,\partial}(M)$ on the set of k-tuples of points in M is transitive. Then by scaling the disks, we shall change f up to isotopy to send $e_1(M)$ into $e_2(M)$. Since the embeddings are locally flat the regions between the disks are homeomorphic to an annulus ([Kir69]; of course in low dimensions, we do not need the full force of the annulus theorem), we can change f up to an isotopy to send $e_1(M)$ to $e_2(M)$. Similarly for the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ on $A^{\delta}_p(M)$, if we have two germs of embeddings e^g_1 and e^g_2 of disks, we first choose representatives of germs and proceed as before. Therefore, the set of the orbits of both actions depend on the centers of the disks (cores of the handles), and since the actions on the centers are transitive, there is a bijection between the set of the orbits. □

To find the homotopy type of $X_p(M)$ and $X^{\delta}_p(M)$, let us first recall a version of Shapiro's lemma. Let G be a discrete group acting on a set X. One could decompose X as

$$\coprod_{\alpha\in\text{orbits}} G/H_\alpha,$$

where $H_\alpha < G$ is a stabilizer subgroup of an element in the orbit α. Then, we have a map

$$\coprod_{\alpha\in\text{orbits}} \mathrm{B}H_\alpha \xrightarrow{\simeq} X /\!\!/ G = |\mathrm{B}_\bullet(X, G, *)|, \tag{3.6}$$

which is a homotopy equivalence.

Now let $e_p \in A^{\mathbf{t}}_p(M)$ be an embedding of $p+1$ disjoint disks and let $[e_p] \in A^{\delta}_p(M)$ denote its germ at its center. Let $\mathrm{Stab}(e_p)$ denote the stabilizer group of e_p under the action of $\mathrm{Homeo}_{0,\partial}(M)$ on $A_p(M)$. We denote the stabilizer of $[e_p]$ under the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ on $A^{\delta}_p(M)$ by $\mathrm{Stab}^{\delta}([e_p])$. Let also $\mathrm{Stab}([e_p])$ denote the same group but with the subspace topology as a subgroup of $\mathrm{Homeo}_{0,\partial}(M)$.

Lemma 3.7. *There is a map* $\mathrm{BStab}^{\delta}([e_p]) \to X^{\delta}_p(M)$ *which is a homotopy equivalence.*

Proof. This is implied by Shapiro's lemma and the fact that the action is transitive in this case. □

Lemma 3.8. *There is a zig-zag of weak equivalences between* $\mathrm{BStab}(e_p)$ *and* $X_p(M)$.

Proof. By the parametrized isotopy extension theorem in the topological setting ([BL74, p. 19]), we know that the map

$$S_\bullet(\mathrm{Homeo}_{0,\partial}(M)) \xrightarrow{\mathrm{ev}} A_p(M)_\bullet,$$

which is induced by the action on a fixed element, is a Kan fibration.[4] The fiber of this map is $S_\bullet(\mathrm{Stab}(e_p))$. Thus, for each k, we have a bijection between the set $A_p(M)_k$ and the coset

$$S_k(\mathrm{Homeo}_{0,\partial}(M))/S_k(\mathrm{Stab}(e_p)).$$

[4]Note that in this case since $A_p(M)_\bullet = S_\bullet(A^{\mathbf{t}}_p(M))$, the fact that the map ev is a Kan fibration implies that the map $\mathrm{Homeo}_{0,\partial}(M) \to A^{\mathbf{t}}_p(M)$ is a Serre fibration.

So again by Shapiro's lemma, we have a simplicial map

$$\begin{aligned}\mathrm{B}S_k(\mathrm{Stab}(e_p)) &\to A_p(M)_k /\!\!/ S_k(\mathrm{Homeo}_{0,\partial}(M))\\ &= |\mathrm{B}_\bullet(A_p(M)_k, S_k(\mathrm{Homeo}_{0,\partial}(M)), *)|,\end{aligned}$$

which is a weak equivalence for all k. So again by using [Qui73, Lemma, p. 86] that the order of realizations for a bisimplicial set does not matter, if we realize in the k-direction, we obtain

$$\mathrm{BStab}(e_p) \xleftarrow{\simeq} \mathrm{B}|S_\bullet(\mathrm{Stab}(e_p))| \to X_p(M) = A_p(M) /\!\!/ |S_\bullet(\mathrm{Homeo}_{0,\partial}(M))|,$$

which is a weak equivalence. □

Now we need a lemma from homotopy theory to show that the weak homotopy type of $|X_\bullet(M)|$ and $|X^\delta_\bullet(M)|$ are the same as $\mathrm{BHomeo}_{0,\partial}(M)$ and $\mathrm{BHomeo}^\delta_{0,\partial}(M)$, respectively.

3.1.2. *A lemma in homotopy theory.* Here the goal is to show that $|\overline{A}^\delta_\bullet(M)|$ and $|A_\bullet(M)|$ are weakly contractible. Proving that the realization of the discrete version $|\overline{A}^\delta_\bullet(M)|$ is contractible is easier. Using a lemma in homotopy theory, we show that the contractibility of $|\overline{A}^\delta_\bullet(M)|$ implies the weak contractibility of $|A_\bullet(M)|$. This technique is originally due to Segal ([Seg78, Appendix]) and it is reformulated by Weiss in [Wei05, Lemma 2.2]. In particular, in the setting of semisimplicial spaces, we use an application of this technique ([GRW18, Proposition 2.8]) due to Galatius and Randal-Williams.

Proposition 3.9. *The realizations $|A^\delta_\bullet(M)|$ and $|\overline{A}^\delta_\bullet(M)|$ are weakly contractible.*

Proof. We give a proof for weak contractibility of $|A^\delta_\bullet(M)|$; the case of $|\overline{A}^\delta_\bullet(M)|$ is similar. Let $f: S^k \to |A^\delta_\bullet(M)|$ be an element in the k-th homotopy group of $|A^\delta_\bullet(M)|$. Since $|A^\delta_\bullet(M)|$ is a CW-complex and S^k is compact, the map f hits finitely many simplices of $|A^\delta_\bullet(M)|$. Hence, there exists a point $\mathbf{p}$ and an embedded disk $e(D^2)$ around it such that as an element of $A^\delta_0(M)$ it is not hit by the map f. Thus, we have $f(S^k) \subset |A^\delta_\bullet(M\backslash e(D^2))|$. Adding the germ of e at $\mathbf{p}$ to the list of germs of embeddings of disks in $M\backslash e(D^2)$ gives a semisimplicial nullhomotopy for the inclusion $A^\delta_\bullet(M\backslash e(D^2)) \hookrightarrow A^\delta_\bullet(M)$. Therefore, the element $f(S^k)$ can be coned off inside $|A^\delta_\bullet(M)|$. □

Remark 3.10. Note that because $|A^\delta_\bullet(M)|$ and $|\overline{A}^\delta_\bullet(M)|$ have CW structures, they are in fact contractible.

Using Proposition 2.18 in [Kup15], to prove that the maps

$$|\theta|\colon |X^\delta_\bullet(M)| \to \mathrm{BHomeo}^\delta_0(M), \tag{3.11}$$

$$||\epsilon||\colon |X_\bullet(M)| \to \mathrm{B}|S_\bullet(\mathrm{Homeo}_{0,\partial}(M))| \tag{3.12}$$

are weak equivalences, it is enough to show that their fibers $|A^\delta_\bullet(M)|$ and $|A_\bullet(M)|$, respectively, are contractible.

Therefore, by Proposition 3.9 the first map $|X^\delta_\bullet(M)| \xrightarrow{\simeq} \mathrm{BHomeo}^\delta_0(M)$ is a weak homotopy equivalence. To prove that the second map is also a weak homotopy equivalence, we need to show that $|A_\bullet(M)|$ is weakly contractible. To do so, we use the bisimplicial technique due to Quillen [Qui73, Proof of Theorem A]. First note that since the map

$$A_\bullet(M) \xrightarrow{\simeq} \overline{A}_\bullet(M)$$

is a levelwise weak equivalence, it induces a weak homotopy equivalence between the realizations in semisimplicial directions. Hence, to show that $|A_\bullet(M)|$ is weakly contractible, it is enough to show that in the zig-zag

$$A_\bullet(M) \xrightarrow{\simeq} \overline{A}_\bullet(M) \xleftarrow{\beta} \overline{A}^\delta_\bullet(M) \tag{3.13}$$

the second map β induces a weak homotopy equivalence after realizations in semisimplicial directions.

Definition 3.14. Let $A_{\bullet,\bullet}(M)_k$ be the bisemisimplicial simplicial set such that $A_{p,q}(M)_k$ is the subset of $\overline{A}^\delta_p(M) \times \overline{A}_q(M)_k$ consisting of those $(p+q+2)$-tuples

$$(a_0, \dots, a_p, c_0, \dots, c_q),$$

where the centers of the disks a_i and the disks $c_j(t)$ are pairwise disjoint for all $t \in \Delta^k$.

The bisemisimplicial simplicial set $A_{\bullet,\bullet}(M)_k$ is augmented in two different semisimplicial directions

$$\epsilon_{p,k} \colon A_{p,\bullet}(M)_k \to \overline{A}^\delta_p(M),$$
$$\delta_{q,k} \colon A_{\bullet,q}(M)_k \to \overline{A}_q(M)_k.$$

Let $A_{\bullet,\bullet}(M)$ be the bisemisimplicial space obtained by realizing $A_{\bullet,\bullet}(M)_k$ in the simplicial direction. Similar to [GRW18, Lemma 5.8], one can show that the following diagram is homotopy commutative:

$$\begin{array}{ccc} |\overline{A}^\delta_\bullet(M)| & \longrightarrow & |\overline{A}_\bullet(M)| \\ & \nwarrow^{\epsilon} \quad {}^{\delta}\nearrow & \\ & |A_{\bullet,\bullet}(M)|. & \end{array} \tag{3.15}$$

Proposition 3.16. *The realization $|A_\bullet(M)|$ is weakly contractible.*

Proof. Since $|A_\bullet(M)| \xrightarrow{\simeq} |\overline{A}_\bullet(M)|$, we instead show that $|\overline{A}_\bullet(M)|$ is weakly contractible. Because the diagram (3.15) is homotopy commutative and $|\overline{A}^\delta_\bullet(M)|$ is weakly contractible, if we show that the map δ is a weak homotopy equivalence, we then deduce that $|\overline{A}_\bullet(M)|$ is also weakly contractible. To do so, it suffices to prove that

$$|\delta_q| \colon |A_{\bullet,q}(M)| \to \overline{A}_q(M)$$

is a weak equivalence. The idea is to show that $|\delta_q|$ is a microfibration with a contractible fiber. But since $\overline{A}_q(M)$ is the realization of a simplicial set, we shall apply a simplicial approximation similar to [Kup15, Proposition 2.44] to exhibit the map $|\delta_q|$ as the realization of a map between simplicial sets. Note that $A_{\bullet,q}(M)_k$ is a semisimplicial set for a fixed q and k. We can freely add all degeneracies (see [ERW19, Section 1]) to obtain a bisimplicial set $EA_{\bullet,q}(M)_\bullet$ whose realization is homotopy equivalent to $|A_{\bullet,q}(M)|$. More concretely, we have

$$EA_{p,q}(M)_k = \coprod_{[p]\twoheadrightarrow[p']} A_{p',q}(M)_k.$$

The realization of the bisimplicial set $EA_{\bullet,q}(M)_\bullet$ is homeomorphic to the realization of its diagonal $\mathrm{diag}(EA_{\bullet,q}(M)_\bullet)$. Therefore, it is enough to show that the augmentation map

$$\delta_{q,\bullet}\colon \mathrm{diag}(EA_{\bullet,q}(M)_\bullet) \to \overline{A}_q(M)_\bullet$$

induces a weak equivalence after realization. By the simplicial approximation, it is enough to show that for each pair $(K, \partial K)$ of simplicial sets where $(|K|, |\partial K|) \cong (D^i, S^{i-1})$ and for each diagram

$$\begin{array}{ccc} \partial K & \xrightarrow{\ g\ } & \mathrm{diag}(EA_{\bullet,q}(M)_\bullet) \\ \downarrow & & \downarrow \delta_{q,\bullet} \\ K & \xrightarrow{\ G\ } & \overline{A}_q(M)_\bullet \end{array}$$

we have a lift $\tilde{G}\colon |K| \to |\mathrm{diag}(EA_{\bullet,q}(M)_\bullet)|$ so that $\tilde{G}|_{|\partial K|} = |g|$. The map G can be represented by a locally flat immersion $f\colon D^i \times \coprod_{[q]} D^2 \to D^i \times M$ over D^i so that the centers of the embedded disks are disjoint.

On the other hand, by Definition 3.14, the map $|g|$ gives a map $h\colon S^{i-1} \to |\overline{A}^{\delta}_{\bullet}(M)|$ where for each $t \in S^{i-1}$ the center of $g(t)$ and the centers $\mathbf{c}_t$ of the embedded disks $f|_t$ are disjoint. We want to show that h can be extended to a map $\tilde{h}\colon D^i \to |\overline{A}^{\delta}_{\bullet}(M)|$ where for each $t \in D^i$ the center of $g(t)$ and $\mathbf{c}_t$ are disjoint. Note that h gives an element of the homotopy group of the space of pairs

$$X = \{(t,x) \in D^i \times |\overline{A}^{\delta}_{\bullet}(M)| \mid \text{the center of } x \text{ and } \mathbf{c}_t \text{ are disjoint}\}.$$

Hence, it suffices to show that X is contractible. Note that the projection $X \to D^i$ is a microfibration by the openness of the condition of centers being disjoint and the fiber over t is homeomorphic to $|\overline{A}^{\delta}_{\bullet}(M\backslash \mathbf{c}_t)|$, which is contractible by Proposition 3.9. Therefore, the projection is a fibration (see [Wei05, Lemma 2.2] or [GRW18, Proposition 2.6]) with a contractible fiber so X is contractible. Hence, since $|\delta_q|$ is a weak equivalence for all q, so is δ. □

3.1.3. *Reducing Theorem* 2.3 *to the case of manifolds with boundary.* Recall that the goal is to compare the spectral sequences for the semisimplicial spaces $X_\bullet(M)$ and $X^{\delta}_{\bullet}(M)$. For these 0-handle resolutions (unlike the 1-handle resolutions, as we shall see later), there is no direct semisimplicial map from $X^{\delta}_{\bullet}(M)$ to $X_\bullet(M)$. But we shall find a zig-zag of semisimplicial maps between them and show that our zig-zag of map induces a map between their spectral sequences.

Since $A^{\mathbf{t},\delta}_{\bullet}(M) = A_\bullet(M)_0$, the inclusion $A^{\mathbf{t},\delta}_{\bullet}(M) \to A_\bullet(M)_\bullet$ is equivariant with respect to the map $\mathrm{Homeo}^{\delta}_{0,\partial}(M) \to S_\bullet(\mathrm{Homeo}_{0,\partial}(M))$. Therefore, we have an induced map between homotopy quotients

$$\alpha_\bullet\colon A^{\mathbf{t},\delta}_{\bullet}(M) /\!\!/ \mathrm{Homeo}^{\delta}_{0}(M) \to X_\bullet(M). \tag{3.17}$$

On the other hand, the map $A^{\mathbf{t},\delta}_{\bullet}(M) \to A^{\delta}_{\bullet}(M)$ is equivariant with respect to the action $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$. Therefore, we have an induced map between homotopy quotients

$$A^{\mathbf{t},\delta}_{\bullet}(M) /\!\!/ \mathrm{Homeo}^{\delta}_{0,\partial}(M) \to X^{\delta}_{\bullet}(M). \tag{3.18}$$

So we have the homotopy commutative diagram

$$
\begin{array}{ccccc}
X_p^{\delta}(M) & \longleftarrow & A_p^{\mathbf{t},\delta}(M) /\!\!/ \operatorname{Homeo}_{0,\partial}^{\delta}(M) & \longrightarrow & X_p(M) \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
\mathrm{BStab}^{\delta}([e_p]) & \longleftarrow & \mathrm{BStab}^{\delta}(e_p) & \longrightarrow & \mathrm{B}|S_\bullet(\mathrm{Stab}(e_p))|,
\end{array} \tag{3.19}
$$

where the first and the last weak equivalences are given by Lemmas 3.7 and 3.8, and the middle weak equivalence is deduced from the transitivity of the action of $\operatorname{Homeo}_{0,\partial}^{\delta}(M)$ on $A_\bullet^{\mathbf{t},\delta}(M)$ and Shapiro's lemma.

Lemma 3.20. *The map* $\mathrm{BStab}^{\delta}(e_p) \to \mathrm{BStab}^{\delta}([e_p])$ *induces a homology isomorphism.*

Note that Lemma 3.20 implies that the spectral sequences of the semisimplicial spaces $X_\bullet^{\delta}(M)$ and $A_\bullet^{\mathbf{t},\delta}(M) /\!\!/ \operatorname{Homeo}_{0,\partial}^{\delta}(M)$ are isomorphic as the map (3.18) induces an isomorphism on E^1-page. On the other hand, $\alpha_\bullet$ induces a map between spectral sequences for $A_\bullet^{\mathbf{t},\delta}(M) /\!\!/ \operatorname{Homeo}_{0,\partial}^{\delta}(M)$ and $X_\bullet(M)$. With abuse of notation, let α_* denote the induced map between spectral sequences for $X_\bullet^{\delta}(M)$ and $X_\bullet(M)$:

$$
\begin{array}{ccc}
H_q(X_p^{\delta}(M)) & \xrightarrow{\alpha_*} & H_q(X_p(M)) \\
\Downarrow & & \Downarrow \\
H_{p+q}(|X_\bullet^{\delta}(M)|) & \longrightarrow & H_{p+q}(|X_\bullet(M)|) \\
\downarrow\cong & & \downarrow\cong \\
H_{p+q}(\mathrm{BHomeo}_{0,\partial}^{\delta}(M)) & \longrightarrow & H_{p+q}(\mathrm{B}|S_\bullet(\operatorname{Homeo}_{0,\partial}(M))|).
\end{array} \tag{3.21}
$$

Before proving Lemma 3.20, let us show that this comparison of spectral sequences reduces Theorem 2.3 from the case of closed surfaces to surfaces with nonempty boundary.

Definition 3.22. Let e_p be an element in $A_p(M)$. Let $M\backslash e_p$ denote the manifold obtained from M by removing the interior of the embedded disks in M given by e_p. Also let $M\backslash c(e_p)$ denote the punctured manifold obtained from M by removing the centers of the embedded disks given by e_p.

Proposition 3.23. *Suppose Theorem* 2.3 *holds for* $M\backslash e_p$ *for all* $e_p \in A_p(M)$ *and all* p. *Then it also holds for* M.

Proof. Given the spectral sequence (3.21), it suffices to prove that α_* induces an isomorphism between E^1-pages. Using the commutative diagram (3.19) and Lemma 3.20, it is enough to show that the hypothesis of the proposition implies that the map

$$\mathrm{BStab}^{\delta}(e_p) \to \mathrm{BStab}(e_p)$$

induces a homology isomorphism. Note that the identity component of $\mathrm{Stab}(e_p)$ is $\operatorname{Homeo}_{0,\partial}(M\backslash e_p)$, so we have a short exact sequence of groups

$$1 \to \operatorname{Homeo}_{0,\partial}(M\backslash e_p) \to \mathrm{Stab}(e_p) \to \pi_0(\mathrm{Stab}(e_p)) \to 1.$$

From this short exact sequence, we obtain a homotopy commutative diagram between two fibrations:

$$
\begin{array}{ccc}
\mathrm{BHomeo}^{\delta}_{0,\partial}(M\backslash e_p) & \longrightarrow & \mathrm{BHomeo}_{0,\partial}(M\backslash e_p) \\
\downarrow & & \downarrow \\
\mathrm{BStab}^{\delta}(e_p) & \longrightarrow & \mathrm{BStab}(e_p) \\
\downarrow & & \downarrow \\
\mathrm{B}\pi_0(\mathrm{Stab}(e_p)) & \xrightarrow{\cong} & \mathrm{B}\pi_0(\mathrm{Stab}(e_p)).
\end{array} \tag{3.24}
$$

Now by the hypothesis, the map between fibers induces a homology isomorphism. Since the bases are the same, using the Serre spectral sequence, we conclude that the map between total spaces induces a homology isomorphism. ☐

Proof of Lemma 3.20. Let us first consider $\mathrm{Stab}(e_p)$ and $\mathrm{Stab}([e_p])$ as subgroups of $\mathrm{Homeo}_{0,\partial}(M)$ with the subspace topology. The identity components are respectively $\mathrm{Homeo}_{0,\partial}(M\backslash e_p)$ and $\mathrm{Homeo}_{0,c}(M\backslash c(e_p))$, where the latter is the identity component of $\mathrm{Homeo}_c(M\backslash c(e_p))$, the compactly supported homeomorphisms of the punctured surface $M\backslash c(e_p)$. Hence, we have a map between two fibrations:

$$
\begin{array}{ccc}
\mathrm{BHomeo}^{\delta}_{0,\partial}(M\backslash e_p) & \longrightarrow & \mathrm{BHomeo}^{\delta}_{0,c}(M\backslash c(e_p)) \\
\downarrow & & \downarrow \\
\mathrm{BStab}^{\delta}(e_p) & \longrightarrow & \mathrm{BStab}^{\delta}([e_p]) \\
\downarrow & & \downarrow \\
\mathrm{B}\pi_0(\mathrm{Stab}(e_p)) & \longrightarrow & \mathrm{B}\pi_0(\mathrm{Stab}([e_p])).
\end{array}
$$

The pushing collar lemma in [Nar17, Corollary 2.3] implies that the map between fibers induces a homology isomorphism. So if we show that the map between bases induces a weak equivalence, the lemma follows from a Serre spectral sequence argument again.

Let us first show that the map $\pi_0(\mathrm{Stab}(e_p)) \to \pi_0(\mathrm{Stab}([e_p]))$ is surjective. Let $f \in \mathrm{Stab}([e_p]) < \mathrm{Homeo}_c(M\backslash c(e_p))$. We shall change f up to isotopy so that it fixes the disks $e_p(\coprod_{[p]} D^2)$. Since f is supported away from the punctures $c(e_p)$, there exists a small disk $D^2_\epsilon \subset D^2$ of radius ϵ so that f is supported away from $e_p(\coprod_{[p]} D^2_\epsilon)$. Let M_ϵ denote the manifold obtained by removng the interior of $e_p(\coprod_{[p]} D^2_\epsilon)$ from M. Then there is a self-embedding p of M_ϵ isotopic to the identity such that $p(M_\epsilon) = M\backslash e_p$. Consider the homeomorphism

$$
h_\epsilon(f)(x) := \begin{cases} p \circ f \circ p^{-1}(x) & \text{if } x \in M\backslash e_p, \\ \mathrm{id} & \text{if } x \in e_p(\coprod_{[p]} D^2). \end{cases}
$$

Since $h_\epsilon(f)(x)$ is isotopic to f and is the identity on $e_p(\coprod_{[p]} D^2)$, we have $h_\epsilon(f)(x) \in \mathrm{Stab}(e_p)$.

Proving that $\pi_0(\mathrm{Stab}(e_p)) \to \pi_0(\mathrm{Stab}([e_p]))$ is injective is also the same. If we have f_0 and f_1 in $\mathrm{Stab}(e_p)$ that are isotopic in $\mathrm{Stab}([e_p])$, then the isotopy f_t is supported away from $e_p(\coprod_{[p]} D^2_\epsilon)$ for some small ϵ. Then a similar argument as the surjectivity case would imply that f_0 and f_1 are isotopic with an isotopy which is the identity on $e_p(\coprod_{[p]} D^2)$. □

Remark 3.25. Note that in dimension 1, by Mather's theorem ([Mat71]), we know that Theorem 2.3 holds for the intervals. Using the 0-handle resolution for S^1 and Proposition 3.23, we deduce the Thurston Theorem 2.3 for $M = S^1$ (see [Jek12, Theorem 4] for a similar idea).

3.2. 1**-handle resolutions.** Using Proposition 3.23, to prove Theorem 2.3, we can assume that M is a surface with nonempty boundary. Now, we want to inductively reduce to the case of a simpler surface by removing 1-handles from M. Similarly to the previous section, to do this reduction we need to define augmented semisimplicial sets whose realizations are contractible.

Definition 3.26. Let $\phi\colon D^1 \times \mathbb{R} \hookrightarrow M$ be a fixed 1-handle so that $\phi(D^1 \times \mathbb{R}) \cap \partial M = \phi(S^0 \times \mathbb{R})$ and the core of the handle is the arc $\phi(D^1 \times \{0\})$ in M. We shall write $\vec{e}$ for a unit basis vector in $\mathbb{R}$.

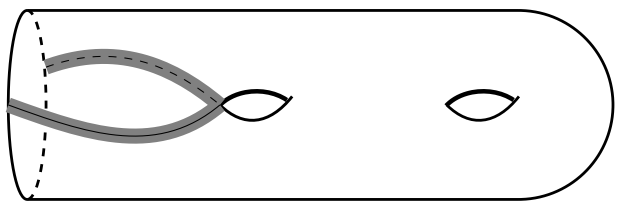

FIGURE 1. The 1-handle $\phi(D^1 \times \mathbb{R})$.

Topological version: We define a semisimplicial simplicial set $B_\bullet(M, \phi)_\bullet$ and a semisimplicial space $B^{\mathbf{t}}_\bullet(M, \phi)$ on which $S_\bullet(\mathrm{Homeo}_{0,\partial}(M))$ and $\mathrm{Homeo}_{0,\partial}(M)$ act respectively.

- We first define the 0-simplices in the semisimplicial direction, $B_0(M, \phi)_\bullet$ to be the simplicial set given by pairs (t, f) where $f \in \mathrm{Emb}^{\mathrm{lf}}_\bullet(D^1, M)$, $t \in \mathbb{R}$ and for all $s \in \Delta^\bullet$, the embedded arc $f(s)$ satisfies

$$f(s)|_{S^0} = \phi|_{S^0 \times \{t\,\vec{e}\}}, \tag{3.27}$$

  and the embedded arc $f(s)$ is isotopic to the arc $\phi(D^1 \times \{t\,\vec{e}\})$ relative to the boundary.

  The set of p-simplices $B_p(M, \phi)_\bullet$ in the semisimplicial direction is given by $(p + 1)$-tuples $((t_0, f_0), (t_1, f_1), \dots, (t_p, f_p))$ so that $t_0 < t_1 < \cdots < t_p$ and for each $s \in \Delta^\bullet$, the arcs $f_i(s)$ are disjoint for all i. The face maps in the semisimplicial direction are given by forgetting the pairs (t_i, f_i). We shall write $B_\bullet(M, \phi)$ for the semisimplicial space obtained by realizing in the simplicial direction.
- Let $B^{\mathbf{t}}_0(M, \phi)$ be the space of pairs (t, f) where t is a real number and $f \in \mathit{Emb}^{\mathrm{lf}}(D^1, M)$ such that f satisfies the equation (3.27). The space of such pairs is topologized as a subspace of $\mathbb{R} \times \mathit{Emb}^{\mathrm{lf}}(D^1, M)$. The space of p-simplices $B^{\mathbf{t}}_0(M, \phi)$ is given by $(p+1)$-tuples $((t_0, f_0), (t_1, f_1), \dots, (t_p, f_p))$ so that $t_0 < t_1 < \cdots < t_p$ and the arcs f_i are disjoint for all i. The face maps for the semisimplicial space $B^{\mathbf{t}}_\bullet(M, \phi)$ are similarly given by forgetting the pairs (t_i, f_i).

Discrete version: Let $B^{\delta}_{\bullet}(M,\phi)$ be the semisimplicial set $B_{\bullet}(M,\phi)_0$ on which the discrete group $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ acts.

Remark 3.28. Since we want to emphasize the methods for the possible applications in higher dimensions, we work with the locally flat embeddings, but in fact in dimension 2 all embedded arcs are locally flat by the Schoenflies theorem.

Note that by definition, for every pair $(t,f)\in B_0(M,\phi)_{\bullet}$, the real number t is uniquely determined by f. We denote this t-coordinate by t_f. Moreover each simplex in $|B^{\delta}_{\bullet}(M,\phi)|$ has a canonical ordering induced by the condition $t_0<t_1<\cdots<t_p$. Therefore, $|B^{\delta}_{\bullet}(M,\phi)|$ has a simplicial complex structure with a natural ordering on each simplex.

Definition 3.29. Similarly to Definition 3.2, we define $Y_{\bullet,n,k}(M,\phi)$ to be an augmented semisimplicial bisimplicial set

$$\begin{aligned} Y_{\bullet,n,k}(M,\phi) &:= B_n(B_{\bullet}(M,\phi)_k, S_k(\mathrm{Homeo}_{0,\partial}(M)),*) \\ &\xrightarrow{\epsilon} B_n(*, S_k(\mathrm{Homeo}_{0,\partial}(M)),*). \end{aligned}$$

We denote the realization of $Y_{p,n,k}(M,\phi)$ in the bisimplicial directions by $Y_p(M,\phi)$. If we realize in the semisimplicial direction, we obtain the map

$$||\epsilon||\colon |Y_{\bullet}(M,\phi)|\to \mathrm{B}|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))|.$$

Definition 3.30. Similarly to Definition 3.4, we define the 1-handle resolution associated to ϕ for $\mathrm{BHomeo}^{\delta}_{0,\partial}(M)$ as the augmented semisimplicial space

$$\theta\colon Y^{\delta}_{\bullet}(M,\phi) := B^{\delta}_{\bullet}(M,\phi) /\!\!/ \mathrm{Homeo}^{\delta}_{0,\partial}(M)\to \mathrm{BHomeo}^{\delta}_{0,\partial}(M).$$

If we realize in the semisimplicial direction, we obtain the map

$$|\theta| : |Y^{\delta}_{\bullet}(M,\phi)|\to \mathrm{BHomeo}^{\delta}_{0,\partial}(M).$$

To show that the maps $||\epsilon||$ and $|\theta|$ are weak equivalences as before, we need to show that $|B^{\delta}_{\bullet}(M,\phi)|$ and $|B_{\bullet}(M,\phi)|$ are weakly contractible.

Lemma 3.31. *The realizations $|B^{\delta}_{\bullet}(M,\phi)|$ and $|B_{\bullet}(M,\phi)|$ are weakly contractible.*

Proof. It is enough to show that $|B^{\delta}_{\bullet}(M,\phi)|$ is contractible since the weak contractibility of $|B_{\bullet}(M,\phi)|$ is deduced from that of $|B^{\delta}_{\bullet}(M,\phi)|$ by the same argument as in Proposition 3.16. To show that the simplicial complex $|B^{\delta}_{\bullet}(M,\phi)|$ is contractible, we prove that all continuous maps $f: S^k\to |B^{\delta}_{\bullet}(M,\phi)|$ are nullhomotopic for all k. Without loss of generality, we can assume that f is a PL map with respect to a triangulation K of S^k. To show that f is nullhomotopic, we show that there exists $\alpha\in B^{\delta}_{\bullet}(M,\phi)$ so that one can change f up to homotopy so that the image $f(K)$ lies in $\mathrm{Star}(\alpha)$.

First we argue that we can assume that the vertices of $f(K)$ are pairwise transverse after changing f up to homotopy. For transversality in the topological category, we need the data of the normal microbundle (see [KS77, Essay 3, Section 1]), but by the Schoenflies theorem all embedded arcs are bicollared, so in this dimension, the embedded arcs have unique normal microbundle data. Therefore, we did not need to add the microbundle data (germ of the cocore around the core of the handle) to the definition of $B^{\delta}_{\bullet}(M,\phi)$.

Claim. After changing f up to homotopy, we can assume that all vertices in $f(K)$ as arcs in M are pairwise transverse.

Doing this in the smooth category is easier, because transversality is an open condition in C^∞-topology, and one can change an arc up to small isotopy so that it becomes simultaneously transverse to several other arcs. But in the topological category one needs to do it inductively. We give an argument in a way that can be generalized to the higher dimensions.

Proof of the Claim. Since the arcs are bicollared, by a parallel copy of an arc ψ we mean an embedded arc close to ψ in its collar neighborhood that is disjoint from ψ and satisfies equation (3.27) for some t. Let us enumerate the vertices of $f(K)$ by $\psi_1, \psi_2, \ldots, \psi_m$. First we choose a parallel copy of ψ_2 and perturb it by a small isotopy to obtain ψ_2' so that it becomes transverse to ψ_1. If the isotopy is small enough ψ_2' is disjoint from ψ_2 and from all vertices in $f(K)$ from which ψ_2 was disjoint. Therefore, there is a homotopy replacing ψ_2 with ψ_2' and fixing the image of other vertices. Thus we may assume that ψ_1 and ψ_2 are transverse. Hence their intersection is a set of points.

Now we move on to ψ_3. Similarly by choosing a nearby copy of ψ_3 and a small perturbation ψ_3' of this nearby copy, we obtain an arc that is disjoint from the points in the intersection of the previous two arcs ψ_1 and ψ_2. Hence we can choose a small neighborhood U of the intersection of ψ_1 and ψ_2 such that ψ_3' is also disjoint from U. Now by Quinn's transversality ([Q$^+$88]), we can find a small isotopy whose support is away from U, and we obtain an arc ψ_3'' that is transverse to the submanifold $(\psi_1(D^1) \cup \psi_2(D^1)) \backslash U$. If we choose the isotopy small enough the arc ψ_3'' is disjoint from ψ_3 and from all the vertices of $f(K)$ from which ψ_3 was disjoint. Hence by a homotopy of the map f, we can replace ψ_3 with ψ_3''. Therefore, we may assume that ψ_3 is transverse to ψ_1 and ψ_2. By continuing this process we can change f up to homotopy to make ψ_i transverse to ψ_j for $j < i$. Hence, we may assume that the vertices of $f(K)$ are pairwise transverse to each other and the t-coordinates are different. $\square$

Similarly, we choose a vertex $\psi \in B^\delta_\bullet(M, \phi)$ that is transverse to all the vertices in $f(K)$. If a vertex $v \in f(K)$ intersects the arc ψ, because of the condition (3.27) it intersects in an even number of points $\{p_1, p_2, \ldots, p_{2i_v}\}$. So the consecutive points of the intersection $\{p_{2i-1}, p_{2i}\}$ give the union of disjoint subintervals on ψ. Let I_v denote these subintervals on ψ associated to its intersection with v. The simplicial complex $f(K)$ has finitely many vertices. From the set of intervals $\{I_v\}_{v\in f(K)}$ on ψ, we choose a maximal family of subintervals that are either disjoint or one includes the other. Such a family has an innermost subinterval D on the arc ψ. Suppose that this innermost subinterval is in I_{ψ_1}. Since the arcs in $\operatorname{Star}(\psi_1)$ are disjoint from ψ_1 and D is an innermost subinterval in a maximal family, D is disjoint from the arcs in $\operatorname{Star}(\psi_1)$. The two ends ∂D on ψ also lie on ψ_1 and they bound a subinterval D' on ψ_1.

Since ψ_1 is isotopic to a parallel copy of ψ, by [FM12, Proposition 1.7] there is a Whitney disk N (or bigon in the context of surgery of arcs on surfaces) that bounds $D \cup D'$. Hence, by doing the Whitney trick, we can choose an arc ψ_1' so that it is disjoint from ψ_1 and all arcs in $\operatorname{Star}(\psi_1)$ and also that it intersects ψ in fewer points. Therefore, there is a simplicial homotopy of f that replaces ψ_1 with ψ_1'. By continuing this process, we can homotope f to the star of the arc ψ. Hence, f is nullhomotopic. $\square$

Corollary 3.32. *The maps* $||\epsilon||$ *and* $|\theta|$ *in Definitions* 3.29 *and* 3.30 *are weak homotopy equivalent.*

Note that unlike the 0-handle case, we have a semisimplicial map

$$B^{\delta}_{\bullet}(M,\phi) \to B_{\bullet}(M,\phi)_{\bullet},$$

which is the inclusion to the 0-simplices in the simplicial direction and is equivariant with respect to the homomorphism $\mathrm{Homeo}^{\delta}_{0,\partial}(M) \to S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))$. Therefore, we obtain a semisimplicial map between 1-handle resolutions

$$\alpha_{\bullet}\colon Y^{\delta}_{\bullet}(M,\phi) \to Y_{\bullet}(M,\phi). \tag{3.33}$$

By comparing the spectral sequences for $Y^{\delta}_{\bullet}(M,\phi)$ and $Y_{\bullet}(M,\phi)$, we want to show that Theorem 2.3 holds true for M if it is true for a surface $M\backslash\phi$ that is obtained from M by removing the 1-handle ϕ.

Definition 3.34. For a p-simplex $\sigma_p \in B^{\delta}_{\bullet}(M,\phi)$. Let $M\backslash\sigma_p$ denote the surface that we obtain by cutting M along the arcs in σ_p.

Proposition 3.35. *Suppose Theorem* 2.3 *holds for* $M\backslash\phi$. *Then it also holds for* M.

Proof. The map $\alpha_{\bullet}$ in (3.33) induces a map of spectral sequences

$$\begin{array}{ccc} H_q(Y^{\delta}_p(M,\phi)) & \xrightarrow{\alpha_*} & H_q(Y_p(M,\phi)) \\ \Downarrow & & \Downarrow \\ H_{p+q}(|Y^{\delta}_{\bullet}(M,\phi)|) & \longrightarrow & H_{p+q}(|Y_{\bullet}(M,\phi)|) \\ \downarrow\cong & & \downarrow\cong \\ H_{p+q}(\mathrm{BHomeo}^{\delta}_{0,\partial}(M)) & \longrightarrow & H_{p+q}(\mathrm{B}|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))|). \end{array} \tag{3.36}$$

So if we show that the hypothesis implies that α_* is an isomorphism, then we have the isomorphism on the E^{∞}-page, which concludes Theorem 2.3 for M.

First, we shall observe that the set of connected components of $Y^{\delta}_p(M,\phi)$ is the same as that of $Y_p(M,\phi)$. Note that the former set is the same as the set of the orbits of the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ on $B^{\delta}_p(M,\phi)$.[5] Hence, by the isotopy extension theorem, this set of orbits can be identified with $\pi_0(B^{\mathbf{t}}_p(M,\phi))$, which comprises the isotopy classes of p-simplices relative to the boundary. On the other hand, similarly to Lemma 3.8, we have a map

$$Y_p(M,\phi) \to B^{\mathbf{t}}_p(M,\phi) /\!\!/ \mathrm{Homeo}_{0,\partial}(M),$$

which is a weak equivalence. Therefore, it induces an isomorphism between set of connected components. But since $\pi_1(\mathrm{BHomeo}_{0,\partial}(M)) = 0$, the long exact sequence for the Borel construction implies that

$$\pi_0(B^{\mathbf{t}}_p(M,\phi)) \xrightarrow{\cong} \pi_0(B^{\mathbf{t}}_p(M,\phi) /\!\!/ \mathrm{Homeo}_{0,\partial}(M)).$$

Therefore, we have $\pi_0(Y^{\delta}_p(M,\phi)) \cong \pi_0(Y_p(M,\phi))$.

[5] In fact, as we shall also use it in Theorem 5.1, the orbit of a p-simplex

$$((t_0,f_0),(t_1,f_1),\dots,(t_p,f_p))$$

is uniquely determined by the real numbers t_i.

For a p-simplex $\sigma_p \in B^{\mathbf{t}}_p(M,\phi)$, let $\mathrm{Stab}(\sigma_p)$ be its stabilizer as a subgroup of $\mathrm{Homeo}_{0,\partial}(M)$ and let $\mathrm{Stab}^{\delta}(\sigma_p)$ denote the same group with the discrete topology. Similarly to Lemmas 3.7 and 3.8, we have a homotopy commutative diagram

$$\begin{array}{ccc} Y^{\delta}_p(M,\phi) & \longrightarrow & Y_p(M,\phi) \\ \simeq\uparrow & & \simeq\uparrow \\ \coprod_{\text{orbits}} \mathrm{BStab}^{\delta}(\sigma_p) & \longrightarrow & \coprod_{\text{orbits}} \mathrm{B}|S_\bullet(\mathrm{Stab}(\sigma_p))|. \end{array} \tag{3.37}$$

Note that if we remove the arcs in σ_p from M, we obtain a surface with p components such that $p-1$ of them are homeomorphic to disks and one of them is homeomorphic to $M\backslash\phi$. Therefore the hypothesis and Mather's theorem ([Mat71]) for disks imply that the map

$$\mathrm{BHomeo}^{\delta}_{0,\partial}(M\backslash\sigma_p) \to \mathrm{BHomeo}_{0,\partial}(M\backslash\sigma_p)$$

induces a homology isomorphism for all p and all σ_p. Since $\mathrm{Homeo}_{0,\partial}(M\backslash\sigma_p)$ is the identity component of $\mathrm{Stab}(\sigma_p)$, the comparison of fibration similar to the diagram (3.24) implies that the map

$$\mathrm{BStab}^{\delta}(\sigma_p) \to \mathrm{BStab}(\sigma_p)$$

induces a homology isomorphism. Therefore, α_* is an isomorphism. □

Proof Theorem 2.3 *for surfaces.* Using Proposition 3.23, we know that in order to prove Theorem 2.3 for a closed surface Σ_g of genus g, it is enough to prove it for a surface $\Sigma_{g,k}$ of genus g and k boundary components for all positive integer k. We induct on $-\chi(\Sigma_{g,k}) = 2g+k-2$. The base case is when $g=0$ and $k=1$, which is homeomorphic to a disk and is given by Mather's theorem ([Mat71]). In general, if $g>0$, we can choose the 1-handle ϕ so that its two ends lie on the same boundary component and $\Sigma_{g,k}\backslash\phi$ is homeomorphic to $\Sigma_{g-1,k+1}$. Since $-\chi(\Sigma_{g-1,k+1}) < -\chi(\Sigma_{g,k})$, our induction hypothesis and Proposition 3.35 imply that Theorem 2.3 holds for $\Sigma_{g,k}$.

So now we suppose that $g=0$ and $k>1$. To reduce the number of boundary components, we can choose the 1-handle ϕ so that its two ends lie on different boundary components and $\Sigma_{0,k}\backslash\phi$ is homeomorphic to $\Sigma_{0,k-1}$. Since $-\chi(\Sigma_{0,k-1}) < -\chi(\Sigma_{0,k})$, the induction hypothesis and Proposition 3.35 imply that Theorem 2.3 also holds for $\Sigma_{0,k}$ for all k. Therefore, it holds for $\Sigma_{g,k}$ for all g and k. □

Remark 3.38. In fact using handle resolutions whose cores have dimensions less than half of the dimension of M, one can show that Thurston's theorem for a manifold M whose dimension is larger than 4 is equivalent to Thurston's theorem for a trivial bordism $N\times D^1$ where N is a manifold whose dimension is $\dim(M)-1$. But it is not known to the author whether for handles of dimension $\dim(M)/2$ or higher, a similar contractibility statement as Lemma 3.31 holds.

4. Cutting three manifolds into disks

To make a similar argument as in the case of surfaces, we need to find contractible semisimplicial spaces that cut the manifold into a union of 3-disks. In this section, disks are 3-dimensional unless mentioned otherwise. Doing an inductive process to cut a 3-manifold into disks, however, is harder than the case of surfaces.

For certain types of 3-manifolds, namely, for Haken 3-manifolds, this process of cutting into disks is well known. Recall that M is Haken if it is irreducible and contains a properly embedded two-sided incompressible surface. Being an irreducible 3-manifold means that every embedded 2-sphere bounds a disk. The existence of this disk allows us to do a similar surgery argument as we did for isotopic arcs in a surface. Recall that a compact connected surface S that is not homeomorphic S^2 in M is an incompressible surface if it is properly embedded $S \cap \partial M = \partial S$, the normal bundle of S is trivial, and the inclusion $S \hookrightarrow M$ is π_1 injective. Given the Haken manifold theory, there is a finite sequence of incompressible surfaces so that as we cut a Haken manifold M along those surfaces, we obtain a disjoint union of disks.

The idea is to induct on the number of prime factors in a prime decomposition of M to reduce Thurston's theorem to the case of Haken manifolds and then use the hierarchy of Haken manifolds to reduce it to the case of disks.

Let $M \cong P_1 \# P_2 \# \cdots \# P_n$ be the connected sum of n prime 3-manifolds. We will define semisimplicial spaces with contractible realizations that encode different ways of cutting M into the union of its prime factors with a certain number of disks removed. By the same argument as in the previous section, a spectral sequence argument shows that Thurston's theorem holds for M if it does for manifolds homeomorphic to P_i with a certain number of disks removed. We then reduce Thurston's theorem for such manifolds to the case of the Haken manifolds.

4.1. **Cutting along separating spheres.** We want to reduce Theorem 2.3 for M to 3-manifolds with fewer prime factors. To do so, we shall define semisimplicial simplicial sets parametrizing *separating spheres* in M. By a separating sphere, we mean an embedded sphere that does not bound a disk in M. If M has a sphere boundary, a separating sphere could be isotopic to a sphere boundary component.

Definition 4.1. Let $S(M)$ be a simplicial complex whose set of vertices of $S(M)$ is given by locally flat embeddings of a 2-sphere $\phi \in \mathrm{Emb}^{\mathrm{lf}}(S^2, M)$ so that its image is a separating sphere. A set of $p+1$ such embeddings constitutes a p-simplex if their images are disjoint.

Definition 4.2. We use $S(M)$ to define a semisimplicial set $S^{\delta}_{\bullet}(M)$ and a semisimplicial simplicial set $S_{\bullet}(M)_{\bullet}$ on which $\mathrm{Homeo}_{0,\partial}(M)$ and $S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))$ act respectively.

- **Discrete version:** Let $S^{\delta}_0(M)$ be the set of the vertices of $S(M)$ and let the set of the p-simplices $S^{\delta}_p(M)$ be all different ways of ordering the p-simplices in $S(M)$. In other words, $S^{\delta}_p(M)$ is the set of $(p+1)$-tuples $(v_0, v_1, \dots, v_p)$ so that the set $\{v_0, v_1, \dots, v_p\}$ is a simplex in $S(M)$.
- **Topological version:** Let $S_{\bullet}(M)_{\bullet}$ be a semisimplicial simplicial set whose k-simplices $S_{\bullet}(M)_k$ in the simplicial direction is given by tuples of vertices $(v_0(t), v_1(t), \dots, v_{\bullet}(t))$ such that $\{v_0(t), v_1(t), \dots, v_{\bullet}(t)\}$ is a simplex in $S(M)$ for all $t \in \Delta^k$.

We shall first prove that $S(M)$ is contractible when it is nonempty and then deduce that realizations of $S^{\delta}_{\bullet}(M)$ and $S_{\bullet}(M)_{\bullet}$ are contractible.

Lemma 4.3. *If M is not a prime manifold, the simplicial complex $S(M)$ is contractible.*

Proof. Similarly to Proposition 3.16, we want to show that for all k, any continuous map $f\colon S^k \to S(M)$ is nullhomotopic. Without loss of generality, we can assume that for a triangulation K of S^k, the map f is PL. To find a nullhomotopy for the map f, it is enough to homotope it so that its image lies in the star of a vertex in $S(M)$.

As in the claim in the proof of Lemma 3.31, we can homotope f so that the vertices in $f(K)$ are pairwise transverse. Let $v_1 \in S(M)$ be an embedding whose image is transverse to the spheres represented by the set of vertices in $f(K)$. To homotope f so that its image lies in $\mathrm{Star}(v_1)$, we inductively remove the circles in the intersection of v_1 and the spheres in $f(K)$.

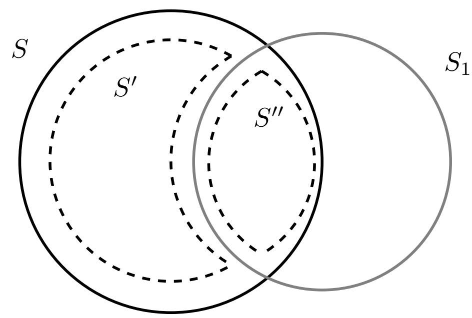

FIGURE 2. Surgery on spheres in one dimension lower.

Let S_1 be the embedded sphere given by the image of v_1. The intersection of the spheres in $f(K)$ and S_1 form a finite number of circles. Among these circles, we choose a maximal family of disjoint circles on S_1. Let C be an innermost circle in this family which is given by the intersection of S_1 and a sphere S given by the image of an embedding $f(x) = v \in f(K)$. The circle C bounds a 2-disk in S_1. We can cut S along the circle C and glue two copies of this 2-disk to obtain two disjoint embedded spheres S' and S'' (see Figure 2). By considering nearby parallel copies, we can assume that S, S', and S'' are disjoint. Note that at least one of the spheres S' and S'' is separating. We assume that S' is separating. Now we shall replace S by S' as the image of the vertex v as follows (see Figure 3). We choose an embedding v' whose image is S'. By choosing nearby parallel copies of the spheres, we can assume that the vertex v' is connected to v; i.e., their corresponding spheres are disjoint.

If we choose S' sufficiently close to the 2-disk in S that bounds C, then any sphere S_2 in the star of v which intersected S' would also intersect this 2-disk. However, this cannot happen: since $S_2 \cap S_1 = \emptyset$ and C was chosen to be an innermost circle among a maximal family of disjoint circles given by intersections with S_1, no disjoint sphere S_2 can intersect the 2-disk bounded by C. Thus, no vertex in the star of v intersects S', so our modified sphere as the image of v' remains disjoint from all the spheres that the image of v is disjoint from. In other words, v' is connected to all vertices in the star of v. Therefore, we have a simplicial homotopy $F\colon K \times [0,1] \to S(M)$ such that $F(-,1)$ is the same as $F(-,0)$ on all vertices but x and $F(x,1) = v'$. Note that the vertices in the image $F(-,1)\colon K \to S(M)$ have fewer circles in their intersection with S_1. By repeating this process, we could

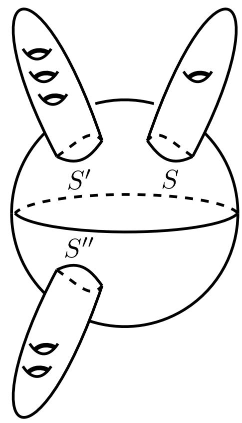

FIGURE 3. Separating spheres S, S', and S'' are depicted in one dimension lower. They bound a 3-sphere with three disks removed.

homotope the map f to a map whose image lies in the star of v_1. Therefore, f is nullhomotopic. ☐

To prove that realizations of $S^{\delta}_{\bullet}(M)$ and $S_{\bullet}(M)_{\bullet}$ are contractible, we need $S(M)$ to have a property that is called *weakly Cohen-Macaulay.*

Definition 4.4. A simplicial complex K is called *weakly Cohen-Macaulay* of dimension at least n and it is denoted by $wCM(K) \geq n$ if it is $(n-1)$-connected and the link of any p-simplex is $(n-p-2)$-connected.

For a simplex σ in the simplicial complex $S(M)$, let $M \backslash \sigma$ denote a manifold that is homeomorphic to the manifold obtained from M by cutting it along the spheres in σ. Note that a link of σ is homeomorphic to $S(M\backslash\sigma)$, which is again contractible by Lemma 4.3. Therefore $S(M)$ is weakly Cohen-Macaulay of dimension infinity.

Proposition 4.5. *If M is not a prime manifold, the realizations of $S^{\delta}_{\bullet}(M)$ and $S_{\bullet}(M)_{\bullet}$ are contractible.*

Proof. Similarly to Proposition 3.16, it is enough to show that the realization of the semisimplicial set $S^{\delta}_{\bullet}(M)$ is contractible. But $S^{\delta}_{\bullet}(M)$ is obtained from $S(M)$ by considering all different orderings on simplices. It is a consequence of the *generalized coloring lemma* ([GRW18, Theorem 2.4]) that if $wCM(S(M)) > n$, then $|S^{\delta}_{\bullet}(M)|$ is at least $(n-1)$-connected (see [Nar17, Theorem 3.9] for a similar argument). Therefore, $|S^{\delta}_{\bullet}(M)|$ is contractible. ☐

As in the case of surfaces, we can use $S^{\delta}_{\bullet}(M)$ and $S_{\bullet}(M)_{\bullet}$ to define semsimplicial resolutions for $\mathrm{BHomeo}^{\delta}_{0,\partial}(M)$ and $\mathrm{B}|S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))|$. Therefore, similar spectral sequence arguments as Propositions 3.23 and 3.35 imply the following reduction of Theorem 2.3.

Proposition 4.6. *If Theorem* 2.3 *holds for* $M\backslash\sigma$ *for all simplices* σ *in* $S(M)$*, then it also holds for* M*.*

By the uniqueness of the prime decomposition for 3-manifolds, it is easy to see that $M\backslash\sigma$ is a disjoint union of pieces that are homeomorphic to either one of the P_i's with a certain number of disks removed or S^3 with a certain number of disks removed. Hence, by repeating Proposition 4.6 we conclude that having the Thurston theorem for prime manifolds or S^3 with a nonzero number of disks removed implies Thurston's theorem for all 3-manifolds that are not prime.

4.2. **Reducing to the case of Haken manifolds.** If M is not prime, by cutting M along sphere systems, we reduced to the case of prime manifolds with a number of disks removed and S^3 with a number of disks removed. To reduce these cases further to the case of the Haken manifold, we remove 1-handles from these pieces. If M is prime, we remove solid tori to reduce it to the case of irreducible manifolds with torus boundary components which are known to be Haken.

First, we shall treat the case where M is prime. Recall that the only prime manifold that is not irreducible is $S^1 \times S^2$ (see [Hat, Proposition 1.4]). To remove solid tori, similarly to the case of 0-handles, we need to consider the germs of embeddings around the core of the solid torus.

Definition 4.7. Let $f\colon S^1 \times D^2 \hookrightarrow M$ and $g\colon S^1 \times D^2 \hookrightarrow M$ be two locally flat embeddings. We say they have the same germ around the cores $f(S^1 \times \{0\})$ and $g(S^1 \times \{0\})$ if there exists some open neighborhood $U \subset D^2$ of the origin such that $f|_{S^1\times U} = g|_{S^1\times U}$. We shall denote the germ of f around its core by $[f]$.

Definition 4.8. Let $\phi : S^1 \times D^2 \hookrightarrow M$ be a π_1-injective embedding. Let $T(M;\phi)$ be a simplicial complex whose vertices are germ of embeddings $f\colon S^1 \times D^2 \hookrightarrow M$ such that the core of f is isotopic to the core of ϕ and $\{[f_0], [f_1], \dots, [f_p]\}$ constitutes a p-simplex if the cores of $[f_i]$ are disjoint.

Lemma 4.9. *The simplicial complex* $T(M;\phi)$ *is weakly Cohen-Macaulay of dimension infinity.*

Proof. We first show that $T(M;\phi)$ is contractible, and then it becomes clear that the links of a simplex are contractible by the same argument. As before, we want to show that for all k, any continuous map $f\colon S^k \to T(M;\phi)$ is nullhomotopic. Without loss of generality, we can assume that f is a PL map with respect to a triangulation K on S^k.

Note that since the codimension of the core of an embedded solid torus in M is 2, if two cores are transverse in M they should be disjoint. Since disjointness is an open condition, we can change f up to homotopy so that all vertices of $f(K)$ are disjoint. We choose another vertex $v \in T(M;\phi)$ whose core is disjoint from the cores of the vertices in $f(K)$. Therefore, $f(K) \subset \mathrm{Star}(v)$, which implies that f is nullhomotopic. □

Definition 4.10. We define a semisimplicial set $T^{\delta}_{\bullet}(M;\phi)$ and a semisimplicial simplicial set $T_{\bullet}(M;\phi)_{\bullet}$ on which $\mathrm{Homeo}_{0,\partial}(M)$ and $S_{\bullet}(\mathrm{Homeo}_{0,\partial}(M))$ act respectively.

- **Discrete version:** The set of the p-simplices of $T^{\delta}_p(M;\phi)$ is given by the set of all different orderings on the set of p-simplices of the simplicial complex $T(M;\phi)$. In other words, the $(p+1)$-tuple $([f_0], [f_1], \dots, [f_p])$ of germs of embeddings of solid tori is a p-simplex if the cores of $[f_i]$'s are disjoint. The i-th face maps are given by omitting $[f_i]$.

- **Topological version:** The set of 0-simplices $T_0(M;\phi)_\bullet$ in the semisimplicial direction is the subset of embeddings $f(t)$ in $\mathrm{Emb}^{\mathrm{lf}}_\bullet(S^1\times D^2, M)$ whose core $f(t)(S^1\times\{0\})$ is isotopic to the core of ϕ for all $t\in\Delta^\bullet$. A p-simplex in the semisimplicial direction is given by $(p+1)$-tuples $(f_0(t), f_1(t),\dots,f_p(t))$ in $T_0(M;\phi)_\bullet^{p+1}$ so that $f_i(t)(S^1\times D^2)$ and $f_j(t)(S^1\times D^2)$ are disjoint for all i,j, and $t\in\Delta^\bullet$. The face maps in both directions are defined as usual.

Proposition 4.11. *The realizations of $T^\delta_\bullet(M;\phi)$ and $T_\bullet(M;\phi)_\bullet$ are contractible.*

Proof. It is similar to Proposition 4.5. □

For a p-simplex σ in $T_p(M;\phi)_0$, let $M\backslash\sigma$ denote a manifold obtained from M by cutting the interior of the solid tori in σ.

Proposition 4.12. *If Theorem* 2.3 *holds for $M\backslash\sigma$ for all simplices σ in $T_p(M;\phi)_0$, then it also holds for M.*

Proof. Similarly to the diagram 3.19 in Section 3, we can define a zig-zag of maps between $T^\delta_\bullet(M;\phi)$ and $T_\bullet(M;\phi)_\bullet$ that would lead to a map of spectral sequences as in Proposition 3.23. The hypothesis implies the isomorphism on the first page of the spectral sequences. Hence, as in Proposition 3.23, it implies Theorem 2.3 for M. □

Remark 4.13. Note that the only thing we used about the solid torus ϕ was that the dimension of its core is less than its codimension. Similarly, if M has boundary, we can define a 1-handle $\phi\colon D^1\times D^2\to M$ so that its two ends $\phi(\{0\}\times D^2)$ and $\phi(\{1\}\times D^2)$ lie on the boundary of M, and then we could define $T^\delta_\bullet(M;\phi)$ and $T_\bullet(M;\phi)_\bullet$ to reduce Theorem 2.3 to the case of M with certain 1-handles removed.

Now we are ready to use Proposition 4.12 to treat the case when M is prime.

Proposition 4.14. *Theorem* 1.1 *holds for closed prime manifolds if it holds for all irreducible* 3*-manifolds whose boundary components are homeomorphic to a torus.*

Proof. We choose a π_1-injective locally flat embedding $\phi\colon S^1\times D^2\hookrightarrow M$. If M is a closed irreducible 3-manifold, then for any p-simplex σ in $T_p(M;\phi)_0$, the manifold $M\backslash\sigma$ is still irreducible. Therefore, this case is easily followed from Proposition 4.12.

The only prime manifold that is not irreducible is $S^1\times S^2$. Since all the essential embedded spheres in $S^1\times S^2$ are isotopic, all the embedded spheres in the manifold $S^1\times S^2\backslash\sigma$ bound a disk. Hence, by Proposition 4.12, the case of $S^1\times S^2$ is also reduced to the case of irreducible 3-manifolds with torus boundary components. □

Therefore, if M is prime, we can reduce Theorem 2.3 to the case of Haken manifolds because irreducible 3-manifolds with torus boundary are Haken. If M is not prime, using Propositions 4.6, 4.12, and 4.14, we can reduce Theorem 2.3 for 3-manifolds to the case of $S^3\backslash\bigcup_{i=1}^k \mathrm{int}(D^3)$ and $P_i\backslash\bigcup_{i=1}^k \mathrm{int}(D^3)$ where the P_i's are irreducible. The goal is to further reduce these cases to the case for Haken manifolds. As the general strategy is to cut along submanifolds, we always get manifolds with boundary. Furthermore, an irreducible 3-manifold with boundary is Haken. So to reduce to the Haken manifolds, we want to cut along submanifolds to get an irreducible 3-manifold with boundary.

Proposition 4.15. *Theorem* 1.1 *holds for* S^3 *with a number of disks removed if it holds for Haken* 3*-manifolds.*

Proof. Let $M = S^3 \backslash \bigcup_{i=1}^k \operatorname{int}(D^3)$. We can assume that $k > 1$ because otherwise M is homeomorphic to a 3-disk and Thurston's theorem in this case is deduced from the Mather theorem ([Mat71]). We choose a 1-handle $\phi : D^1 \times D^2 \hookrightarrow M$ so that $\phi(\{0\} \times D^2)$ and $\phi(\{1\} \times D^2)$ are subsets of different sphere boundary components. Hence, by Remark 4.13, Thurston's theorem holds for M if it holds for $M\backslash\sigma$ for all simplices σ in $T_\bullet(M;\phi)_0$. But for a p-simplex σ, the manifold $M\backslash\sigma$ has fewer boundary components. By repeating this process, we can reduce the theorem for M to a handlebody which has one boundary component. But a handlebody is Haken. $\square$

Note that the sphere boundaries in $P_i \backslash \bigcup_{j=1}^k \operatorname{int}(D^3)$ destroy the irreducibility. So to reduce Thurston's theorem for $P_i \backslash \bigcup_{j=1}^k \operatorname{int}(D^3)$ to the case for Haken manifolds, we first cut along certain 1-handles to reduce the number of sphere boundaries. But unlike the case of S^3 with a number of disks removed, we want to increase genus of each boundary component. Because it is not clear how to do the same procedure as for $S^3 \backslash \bigcup_{i=1}^k \operatorname{int}(D^3)$ to get a handlebody at the end, we first need to show that $P_i \backslash \bigcup_{j=1}^k \operatorname{int}(D^3)$ are not simply connected.

Lemma 4.16. *If a* 3*-manifold* M *with boundary is simply connected, it is obtained from* S^3 *by removing the interior of a union of disjoint disks in* S^3*.*

Proof. It is enough to show that the boundary ∂M is homeomorphic to the union of S^2's, because if we fill in the sphere boundaries by disks, we obtain a simply connected closed 3-manifold which has to be homeomorphic to S^3 by Perelman's theorem ([Per02, Per03]). Since M is simply connected, we have $H_1(M) = 0$, so by the Poincaré-Lefschetz duality, we also have $H_2(M, \partial M) \cong H^1(M) = 0$. The homology long exact sequence for the pair $(M, \partial M)$ implies that $H_2(M, \partial M) \to H_1(\partial M) \to H_1(M)$ is exact. Therefore, $H_1(\partial M) = 0$, which implies that ∂M is homeomorphic to a union of S^2's. $\square$

Let Q be the manifold obtained from P by removing the interior of m disjoint disks in P. To prove Thurston's theorem for Q, we want to cut 1-handles from Q to make it irreducible. Note that since P is not simply connected and is not homeomorphic to the sphere, by Lemma 4.16, the manifold Q is not simply connected either. Let $\partial_i Q$ be the i-th boundary component. We choose an arc γ_i with the two ends on $\partial_i Q$ so that the arc γ_i with a path between its two ends on the boundary is nontrivial in the fundamental group of P.

Let $\phi_i : D^1 \times D^2 \hookrightarrow Q$ be a 1-handle whose core is γ_i. Let us denote the manifold obtained from Q by removing the interior of the handle ϕ_i by $Q \backslash \bigcup_{i=1}^m \phi_i$.

Lemma 4.17. $Q \backslash \bigcup_{i=1}^m \phi_i$ *is irreducible.*

Proof. Given that P is irreducible, every embedded sphere in $Q \backslash \bigcup_{i=1}^m \phi_i$ bounds a disk in P. If this disk contains any of the boundary components with a 1-handle attached to it, then the loop given by the union of the core of the 1-handle with a path between the two ends of the core on the boundary would be trivial in the fundamental group of P, which is a contradiction. $\square$

Proposition 4.18. *Theorem* 2.3 *holds for* Q *if it does for Haken manifolds.*

Proof. For any simplex σ in $T_\bullet(M;\phi_i)_0$, the i-th boundary component of $Q\backslash\sigma$ is no longer a sphere. Hence, by repeating Proposition 4.12 for all boundary components, we could reduce Theorem 2.3 for Q to the case of irreducible manifolds with boundary. □

4.3. **Finishing Theorem 2.3: The case of Haken 3-manifolds with boundary.** By the theory of Haken manifolds ([Hak62]), we know that they have a hierarchy where they can be split up into 3-disks along incompressible surfaces. Let S be a surface with boundary and let $\psi\colon S\times\mathbb{R}\hookrightarrow M$ be a proper locally flat bicollared embedding of an incompressible surface. Given the case of Haken manifolds which are lower compared to M in the Haken hierarchy, we inductively prove Theorem 2.3 for M by cutting along incompressible surfaces.

Definition 4.19. We define a semisimplicial set $K^\delta_\bullet(M,\psi)$ and a semisimplicial simplicial set $K_\bullet(M,\psi)_\bullet$ on which $\mathrm{Homeo}_{0,\partial}(M)$ and $S_\bullet(\mathrm{Homeo}_{0,\partial}(M))$ act respectively.

- **Discrete version:** The 0-simplices of $K^\delta_\bullet(M,\psi)$ are given by pairs (t,ϕ) such that $\phi\in\mathrm{Emb}^{\mathrm{lf}}_0(S,M)$ satisfying

$$\phi(\partial S)=\psi(\partial S\times\{t\,\vec{e}\}), \tag{4.20}$$

  where $\vec{e}$ is the unit basis vector of $\mathbb{R}$ and $\phi(S)$ is isotopic to the surface $\psi(S\times\{t\,\vec{e}\})$.

  The set of p-simplices $K^\delta_p(M,\psi)$ consists of $(p+1)$-tuples

$$((t_0,\phi_0),(t_1,\phi_1),\dots,(t_p,\phi_p))$$

  in $K^\delta_0(M,\psi)^{p+1}$ so that $t_0<t_1<\dots<t_p$ and the embedded surfaces $\phi_i(S)$ are disjoint. The face maps are given by forgetting the embeddings. Given that the t-coordinate in $(t,\phi)\in K^\delta_0(M,\psi)$ is uniquely determined by ϕ, we shall write ϕ for a vertex and refer to its t-coordinate by t_ϕ.
- **Topological version:** The 0-simplices $K_\bullet(M,\psi)_0$ in the simplicial direction constitute the same seimisimplicial set as $K^\delta_\bullet(M,\psi)$. Its k-simplices in the simplicial direction are given by tuples

$$((t_0,\phi_0(s)),(t_1,\phi_1(s)),\dots,(t_\bullet,\phi_\bullet(s)))$$

  in $K^\delta_0(M,\psi)^{\bullet+1}$ for all $s\in\Delta^k$.

Proposition 4.21. *Let M be a Haken manifold with boundary. The realizations $|K^\delta_\bullet(M,\psi)|$ and $|K_\bullet(M,\psi)_\bullet|$ are contractible.*

Proof. Similar to Proposition 3.16, it is enough to show that $|K^\delta_\bullet(M,\psi)|$ is contractible. But since the t-coordinates of vertices of a simplex in $K^\delta_\bullet(M,\psi)$ are ordered, there is a natural order on simplices. Therefore $|K^\delta_\bullet(M,\psi)|$ is a simplicial complex. Let us represent an element of the homotopy group $f:S^k\to|K^\delta_\bullet(M,\psi)|$ by a PL map with respect to some triangulation K on S^k. Similarly to Lemma 3.31 we shall arrange f so that the vertices of $f(K)$ are pairwise transverse.

Now we choose another vertex $\phi\in K^\delta_0(M,\psi)$ so that $\phi(S)$ is transverse to all vertices in $f(K)$ and its t-coordinate is different from that of vertices in $f(K)$. Therefore, the vertices of $f(K)$ do not intersect $\phi(S)$ on the boundary, and each component of the intersections is homeomorphic to a circle. We shall change the map f up to homotopy to remove these circles to arrange $f(K)\subset\mathrm{Star}(\phi)$.

Step 1. We first remove all circles on $\phi(S)$ that are nullhomotopic in M. Since $\phi(S)$ is incompressible, any circle on $\phi(S)$ that is nullhomotopic in M is in fact nullhomotopic in $\phi(S)$. Hence such circles bound a 2-disk on $\phi(S)$.

Among the set of circles in the intersection of $\phi(S)$ and the vertices of $f(K)$, we choose a maximal family of disjoint circles on $\phi(S)$. Since the circles in this family are disjoint, there is an innermost circle C. Suppose C is in the intersection of $\phi(S)$ and $\phi_0(S)$ where $\phi_0 = f(v) \in f(K)$. Since $\phi_0(S)$ is also incompressible, the circle C bounds a 2-disk D' on $\phi_0(S)$ and a 2-disk D on $\phi(S)$. Since M is irreducible, the embedded sphere $D \cup D'$ bounds a disk B in M.

If $\phi' \in \mathrm{Star}(\phi_0)$, then $\phi'(S)$ cannot intersect D and D'. The latter is clear, because $\phi''(S)$ does not even intersect $\phi_0(S)$. But if $\phi'(S)$ intersects D, since it is disjoint from $\partial D' = C$, their intersection would give circles inside D. Given that C was an innermost circle, this is a contradiction.

By pushing D' across the disk B toward D and considering a nearby parallel copy, we obtain a vertex $\phi'' \in K_0^\delta(M,\psi)$ so that $\phi''(S)$ is disjoint from $\phi_0(S)$ and $\phi'(S)$ for all $\phi' \in \mathrm{Star}(\phi_0)$. Hence, we can find a homotopy $F\colon K \times [0,1] \to |K_\bullet^\delta(M,\psi)|$ where $F(-,0) = f$, $F(v,1) = \phi'$, and $F(-,1)$ is the same as f on all vertices other than v. Therefore, by repeating this process, we can assume that the circles in the intersection of $\phi(S)$ and the vertices of $f(K)$ are not nullhomotopic.

Step 2. Now we assume that all circles in the intersection of $\phi(S)$ and the vertices in $f(K)$ are not nullhomotopic in $\phi(S)$. In the previous case to remove circles we used embedded disks in M where we thought of the disk B as a "*pinched product*" (see Figure 4) between two 2-disks D and D'. By a pinched product P over Σ a surface with boundary, we mean the quotient of the product $\Sigma \times [0,1]$ by all segments $\{x\} \times [0,1]$ where $x \in \partial\Sigma$. This pinched product is a handlebody with corner $\partial\Sigma$ whose boundary ∂P is a union of two copies of Σ that we denote by $\partial_- P$ and $\partial_+ P$. These two copies intersect in the corner $\partial\partial_- P = \partial\partial_+ P$. To reduce the number of circles in the intersection of $\phi(S)$ and a vertex $\phi' \in K_0^\delta(M,\psi)$, we shall find a pinched product P so that $\partial_+ P$ lies on $\phi'(S)$ and $\partial_- P$ lies on $\phi(S)$. Then by pushing $\phi'(S)$ across P and a little beyond, we obtain a new vertex $\phi'' \in K_0^\delta(M,\psi)$ that has fewer circles in its intersection with $\phi(S)$.

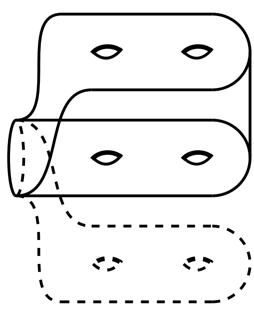

FIGURE 4. Pushing across the pinched product.

Similarly to [Hat76, p. 346] and [Hat99, Step 3], we use the covering $p : (\widetilde{M}, \tilde{x}) \to (M, x)$ corresponding to the subgroup $\pi_1(\phi(S), x)$ in $\pi_1(M, x)$ for a base point $x \in \phi(S)$ to do surgery. There is a homeomorphic lift $\widetilde{\phi}(S)$ of $\phi(S)$ in $\widetilde{M}$ containing $\tilde{x}$. For each vertex $f(v) \in f(K)$, if $f(v)(S)$ intersects $\phi(S)$, we choose a lift $\widetilde{f(v)}(S)$

of the surface $f(v)(S)$ that intersects $\widetilde{\phi}(S)$; otherwise we choose any lift of $f(v)(S)$. Given our choice of covering, the map p restricted to $\widetilde{f(v)}(S)$ is a homeomorphism. In this way, we have a lift $\widetilde{f}(K)$ to $|K^{\delta}_{\bullet}(\widetilde{M};\widetilde{\phi})|$. We shall use $K^{\delta}_{\bullet}(\widetilde{M};\widetilde{\phi})$ as bookkeeping to change f inductively up to a simplicial homotopy so that its image lies in the star of $\phi(S)$.

As Hatcher showed in [Hat99, Step 3], the incompressibility of $\phi(S)$ implies that every connected component of $p^{-1}(\phi(S))$ separates $\widetilde{M}$ into two components. Let $\widetilde{S}$ be a nearby parallel copy of $\widetilde{\phi(S)}$. For each component S_i of $p^{-1}(\phi(S))$, let M_{S_i} denote the component of $\widetilde{M}\backslash S_i$ that does not contain the boundary $\partial\widetilde{S}$. We order these components by inclusion. Let M_{S_i} be a minimal component that intersects the union $\bigcup_{v\in K}\widetilde{f(v)}(S)$.

Let C_v be a component of $\widetilde{f(v)}(S)\cap M_{S_i}$. Laudenbach showed (see [Lau74, Corollary II.4.2] and also [Hat99, p. 8]) that there is a unique pinched product P_v so that $\partial_+P = C_v$ and ∂_-P_v lies on S_i. For those $v\in K$ that $\widetilde{f(v)}(S)$ intersects in S_i, we have a partial order on the subsurfaces ∂_-P_v in S_i given by inclusion.

To be able to use the pinched products P_v to change $\widetilde{f}$ up to simplicial homotopy, we need Lemma 4.22. Given this lemma, the rest of the argument is as follows. We choose a maximal family V of vertices $v\in K$ so that their corresponding subsurfaces ∂_-P_v on S_i are either disjoint or one includes the other. Let $z\in V$ be a vertex for which ∂_-P_z is innermost. Then by moving ∂_+P_z along P_z and a little beyond, we obtain a vertex $y\in K^{\delta}_0(\widetilde{M};\widetilde{\phi})$ that does not intersect S_i in ∂_-P_z anymore. Note that the restriction of the covering map $p:\widetilde{M}\to M$ to the pinched product P_z is a homeomorphism. Hence $p(P_z)$ is also a pinched product.

Now we shall observe that if $w\in \mathrm{Star}(z)$, then $f(w)\in\mathrm{Star}(p(y))$, because if $f(w)(S)$ intersects $p(y)(S)$, then it has to intersect the pinched product $p(P_z)$. Therefore, its lift $\widetilde{f(w)}(S)$ intersects ∂_-P_z, and by Lemma 4.22 below, we would have $\partial_-P_w\subset\partial_-P_z$, which contradicts the fact that ∂_-P_z was innermost. Also by considering a nearby parallel copy, we can assume that $p(y)(S)$ is also disjoint from $f(z)(S)$. Therefore, $p(y)$ is connected to $f(z)$ and $f(w)$ for all $w\in\mathrm{Star}(z)$. Hence, we can find a homotopy $F\colon K\times[0,1]\to|K^{\delta}_{\bullet}(M;\phi)|$ where $F(-,0)=f$, $F(z,1)=p(y)$, and $F(-,1)$ is the same as f on all vertices other than z. Therefore, by repeating this process, we can reduce the number of the circles in the intersection of the vertices of $f(K)$ and $\phi(S)$ which would give a homotopy of f to a map whose image is in $\mathrm{Star}(\phi)$. □

Lemma 4.22. *Suppose $w\in\mathrm{Star}(v)$ and $\widetilde{f(w)}(S)$ intersects S_i. Then ∂_-P_v and ∂_-P_w are either disjoint or one contains the other.*

Proof. Suppose the contrary, so the intersection of ∂_-P_v and ∂_-P_w is a subsurface $\Sigma\subset S_i$ such that its boundary $\partial\Sigma$ decomposes into two parts $\partial_v\Sigma\subset\partial\partial_-P_v$ and $\partial_w\Sigma\subset\partial\partial_-P_w$. Over Σ in P_v, we shall choose a "*partial*" pinched product Q_v which is a submanifold in P_v with corner (see Figure 5). The manifold Q_v is homeomorphic to the quotient of $\Sigma\times[0,1]$ given by pinching $\{x\}\times[0,1]$ for all $x\in\partial_v\Sigma$. The boundary of the partially pinched product Q_v is the union of $\partial_-Q_v=\Sigma$, a piece ∂_0Q_v that is homeomorphic to $\partial_w\Sigma\times[0,1]$, and ∂_+Q_v that lies on ∂_+P_v.

We shall change ∂_0Q_v up to isotopy to make it disjoint from P_w. Since $w\in\mathrm{Star}(v)$, we know ∂_+P_v and ∂_+P_w are disjoint. Therefore, ∂_+Q_v either lies inside

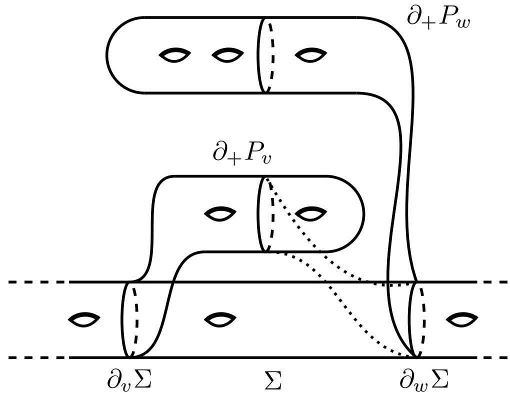

FIGURE 5. The dotted line is $\partial_0 Q_v$.

or outside P_w. First, let us assume that it is inside P_w. Each connected component L_i of $\partial_0 Q_v$ is homeomorphic to a cylinder. We first change $\partial_0 Q_v$ relative to its boundary up to isotopy to make it transverse to the surface $\partial_+ P_w$. The intersection on L_i are circles. If a circle is nullhomotopic on L_i, by irreducibility of $\widetilde{M}$ and incompressibility of $\partial_+ P_w$, we can remove it by an isotopy. So we assume all the circles in the intersection $L_i \cap \partial_+ P_w$ are disjoint isotopic circles on the cylinder L_i.

Let L_i^α be a piece of the cylinder L_i that lies outside P_w and bounds two consecutive circles C_1^α and C_2^α on L_i. Since C_1^α and C_2^α are also isotopic on $\partial_+ P_w$, they bound a cylinder L_w^α on $\partial_+ P_w$. Hence, by the Laudenbach theorem ([Lau74, Corollary II.4.2]) again these two cylinders are isotopic. So we can push $\partial_0 Q_v$ by an isotopy inside P_w. Therefore, the surface $\partial_+ P_v$ lies inside P_w, but this contradicts the incompressibility of $\partial_+ P_v$, because the fundamental group of P_v and P_w are the same as that of $\partial_- P_v$ and $\partial_- P_w$. Since the surfaces $\partial_- P_v$ and $\partial_- P_w$ are not nested, incompressibility implies that the fundamental groups P_v and P_w as subgroups of $\pi_1(\widetilde{M}, \tilde{x})$ are not nested either. Hence, the surface $\partial_+ P_v$ cannot lie inside P_w, which is a contradiction. Similarly if $\partial_+ Q_v$ lies outside, we arrive at a contradiction by showing that $\partial_+ P_w$ lies inside P_v. □

Using Proposition 4.21, we can resolve $\mathrm{BHomeo}^\delta_{0,\partial}(M)$ and $\mathrm{B}|S_\bullet(\mathrm{Homeo}_{0,\partial}(M))|$ and we have a natural map between them. Therefore, exactly the same argument as Proposition 3.35 implies that Theorem 2.3 holds if it holds for $M \backslash \sigma_p$ for all $\sigma_p \in K_p^\delta(M; \psi)$. So, now we can do an inductive argument to prove the Thurston theorem for Haken manifolds with boundary.

Theorem 4.23. *Theorem* 2.3 *holds for Haken manifolds with boundary.*

Proof. First we assume that M is a handlebody and we induct on the genus. The base case is the disk which is the Mather theorem. We choose an incompressible disk $\psi\colon D^2 \to M$. Note that for a p-simplex $\sigma_p \in K_p^\delta(M; \psi)$, the manifold $M \backslash \sigma_p$ is homeomorphic to the disjoint union of p disks and a handlebody of lower genus. So by induction Thurston's theorem holds for $M \backslash \sigma_p$ for all p and all p-simplices σ_p. Therefore, it also holds for M.

Now for a general Haken manifold M, we have a finite Haken hierarchy; i.e., there are a finite number of incompressible surfaces in M such that if we cut M along those surfaces, we obtain a disjoint union of disks. We shall induct on the Haken hierarchy. For an incompressible surface $\psi\colon S \to M$ and a p-simplex $\sigma_p \in K_p^\delta(M;\psi)$, the manifold $M\backslash\sigma_p$ is homeomorphic to the disjoint union of p handlebodies, each homeomorphic to $S\times[0,1]$, and the manifold $M\backslash\psi$ which is lower in the hierarchy. Therefore, by induction and the previous case, Thurston's theorem holds for all $M\backslash\sigma_p$. Hence, it also holds for M. □

5. On the homotopy type of $\mathrm{homeo}_{0,\partial}(M)$

In this section, we use some of the semisimplicial resolutions in previous sections to give new proofs of the contractibility of $\mathrm{Homeo}_{0,\partial}(M)$ when M is a hyperbolic surface ([Ham74]) or when it is a Haken 3-manifold with boundary ([Hat99,Hat76]).

Our strategy is to show that in these cases the classifying space $\mathrm{BHomeo}_{0,\partial}(M)$ is acyclic. Since it is simply connected, Whitehead's theorem implies that it should be weakly contractible. Therefore, the weak contractibility of $\mathrm{Homeo}_{0,\partial}(M)$ follows from the weak contractibility of its delooping.

The statements hold for diffeomorphism groups of these manifolds, but since we defined resolutions for homeomorphism groups we give the argument for homeomorphism groups. Also another convenience of working with homeomorphism groups is that we can use the Thurston Theorem 2.3 that the map

$$\mathrm{BHomeo}^\delta_{0,\partial}(M) \to \mathrm{BHomeo}_{0,\partial}(M)$$

induces a homology isomorphism. Therefore, if we want to show that $\mathrm{Homeo}_{0,\partial}(M)$ is weakly contractible, it is enough to show that $\mathrm{Homeo}^\delta_{0,\partial}(M)$ is an acyclic group.

5.1. **The case of hyperbolic surfaces.** In this section we prove

Theorem 5.1. *The group* $\mathrm{Homeo}^\delta_{0,\partial}(\Sigma)$ *is an acyclic group when* Σ *is a hyperbolic surface.*

Proof. We consider the case of closed surfaces and surfaces with boundary separately.

Case 1. Suppose Σ has a nonempty boundary. We induct on $-\chi(\Sigma)$. The base case is when Σ is a disk which is the Mather theorem ([Mat71]). If the genus $g(\Sigma)$ is not zero, we choose an arc ϕ whose two ends lie on the same boundary component so that cutting along ϕ decreases the genus. If the genus is zero, we choose an arc ϕ whose two ends lie on different boundary components so that cutting along ϕ decreases the number of boundary components. In either case, the realization of $B^\delta_\bullet(\Sigma,\phi)$ is contractible (see Lemma 3.31), which gives a semisimplicial resolution

$$Y^\delta_\bullet(\Sigma,\phi) \to \mathrm{BHomeo}^\delta_{0,\partial}(\Sigma).$$

Therefore, as we discussed in Proposition 3.35, we have a spectral sequence

$$E^1_{p,q} = H_q(Y^\delta_p(\Sigma,\phi)) \Rightarrow H_{p+q}(\mathrm{BHomeo}^\delta_{0,\partial}(\Sigma)).$$

The homotopy type of $Y^\delta_p(\Sigma,\phi)$ is the same as $\coprod_{\text{orbits}} \mathrm{BStab}^\delta(\sigma_p)$, where σ_p varies over the set of the orbits of the action of $\mathrm{Homeo}^\delta_{0,\partial}(\Sigma)$ on $B^\delta_p(\Sigma,\phi)$.

Claim. For all σ_p, the group $\mathrm{Stab}^\delta(\sigma_p)$ is an acyclic group.

Proof of the Claim: Let $\Sigma\backslash\sigma_p$ denote a surface obtained from Σ by cutting along the arcs in σ_p. This surface is homeomorphic to the disjoint union of $\Sigma\backslash\phi$ with p disjoint disks. Hence using the induction hypothesis and the Mather theorem for disks, the group $\mathrm{Homeo}^{\delta}_{0,\partial}(\Sigma\backslash\sigma_p)$ is acyclic. Recall from diagram (3.24) that we have a fibration

$$\mathrm{BHomeo}^{\delta}_{0,\partial}(\Sigma\backslash\sigma_p) \to \mathrm{BStab}^{\delta}(\sigma_p) \to \mathrm{B}\pi_0(\mathrm{Stab}(\sigma_p)).$$

Therefore, if we show that $\pi_0(\mathrm{Stab}(\sigma_p))$ is trivial, we can conclude that $\mathrm{Stab}^{\delta}(\sigma_p)$ is an acyclic group. Note that $\pi_0(\mathrm{Stab}(\sigma_p))$ is the kernel of the map

$$\pi_0(\mathrm{Homeo}_{0,\partial}(\Sigma\backslash\sigma_p)) \to \pi_0(\mathrm{Homeo}_{0,\partial}(\Sigma)).$$

But this kernel is trivial (see [PR00, Corollary 4.2] for an elementary proof); i.e., if $f \in \mathrm{Homeo}_{0,\partial}(\Sigma)$ fixes σ, then it is isotopic to the identity relative to the arcs in Σ. Hence $H_q(\mathrm{BStab}^{\delta}(\sigma_p)) = 0$ unless $q = 0$, in which case it is isomorphic to $\mathbb{Z}$. □

On the other hand, two p-simplices $\sigma = (\phi_0, \phi_1, \dots, \phi_p)$ and $\sigma' = (\phi'_0, \phi'_1, \dots, \phi'_p)$ are on the same orbit if and only if the corresponding t-coordinates $t_{\phi_i} = t_{\phi'_i}$ are the same for all i. Therefore, the set of orbits of the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(\Sigma)$ on $B_p(\Sigma, \phi)$ is the same as $\mathrm{Conf}_{p+1}(\mathbb{R})$, the set of configurations of $p+1$ points on the real line.

Hence, E^1-page is concentrated in the first line when $q = 0$ and $E^1_{0,p} = \mathbb{Z}[\mathrm{Conf}_{p+1}(\mathbb{R})]$. But the chain complex $(\mathbb{Z}[\mathrm{Conf}_{p+1}(\mathbb{R})], d_1)$ calculates the homology of $|\mathrm{Conf}_{\bullet+1}(\mathbb{R})|$, which is an infinite simplex on $\mathbb{R}$. Therefore, the spectral sequence converges to zero in positive degrees, which implies that $\mathrm{Homeo}^{\delta}_{0,\partial}(\Sigma)$ is an acyclic group.

Case 2. Suppose Σ is a closed surface. Reducing the case of closed surfaces to the case of surfaces with boundary could be done using the long exact sequence for the fibration $\mathrm{Homeo}(\Sigma) \to \mathrm{Emb}(D^2, \Sigma)$ and Birman's exact sequence. But here, we give an argument using the 0-handle resolution which might be useful for closed hyperbolic 3-manifolds as we shall discuss in Section 5.3.

In this case we use the semisimplicial set $A^{\delta}_{\bullet}(\Sigma)$ (see Definition 3.4) and the resolution

$$X^{\delta}_{\bullet}(\Sigma) \to \mathrm{BHomeo}^{\delta}_0(\Sigma).$$

Let $e_p \in A^{\mathbf{t}}_p(M)$ be an embedding of $p+1$ disjoint disks and let $[e_p] \in A^{\delta}_p(M)$ denote its germ. In Lemma 3.7, we showed that $X^{\delta}_p(\Sigma)$ has the same homotopy type as $\mathrm{BStab}^{\delta}([e_p])$. But given Lemma 3.20, the spectral sequence for the semisimplicial resolution $X^{\delta}_{\bullet}(\Sigma)$ can be written as

$$E^1_{p,q} = H_q(\mathrm{BStab}^{\delta}(e_p)) \Rightarrow H_{p+q}(\mathrm{BHomeo}^{\delta}_0(\Sigma)). \tag{5.2}$$

Again we have a fibration

$$\mathrm{BHomeo}^{\delta}_{0,\partial}(\Sigma\backslash e_p) \to \mathrm{BStab}^{\delta}(e_p) \to \mathrm{B}\pi_0(\mathrm{Stab}(e_p)),$$

where the group $\mathrm{Homeo}^{\delta}_{0,\partial}(\Sigma\backslash e_p)$ is acyclic by Case 1. Therefore, we have

$$\mathrm{BStab}^{\delta}(e_p) \to \mathrm{B}\pi_0(\mathrm{Stab}(e_p)) \tag{5.3}$$

is a homology isomorphism.

On the other hand, the group $\pi_0(\mathrm{Stab}(e_p))$ is the kernel of

$$\pi_0(\mathrm{Homeo}_{\partial}(\Sigma\backslash e_p)) \to \pi_0(\mathrm{Homeo}_{\partial}(\Sigma)).$$

To determine $\pi_0(\mathrm{Stab}(e_p))$, consider the Kan fibration given by the parametrized isotopy extension and Remark 2.7:

$$S_\bullet(\mathrm{Homeo}_\partial(\Sigma\backslash e_p)) \to S_\bullet(\mathrm{Homeo}_\partial(\Sigma)) \to S_\bullet(A_p^{\mathbf{t}}(\Sigma)).$$

The long exact sequence of homotopy groups for this fibration implies that we have a short exact sequence

$$\pi_1(A_p^{\mathbf{t}}(\Sigma)) \to \pi_0(\mathrm{Homeo}_\partial(\Sigma\backslash e_p)) \to \pi_0(\mathrm{Homeo}(\Sigma)). \tag{5.4}$$

Claim. $\pi_1(A_p^{\mathbf{t}}(\Sigma)) \cong \pi_0(\mathrm{Stab}(e_p))$

Proof of the Claim: Given the exact sequence (5.4), we only need to show that the map $\pi_1(A_p^{\mathbf{t}}(\Sigma)) \to \pi_0(\mathrm{Homeo}_\partial(\Sigma\backslash e_p))$ is injective. The group $\pi_1(A_p^{\mathbf{t}}(\Sigma))$ is known as the framed pure surface braid group of Σ. The pure surface braid group (not framed) is the fundamental group of the space of ordered configurations of points $\mathrm{Conf}_{p+1}(\Sigma)$. These groups sit in a short exact sequence

$$1 \to \mathbb{Z}^{p+1} \to \pi_1(A_p^{\mathbf{t}}(\Sigma)) \to \pi_1(\mathrm{Conf}_{p+1}(\Sigma)) \to 1,$$

where $\mathbb{Z}^{p+1}$ is the group generated by the Dehn twists around the boundary components in $\Sigma\backslash e_p$. These twists and all their nonzero powers induce nontrivial inner automorphisms of the fundamental group $\Sigma\backslash e_p$ which is a free group. Therefore, $\mathbb{Z}^{p+1}$ as a subgroup of $\pi_1(A_p^{\mathbf{t}}(\Sigma))$ injects into $\pi_0(\mathrm{Homeo}_\partial(\Sigma\backslash e_p))$, and in fact we have a short exact sequence

$$1 \to \mathbb{Z}^{p+1} \to \pi_0(\mathrm{Homeo}_\partial(\Sigma\backslash e_p)) \to \pi_0(\mathrm{Homeo}(\Sigma, c(e_p))) \to 1,$$

where $\pi_0(\mathrm{Homeo}(\Sigma, c(e_p)))$ is the mapping class group of Σ with marked points, $c(e_p)$, the centers of the disks in e_p. Therefore, to show that

$$\pi_1(A_p^{\mathbf{t}}(\Sigma)) \to \pi_0(\mathrm{Homeo}_\partial(\Sigma\backslash e_p))$$

is injective, it is enough to show the natural map

$$\pi_1(\mathrm{Conf}_{p+1}(\Sigma)) \to \pi_0(\mathrm{Homeo}(\Sigma, c(e_p))) \tag{5.5}$$

is injective. This map is induced by the long exact sequence of homotopy groups for the fibration (see also [McC63, Section 4]) given by an evaluation map

$$\mathrm{Homeo}_0(\Sigma) \to \mathrm{Conf}_{p+1}(\Sigma).$$

It is known that $\pi_1(\mathrm{Conf}_{p+1}(\Sigma))$ does not have a center for hyperbolic surfaces (see [PR99, Proposition 1.6]). Therefore, the composite of the maps

$$\pi_1(\mathrm{Conf}_{p+1}(\Sigma)) \to \pi_0(\mathrm{Homeo}_0(\Sigma, c(e_p))) \to \mathrm{Aut}(\pi_1(\mathrm{Conf}_{p+1}(\Sigma))),$$

which is given by inner automorphisms, is injective. Therefore, the map in (5.5) is injective. $\square$

Since a hyperbolic surface and its frame bundle are aspherical, one can use the fibration $A_p^{\mathbf{t}}(\Sigma) \to A_{p-1}^{\mathbf{t}}(\Sigma)$ inductively to conclude that $A_p^{\mathbf{t}}(\Sigma)$ is also aspherical. Therefore, we have

$$A_p^{\mathbf{t}}(\Sigma) \xrightarrow{\simeq} \mathrm{B}\pi_1(A_p^{\mathbf{t}}(\Sigma)) \xrightarrow{\simeq} \mathrm{B}\pi_0(\mathrm{Stab}(e_p)).$$

Given that the map (5.3) is a homology isomorphism, $A_p^{\mathbf{t}}(\Sigma)$ and $\mathrm{BStab}^\delta(e_p)$ have the same homology groups. We have a zig-zag of semisimplicial maps

$$X_\bullet^\delta(\Sigma) \leftarrow A_\bullet^{\mathbf{t},\delta}(\Sigma) /\!\!/ \mathrm{Homeo}_0^\delta(\Sigma) \to A_\bullet^{\mathbf{t}}(\Sigma) /\!\!/ \mathrm{Homeo}_0(\Sigma) \leftarrow A_\bullet^{\mathbf{t}}(\Sigma),$$

where the first two maps induce isomorphisms on E^1-pages (see Section 3.1.3). Hence, we obtain a comparison map of the spectral sequence (5.2) with

$$E^1_{p,q} = H_q(A^{\mathbf{t}}_p(\Sigma)) \Rightarrow H_{p+q}(|A^{\mathbf{t}}_\bullet(\Sigma)|),$$

which is an isomorphism on the E^1-page. Since $|A^{\mathbf{t}}_\bullet(\Sigma)|$ is weakly contractible (see Proposition 3.9), the acyclity of the space $\mathrm{BHomeo}^{\delta}_0(\Sigma)$ follows. □

5.2. **The case of Haken 3-manifolds with boundary.** Hatcher computed the homotopy type of the group of homemorphisms of Haken manifolds in [Hat76]. Given his proof of Smale's conjecture, his computation of homeomorphisms carries over to diffeomorphisms of the Haken manifolds ([Hat99]). Hatcher developed a subtle disjunction technique to study embedding spaces ([Hat83, Hat76, Hat81]). In the case of Haken 3-manifolds, he improved Laudenbach's surgery techniques ([Lau74, Chapter 2.5]) to do a parametrized surgery on the space of incompressible surfaces. Here, we also use Laudenbach's results, but instead of Hatcher's disjunction argument, we prove that $\mathrm{BHomeo}^{\delta}_{0,\partial}(M)$ is acyclic for Haken 3-manifolds with boundary similar to the case of surfaces.

Theorem 5.6. *Let M be a Haken 3-manifold with nonempty boundary. The space $\mathrm{BHomeo}^{\delta}_{0,\partial}(M)$ is acyclic.*

Proof. We induct on the Haken hierarchy. We know that there are a finite number of incompressible surfaces and that if we cut along those surfaces, we obtain a disjoint union of disks. Let $\psi : S \hookrightarrow M$ be an incompressible surface in M. Since $K^{\delta}_\bullet(M,\psi)$ has a contractible realization (see Proposition 4.21), similar to the case of surfaces with boundary, we obtain a spectral sequence

$$E^1_{p,q} = H_q(\coprod_{\text{orbits}} \mathrm{BStab}^{\delta}(\sigma_p)) \Rightarrow H_{p+q}(\mathrm{BHomeo}^{\delta}_{0,\partial}(M)),$$

where σ_p varies over a representative of the set of orbits of the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ on $K^{\delta}_p(M,\psi)$. First note that the orbit of σ_p is completely determined by the t-coordinates of the surfaces in σ_p. Therefore, the set of orbits of the action of $\mathrm{Homeo}^{\delta}_{0,\partial}(M)$ on $K^{\delta}_p(M,\psi)$ is identified with $\mathrm{Conf}_{p+1}(\mathbb{R})$.

Claim. $\mathrm{Stab}^{\delta}(\sigma_p) \cong \mathrm{Homeo}^{\delta}_{0,\partial}(M\backslash\sigma_p)$.

This is a consequence of the Laudenbach result ([Lau74, pp. 48–62]). In other words, we need to show that $\pi_0(\mathrm{Stab}(\sigma_p))$ is trivial. Laudenbach showed that if $f \in \mathrm{Homeo}_{0,\partial}(M)$ fixes an incompressible surface S, then f is isotopic to the identity relative to S, which implies that $\pi_0(\mathrm{Stab}(\sigma_p))$ is trivial for all p.

On the other hand, note that $M\backslash\sigma_p$ is homeomorphic to the disjoint union of $M\backslash\psi$ with p disjoint handlebodies homeomorphic to $S \times [0,1]$. By the induction hypothesis, we know that $\mathrm{Homeo}^{\delta}_{0,\partial}(M\backslash\psi)$ is an acyclic group. For the handlebodies we can induct on the genus and do exactly the same argument to cut along incompressible disks to deduce that $\mathrm{Homeo}^{\delta}_{0,\partial}(S \times [0,1])$ is also acyclic. Hence, $\mathrm{Stab}^{\delta}(\sigma_p)$ is an acyclic group. So the E^1-page is concentrated in the first line when $q = 0$ and is isomorphic to the chain complex $(\mathbb{Z}[\mathrm{Conf}_{\bullet+1}(\mathbb{R})], d_1)$ which calculates the homology of an infinite simplex. Therefore, the spectral sequence converges to zero in positive degrees. □

5.3. **Further discussion.** We end with a question about hyperbolic 3-manifolds. Let M be a closed hyperbolic 3-manifold. Gabai in [Gab01] used his high-powered "insulator" machinary (see [Gab97]) and minimal surface theory to prove that $\mathrm{Homeo}_0(M)$ is contractible by reducing to the case of Haken manifolds with boundary. It would be interesting if the techniques of this paper could prove Gabai's theorem without using high-powered tools in geometry. It is desirable to have an argument similar to the case of closed surfaces (Case 2 in the proof of Theorem 5.1). In that case, we used a semisimplicial resolution to reduce to the case of surfaces with boundary. Surfaces with boundary behave like Haken 3-manifolds. It would be interesting to define a semisimplicial resolution by cutting certain submanifolds, like solid tori, to reduce to the case of Haken 3-manifolds with boundary. One candidate for a semisimplicial resolution for $\mathrm{BHomeo}_0(M)$ could be as follows. Let γ be a closed geodesic in M. Fix a parametrized tubular neighborhood of γ by embedding $\phi : D^2 \times S^1 \hookrightarrow M$ so that $\phi(\{(0,0)\} \times S^1) = \gamma$.

Definition 5.7. Let $C_\bullet(M)$ be a semisimplicial set whose set of 0 simplices is given by oriented closed curves that are isotopic to γ. We define $C_p(M)$ as a subset of $C_0(M)^{p+1}$ to be the set of $(p+1)$-tuples $\sigma_p = (\gamma_0, \gamma_1, \dots, \gamma_p)$ so that there exists a homeomorphism $f_{\sigma_p} \in \mathrm{Homeo}_0(M)$ where $f_{\sigma_p}(\gamma_i) = \phi(\{(t_i, 0)\} \times S^1)$ for a t_i such that $t_0 < t_1 < \cdots < t_p$. The i-th face maps are given by forgetting the i-th curve.

Question 5.8. Is $|C_\bullet(M)|$ weakly contractible?

We need to show that realization of the semisimplicial set $C_\bullet(M)$ is contractible. If the answer to this question is affirmative, one could give a simpler proof of Gabai's theorem as follows: Consider the semisimplicial resolution

$$C_\bullet(M) /\!\!/ \mathrm{Homeo}_0^\delta(M) \to \mathrm{BHomeo}_0^\delta(M).$$

Since the action of $\mathrm{Homeo}_0^\delta(M)$ on $C_\bullet(M)$ is transitive, for a p-simplex σ_p in $C_p(M)$, we have $C_p(M) /\!\!/ \mathrm{Homeo}_0^\delta(M) \simeq \mathrm{BStab}(\sigma_p)$. Given that the complement of σ_p in M is a Haken manifold, the identity component of $\mathrm{Stab}(\sigma_p)$ is contractible; therefore $\mathrm{BStab}(\sigma_p) \simeq \mathrm{B}\pi_0(\mathrm{Stab}(\sigma_p))$. On the other hand, using JSJ decomposition and some hyperbolic geometry, it is not hard to show that $\pi_0(\mathrm{Stab}(\sigma_p))$ is isomorphic to the pure braid group PBr_{p+1}. Hence, one might have a spectral sequence

$$E^1_{p,q} = H_q(\mathrm{BPBr}_{p+1}) \Rightarrow H_{p+q}(\mathrm{BHomeo}_0^\delta(M); \mathbb{Z}),$$

but recall that a model for BPBr_{p+1} is an ordered configuration space $\mathrm{Emb}([p], D^2)$. Thus the above spectral sequence converges to the realization of the semisimplicial space $\mathrm{Emb}([\bullet], D^2)$. Now from Proposition 3.16 we know that the realization of $\mathrm{Emb}([\bullet], D^2)$ is weakly contractible; therefore the above spectral sequence converges to zero in positive degrees.

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Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, Illinois 60208; and University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark

Email address: sam@math.northwestern.edu, sam@math.ku.dk