LINEAR STABILITY OF HIGHER DIMENSIONAL SCHWARZSCHILD SPACETIMES: DECAY OF MASTER QUANTITIES

PEI-KEN HUNG, JORDAN KELLER, AND MU-TAO WANG

ABSTRACT. In this paper, we study solutions to the linearized vacuum Einstein equations centered at higher-dimensional Schwarzschild metrics. We employ Hodge decomposition to split solutions into scalar, co-vector, and two-tensor pieces; the first two portions respectively correspond to the closed and co-closed, or polar and axial, solutions in the case of four spacetime dimensions, while the two-tensor portion is a new feature in the higher-dimensional setting. Rephrasing earlier work of Kodama-Ishibashi-Seto in the language of our Hodge decomposition, we produce decoupled gauge-invariant master quantities satisfying Regge-Wheeler type wave equations in each of the three portions. The scalar and co-vector quantities respectively generalize the Moncrief-Zerilli and Regge-Wheeler quantities found in the setting of four spacetime dimensions; beyond these quantities, we discover a higher-dimensional analog of the Cunningham-Moncrief-Price quantity in the co-vector portion. In addition, our work provides the first verification that the scalar master quantity satisfies its putative Regge-Wheeler equation. In the analysis of the master quantities, we strengthen the mode stability result of Kodama-Ishibashi to a uniform boundedness estimate in all dimensions; further, we prove decay estimates in the case of six or fewer spacetime dimensions. In the case of more than six spacetime dimensions, we discover an obstruction to Morawetz type estimates arising from negative potential terms growing quadratically in spacetime dimension. Finally, we provide a rigorous argument that linearized solutions of low angular frequency are decomposable as a sum of pure gauge solution and linearized Myers-Perry solution, the latter solutions generalizing the linearized Kerr solutions in four spacetime dimensions.

1. Introduction

The Schwarzschild-Tangherlini black holes are higher-dimensional generalizations of the Schwarzschild spacetimes, comprising a static, spherically

Portions of this work were carried out during visits to the Department of Mathematics and The Institute of Mathematical Sciences at the Chinese University of Hong Kong. The authors wish to thank these institutions for their hospitality and support. The second author wishes to thank the John Templeton Foundation for its support. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1810856 (Mu-Tao Wang).

symmetric family of black hole solutions to higher-dimensional vacuum gravity:

$$Ric(g) = 0. \tag{1}$$

Non-linear stability of the Schwarzschild-Tangherlini black holes as solutions of (1) is a matter of considerable mathematical interest, owing to the developments in geometric analysis necessary in the problem's resolution. Such work would add to non-linear stability results in four spacetime dimensions, in particular that of Christodoulou-Klainerman [7] for the Minkowski spacetime, in addition to the more recent non-linear stability results of Hintz-Vasy [14] for the slowly rotating Kerr-de Sitter spacetimes and Klainerman-Szeftel [20] for the Schwarzschild spacetime subject to polarized axisymmetric perturbations.

In this paper, we consider the simpler matter of linear stability of the Schwarzschild-Tangherlini solutions, concerning solutions δg of the linearization of the vacuum Einstein equations about a member of the Schwarzschild-Tangherlini family $(\mathcal{M}, g_M)$, with mass $M > 0$:

$$\delta Ric_{g_M}(\delta g) = 0. \tag{2}$$

Owing to diffeomorphism-invariance of the Einstein equation, infinitesimal deformations of the background spacetime via smooth co-vector fields X

$$\pi_X := \mathcal{L}_X g_M, \tag{3}$$

referred to as pure gauge solutions, are solutions to the linearized equation (2). Moreover, the Schwarzschild-Tangherlini family is contained within the larger family of Myers-Perry solutions, yielding solutions to (2) corresponding to infinitesimal changes in mass and angular velocity. To demonstrate linear stability it suffices to show that, with a choice of well-posed pure gauge solution π_X, the normalized solution

$$\widehat{\delta g} = \delta g - \pi_X$$

decays through a suitable foliation to a Myers-Perry perturbation under appropriate initial conditions.

In both the physics and mathematics literature, the identification and analysis of gauge-invariant quantities satisfying decoupled wave equations forms the basis of linear stability. Building upon our earlier work [16] in four spacetime dimensions, we utilize the spherical symmetry of the background Schwarzschild-Tangherlini spacetimes to split linearized solutions into scalar, co-vector, and two-tensor portions in a spacetime Hodge decomposition. Identification of gauge-invariant master quantities satisfying decoupled Regge-Wheeler type wave equations for each of the three portions appears in the work of Kodama-Ishibashi-Seto [22] and Kodama-Ishibashi [21], with the scalar and co-vector quantities respectively generalizing the Moncrief-Zerilli [26, 32] and Regge-Wheeler [29] quantities in four spacetime dimensions. Beyond recasting the quantities of Kodama-Ishibashi-Seto, we identify a higher-dimensional analog of the Cunningham-Moncrief-Price quantity [8] in the co-vector portion. In addition, we verify that the scalar quantity is indeed a solution of its putative Regge-Wheeler equation, such an argument being absent in [21]. Our work is the first of numerous recent results on linear stability [16, 9, 24, 2, 18, 15] to consider higher-dimensional gravity. We remark that a generalization of the four dimensional approach of Dafermos-Holzegel-Rodnianski [9], involving decoupled Weyl curvature components satisfying the Bardeen-Press equation [3], could provide another avenue towards higher-dimensional linear stability.

The analysis of the Regge-Wheeler type equations (129, 144, 148, 169) satisfied by the master quantities is informed by the study of the scalar wave equation, regarding as a “poor man’s” linearization of the vacuum Einstein equations. We draw upon the pioneering efforts and later refinements of many authors in four spacetime dimensions for the Schwarzschild and Kerr spacetimes [19, 4, 10, 11, 23, 25, 13, 31, 1, 12], in addition to the higher-dimensional generalization of Schlue [30], utilizing the red-shift, Morawetz, and r^p estimates described in these works. The equations we consider differ from the scalar case owing to their tensorial nature, the associated solutions being symmetric traceless two-tensors on the unit n-sphere, and owing to the presence of potentials. Roughly speaking, we benefit from the tensorial nature of the equations, as we are able to borrow from angular derivatives to control base terms upon integrating over orbit spheres, with the beneficial terms scaling linearly in the spacetime dimension. On the other hand, all but one of the potentials we consider contains negative terms scaling quadratically with spacetime dimension. Taken together, the equations we consider are more challenging to analyze than the standard wave equation. We overcome the difficulties presented by the potentials with respect to uniform boundedness, using Hardy estimates in the spirit of Kodama-Ishibashi [17] to strengthen their mode stability result to a uniform boundedness estimate in all dimensions. Turning to uniform decay, we find an obstruction in proving Morawetz estimates as spacetime dimension grows large. See the end of subsection 9.3.2 for details. For the lower spacetime dimensions, six or fewer, we succeed in establishing uniform decay estimates.

Spherical symmetry of the background Schwarzschild-Tangherlini spacetime allows for an additional decomposition of the metric perturbation into tensor spherical harmonics. In particular, we decompose a linearized solution into portions of lower and higher angular frequency:

$$\delta g = \delta g^{\ell<2} + \delta g^{\ell\geq 2}, \tag{4}$$

per Proposition 6. The master quantities discussed above are central to controlling the higher angular frequency portion $\delta g^{\ell\geq 2}$, but have no control over the lower angular frequency portion $\delta g^{\ell<2}$. Generalizing the situation in four spacetime dimensions, we prove that $\delta g^{\ell<2}$ splits as the sum of a pure gauge solution and a linearized Myers-Perry solution. We provide the first rigorous treatment of these lower angular modes in higher dimensions, formalizing the same claim in Kodama-Ishibashi [21], which the authors base upon an enumeration of degrees of freedom.

We summarize our results in the two main theorems below. First, we have the analysis of the lower angular frequency portion $\delta g^{\ell<2}$:

Theorem 1. *Let δg be a smooth, symmetric two-tensor on a Schwarzschild-Tangherlini spacetime, satisfying the linearized Einstein equation* (2)*. For the lower frequency portion $\delta g^{\ell<2}$ of δg, there exists a smooth co-vector $X^{\ell<2}$ on the Schwarzschild-Tangherlini background, unique modulo Killing fields, and constants c, d_m such that*

$$\delta g^{\ell<2} = \pi_{X^{\ell<2}} + cK + \sum_{m=1}^{\frac{1}{2}n(n+1)} d_m K_m, \tag{5}$$

where K, K_m are the basis solutions of the linearized Myers-Perry family in Definition 8.

Next, we have the analysis of the gauge-invariant master quantities for the higher angular frequency portion $\delta g^{\ell\geq 2}$:

Theorem 2. *Making the same assumption on δg as in Theorem 1, there exist gauge-invariant master quantities satisfying decoupled Regge-Wheeler type equations (129, 144, 148, 169) for the scalar, co-vector, and two-tensor portions of $\delta g^{\ell\geq 2}$. With further specification of an initial data slice Σ_0 which extends to null infinity and a decay foliation $\Sigma_\tau := \phi_\tau(\Sigma_0)$ formed by flowing along the static Killing vector field, a symmetric traceless two-tensor Ψ solving any one of the equations (129, 144, 148, 169) satisfies the uniform boundedness estimate*

$$\check{E}_\Psi(\Sigma_\tau) \leq C(n, M)\check{E}_\Psi(\Sigma_0), \tag{6}$$

and, in spacetime dimension six and below, the uniform decay estimate

$$\check{E}_\Psi(\Sigma_\tau) \leq C(n, M)\frac{I_\Psi(\Sigma_0)}{\tau^2}, \tag{7}$$

with $\check{E}_\Psi$ and $I_\Psi(\Sigma_0)$ representing Sobolev data for Ψ on members of the decay foliation Σ_τ and $C(n, M)$ being a universal constant depending upon the orbit sphere dimension n and the background mass M.

We remark that further pointwise uniform boundedness and uniform decay estimates can be derived from those above by means of commutation with the angular Killing fields and application of Sobolev estimates on the orbit spheres.

It is expected that, after estimating the master quantities and making a suitable choice of linearized gauge, the remaining linearized metric components can be controlled by the gauge-invariant master quantities to ensure their uniform boundedness and decay, yielding a complete proof of linear stability. We do not treat this matter in the current paper, deferring it to later work. We remark that such efforts have borne fruit in the case of four spacetime dimensions. In particular, our earlier work [16] controls the linearized metric via a rather irregular combination of the Regge-Wheeler and Chandrasekhar gauges, while Johnson [18] controls the linearized metric uniformly in the Regge-Wheeler gauge, after an intermediate passage through the wave-coordinate gauge. Inasmuch as these results depend upon corresponding gauge choices, each approach has drawbacks in extending to the non-linear regime. In this direction, there is also work of the first author [15], wherein control of the odd portion of the linearized metric in the wave-map gauge is accomplished.

The paper is organized as follows. In Section 2, we present the Schwarzschild-Tangherlini black holes. In Section 3, we discuss the more general class of spherically symmetric spacetimes; in particular, we present Hodge decomposition and tensor spherical harmonic decomposition. In Section 4 we discuss gravitational perturbations of spherically symmetric spacetimes, identifying pure gauge solutions and linearized Myers-Perry solutions in Section 5. We prove Theorem 1, decomposing $\delta g^{\ell<2}$ as a sum of a pure gauge and linearized Myers-Perry solution, in Section 6. In Section 7, we discuss general estimates for Regge-Wheeler type equations sufficient to prove uniform boundedness and uniform decay. In Section 8, we identify and analyze the master quantity for the two-tensor portion, proving uniform boundedness and decay in all spacetime dimensions. Similar analyses for the co-vector and scalar portions are carried out in Sections 9 and 10, respectively; in each case, we prove uniform boundedness estimates in all spacetime dimensions and uniform decay estimates in spacetime dimension six and fewer. We summarize our results on the master quantities of $\delta g^{\ell\geq 2}$ in Section 11, wherein we prove Theorem 2.

2. Higher Dimensional Schwarzschild Spacetimes

The higher dimensional Schwarzschild-Tangherlini black holes $(\mathcal{M}^{2+n}, g_M)$ generalize the well-known four-dimensional spacetimes, with the $(2+n)$-dimensional family comprised of static, spherically-symmetric (i.e., $SO(n+1)$-invariant) members, parametrized by mass $M>0$. Each such member is a solution of vacuum gravity; i.e., each metric g_M satisfies $Ric(g_M)=0$.

In standard Schwarzschild coordinates (t, r, x^α), x^α coordinates on S^n, the Schwarzschild metric takes the form

$$g_M = -(1-\mu)dt^2 + (1-\mu)^{-1}dr^2 + r^2 \mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta, \tag{8}$$

with

$$\mu := \frac{2M}{r^{n-1}} \tag{9}$$

and

$$\mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta \tag{10}$$

understood to be the standard round metric of the unit n-sphere.

Defining the Regge-Wheeler coordinate via

$$r_* = \int^r \left(1 - \frac{2M}{s^{n-1}}\right)^{-1} ds, \tag{11}$$

we find

$$g_M = -(1-\mu)dt^2 + (1-\mu)dr_*^2 + r^2\mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta. \tag{12}$$

Using the Regge-Wheeler coordinates, we specify the Eddington-Finkelstein double-null coordinates by

$$\begin{aligned} u &= \frac{1}{2}(t - r_*), \\ v &= \frac{1}{2}(t + r_*), \end{aligned} \tag{13}$$

such that

$$g_M = -4(1-\mu)dudv + r^2\mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta. \tag{14}$$

We remark that r_* is defined up to normalization. All three of the coordinate systems cover the exterior region of the spacetime and degenerate at the event horizon.

We also make use of the ingoing Eddington-Finkelstein coordinate system

$$\begin{aligned} \bar{v} &= t + r_*, \\ R &= r \end{aligned} \tag{15}$$

with

$$g_M = -(1-\mu)d\bar{v}^2 + 2d\bar{v}dR + R^2\mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta. \tag{16}$$

Finally, a variant of the Regge-Wheeler coordinates takes

$$t_* = t - r + r_*, \tag{17}$$

such that

$$g_M = -(1-\mu)dt_*^2 + 2\mu dt_* dr + (1+\mu)dr^2 + r^2\mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta. \tag{18}$$

These two coordinate systems remain regular up to and on the future event horizon.

The event horizon appears in this coordinate system as the null hypersurface

$$r = r_h := (2M)^{1/(n-1)}. \tag{19}$$

Along the event horizon, the Schwarzschild-Tangherlini solution has positive surface gravity

$$\kappa_n := \frac{(n-1)}{2r_h}, \tag{20}$$

in addition to simple trapping at the timelike hypersurface

$$r = r_P := ((n+1)M)^{1/(n-1)}. \tag{21}$$

This hypersurface is referred to as the photon sphere. With it, we normalize the Regge-Wheeler coordinate by

$$r_*(r_P) = 0. \tag{22}$$

For a detailed discussion of these and other issues related to the geometry of higher dimensional Schwarzschild spacetimes, we refer the reader to Schlue [30].

3. Spherically Symmetric Spacetimes

3.1. **General Considerations.** Let $(\mathcal{Q}, \tilde{g})$ be a two-dimensional Lorentzian manifold with local coordinates $x^A, A = 0, 1$, and let $(S^n, \mathring{\sigma})$ be the unit n-sphere with the standard round metric in local coordinates $x^\alpha, \alpha = 2, \dots, n+1$. Each point on $\mathcal{Q}$ represents an orbit sphere, with r a positive function which represents the areal radius of each orbit sphere. We consider a general spherically symmetric spacetime in local coordinates $x^0, x^1, x^2, \dots x^{n+1}$:

$$g_{ab}dx^a dx^b = \tilde{g}_{AB}dx^A dx^B + r^2 \mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta. \tag{23}$$

The index notations above are adopted throughout the paper: $A, B, C, \dots = 0, 1$ for quotient indices, $\alpha, \beta, \gamma, \dots = 2, \dots, n+1$ for spherical indices, and $a, b, c, \dots = 0, 1, 2, \dots, n+1$ for spacetime indices.

The Christoffel symbols Γ^c_{ab} of a spherically symmetric spacetime are

$$\begin{aligned}
\Gamma^C_{AB} &= \tilde{\Gamma}^C_{AB},\\
\Gamma^\gamma_{\alpha\beta} &= \mathring{\Gamma}^\gamma_{\alpha\beta},\\
\Gamma^\beta_{\alpha A} &= r^{-1}\partial_A r(\delta^\beta_\alpha),\\
\Gamma^D_{\alpha\beta} &= -r\partial^D r(\mathring{\sigma}_{\alpha\beta}),
\end{aligned}$$

where Γ^C_{AB} and $\mathring{\Gamma}^\gamma_{\alpha\beta}$ are the Christoffel symbols of $\tilde{g}_{AB}$ and $\mathring{\sigma}_{\alpha\beta}$, respectively.

Using the Christoffel symbols, it is possible to calculate the curvature of the quotient $\mathcal{Q}$ and the n-sphere S^n directly. On the other hand, as $\mathcal{Q}$ is a two-manifold, we have immediately

$$\begin{aligned}
\tilde{R}_{ABCD} &= \tilde{K}\left(\tilde{g}_{AC}\tilde{g}_{BD} - \tilde{g}_{AD}\tilde{g}_{BC}\right),\\
\tilde{R}_{AB} &= \tilde{K}\tilde{g}_{AB},\\
\tilde{R} &= 2\tilde{K},
\end{aligned} \tag{24}$$

relating the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature of the quotient to its sectional curvature $\tilde{K}$. Likewise, as the n-sphere is a space form with constant sectional curvature $\mathring{K} = 1$, we find

$$\begin{aligned}
\mathring{R}_{\alpha\beta\gamma\eta} &= \left(\mathring{\sigma}_{\alpha\gamma}\mathring{\sigma}_{\beta\eta} - \mathring{\sigma}_{\alpha\eta}\mathring{\sigma}_{\beta\gamma}\right),\\
\mathring{R}_{\alpha\beta} &= (n-1)\mathring{\sigma}_{\alpha\beta},\\
\mathring{R} &= n(n-1).
\end{aligned} \tag{25}$$

With respect to the $2+n$ decomposition into quotient and spherical parts, we consider two types of differential operators, $\tilde{\nabla}_A$ and $\mathring{\nabla}_\alpha$. When applied to functions, $\tilde{\nabla}_A$ and $\mathring{\nabla}_\alpha$ are just differentiation with respect to coordinate variables $x^A, A = 0, 1$ and $x^\alpha, \alpha = 2, \ldots, n+1$, respectively. For co-vectors, we define

$$\begin{aligned}
\tilde{\nabla}_A dx^B &= -\tilde{\Gamma}^B_{AC} dx^C, \\
\tilde{\nabla}_A dx^\alpha &= 0, \\
\mathring{\nabla}_\alpha dx^B &= 0, \\
\mathring{\nabla}_\alpha dx^\beta &= -\mathring{\Gamma}^\beta_{\alpha\gamma} dx^\gamma,
\end{aligned} \tag{26}$$

with an obvious extension of the operators to more elaborate tensor bundles.

We use the notation $\tilde{\Box}$ and $\mathring{\Delta}$ for the quotient d'Alembertian and the spherical Laplacian operators. Furthermore, we denote the volume form for the quotient space by ϵ_{AB}.

Later in the work, we will make use of the commutation identities

$$\begin{aligned}
\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c &= R_{abc}{}^d v_d, \\
\nabla_a \nabla_b v_{cd} - \nabla_b \nabla_a v_{cd} &= R_{abc}{}^e v_{ed} + R_{abd}{}^e v_{ce},
\end{aligned} \tag{27}$$

which apply to either the quotient or the orbit spheres.

Specializing to the Schwarzschild spacetime, we note the formulae

$$\begin{aligned}
&\tilde{\nabla}_A \tilde{\nabla}_B r = \frac{M(n-1)}{r^n} \tilde{g}_{AB} = \frac{(n-1)}{2r} \mu \tilde{g}_{AB}, \\
&\tilde{\nabla}_A \tilde{\nabla}_B t = -(1-\mu)^{-1} \left(\tilde{\Box} r\right) t_{(A} r_{B)}, \\
&r^A r_A = |\tilde{\nabla} r|^2 = 1 - \frac{2M}{r^{n-1}} = 1 - \mu, \\
&\tilde{K} = \frac{n(n-1)M}{r^{n+1}} = \frac{n(n-1)}{2r^2} \mu.
\end{aligned} \tag{28}$$

3.2. **Tensors on S^n.** Specializing to the n-sphere, with the curvature calculations and commutation relations above taken into account, we find

$$\begin{aligned}
\mathring{\nabla}_\alpha \mathring{\nabla}_\beta v_\gamma - \mathring{\nabla}_\beta \mathring{\nabla}_\alpha v_\gamma &= \mathring{\sigma}_{\alpha\gamma} v_\beta - \mathring{\sigma}_{\beta\gamma} v_\alpha, \\
\mathring{\nabla}_\alpha \mathring{\nabla}_\beta v_{\gamma\delta} - \mathring{\nabla}_\beta \mathring{\nabla}_\alpha v_{\gamma\delta} &= \mathring{\sigma}_{\alpha\gamma} v_{\beta\delta} - \mathring{\sigma}_{\beta\gamma} v_{\alpha\delta} + \mathring{\sigma}_{\alpha\delta} v_{\gamma\beta} - \mathring{\sigma}_{\beta\delta} v_{\gamma\alpha}.
\end{aligned} \tag{29}$$

3.2.1. *Tensor Spherical Harmonics.* In this subsection we outline tensor spherical harmonics on S^n, following closely the discussion in Chodos-Myers [6].

The scalar spherical harmonics $Y^{\ell m_s(n,\ell)}$ form an $L^2(S^2)$ basis of eigenfunctions of the spherical Laplacian, satisfying

$$\mathring{\Delta} Y^{\ell m_s(n,\ell)} = -\ell(\ell+n-1) Y^{\ell m_s(n,\ell)}, \tag{30}$$

with indices $\ell \geq 0$ and $m_s(n,\ell) \in \{1, \ldots, d_s(n,\ell)\}$, where

$$d_s(n,\ell) = \binom{n+\ell}{\ell} - \binom{n+\ell-2}{\ell-2}. \tag{31}$$

Note that the formula gives

$$\begin{aligned} d_s(n,0) &= 1, \\ d_s(n,1) &= n+1 \\ d_s(n,2) &= \frac{1}{2}(n+2)(n+1)-1. \end{aligned} \tag{32}$$

Using the scalar spherical harmonics, we obtain $L^2(S^2)$ bases of eigensections for the sub-bundles of co-vectors and symmetric traceless two-tensors given by scalar potentials. Namely, we have eigensections

$$Y_\alpha^{\ell m_s(n,\ell)} := \mathring{\nabla}_\alpha Y^{\ell m_s(n,\ell)}, \tag{33}$$

satisfying

$$\mathring{\Delta} Y_\alpha^{\ell m_s(n,\ell)} = \left((n-1)-\ell(\ell+n-1)\right) Y_\alpha^{\ell m_s(n,\ell)}, \tag{34}$$

and eigensections

$$Y_{\alpha\beta}^{\ell m_s(n,\ell)} := \mathring{\nabla}_\alpha \mathring{\nabla}_\beta Y^{\ell m_s(n,\ell)} - \frac{1}{n}\mathring{\sigma}_{\alpha\beta}\mathring{\Delta} Y^{\ell m_s(n,\ell)}, \tag{35}$$

such that

$$\mathring{\Delta} Y_{\alpha\beta}^{\ell m_s(n,\ell)} = \left(2n-\ell(\ell+n-1)\right) Y_{\alpha\beta}^{\ell m_s(n,\ell)}, \tag{36}$$

with $\ell \geq 2$. For a detailed derivation of these spectra, see Chodos-Myers [6].

In addition, the spherical Laplacian acts as an endomorphism on the sub-bundles of divergence-free co-vectors and divergence-free symmetric traceless two-tensors. Regarding such co-vectors, we have an $L^2(S^2)$ basis of eigensections $X_\alpha^{\ell m_v(n,\ell)}$ satisfying

$$\mathring{\Delta} X_\alpha^{\ell m_v(n,\ell)} = \left(1-\ell(\ell+n-1)\right) X_\alpha^{\ell m_v(n,\ell)}, \tag{37}$$

for $\ell \geq 1$ and $m_v(n,\ell) \in \{1,\dots,d_v(n,\ell)\}$, where

$$\begin{aligned} d_v(n,\ell) &= (n+1)d_s(n,\ell) - d_s(n,\ell+1) - d_s(n,\ell-1) \\ &= (n+1)\left(\binom{n+\ell}{\ell} - \binom{n+\ell-2}{\ell-2}\right) \\ &\quad - \binom{n+\ell+1}{\ell+1} + \binom{n+\ell-3}{\ell-3}. \end{aligned} \tag{38}$$

Note that the formula above together with (32) gives

$$d_v(n,1) = \frac{1}{2}n(n+1). \tag{39}$$

Again, we refer the reader to Chodos-Myers [6] for further discussion of this spectrum.

For those symmetric traceless two-tensors given by divergence-free co-vector potentials, we have an $L^2(S^2)$ basis of eigensections

$$X_{\alpha\beta}^{\ell m_v(n,\ell)} := \mathring{\nabla}_\alpha X_\beta^{\ell m_v(n,\ell)} + \mathring{\nabla}_\beta X_\alpha^{\ell m_v(n,\ell)}, \tag{40}$$

with

$$\mathring{\Delta} X_{\alpha\beta}^{\ell m_v(n,\ell)} = \left(n+2-\ell(\ell+n-1)\right) X_{\alpha\beta}^{\ell m_v(n,\ell)} \tag{41}$$

for $\ell \geq 2$, per Chodos-Myers [6].

On the sub-bundle of divergence-free symmetric traceless two-tensors, we find

$$\mathring{\Delta} U_{\alpha\beta}^{\ell m_t(n,\ell)} = (2 - \ell(\ell + n - 1))\, U_{\alpha\beta}^{\ell m_t(n,\ell)}, \tag{42}$$

for eigensections $U_{\alpha\beta}^{\ell m_t(n,\ell)}$ forming an $L^2(S^2)$ basis, with $\ell \geq 2$ and $m_t(n,\ell) \in \{1, \ldots, d_t(n,\ell)\}$, where

$$\begin{aligned} d_t(n,\ell) = {} & \frac{1}{2}(n+1)(n+2) d_s(n,\ell) - (d_v(n,\ell+1) + d_v(n,\ell-1)) \\ & - d_s(n,\ell+2) - 2d_s(n,\ell) - d_s(n,\ell-2) \\ = {} & \frac{1}{2}(n+1)(n+2)\left(\binom{n+\ell}{\ell} - \binom{n+\ell-2}{\ell-2}\right) \\ & - (n+1)\left(\binom{n+\ell+1}{\ell+1} - \binom{n+\ell-3}{\ell-3}\right), \end{aligned} \tag{43}$$

per Chodos-Myers [6].

We remark that the $\ell = 1$ scalar co-vectors correspond to conformal Killing vectors,

$$\mathring{\nabla}_\alpha Y_\beta^{1m_s(n,1)} + \mathring{\nabla}_\beta Y_\alpha^{1m_s(n,1)} = -2Y^{1m_s(n,1)} \mathring{\sigma}_{\alpha\beta}, \tag{44}$$

and the $\ell = 1$ divergence-free co-vectors correspond to Killing vectors

$$\mathring{\nabla}_\alpha X_\beta^{1m_v(n,1)} + \mathring{\nabla}_\beta X_\alpha^{1m_v(n,1)} = 0. \tag{45}$$

Concretely, the conformal Killing vector fields can be realized by considering a Cartesian coordinate system $(X^1, \ldots, X^{n+1})$ on $\mathbb{R}^{n+1}$. Denoting the restriction of the coordinate functions to the unit sphere S^n by $\tilde{X}^A, A = 1 \cdots n+1$, the set of Killing vector fields $\{X_\alpha^{1m_v(n,1)}\}_{m_v(n,1)=1,\ldots,d_v(n,1)}$ is given by

$$\tilde{X}^A \mathring{\nabla}_\alpha \tilde{X}^B - \tilde{X}^B \mathring{\nabla}_\alpha \tilde{X}^A, \tag{46}$$

where $1 \leq A < B \leq n+1$, and the set of conformal Killing vector fields $\{Y_\alpha^{1m_s(n,1)}\}_{m_s(n,1)=1,\ldots,d_s(n,1)}$ is given by

$$\mathring{\nabla}_\alpha \tilde{X}^A, \tag{47}$$

with $1 \leq A \leq n+1$.

3.2.2. *Tensor Decomposition.* The following decomposition lemmae, generalizing the situation for the two-sphere, are fundamental to the remainder of the work.

Lemma 3. *Given any co-vector v_α on S^n, there exist a scalar function V and a divergence-free co-vector $\hat{v}_\alpha$ (i.e. $\mathring{\nabla}^\alpha \hat{v}_\alpha = 0$) such that*

$$v_\alpha = \mathring{\nabla}_\alpha V + \hat{v}_\alpha.$$

The decomposition is unique modulo the $\ell = 0$ mode of V.

Proof. Let V be a solution of Poisson's equation

$$\mathring{\Delta} V = \mathring{\nabla}^\alpha v_\alpha. \tag{48}$$

The difference $\hat{v}_\alpha := v_\alpha - \mathring{\nabla}_\alpha V$ is manifestly divergence-free, and we have

$$v_\alpha = \mathring{\nabla}_\alpha V + \hat{v}_\alpha.$$

Assuming that v_α has such a decomposition, it must be that the scalar function satisfies Poisson's equation above, so that V is determined up to its constant $\ell = 0$ mode and $\hat{v}_\alpha$ is uniquely determined. □

Lemma 4. *Given any symmetric traceless $(0,2)$-tensor $t_{\alpha\beta}$ on S^n, there exists a co-vector v_α and a divergence-free symmetric traceless $(0,2)$-tensor $\hat{t}_{\alpha\beta}$ such that*

$$t_{\alpha\beta} = \mathring{\nabla}_\alpha v_\beta + \mathring{\nabla}_\beta v_\alpha - \frac{2}{n}\mathring{\sigma}_{\alpha\beta}\mathring{\nabla}^\gamma v_\gamma + \hat{t}_{\alpha\beta}.$$

Combined with the previous lemma, we have

$$t_{\alpha\beta} = \left(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta V - \frac{1}{n}\mathring{\Delta} V\mathring{\sigma}_{\alpha\beta}\right) + \left(\mathring{\nabla}_\alpha\hat{v}_\beta + \mathring{\nabla}_\beta\hat{v}_\alpha\right) + \hat{t}_{\alpha\beta}. \tag{49}$$

The decomposition is unique modulo the $\ell < 2$ modes of V and the $\ell = 1$ mode of $\hat{v}_\alpha$.

Proof. The divergence of $t_{\alpha\beta}$ has co-vector decomposition

$$\mathring{\nabla}^\alpha t_{\alpha\beta} =: d_\beta = \mathring{\nabla}_\beta D + \hat{d}_\beta,$$

per the previous lemma. Note that d_β is supported in $\ell \geq 2$, as

$$\begin{aligned}
\int_{S^n}\mathring{\nabla}^\alpha t_\alpha^\beta Y_\beta^{1m} &= -\int_{S^n} t_\alpha^\beta\mathring{\nabla}^\alpha Y_\beta^{1m} = 2\int_{S^n} t_{\alpha\beta}\mathring{\sigma}^{\alpha\beta} = 0\\
\int_{S^n}\mathring{\nabla}^\alpha t_{\alpha\beta} X_\beta^{1m} &= -\int_{S^n} t_\alpha^\beta\mathring{\nabla}^\alpha X_\beta^{1m} = -\int_{S^n} t^{\alpha\beta}\mathring{\nabla}_{(\alpha}X_{\beta)}^{1m} = 0,
\end{aligned}$$

using the conformal Killing and Killing equations (44, 45) and the symmetry and tracelessness of $t_{\alpha\beta}$.

We solve for V and $\hat{v}_\alpha$ such that

$$\begin{aligned}
&\frac{n-1}{n}\mathring{\Delta}\left(\mathring{\Delta} + n\right) V = \mathring{\Delta} D = \mathring{\nabla}^\alpha\mathring{\nabla}^\beta t_{\alpha\beta},\\
&\left(\mathring{\Delta} + (n-1)\right)\hat{v}_\beta = \hat{d}_\beta.
\end{aligned}$$

Each of the elliptic operators in the left-hand side is self-adjoint, with kernels being the $\ell < 2$ scalar modes and the $\ell = 1$ co-vector modes, respectively. Owing to support of d_β and its constituents in $\ell \geq 2$, we have orthogonality of the right-hand side and the kernel, from which existence of solutions to the equations follows.

The quantity

$$\hat{t}_{\alpha\beta} := t_{\alpha\beta} - \left(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta V - \frac{1}{n}\mathring{\Delta} V\mathring{\sigma}_{\alpha\beta}\right) - \left(\mathring{\nabla}_\alpha\hat{v}_\beta + \mathring{\nabla}_\beta\hat{v}_\alpha\right),$$

satisfies the divergence-free property owing to the choices of V and $\hat{v}_\alpha$, and we have

$$t_{\alpha\beta} = \left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta V - \frac{1}{n}\mathring{\Delta} V \mathring{\sigma}_{\alpha\beta}\right) + \left(\mathring{\nabla}_\alpha \hat{v}_\beta + \mathring{\nabla}_\beta \hat{v}_\alpha\right) + \hat{t}_{\alpha\beta}$$

by definition.

Given such a decomposition, the constituents V and $\hat{v}_\alpha$ necessarily satisfy the equations above; hence they are uniquely determined up to the $\ell < 2$ modes of V and the $\ell = 1$ mode of $\hat{v}_\alpha$. □

In the first lemma, the divergence-free co-vector $\hat{v}_\alpha$ generalizes the co-closed potentials in our earlier work [16].

As outlined in the second lemma, the situation for symmetric traceless two-tensors is more interesting. Subtracting off the piece involving the scalar potential V, the remainder of the tensor satisfies the double-divergence free property, analogous to the co-closed potential in our earlier work. However, this remainder admits a further decomposition, amounting to the term $\hat{t}_{\alpha\beta}$ with the stronger divergence-free property. The term $\hat{t}_{\alpha\beta}$ is a novel feature in this higher dimensional setting.

3.3. **The Projected Covariant Derivative.** In what follows, we consider quantities which are scalars, co-vectors, or symmetric traceless two-tensors on the spheres of symmetry. The associated sphere bundles, respectively referred to as $\mathcal{L}(0), \mathcal{L}(-1)$, and $\mathcal{L}(-2)$, come equipped with projected covariant derivative operators $\nabla\!\!\!/$, defined for scalars by ordinary differentiation and for co-vectors by

$$\nabla\!\!\!/_a dx^\alpha = -\Gamma^\alpha_{a\gamma} dx^\gamma, \text{ for } a = 0, 1, 2, \cdots, n+1,$$

extending to symmetric traceless two-tensors via the product rule. We denote the associated d'Alembertian operators by

$$\Box\!\!\!/_{\mathcal{L}(-s)} := \nabla\!\!\!/^a \nabla\!\!\!/_a, \tag{50}$$

with $s = 0, 1, 2$ and the appropriate covariant derivative operator. Note that $\Box\!\!\!/_{\mathcal{L}(0)} = \Box$ is the standard d'Alembertian operator on $\mathcal{M}$.

The projected connection, as well as the associated d'Alembertian and Laplacian operators, are related to the quotient and spherical operators of the first subsection in a straightforward fashion. We illustrate the procedure on the bundle $\mathcal{L}(-2)$:

$$\begin{aligned}\nabla\!\!\!/_A t_{\alpha\beta} &= \partial_A t_{\alpha\beta} - \Gamma^\gamma_{A\alpha} t_{\gamma\beta} - \Gamma^\gamma_{A\beta} t_{\alpha\gamma} \\ &= \tilde{\nabla}_A t_{\alpha\beta} - 2r^{-1} r_A t_{\alpha\beta},\end{aligned}$$

$$\begin{aligned}\not\nabla_B \not\nabla_A t_{\alpha\beta} &= \partial_B \left(\not\nabla_A t_{\alpha\beta}\right) - \Gamma^C_{BA} \not\nabla_C t_{\alpha\beta} \\ &- \Gamma^\gamma_{BA} \not\nabla_\gamma t_{\alpha\beta} - \Gamma^\gamma_{B\alpha} \not\nabla_A t_{\gamma\beta} - \Gamma^\gamma_{B\beta} \not\nabla_A t_{\alpha\gamma} \\ &= \tilde{\nabla}_B \left(\not\nabla_A t_{\alpha\beta}\right) - 2r^{-1} r_B \left(\not\nabla_A t_{\alpha\beta}\right) \\ &= \tilde{\nabla}_B \tilde{\nabla}_A t_{\alpha\beta} - 2r^{-1} r_A \tilde{\nabla}_B t_{\alpha\beta} - 2r^{-1} r_B \tilde{\nabla}_A t_{\alpha\beta} \\ &+ 6r^{-2} r_A r_B t_{\alpha\beta} - 2r^{-1} \left(\tilde{\nabla}_A \tilde{\nabla}_B r\right) t_{\alpha\beta},\end{aligned}$$

$$\not\nabla_\gamma t_{\alpha\beta} = \mathring{\nabla}_\gamma t_{\alpha\beta},$$

$$\begin{aligned}\not\nabla_\lambda \not\nabla_\gamma t_{\alpha\beta} &= \partial_\lambda \left(\not\nabla_\gamma t_{\alpha\beta}\right) - \Gamma^\delta_{\lambda\gamma} \not\nabla_\delta t_{\alpha\beta} \\ &- \Gamma^\delta_{\lambda\alpha} \not\nabla_\gamma t_{\delta\beta} - \Gamma^\delta_{\lambda\beta} \not\nabla_\gamma t_{\alpha\delta} - \Gamma^A_{\lambda\gamma} \not\nabla_A t_{\alpha\beta} \\ &= \mathring{\nabla}_\lambda \mathring{\nabla}_\gamma t_{\alpha\beta} + r r^A \mathring{\sigma}_{\lambda\gamma} \left(\not\nabla_A t_{\alpha\beta}\right) \\ &= \mathring{\nabla}_\lambda \mathring{\nabla}_\gamma t_{\alpha\beta} + r r^A \mathring{\sigma}_{\lambda\gamma} \left(\tilde{\nabla}_A t_{\alpha\beta} - 2r^{-1} r_A t_{\alpha\beta}\right).\end{aligned}$$

Contracting the above, we deduce the relation

$$\begin{aligned}\not\Box_{\mathcal{L}(-2)} t_{\alpha\beta} &= \tilde{\Box} t_{\alpha\beta} + (n-4) r^{-1} r^A \tilde{\nabla}_A t_{\alpha\beta} + r^{-2} \mathring{\Delta} t_{\alpha\beta} \\ &+ (6-2n) r^{-2} r^A r_A t_{\alpha\beta} - 2r^{-1} \left(\tilde{\Box} r\right) t_{\alpha\beta}.\end{aligned} \tag{51}$$

Likewise, we calculate

$$\begin{aligned}\not\Box_{\mathcal{L}(-1)} v_\alpha &= \tilde{\Box} v_\alpha + (n-2) r^{-1} r^A \tilde{\nabla}_A v_\alpha + r^{-2} \mathring{\Delta} v_\alpha \\ &+ (2-n) r^{-2} r^A r_A v_\alpha - r^{-1} \left(\tilde{\Box} r\right) v_\alpha,\end{aligned} \tag{52}$$

$$\not\Box_{\mathcal{L}(0)} V = \Box V = \tilde{\Box} V + n r^{-1} r^A \tilde{\nabla}_A V + r^{-2} \mathring{\Delta} V. \tag{53}$$

4. Gravitational Perturbations

4.1. **Decomposition of the Linearized Metric.** As a $(0,2)$-tensor on the background, a linear perturbation $h_{ab} = \delta g_{ab}$ admits the $(2{+}n)$-decomposition

$$\delta g = h_{AB} dx^A dx^B + 2h_{A\alpha} dx^A dx^\alpha + h_{\alpha\beta} dx^\alpha dx^\beta, \tag{54}$$

with each component of $h_{AB}, h_{A\alpha}, h_{\alpha\beta}$ depending upon all spacetime variables.

Applying the above lemmae, we have:

Proposition 5. *Any symmetric two-tensor δg of the form* (54) *can be decomposed as $\delta g = h_1 + h_2 + h_3$, with*

$$\begin{aligned}h_1 &= h_{AB} dx^A dx^B + 2(\mathring{\nabla}_\alpha H_A) dx^\alpha dx^A + H \mathring{\sigma}_{\alpha\beta} dx^\alpha dx^\beta \\ &+ \left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta H_2 - \frac{1}{n} \mathring{\sigma}_{\alpha\beta} \mathring{\Delta} H_2\right) dx^\alpha dx^\beta,\end{aligned} \tag{55}$$

$$h_2 = 2\hat{h}_{A\alpha} dx^\alpha dx^A + (\mathring{\nabla}_\alpha \hat{h}_\beta + \mathring{\nabla}_\beta \hat{h}_\alpha) dx^\alpha dx^\beta, \tag{56}$$

$$h_3 = \hat{h}_{\alpha\beta} dx^\alpha dx^\beta, \tag{57}$$

where the following equations are satisfied:

$$\mathring{\nabla}^\alpha \hat{h}_{A\alpha} = 0, \mathring{\nabla}^\alpha \hat{h}_\alpha = 0, \mathring{\nabla}^\alpha \hat{h}_{\alpha\beta} = 0.$$

When $n = 2$, both $\hat{h}_{A\alpha}$ and $\hat{h}_\alpha$ have potentials ($\underline{H}_A$ and $\underline{H}_2$) and $\hat{h}_{\alpha\beta}$ must vanish, recovering the Hodge type decomposition in our earlier work [16].

To summarize, in general dimension, we have a splitting of δg into three pieces, scalar, co-vector, and two-tensor, as objects on the orbit spheres.

It is also possible to subdivide the linearized metric with respect to the spherical harmonic decomposition outlined in the previous section. Namely, we split the linearized metric as

$$\delta g = \delta g^{\ell<2} + \delta g^{\ell\geq 2}, \tag{58}$$

according to the following proposition.

Proposition 6. *Any symmetric two-tensor δg on a spherically symmetric spacetime can be split into $\delta g = \delta g^{\ell<2} + \delta g^{\ell\geq 2}$, in which the components of $\delta g^{\ell\geq 2}$, further decomposed according to Proposition 5, satisfy*

$$\int_{S^2} h_{AB} Y^{\ell m_s(n,\ell)} = 0,$$
$$\int_{S^2} H Y^{\ell m_s(n,\ell)} = 0,$$

with respect to the scalar harmonics $Y^{\ell m_s(n,\ell)}$ having $\ell < 2$, and

$$\int_{S^2} H_A Y^{1 m_s(n,1)} = 0,$$
$$\int_{S^2} \hat{h}_{A\alpha} X_\beta^{1 m_v(n,1)} \mathring{\sigma}^{\alpha\beta} = 0,$$

with respect to the $\ell = 1$ scalar harmonics $Y^{1m_s(n,1)}$ and the $\ell = 1$ co-vector harmonics $X_\alpha^{1m_v(n,1)}$.

We remark the components H_2, $\hat{h}_\alpha$, and $\hat{h}_{\alpha\beta}$ are necessarily supported in $\ell \geq 2$.

4.2. **Decomposition of the Linearized Ricci Tensor.** We recall that a perturbation of the Ricci curvature δR_{bd} satisfies

$$2\delta R_{bd} = g^{ae}(\nabla_a \nabla_d h_{eb} + \nabla_a \nabla_b h_{ed} - \nabla_d \nabla_b h_{ea} - \nabla_a \nabla_e h_{bd}). \tag{59}$$

Perturbing about a spherically symmetric spacetime, with radial function r, we record the calculations in Appendix B of Kodama-Ishibashi-Seto [22]. We remark that verification of these expressions amounts to relating the spacetime covariant derivative in (59) with those on the quotient space and unit n-sphere; in this direction, we refer the reader to the connection and curvature calculations presented in Subsection 3.1.

$$
\begin{aligned}
2\delta R_{AB} = & -\tilde{\Box} h_{AB} - \tilde{\nabla}_A \tilde{\nabla}_B \left(g^{CD} h_{CD}\right) + \tilde{\nabla}_A \tilde{\nabla}^C h_{CB} + \tilde{\nabla}_B \tilde{\nabla}^C h_{CA} \\
& + \tilde{R}_A^C h_{CB} + \tilde{R}_B^C h_{CA} - 2\tilde{R}_{ACBD} h^{CD} - r^{-2} \mathring{\Delta} h_{AB} \\
& + n r^{-1} r^C \left(\tilde{\nabla}_B h_{CA} + \tilde{\nabla}_A h_{CB} - \tilde{\nabla}_C h_{AB}\right) \\
& + r^{-2} \left(\tilde{\nabla}_A \mathring{\nabla}^\alpha h_{B\alpha} + \tilde{\nabla}_B \mathring{\nabla}^\alpha h_{A\alpha}\right) \\
& - n r^{-3} r_B \tilde{\nabla}_A H - n r^{-3} r_A \tilde{\nabla}_B H \\
& + 4 n r^{-4} r_A r_B H - n \tilde{\nabla}_A \tilde{\nabla}_B \left(r^{-2} H\right),
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
2\delta R_{A\alpha} = & \mathring{\nabla}_\alpha \tilde{\nabla}^B h_{AB} + (n-2) r^{-1} r^B \mathring{\nabla}_\alpha h_{AB} - r \mathring{\nabla}_\alpha \tilde{\nabla}_A \left(r^{-1} g^{BC} h_{BC}\right) \\
& - r \tilde{\Box} \left(r^{-1} h_{A\alpha}\right) - n r^{-1} r^B \tilde{\nabla}_B h_{A\alpha} - r^{-2} \mathring{\Delta} h_{A\alpha} \\
& + \left[(n+1) r^{-2} r^B r_B + (n-1) r^{-2} \left(1 - r^B r_B\right) - r^{-1} \left(\tilde{\Box} r\right)\right] h_{A\alpha} \\
& - r_A \tilde{\nabla}^B \left(r^{-1} h_{B\alpha}\right) + (n+1) r^{-1} r^B \tilde{\nabla}_A h_{B\alpha} + r^{-2} r_A r^B h_{B\alpha} \\
& + r \tilde{\nabla}_A \tilde{\nabla}^B \left(r^{-1} h_{B\alpha}\right) + \tilde{R}_A^B h_{B\alpha} \\
& + (n+1) r \tilde{\nabla}_A \left(r^{-2} r^B\right) h_{B\alpha} - (n+2) r^{-1} \left(\tilde{\nabla}_A \tilde{\nabla}^B r\right) h_{B\alpha} \\
& + r^{-2} \mathring{\nabla}_\alpha \mathring{\nabla}^\beta h_{A\beta} + r \tilde{\nabla}_A \left(r^{-3} \mathring{\nabla}^\beta h_{\alpha\beta}\right) + r^{-3} r_A \mathring{\nabla}^\beta h_{\alpha\beta} \\
& - n r^{-3} r_A \mathring{\nabla}_\alpha H - n r \mathring{\nabla}_\alpha \tilde{\nabla}_A \left(r^{-3} H\right),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
2\delta R_{\alpha\beta} = & \left[2 r r^A \tilde{\nabla}^B h_{AB} + 2(n-1) r^A r^B h_{AB} + 2r \left(\tilde{\nabla}^A \tilde{\nabla}^B r\right) h_{AB}\right] \mathring{\sigma}_{\alpha\beta} \\
& - \mathring{\nabla}_\alpha \mathring{\nabla}_\beta \left(h_{AB} g^{AB}\right) - r r^A \tilde{\nabla}_A \left(h_{BC} g^{BD}\right) \mathring{\sigma}_{\alpha\beta} \\
& + r \tilde{\nabla}^A \left(r^{-1} \left(\mathring{\nabla}_\alpha h_{A\beta} + \mathring{\nabla}_\beta h_{A\alpha}\right)\right) \\
& + (n-1) r^{-1} r^A \left(\mathring{\nabla}_\alpha h_{A\beta} + \mathring{\nabla}_\beta h_{A\alpha}\right) + 2 r^{-1} r^A \mathring{\nabla}^\gamma h_{A\gamma} \mathring{\sigma}_{\alpha\beta} \\
& - r^2 \tilde{\Box} (r^{-2} h_{\alpha\beta}) - n r^{-1} r^A \tilde{\nabla}_A h_{\alpha\beta} - r^{-2} \mathring{\Delta} h_{\alpha\beta} \\
& + r^{-2} \left(\mathring{\nabla}_\alpha \mathring{\nabla}^\gamma h_{\gamma\beta} + \mathring{\nabla}_\beta \mathring{\nabla}^\gamma h_{\gamma\alpha}\right) \\
& + 2 \left[(n-1) r^{-2} + 2 r^{-2} r^A r_A - r^{-1} \left(\tilde{\Box} r\right)\right] h_{\alpha\beta} \\
& - 2 r^{-2} (1 - r^A r_A) \left(n H \mathring{\sigma}_{\alpha\beta} - h_{\alpha\beta}\right) - 2 n r^{-2} r^A r_A H \mathring{\sigma}_{\alpha\beta} \\
& - n r^{-2} \mathring{\nabla}_\alpha \mathring{\nabla}_\beta H - n r r^A \tilde{\nabla}_A \left(r^{-2} H\right) \mathring{\sigma}_{\alpha\beta}.
\end{aligned}
\tag{62}
$$

In the remainder of the subsection, we rewrite the above expressions, with the dual aims of expanding in the scalar, co-vector, and two-tensor metric perturbation components of Proposition 5 and of writing the linearized Ricci tensor in such a form, with scalar, co-vector, and two-tensor portions. As we shall see, the scalar, co-vector, and two-tensor portions of the linearized Ricci tensor are determined entirely by their respective metric portions. This correspondence first appeared in the work of Regge-Wheeler [29] in four spacetime dimensions and was later generalized to higher dimensions by Kodama-Ishibashi-Seto [22].

4.2.1. *Pure Quotient Term.* The pure quotient portion (60) needs little modification. Expanding, we find

$$
\begin{aligned}
2\delta R_{AB} = & -\tilde{\Box} h_{AB} - \tilde{\nabla}_A \tilde{\nabla}_B \left(g^{CD} h_{CD}\right) + \tilde{\nabla}_A \tilde{\nabla}^C h_{CB} + \tilde{\nabla}_B \tilde{\nabla}^C h_{CA} \\
& + \tilde{R}_A^C h_{CB} + \tilde{R}_B^C h_{CA} - 2\tilde{R}_{ACBD} h^{CD} - r^{-2} \mathring{\Delta} h_{AB} \\
& + n r^{-1} r^C \left(\tilde{\nabla}_B h_{CA} + \tilde{\nabla}_A h_{CB} - \tilde{\nabla}_C h_{AB}\right) \\
& + r^{-2} \left(\tilde{\nabla}_A \mathring{\Delta} H_B + \tilde{\nabla}_B \mathring{\Delta} H_A\right) \\
& - n r^{-3} r_B \tilde{\nabla}_A H - n r^{-3} r_A \tilde{\nabla}_B H \\
& + 4 n r^{-4} r_A r_B H - n \tilde{\nabla}_A \tilde{\nabla}_B \left(r^{-2} H\right),
\end{aligned}
\tag{63}
$$

involving only the scalar piece of δg.

4.2.2. *Cross Term.* Writing the cross-term of the linearized Ricci tensor (61) in terms of a scalar potential and a divergence-free co-vector, we find

$$
\begin{aligned}
2\delta R_{A\alpha} = & \mathring{\nabla}_\alpha \Big[\tilde{\nabla}^B h_{AB} + (n-2) r^{-1} r^B h_{AB} + r^{-1} r_A \left(g^{BC} h_{BC}\right) - \tilde{\nabla}_A \left(g^{BC} h_{BC}\right) \\
& - \tilde{\Box} H_A + (2-n) r^{-1} r^B \tilde{\nabla}_B H_A + \tilde{\nabla}^B \tilde{\nabla}_A H_B - 2 r^{-1} \left(\tilde{\nabla}_A \tilde{\nabla}^B r\right) H_B \\
& + 2(1-n) r^{-2} r_A r^B H_B - 2 r^{-1} r_A \tilde{\nabla}^B H_B + n r^{-1} r^B \tilde{\nabla}_A H_B \\
& + (1-n) \tilde{\nabla}_A \left(r^{-2} H\right) + (n-1) \tilde{\nabla}_A \left(r^{-2} \left(H_2 + \frac{1}{n} \mathring{\Delta} H_2\right)\right) \Big] \\
& + \Big[- \tilde{\Box} \hat{h}_{A\alpha} - r^{-2} \mathring{\Delta} \hat{h}_{A\alpha} + (2-n) r^{-1} r^B \tilde{\nabla}_B \hat{h}_{A\alpha} \\
& + (n-1) r^{-2} \hat{h}_{A\alpha} + \tilde{\nabla}^B \tilde{\nabla}_A \hat{h}_{B\alpha} - 2 r^{-1} \left(\tilde{\nabla}_A \tilde{\nabla}^B r\right) \hat{h}_{B\alpha} \\
& + 2(1-n) r^{-2} r_A r^B \hat{h}_{B\alpha} - 2 r^{-1} r_A \tilde{\nabla}^B \hat{h}_{B\alpha} + n r^{-1} r^B \tilde{\nabla}_A \hat{h}_{B\alpha} \\
& + (n-1) \tilde{\nabla}_A \left(r^{-2} \hat{h}_\alpha\right) + \tilde{\nabla}_A \left(r^{-2} \mathring{\Delta} \hat{h}_\alpha\right) \Big],
\end{aligned}
\tag{64}
$$

where we have likewise expanded the metric perturbation in terms of the decomposition of Proposition 5. Note that the scalar potential of $\delta R_{A\alpha}$ involves only the scalar piece of δg, with the co-vector pieces being similarly related.

We remark that, although the reduction seems quite complex, much of the work in obtaining (64) from (61) is straightforward. Namely, the only terms in (61) which are difficult to rewrite in terms of a scalar potential and a divergence-free co-vector constitute

$$-r^{-2}\mathring{\Delta}h_{A\alpha}+r^{-2}\mathring{\nabla}_\alpha\mathring{\nabla}^\beta h_{A\beta}+r\tilde{\nabla}_A\left(r^{-3}\mathring{\nabla}^\beta h_{\alpha\beta}\right)+r^{-3}r_A\mathring{\nabla}^\beta h_{\alpha\beta}.$$

To deal with such terms, we use the angular commutation relations (29); in particular, we rely upon the identity

$$\begin{aligned}\mathring{\Delta}h_{A\alpha}&=\mathring{\nabla}_\alpha\mathring{\Delta}H_A+(n-1)\mathring{\nabla}_\alpha H_A+\mathring{\Delta}\hat{h}_{A\alpha},\\ \mathring{\nabla}^\gamma h_{\alpha\gamma}&=\mathring{\nabla}_\alpha H+\frac{n-1}{n}\mathring{\nabla}_\alpha\left(\mathring{\Delta}+n\right)H_2+\left(\mathring{\Delta}+(n-1)\right)\hat{h}_\alpha\end{aligned}$$

4.2.3. *Pure Angular Term.* The decomposition of the angular term (62) is lengthiest of all. We begin by calculating the trace

$$\begin{aligned}2\delta R_{\alpha\beta}\mathring{\sigma}^{\alpha\beta}&=n\Big(2rr^A\tilde{\nabla}^Bh_{AB}+2(n-1)r^Ar^Bh_{AB}+2r\left(\tilde{\nabla}^A\tilde{\nabla}^Br\right)h_{AB}\\&-rr^A\tilde{\nabla}_A\left(h_{BC}g^{BC}\right)\Big)-\mathring{\Delta}(h_{AB}g^{AB})\\&+4(n-1)r^{-1}r^A\mathring{\Delta}H_A+2\tilde{\nabla}^A\mathring{\Delta}H_A\\&-nr^2\tilde{\Box}\left(r^{-2}H\right)+2(1-n)r^{-2}\mathring{\Delta}H-2n^2r^{-1}r^A\tilde{\nabla}_AH\\&-2nr^{-1}\left(\tilde{\Box}r\right)H+2n(n+1)r^{-2}r^Ar_AH\\&+2(n-1)r^{-2}\left(\mathring{\Delta}H_2+\frac{1}{n}\mathring{\Delta}\mathring{\Delta}H_2\right).\end{aligned}\tag{65}$$

In calculating the traceless portion, we introduce the notation

$$\begin{aligned}\check{h}_{\alpha\beta}&:=h_{\alpha\beta}-H\mathring{\sigma}_{\alpha\beta},\\ 2\check{\delta}R_{\alpha\beta}&:=2\delta R_{\alpha\beta}-\frac{1}{n}\left(2\delta R_{\gamma\delta}\mathring{\sigma}^{\gamma\delta}\right)\mathring{\sigma}_{\alpha\beta}.\end{aligned}\tag{66}$$

Concretely,

$$\check{h}_{\alpha\beta}=\left(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta H_2-\frac{1}{n}\mathring{\Delta}H_2\mathring{\sigma}_{\alpha\beta}\right)+\left(\mathring{\nabla}_\alpha\hat{h}_\beta+\mathring{\nabla}_\beta\hat{h}_\alpha\right)+\hat{h}_{\alpha\beta}.\tag{67}$$

A preliminary calculation yields

$$\begin{aligned}
2\check{\delta} R_{\alpha\beta} = &-\mathring{\nabla}_\alpha \mathring{\nabla}_\beta \left(h_{AB} g^{AB}\right) + \frac{1}{n}\mathring{\Delta}(h_{AB}g^{AB})\mathring{\sigma}_{\alpha\beta} \\
&+ 2\tilde{\nabla}^A \left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta H_A - \frac{1}{n}\mathring{\Delta} H_A \mathring{\sigma}_{\alpha\beta}\right) \\
&+ 2(n-2)r^{-1}r^A \left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta H_A - \frac{1}{n}\mathring{\Delta} H_A \mathring{\sigma}_{\alpha\beta}\right) \\
&+ \tilde{\nabla}^A \left(\mathring{\nabla}_\alpha \hat{h}_{A\beta} + \mathring{\nabla}_\beta \hat{h}_{A\alpha}\right) \\
&+ (n-2)r^{-1}r^A \left(\mathring{\nabla}_\alpha \hat{h}_{A\beta} + \mathring{\nabla}_\beta \hat{h}_{A\alpha}\right) \\
&- r^2 \tilde{\Box}\left(r^{-2}\check{h}_{\alpha\beta}\right) - nr^{-1}r^A\tilde{\nabla}_A \check{h}_{\alpha\beta} - r^{-2}\mathring{\Delta}\check{h}_{\alpha\beta} \\
&+ 2\left[(n-1)r^{-2} + 2r^{-2}r^A r_A - r^{-1}\left(\tilde{\Box} r\right)\right]\check{h}_{\alpha\beta} \\
&+ 2r^{-2}(1 - r^A r_A)\check{h}_{\alpha\beta} - nr^{-2}\left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta H - \frac{1}{n}\mathring{\Delta} H \mathring{\sigma}_{\alpha\beta}\right) \\
&+ r^{-2}\left[\left(\mathring{\nabla}_\alpha \mathring{\nabla}^\gamma h_{\gamma\beta} + \mathring{\nabla}_\beta \mathring{\nabla}^\gamma h_{\gamma\alpha}\right) - \frac{2}{n}\left(\mathring{\nabla}^\gamma \mathring{\nabla}^\delta h_{\gamma\delta}\right)\mathring{\sigma}_{\alpha\beta}\right].
\end{aligned} \tag{68}$$

It remains to expand the traceless part of the metric perturbation, and to collect the terms of the linearized Ricci quantity with respect to such a decomposition. We emphasize that this amounts to expressing the symmetric traceless two-tensor $\check{\delta R}_{\alpha\beta}$ as the sum of the traceless Hessian of a scalar function, the symmetrized gradient of a divergence-free co-vector, and a divergence-free two-tensor. As with the cross-term, there are few troublesome terms, not admitting a straightforward expansion and collection. Such terms constitute

$$r^{-2}\left[\left(\mathring{\nabla}_\alpha \mathring{\nabla}^\gamma h_{\gamma\beta} + \mathring{\nabla}_\beta \mathring{\nabla}^\gamma h_{\gamma\alpha}\right) - \frac{2}{n}\left(\mathring{\nabla}^\gamma \mathring{\nabla}^\delta h_{\gamma\delta}\right)\mathring{\sigma}_{\alpha\beta}\right] - r^{-2}\mathring{\Delta}\check{h}_{\alpha\beta}.$$

Again, these troublesome terms can be expanded and collected by careful application of the angular commutation relations (29). In particular, it is useful to note

$$\begin{aligned}
&\mathring{\nabla}^\gamma \check{h}_{\alpha\gamma} = \frac{n-1}{n}\mathring{\nabla}_\alpha\left(\mathring{\Delta} + n\right)H_2 + \left(\mathring{\Delta} + (n-1)\right)\hat{h}_\alpha, \\
&\mathring{\nabla}^\alpha \mathring{\nabla}^\gamma \check{h}_{\alpha\gamma} = \frac{n-1}{n}\mathring{\Delta}\left(\mathring{\Delta} + n\right)H_2, \\
&\mathring{\Delta}\left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta H_2 - \frac{1}{n}\mathring{\Delta} H_2 \mathring{\sigma}_{\alpha\beta}\right) \\
&= \left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta \mathring{\Delta} H_2 - \frac{1}{n}\mathring{\Delta}\mathring{\Delta} H_2 \mathring{\sigma}_{\alpha\beta}\right) + 2n\left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta H_2 - \frac{1}{n}\mathring{\Delta} H_2 \mathring{\sigma}_{\alpha\beta}\right), \\
&\mathring{\Delta}\left(\mathring{\nabla}_\alpha \hat{h}_\beta + \mathring{\nabla}_\beta \hat{h}_\alpha\right) = \left(\mathring{\nabla}_\alpha \mathring{\Delta}\hat{h}_\beta + \mathring{\nabla}_\beta \mathring{\Delta}\hat{h}_\alpha\right) + (n+1)\left(\mathring{\nabla}_\alpha \hat{h}_\beta + \mathring{\nabla}_\beta \hat{h}_\alpha\right).
\end{aligned}$$

In the end, we find

$$
\begin{aligned}
2\check{\delta} R_{\alpha\beta} = & \left(\mathring{\nabla}_\alpha \mathring{\nabla}_\beta - \frac{1}{n}\mathring{\Delta}\mathring{\sigma}_{\alpha\beta}\right)\Big[\left(-h_{AB}g^{AB}\right) + 2\tilde{\nabla}^A H_A \\
& + 2(n-2)r^{-1}r^A H_A - r^2\tilde{\Box}\left(r^{-2}H_2\right) - nr^{-1}r^A\tilde{\nabla}_A H_2 \\
& + 2r^{-2}r^A r_A H_2 - 2r^{-1}\left(\tilde{\Box} r\right)H_2 + 2(n-1)r^{-2}H_2 \\
& + \left(\frac{n-2}{n}\right)r^{-2}\mathring{\Delta}H_2 + (2-n)r^{-2}H\Big] \\
& + \Big[\mathring{\nabla}_\alpha\Big(\tilde{\nabla}^A\hat{h}_{A\beta} + (n-2)r^{-1}r^A\hat{h}_{A\beta} - r^2\tilde{\Box}\left(r^{-2}\hat{h}_\beta\right) \\
& - nr^{-1}r^A\tilde{\nabla}_A\hat{h}_\beta + 2(n-1)r^{-2}\hat{h}_\beta + 2r^{-2}r^A r_A\hat{h}_\beta \\
& - 2r^{-1}\left(\tilde{\Box} r\right)\hat{h}_\beta\Big) \\
& + \mathring{\nabla}_\beta\Big(\tilde{\nabla}^A\hat{h}_{A\alpha} + (n-2)r^{-1}r^A\hat{h}_{A\alpha} - r^2\tilde{\Box}\left(r^{-2}\hat{h}_\alpha\right) \\
& - nr^{-1}r^A\tilde{\nabla}_A\hat{h}_\alpha + 2(n-1)r^{-2}\hat{h}_\alpha + 2r^{-2}r^A r_A\hat{h}_\alpha \\
& - 2r^{-1}\left(\tilde{\Box} r\right)\hat{h}_\alpha\Big)\Big] \\
& + \Big[-r^2\tilde{\Box}\left(r^{-2}\hat{h}_{\alpha\beta}\right) - nr^{-1}r^A\tilde{\nabla}_A\hat{h}_{\alpha\beta} - r^{-2}\mathring{\Delta}\hat{h}_{\alpha\beta} \\
& + 2r^{-2}\left(n + r^A r_A - r\tilde{\Box} r\right)\hat{h}_{\alpha\beta}\Big].
\end{aligned}
\tag{69}
$$

As mentioned prior to its decomposition, the scalar, co-vector, and two-tensor portions of the linearized Ricci tensor are determined by the corresponding portions of the metric perturbation. Assuming the linearized vacuum Einstein equations are satisfied, so that each of the scalar, co-vector, and two-tensor portions of the linearized Ricci tensor vanish, we obtain separate linear subsystems for the scalar, co-vector, and two-tensor portions of the linearized metric. This simplification allows us to study these portions of the linearized metric separately in the remainder of our work.

5. Special Solutions of Linearized Gravity

5.1. **Pure Gauge Solutions.** Diffeomorphism invariance of the Einstein equations reduces in the linear theory to invariance under infinitesimal deformations of the underlying spacetime metric. That is, with X a co-vector and δg a metric perturbation, satisfying the linearized vacuum Einstein equations, the concatenation $\delta g + \mathcal{L}_X g$ yields a new solution of the same. In this subsection, we record how these gauge transformations affect the various components of δg with respect to the decomposition outlined in the first subsection. Such solutions feature in the analysis of the lower angular mode solution $\delta g^{\ell<2}$ in the next section and in the definition of gauge-invariant quantities, used to control the higher angular mode solution $\delta g^{\ell\geq 2}$.

We decompose X as

$$X = X_a dx^a = X_A dx^A + X_\alpha dx^\alpha = X_A dx^A + \left(\mathring{\nabla}_\alpha X_2\right) dx^\alpha + \hat{X}_\alpha dx^\alpha, \quad (70)$$

with $\hat{X}$ a divergence-free co-vector on the orbit spheres.

Calculating the deformation tensor π_X, and splitting according to the decomposition of Proposition 5, we have $\pi_X = \pi_1 + \pi_2$ with

$$\begin{aligned}\pi_1 = &\left[\tilde{\nabla}_A X_B + \tilde{\nabla}_B X_A\right] dx^A dx^B \\ &+ 2\mathring{\nabla}_\alpha \left[X_A + \tilde{\nabla}_A X_2 - 2r^{-1} r_A X_2\right] dx^A dx^\alpha \\ &+ 2\left[r r^A X_A + \frac{1}{n}\mathring{\Delta} X_2\right] \mathring{\sigma}_{\alpha\beta} dx^\alpha dx^\beta \\ &+ 2\left[\mathring{\nabla}_\alpha \mathring{\nabla}_\beta X_2 - \frac{1}{n}\mathring{\Delta} X_2 \mathring{\sigma}_{\alpha\beta}\right] dx^\alpha dx^\beta,\end{aligned} \quad (71)$$

$$\begin{aligned}\pi_2 = &\, 2\left[\tilde{\nabla}_A \hat{X}_\alpha - 2r^{-1} r_A \hat{X}_\alpha\right] dx^A dx^\alpha \\ &+ \left[\mathring{\nabla}_\alpha \hat{X}_\beta + \mathring{\nabla}_\beta \hat{X}_\alpha\right] dx^\alpha dx^\beta.\end{aligned} \quad (72)$$

Gauge-invariant quantities are defined in terms of pure-gauge solutions as follows:

Definition 7. A quantity $\mathcal{P}[h]$, which depends linearly on a symmetric two-tensor h is gauge-invariant if $\mathcal{P}[\pi_X] = 0$ for any co-vector X.

In particular, we observe the divergence-free portion $\hat{h}_{\alpha\beta}$ in the decomposition of Proposition 5 remains unchanged; that is, $\hat{h}_{\alpha\beta}$ is a gauge-invariant quantity.

5.2. **Linearized Myers-Perry Solutions.** We briefly describe the standard presentation of the Myers-Perry solutions, generalizing the Kerr solution to higher dimensions, following [28]. These solutions likewise feature in the analysis of the lower angular mode solution $\delta g^{\ell<2}$ in the next section. Assuming an odd number of spacetime dimensions $d = 2q+1$, the Minkowski metric can be written as

$$\begin{aligned}\bar{g} &= -dt^2 + \sum_{i=1}^{q} \left(dx_i^2 + dy_i^2\right) \\ &= -dt^2 + dr^2 + r^2 \sum_{i=1}^{q} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right).\end{aligned}$$

Here, we have expressed the even number of spatial coordinates as paired Cartesian coordinates (x_i, y_i) for q mutually orthogonal planes. Rewritten in generalized polar coordinates, we have the relations

$$\begin{aligned}x_i &= r\mu_i \cos\phi_i, \\ y_i &= r\mu_i \sin\phi_i,\end{aligned} \quad (73)$$

with the constraint

$$\sum_{i}^{q} \mu_i^2 = 1.$$

With even spacetime dimension $d = 2q+2$, there is an extra unpaired spatial coordinate, also regarded as an azimuthal polar coordinate,

$$z = r\alpha,$$

with $\alpha \in [-1, 1]$, such that the Minkowski metric has the polar form

$$\bar{g} = -dt^2 + dr^2 + r^2 \sum_{i=1}^{q} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right) + r^2 d\alpha^2,$$

with the constraint

$$\sum_{i}^{q} \mu_i^2 + \alpha^2 = 1.$$

Likewise, the Myers-Perry solutions have Boyer-Lindquist type coordinates featuring the generalized polar coordinates above. Defining

$$F := 1 - \sum_{i=1}^{q} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2},$$

$$\Pi := \prod_{i=1}^{q} (r^2 + a_i^2),$$

in even spacetime dimension $d = 2q + 2$ the metric takes the form

$$\begin{aligned} g_{M,a_i} = &- dt^2 + \frac{2Mr}{\Pi F} \left(dt + \sum_{i=1}^{q} a_i \mu_i^2 d\phi_i \right)^2 + \frac{\Pi F}{\Pi - 2Mr} dr^2 \\ &+ \sum_{i=1}^{q} (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + r^2 d\alpha^2, \end{aligned}$$

whereas in odd spacetime dimension $d = 2q + 1$,

$$\begin{aligned} g_{M,a_i} = &- dt^2 + \frac{2Mr}{\Pi F} \left(dt + \sum_{i=1}^{q} a_i \mu_i^2 d\phi_i \right)^2 + \frac{\Pi F}{\Pi - 2Mr} dr^2 \\ &+ \sum_{i=1}^{q} (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2). \end{aligned}$$

Note that both metrics are parametrized by the mass $M > 0$ and the q angular velocity parameters a_i, with parameter increases occurring only in odd dimension.

In each case, the metric can be rewritten in a manner suggestive of our perturbative framework:

$$\begin{aligned} g_{M,a_i} =& -\left(1-\frac{2M}{r^{n-1}}\right)dt^2+\left(1-\frac{2M}{r^{n-1}}\right)^{-1}dr^2+r^2\mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta \\ &+\sum_{i=1}^{q}\frac{4M}{r^{n-1}}\mu_i^2 a_i dt d\phi_i+O(a_i)^2, \end{aligned} \tag{74}$$

where $n=2q$ in even dimension and $n=2q-1$ in odd dimension as given above. Note that we have rewritten the top-order polar terms using our earlier concise notation for the round metric on the unit sphere.

We treat separately the linearized change in mass and change in angular velocity arising from (74) below.

5.3. **Linearized Change in Mass.** Linearized mass solutions are generated by constant multiples of

$$h_{ab}dx^a dx^b=\frac{1}{r^{n-1}}dt^2+\frac{r^{n-1}}{(r^{n-1}-2M)^2}dr^2. \tag{75}$$

By direct calculation, one can verify that (75) satisfies the linearized vacuum Einstein equations. With respect to the aforementioned Hodge and spherical harmonic decompositions, such solutions are scalar with support at $\ell=0$.

5.4. **Linearized Change in Angular Velocity.** Each pair (x_i,y_i) in the construction of the Myers-Perry solution gives a rotation Killing vector field

$$\frac{x_i}{r}d\left(\frac{y_i}{r}\right)-\frac{y_i}{r}d\left(\frac{x_i}{r}\right)=\mu_i^2 d\phi_i$$

in the background Schwarzschild spacetime, where we have used the coordinate relation (73). In view of this, the Myers-Perry expansion (74) gives rise to linearized solutions

$$h_{ab}dx^a dx^b=\frac{1}{r^{n-1}}\mu_i^2 dt d\phi_i, i=1\cdots q$$

Since $\frac{x_i}{r}d\left(\frac{y_i}{r}\right)-\frac{y_i}{r}d\left(\frac{x_i}{r}\right)$ is dual to a rotation Killing field and the duals of $X_\alpha^{1m_v(n,1)}dx^\alpha$, $1\le m_v(n,1)\le\frac{1}{2}n(n+1)$ form a basis of the space of rotation Killing fields per (39). For each $i=1\cdots q$, there exist $c_{m_v(n,1)}, m_v(n,1)=1\cdots\frac{1}{2}n(n+1)$ such that

$$\mu_i^2 d\phi_i=\sum_{m_v(n,1)=1}^{\frac{1}{2}n(n+1)}c_{m_v(n,1)}X_\alpha^{1m_v(n,1)}dx^\alpha.$$

On the other hand, we claim that each $X_\alpha^{1m_v(n,1)}dx^\alpha$ on S^n can be expressed as a linear combination of $\mu_i^2 d\phi_i, i=1\cdots q$ by choosing a suitable coordinate system. We assume that $n=2q-1$ and take an arbitrary Cartesian coordinate system $X^A, A=1\cdots 2q$ on $\mathbb{R}^{2q}$. Recall from (39) that

$\tilde{X}^A d\tilde{X}^B - \tilde{X}^B d\tilde{X}^A, 1 \leq A < B \leq 2q$ form a basis of the space of rotation Killing fields. In particular, there exists an alternating constant $2q \times 2q$ matrix P_{AB}, $P_{AB} = -P_{BA}$ such that $X_\alpha^{1m_v(n,1)} dx^\alpha = \sum_{A,B=1}^{2q} P_{AB} \tilde{X}^A d\tilde{X}^B$. Applying the spectral theorem to P_{AB}, we deduce that there exist a new coordinate system $x_i, y_i, i = 1 \cdots q$ of $\mathbb{R}^{2q}$ and constants $\lambda_i, i = 1 \cdots q$ such that

$$X_\alpha^{1m_v(n,1)} dx^\alpha = \sum_{i=1}^{q} \lambda_i [\frac{x_i}{r} d\left(\frac{y_i}{r}\right) - \frac{y_i}{r} d\left(\frac{x_i}{r}\right)],$$

The coordinate change (73) turns the last expression into $\sum_{i=1}^q \lambda_i \mu_i^2 d\phi_i$. The case of $n = 2q$ can be derived similarly.

Infinitesimal change in angular velocity is encoded in the basis solutions

$$h_{ab} dx^a dx^b = \frac{1}{r^{n-1}} X_\alpha^{1m_v(n,1)} dt dx^\alpha, \tag{76}$$

with the $X_\alpha^{1m_v(n,1)}$ spanning the $\ell = 1$ eigenspace of divergence-free co-vectors, $1 \leq m_v(n,1) \leq \frac{1}{2} n(n+1)$ (39). These basis two-tensors are co-vector solutions supported at $\ell = 1$, satisfying the linearized vacuum Einstein equations.

Together, linear combinations of the above linear perturbations of the Schwarzschild metric form the family of linearized Myers-Perry solutions.

Definition 8. The linearized Myers-Perry solutions of the linearized vacuum Einstein equation on the Schwarzschild spacetime in $n + 2$ dimensions are linear combinations of the basis solutions

$$\begin{aligned} K &= \frac{1}{r^{n-1}} dt^2 + \frac{r^{n-1}}{(r^{n-1} - 2M)^2} dr^2, \\ K_m &= \frac{1}{r^{n-1}} X_\alpha^{1m} dt dx^\alpha, \end{aligned} \tag{77}$$

where $X_\alpha^{1m} = X_\alpha^{1m_v(n,1)}$, with eigenvalue given by (37), and

$$1 \leq m = m_v(n,1) \leq \frac{1}{2} n(n+1).$$

6. Analysis of the Lower Angular Modes and Proof of Theorem 1

In this section we analyze the lower angular mode solution $\delta g^{\ell<2}$ and prove its decomposition into a pure gauge solution and a linearized Myers-Perry solution, the content of Theorem 1. In doing so, we provide the first rigorous treatment of the lower angular modes in higher dimensions, formalizing the same claim in Kodama-Ishibashi [21], which the authors base upon an enumeration of degrees of freedom.

6.1. **The $\ell = 0$ Scalar Mode.** In this case the scalar portion h_1 has the form

$$h_1 = h_{AB} dx^A dx^B + H \mathring{\sigma}_{\alpha\beta} dx^\alpha dx^\beta,$$

and the linearized Ricci tensor reduces to

$$\begin{aligned} 2\delta R_{AB} = {} & g_{AB} \left(\tilde{\nabla}^C \tilde{\nabla}^D h_{CD} \right) + 2\tilde{K} h_{AB} - \tilde{K} S g_{AB} - \left(\tilde{\Box} S \right) g_{AB} \\ & + n r^{-1} r^C \left(\tilde{\nabla}_B h_{CA} + \tilde{\nabla}_A h_{CB} - \tilde{\nabla}_C h_{AB} \right) \\ & - n r^{-3} r_B \tilde{\nabla}_A H - n r^{-3} r_A \tilde{\nabla}_B H \\ & + 4 n r^{-4} r_A r_B H - n \tilde{\nabla}_A \tilde{\nabla}_B \left(r^{-2} H \right), \end{aligned}$$

$$\begin{aligned} 2\delta R_{\alpha\beta} \mathring{\sigma}^{\alpha\beta} = {} & n \Big(2 r r^A \tilde{\nabla}^B h_{AB} + 2(n-1) r^A r^B h_{AB} + r \left(\tilde{\Box} r \right) S \\ & - r r^A \tilde{\nabla}_A S \Big) - n r^2 \tilde{\Box} \left(r^{-2} H \right) - 2 n^2 r^{-1} r^A \tilde{\nabla}_A H \\ & - 2 n r^{-1} \left(\tilde{\Box} r \right) H + 2n(n+1) r^{-2} r^A r_A H, \end{aligned}$$

where we have used $S := g^{AB} h_{AB}$ and the identity

$$\begin{aligned} & - \tilde{\Box} h_{AB} + \tilde{\nabla}^C \tilde{\nabla}_A h_{CB} + \tilde{\nabla}^C \tilde{\nabla}_B h_{CA} \\ & = g_{AB} \left(\tilde{\nabla}^C \tilde{\nabla}^D h_{CD} \right) + 2\tilde{K} h_{AB} - \tilde{K} S g_{AB} + \tilde{\nabla}_A \tilde{\nabla}_B S - \left(\tilde{\Box} S \right) g_{AB}. \end{aligned} \tag{78}$$

Given a co-vector $X = X_A dx^A$, the associated pure gauge solution π_X modifies h_1 as

$$\begin{aligned} h_1 & \to h_1 - \pi_X, \\ h_{AB} & \to h_{AB} - \tilde{\nabla}_A X_B - \tilde{\nabla}_B X_A, \\ H & \to H - 2 r r^A X_A. \end{aligned} \tag{79}$$

We choose X to eliminate S and H; that is, X satisfies

$$\begin{aligned} 2 \tilde{\nabla}^A X_A & = S, \\ 2 r r^A X_A & = H. \end{aligned} \tag{80}$$

Rewriting the first equation as

$$-2 T^A \tilde{\nabla}_A \left(T^B X_B \right) + 2 r^A \tilde{\nabla}_A \left(r^B X_B \right) = r^B r_B S,$$

we observe that $r^A X_A$ is determined by the second equation, while $T^A X_A$ is determined by the first equation only up to specification on an initial slice, say the hypersurface Σ_0 appearing in the decay foliation of Subsection 7.3 to follow. We utilize this additional gauge freedom to set

$$\begin{aligned} & T^A r^B \left(h_{AB} - \tilde{\nabla}_A X_B - \tilde{\nabla}_B X_A \right) \\ & = T^A r^B h_{AB} - T^A \tilde{\nabla}_A \left(r^B X_B \right) \\ & - (1-\mu) r^A \tilde{\nabla}_A \left((1-\mu)^{-1} T^B X_B \right) = 0 \end{aligned} \tag{81}$$

on Σ_0. Note that there is still residual gauge freedom of the form $T^B X_B = c(1-\mu)$; these gauge transformations correspond to scalar multiples of the static Killing vector field T. To summarize, we have performed a change of gauge eliminating S and H globally and $T^A r^B h_{AB}$ on an initial slice Σ_0.

The Einstein equations for the gauge-normalized solution $h_1^* = h_1 - \pi_X$ amount to

$$g_{AB}\left(\tilde{\nabla}^C\tilde{\nabla}^D h^*_{CD}\right) + 2\tilde{K}h^*_{AB} + nr^{-1}r^C\left(\tilde{\nabla}_B h^*_{CA} + \tilde{\nabla}_A h^*_{CB} - \tilde{\nabla}_C h^*_{AB}\right) = 0,$$
$$2rr^A\tilde{\nabla}^B h^*_{AB} + 2(n-1)r^A r^B h^*_{AB} = 0.$$

Taking the trace of the first equation and comparing with the second equation, we can rewrite the first equation as

$$\begin{aligned} &nr^{-1}r^C\left(\tilde{\nabla}_B h^*_{CA} + \tilde{\nabla}_A h^*_{CB} - \tilde{\nabla}_C h^*_{AB}\right) + 2\tilde{K}h^*_{AB} \\ &+ n(n-1)r^{-2}\left(r^C r^D h^*_{CD}\right) g_{AB} = 0. \end{aligned} \tag{82}$$

Contracting (82) with $r^A r^B$ and noting the identity

$$r^A r^B r^C\left(\tilde{\nabla}_B h^*_{CA} + \tilde{\nabla}_A h^*_{CB} - \tilde{\nabla}_C h^*_{AB}\right) = r^A\tilde{\nabla}_A\left(r^B r^C h^*_{BC}\right) - \frac{2r}{n}\tilde{K}\left(r^B r^C h^*_{BC}\right),$$

we deduce

$$r^A\tilde{\nabla}_A\left(r^{n-1}\left(r^B r^C h^*_{BC}\right)\right) = 0,$$

such that

$$r^A r^B h^*_{AB} = c(t) r^{-(n-1)},$$

with $c(t)$ an arbitrary function of time. Contracting (82) with $T^A r^B$ instead, we find

$$T^A\tilde{\nabla}_A\left(r^B r^C h^*_{BC}\right) = 0,$$

so that $c(t)$ is constant. We add a linearized Schwarzschild solution of the form (75) to eliminate $r^A r^B h^*_{AB}$; note that this addition preserves the global vanishing of S^* and H^*, as well as that of $T^A r^B h^*_{AB}$ on Σ_0. In particular, owing to the vanishing of S^*, this addition will also eliminate $T^A T^B h^*_{AB}$. The solution $h_1^{**} = h_1^* - cK = h_1 - \pi_X - cK$ has only the cross-term $T^A r^B h^{**}_{AB}$ to be accounted for. The solution still satisfies (82), the contraction of which with $T^A T^B$ leads to

$$T^A\tilde{\nabla}_A\left(T^B r^C h^{**}_{BC}\right) = 0.$$

Together with the vanishing of $T^A r^B h^{**}_{AB}$ on the initial slice Σ_0, the above gives vanishing of $T^A r^B h^{**}_{AB}$ globally. In summary, we have $h_1 = \pi_X + cK$.

6.2. The $\ell = 1$ Scalar Mode.

The scalar portion h_1 has the form

$$h_1 = h_{AB}dx^A dx^B + 2\mathring{\nabla}_\alpha H_A dx^A dx^\alpha + H\mathring{\sigma}_{\alpha\beta}dx^\alpha dx^\beta,$$

and the linearized Ricci tensor appears as

$$
\begin{aligned}
2\delta R_{AB} &= g_{AB}\left(\tilde\nabla^C\tilde\nabla^D h_{CD}\right) + 2\tilde K h_{AB} - \tilde K S g_{AB} - \left(\tilde\Box S\right) g_{AB}\\
&\quad + nr^{-2}h_{AB} + nr^{-1}r^C\left(\tilde\nabla_B h_{CA} + \tilde\nabla_A h_{CB} - \tilde\nabla_C h_{AB}\right)\\
&\quad - nr^{-2}\left(\tilde\nabla_A H_B + \tilde\nabla_B H_A\right) - nr^{-3}r_B\tilde\nabla_A H - nr^{-3}r_A\tilde\nabla_B H\\
&\quad + 4nr^{-4}r_Ar_BH - n\tilde\nabla_A\tilde\nabla_B\left(r^{-2}H\right),
\end{aligned}
$$

$$
\begin{aligned}
2\delta R_{A\alpha} &= \mathring\nabla_\alpha\Big[\tilde\nabla^B h_{AB} + (n-2)r^{-1}r^B h_{AB} + r^{-1}r_A S - \tilde\nabla_A S\\
&\quad - \tilde\Box H_A + (2-n)r^{-1}r^B\tilde\nabla_B H_A + \tilde\nabla^B\tilde\nabla_A H_B - 2r^{-1}\left(\tilde\nabla_A\tilde\nabla^B r\right)H_B\\
&\quad + 2(1-n)r^{-2}r_Ar^BH_B - 2r^{-1}r_A\tilde\nabla^B H_B + nr^{-1}r^B\tilde\nabla_A H_B\\
&\quad + (1-n)\tilde\nabla_A\left(r^{-2}H\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
2\delta R_{\alpha\beta}\mathring\sigma^{\alpha\beta} &= n\Big(2rr^A\tilde\nabla^B h_{AB} + 2(n-1)r^Ar^Bh_{AB} + r\left(\tilde\Box r\right)S - rr^A\tilde\nabla_A S\Big) + nS\\
&\quad - 4n(n-1)r^{-1}r^AH_A - 2n\tilde\nabla^A H_A\\
&\quad - nr^2\tilde\Box\left(r^{-2}H\right) + 2n(n-1)r^{-2}H - 2n^2r^{-1}r^A\tilde\nabla_A H\\
&\quad - 2nr^{-1}\left(\tilde\Box r\right)H + 2n(n+1)r^{-2}r^Ar_AH,
\end{aligned}
$$

where we have used the notation $S := g^{AB}h_{AB}$ and (78).

Given a co-vector $X = X_A dx^A + \mathring\nabla_\alpha X_2 dx^\alpha$, the associated pure gauge solution π_X modifies h_1 as

$$
\begin{aligned}
h_1 &\to h_1 - \pi_X,\\
h_{AB} &\to h_{AB} - \tilde\nabla_A X_B - \tilde\nabla_B X_A,\\
H_A &\to H_A - X_A - r^2\tilde\nabla_A\left(r^{-2}X_2\right),\\
H &\to H - 2rr^AX_A + 2X_2.
\end{aligned}
\tag{83}
$$

We choose X' to eliminate the quantities $H - 2rr^AH_A$ and H_A. This reduction amounts to solving

$$-2r^3r^A\tilde\nabla_A\left(r^{-2}X_2\right) - 2X_2 = H - 2rr^AH_A$$

for X_2, then solving

$$X_A + r^2\tilde\nabla_A\left(r^{-2}X_2\right) = H_A$$

for X_A. Note that there is residual freedom in the form

$$
\begin{aligned}
X_2 &= c(t)r(1-\mu)^{-1/(n-1)},\\
X_A &= -r^2\tilde\nabla_A\left(c(t)r^{-1}(1-\mu)^{-1/(n-1)}\right).
\end{aligned}
$$

In particular, we note the transformation

$$r^Ar^Bh_{AB} \to r^Ar^Bh_{AB} + 2c(t)\mu r^{-1}(1-\mu)^{-1/(n-1)}$$

of the component $r^A r^B h_{AB}$ under this residual gauge freedom.

The linearized Einstein equations for the gauge-normalized solution $h_1^* = h_1 - \pi_{X'}$ amount to

$$g_{AB}\left(\tilde{\nabla}^C\tilde{\nabla}^D h^*_{CD}\right) + 2\tilde{K}h^*_{AB} - \tilde{K}S^* g_{AB} - \left(\tilde{\Box}S^*\right) g_{AB}$$
$$+ nr^{-2}h^*_{AB} + nr^{-1}r^C\left(\tilde{\nabla}_B h^*_{CA} + \tilde{\nabla}_A h^*_{CB} - \tilde{\nabla}_C h^*_{AB}\right) = 0,$$

$$\tilde{\nabla}^B h^*_{AB} + (n-2)r^{-1}r^B h^*_{AB} + r^{-1}r_A S^* - \tilde{\nabla}_A S^* = 0,$$
$$2rr^A\tilde{\nabla}^B h^*_{AB} + 2(n-1)r^A r^B h^*_{AB} + r\left(\tilde{\Box}r\right)S^* - rr^A\tilde{\nabla}_A S^* + S^* = 0.$$

Replacing the first term in the third equation by means of the second equation, we find

$$rr^A\tilde{\nabla}_A S^* - 2r^A r_A S^* + r(\tilde{\Box}r)S^* + S^* + 2\left(r^A r^B h^*_{AB}\right) = 0.$$

Taking the divergence of the second equation, we deduce the relation

$$\tilde{\nabla}^A\tilde{\nabla}^B h^*_{AB} = \tilde{\Box}S^* + (1-n)r^{-1}r^A\tilde{\nabla}_A S^* + (n-1)r^{-2}r^B r_B S^*$$
$$- \frac{n}{2}r^{-1}(\tilde{\Box}r)S^* + (n-2)(n-1)r^{-2}\left(r^A r^B h^*_{AB}\right).$$

Rewriting the double divergence term in the first equation in this way, contracting with $r^A r^B$, and applying the previous relation between S^* and $r^A r^B h^*_{AB}$, we find an autonomous equation for $r^A r^B h^*_{AB}$

$$r^{-1}r^C\tilde{\nabla}_C\left(r^A r^B h^*_{AB}\right) + r^{-2}\left(1 + (n-1)r^C r_C\right)\left(r^A r^B h^*_{AB}\right) = 0,$$

with general solution

$$r^A r^B h^*_{AB} = d(t)\mu r^{-1}(1-\mu)^{-1/(n-1)}.$$

Contracting the first equation with $T^A r^B$ instead, we find

$$nr^{-1}T^C\tilde{\nabla}_C\left(r^A r^B h^*_{AB}\right) + nr^{-2}\left(T^A r^B h^*_{AB}\right) = 0.$$

Finally, contracting the second equation with r^A, we have

$$\tilde{\nabla}^B\left(r^A h^*_{AB}\right) - \frac{\tilde{\Box}r}{2}S^* + (n-2)r^{-1}\left(r^A r^B h^*_{AB}\right) + r^{-1}r^A r_A S^* - r^A\tilde{\nabla}_A S^* = 0.$$

Exercising our residual freedom by the choice $c(t) = -\frac{1}{2}d(t)$ above, with associated co-vector field $\bar{X}$, and letting $X = X' + \bar{X}$, the normalized solution $h_1^{**} = h_1 - \pi_X$ has vanishing $r^A r^B h^{**}_{AB}$ component. The equations above immediately imply vanishing of the component $T^A r^B h^{**}_{AB}$. It remains to consider the component S^{**}, which satisfies

$$rr^A\tilde{\nabla}_A S^{**} - 2r^A r_A S^{**} + r(\tilde{\Box}r)S^{**} + S^{**} = 0,$$

$$-\frac{\tilde{\Box}r}{2}S^{**} + r^{-1}r^A r_A S^{**} - r^A\tilde{\nabla}_A S^{**} = 0.$$

Taken together, the two equations imply $S^{**} = 0$. In this way, we have shown that $h_1^{**} = 0$; that is, $h_1 = \pi_X$ is a pure gauge solution.

6.3. **The $\ell = 1$ Co-Vector Mode.** The co-vector portion amounts to

$$h_2 = 2\hat{h}_{A\alpha} dx^A dx^\alpha,$$

and the linearized Ricci tensor takes the form

$$2\delta R_{A\alpha} = -r^{-n}\epsilon_{AB}\tilde{\nabla}^B \left(r^{n+2}\epsilon^{CD}\tilde{\nabla}_D \left(r^{-2}\hat{h}_{C\alpha} \right) \right).$$

Given a co-vector $X = \hat{X}_\alpha dx^\alpha$, with $\hat{X}_\alpha$ satisfying the divergence-free condition $\mathring{\nabla}^\alpha \hat{X}_\alpha = 0$, the associated pure gauge solution π_X modifies h_2 as

$$\begin{aligned} h_2 &\to h_2 - \pi_X, \\ \hat{h}_{A\alpha} &\to \hat{h}_{A\alpha} - r^2 \tilde{\nabla}_A \left(r^{-2}\hat{X}_\alpha \right). \end{aligned} \tag{84}$$

We choose X' to eliminate $r^A \hat{h}_{A\alpha}$, such that

$$r^A \tilde{\nabla}_A \hat{X}_\alpha - 2r^{-1} r^A r_A \hat{X}_\alpha = r^A \hat{h}_{A\alpha}. \tag{85}$$

In accordance with the homogeneous solutions of the equation above, there remains residual gauge freedom in the form $\hat{X}_\alpha = \bar{c}_{m_v(n,1)}(t) r^2 X_\alpha^{1 m_v(n,1)}$.

We consider the normalized $h_2^* = h_2 - \pi_{X'}$ and decompose $\hat{h}^*_{A\alpha} = \sum_{m_v(n,1)} \hat{h}^{*1m_v(n,1)}_{A\alpha}$. Contracting the linearized Einstein equation

$$-r^{-n}\epsilon_{AB}\tilde{\nabla}^B \left(r^{n+2}\epsilon^{CD}\tilde{\nabla}_D \left(r^{-2}\hat{h}^{*1m_v(n,1)}_{C\alpha} \right) \right) = 0$$

with r^A and T^A, we deduce

$$r^{n+2}\epsilon^{CD}\tilde{\nabla}_D \left(r^{-2}\hat{h}^{*1m_v(n,1)}_{C\alpha} \right) = d_{m_v(n,1)},$$

with $d_{m_v(n,1)}$ a constant. As $r^A \hat{h}^{*1m_v(n,1)}_{A\alpha} = 0$, the above reduces to

$$r^{n+2} r^A \tilde{\nabla}_A \left(r^{-2} T^B \hat{h}^{*1m_v(n,1)}_{B\alpha} \right) = d_{m_v(n,1)} r^B r_B,$$

with general solution

$$T^A \hat{h}^{*1m_v(n,1)}_{A\alpha} = c_{m_v(n,1)}(t) r^2 X_\alpha^{1m_v(n,1)} + \frac{d_{m_v(n,1)}}{r^{n-1}}.$$

Taking $\bar{X} = \bar{c}_{m_v(n,1)}(t) r^2 X_\alpha^{1m_v(n,1)}$ with $\bar{c}'_{m_v(n,1)}(t) = c_{m_v(n,1)}(t)$ and letting $X = X' + \bar{X}$, we have shown

$$h_2 = \pi_X + \sum_{m=1}^{\frac{1}{2}n(n+1)} d_m K_m,$$

with K_m being the Myers-Perry solutions of Definition 8 and $d_m = d_{m_v(n,1)}$. Note that there remains gauge freedom in the form $\bar{c}(t) \equiv \bar{c}$; such transformations correspond to scalar multiples of the angular Killing fields Ω_i.

6.4. **Proof of Theorem 1.** Combining the results of the subsections above, we have a proof of Theorem 1. In particular, adding the various linearized solutions, we obtain a smooth co-vector $X^{\ell<2}$ on the Schwarzschild-Tangherlini background, unique modulo Killing fields, and constants c, d_m such that

$$\delta g^{\ell<2} = \pi_{X^{\ell<2}} + cK + \sum_{m=1}^{\frac{1}{2}n(n+1)} d_m K_m, \tag{86}$$

where K, K_m are the basis solutions for the linearized Myers-Perry family of Definition 8.

The remainder of the paper concerns the identification and analysis of the gauge-invariant master quantities of the higher angular frequency portion $\delta g^{\ell\geq 2}$.

7. Analysis of Regge-Wheeler Type Equations

The remainder of this work is concerned with the higher angular modes, encoded by $\delta g^{\ell\geq 2}$. For each portion of $\delta g^{\ell\geq 2}$, we decouple gauge-invariant quantities satisfying Regge-Wheeler type equations, the analysis of which is expected to provide an avenue towards proving decay of the solution.

To eliminate redundancy, we present in this section a general theory for the analysis of such equations, specialized as necessary in subsequent sections. We consider solutions within the sub-bundle $\mathcal{L}(-2)$ of symmetric traceless two-tensors on the spheres of symmetry, with such solutions either being divergence-free or possessing scalar potentials or divergence-free co-vector potentials. In this language, we define a solution of a Regge-Wheeler type equation as follows.

Definition 9. Let Ψ be a symmetric traceless two-tensor, regarded as a section of $\mathcal{L}(-2)$. We say that Ψ is a solution of a Regge-Wheeler type equation with potential V if Ψ satisfies

$$\not\Box_{\mathcal{L}(-2)}\Psi = V\Psi. \tag{87}$$

We further assume that V is a radial function with the form

$$V = \frac{a_0}{r^2} + \frac{b_0 M}{r^{n+1}} + O_1\left(\frac{M^2}{r^{2n}}\right),\ a_0 > -\frac{n^2-2n}{4}. \tag{88}$$

We remark that V needs not be non-negative; indeed, this will be the case for two of the potentials derived subsequently.

The argument in this section goes as follows. For a solution Ψ of equation (87) with potential V of the form (88), we further assume the T-energy comparison (112) and the Morawetz estimate in Assumption 14 and derive various estimates based on these assumptions. Whether these assumptions hold or not would depend on the more refined structure of V. We then verify these assumptions for each case in subsequent sections. For example, the potential for the two-tensor portion is given in (130), the T-energy comparison is verified in 8.3.1 and the Morawetz estimate is verified in 8.3.2.

7.1. **Stress-Energy Tensors.** We consider the natural stress-energy tensor

$$T_{ab}[\Psi] := \not\nabla_a \Psi \cdot \not\nabla_b \Psi - \frac{1}{2} g_{ab}((\not\nabla \Psi)^2 + V|\Psi|^2), \tag{89}$$

where we emphasize that

$$\begin{aligned} \not\nabla_a \Psi \cdot \not\nabla_b \Psi &= g^{\alpha\beta} g^{\gamma\delta} (\not\nabla_a \Psi)_{\alpha\gamma} (\not\nabla_b \Psi)_{\beta\delta}, \\ |\Psi|^2 &= g^{\alpha\beta} g^{\gamma\delta} \Psi_{\alpha\gamma} \Psi_{\beta\delta}, \end{aligned}$$

and

$$\begin{aligned} (\not\nabla \Psi)^2 &= g^{ab} g^{\alpha\beta} g^{\gamma\delta} (\not\nabla_a \Psi)_{\alpha\gamma} (\not\nabla_b \Psi)_{\beta\delta}, \\ &= (\tilde{\not\nabla} \Psi)^2 + r^{-2} |\mathring{\not\nabla} \Psi|^2, \\ (\tilde{\not\nabla} \Psi)^2 &= g^{AB} g^{\alpha\beta} g^{\gamma\delta} (\not\nabla_A \Psi)_{\alpha\gamma} (\not\nabla_B \Psi)_{\beta\delta}, \\ |\mathring{\not\nabla} \Psi|^2 &= \mathring{\sigma}^{\eta\nu} g^{\alpha\beta} g^{\gamma\delta} (\not\nabla_\eta \Psi)_{\alpha\gamma} (\not\nabla_\nu \Psi)_{\beta\delta}. \end{aligned}$$

In addition, we will use the virtual stress-energy tensor

$$\check{T}_{ab}[\Psi] := \not\nabla_a \Psi \cdot \not\nabla_b \Psi - \frac{1}{2} g_{ab} (\not\nabla^c \Psi \cdot \not\nabla_c \Psi). \tag{90}$$

Estimates are obtained by contracting with a vector-field multiplier X^b and applying the spacetime Stokes' theorem

$$\int_{\partial\mathcal{D}} T_{ab}[\Psi] n^a_{\partial\mathcal{D}} X^b = \int_{\mathcal{D}} \nabla^a (T_{ab}[\Psi] X^b), \tag{91}$$

over a spacetime region $\mathcal{D}$ with boundary $\partial\mathcal{D}$. A similar identity holds for the virtual stress tensor (90).

We remark that the natural stress-energy tensor (89) has non-trivial divergence

$$\nabla^a T_{ab}[\Psi] = -\frac{1}{2} \nabla_b V |\Psi|^2 + \not\nabla^a \Psi [\not\nabla_a, \not\nabla_b] \Psi, \tag{92}$$

with the commutator $[\not\nabla_a, \not\nabla_b]$ vanishing when contracted with a multiplier invariant under the angular Killing fields. In particular, all such multipliers considered in the analysis below have this property.

Although the virtual stress-energy tensor satisfies a positive energy condition, this is not necessarily true for the natural stress-energy tensor. Indeed, in what follows it will be the case that the stress-energy tensor fails to satisfy a pointwise positivity condition, owing to lack of non-negativity of the potential V, at which point it becomes necessary to incorporate integral estimates.

7.2. **Comparisons.** Given A and B, we use the notation

$$A \approx B$$

if the two quantities are comparable up to constants depending upon the orbit sphere dimension n and the mass M. That is, there exists $C(n, M)$ such that

$$\frac{1}{C(n,M)} A \leq B \leq C(n,M) A.$$

Likewise, we use

$$A \gtrsim B$$

for one-sided comparisons up to constants with such dependence.

7.3. **Decay Foliation.** Recalling the Eddington-Finkelstein double null coordinates

$$u = \frac{1}{2}(t - r_*),$$
$$v = \frac{1}{2}(t + r_*),$$

we further define

$$\begin{aligned} L &:= \frac{\partial}{\partial v} = \frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}, \\ \underline{L} &:= \frac{\partial}{\partial u} = \frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}. \end{aligned} \tag{93}$$

Fixing radii r_1 and R_1 satisfying $r_h < r_1 < r_P < R_1$, we choose a Lipschitz hypersurface Σ_0 with the following properties:

- Σ_0 intersects the future horizon $\mathcal{H}^+$ transversely and Σ_0 is spacelike in $r_h \leq r \leq R_1$,
- $\Sigma_0 \cap \{r_1 \leq r \leq R_1\} = \{t = 0\} \cap \{r_1 \leq r \leq R_1\}$,
- $\Sigma_0 \cap \{R_1 \leq r\} = \{u = -\frac{1}{2}(R_1)_*\} \cap \{R_1 \leq r\}$,

where $(R_1)_*$ is given by (11) with normalization (22).

Flowing along the static Killing vector field T, we construct our decay foliation $\Sigma_\tau := \phi_\tau(\Sigma_0)$. We define n_{Σ_τ} as the unit timelike normal vector for Σ_τ for $r \leq R_1$ and $n_{\Sigma_\tau} := L$ for $r > R_1$. Further, we denote the volume form on Σ_τ corresponding to n_{Σ_τ} by $dVol_{\Sigma_\tau}$, and the volume form on the unit round n-sphere S^n by $dVol_{S^n}$.

We denote by $D(\tau_1, \tau_2)$ the spacetime region between the hypersurfaces Σ_{τ_1} and Σ_{τ_2}, with $\tau_1 \leq \tau_2$. To be more precise, we define

$$D(\tau_1, \tau_2) := J^+(\Sigma_{\tau_1}) \cap J^-(\Sigma_{\tau_2}), \tag{94}$$

with volume form $dVol$. We denote the null hypersurface bounding $D(\tau_1, \tau_2)$ at the future event horizon by

$$\mathcal{H}^+(\tau_1, \tau_2) := D(\tau_1, \tau_2) \cap \mathcal{H}^+, \tag{95}$$

with null tangential chosen to be $n_{\mathcal{H}^+} := T$ and the associated volume form denoted $dVol_{\mathcal{H}^+}$. In addition, we define the null hypersurfaces

$$C_{v_0}(u_1, u_2) := \{v = v_0, u_1 \leq u \leq u_2\}, \tag{96}$$

with the choice of null tangential $\underline{L}$ and the associated volume form $r^n du dVol_{S^n}$. Intersecting the constant v-hypersurface with the spacetime region $D(\tau_1, \tau_2)$, we find

$$C_v(-\infty, \infty) \cap D(\tau_1, \tau_2) = C_v(\tau_1 - \frac{1}{2}(R_1)_*, \tau_2 - \frac{1}{2}(R_1)_*),$$

and define the limit of such hypersurfaces

$$\mathcal{I}^+(\tau_1, \tau_2) := \lim_{v \to \infty} C_v(\tau_1 - \frac{1}{2}(R_1)_*, \tau_2 - \frac{1}{2}(R_1)_*). \tag{97}$$

Often we write boundary integrals of the form

$$\int_{\mathcal{I}^+(\tau_1,\tau_2)} J^X_a[\Psi]\underline{L}^a \left(r^n du dVol_{S^n}\right),$$

where it is understood that we are evaluating the limit as v approaches infinity of the boundary integrals

$$\int_{C_v(\tau_1 - \frac{1}{2}(R_1)_*, \tau_2 - \frac{1}{2}(R_1)_*)} J^X_a[\Psi]\underline{L}^a \left(r^n du dVol_{S^n}\right).$$

7.4. **Poincaré Inequalities.** The spherical Laplacian operator acts as an endomorphism on each of the aforementioned sub-bundles of $\mathcal{L}(-2)$, with spectra described in (36, 41, 42).

We assume that, in acting on the sub-bundle associated with Ψ, the spherical Laplacian $\mathring{\Delta}$ has least eigenvalue λ. The identity

$$\mathring{\Delta}|\Psi|^2 = 2\mathring{\not{\Delta}}\Psi \cdot \Psi + 2|\mathring{\not{\nabla}}\Psi|^2 \tag{98}$$

yields the Poincaré inequality

$$\int_{S^n} |\mathring{\not{\nabla}}\Psi|^2 \geq \lambda \int_{S^n} |\Psi|^2. \tag{99}$$

For those sections of $\mathcal{L}(-2)$ with scalar potential, we have $\lambda = 2$, whereas those with divergence-free co-vector potential have $\lambda = n$. Finally, on the sub-bundle of divergence-free symmetric traceless two-tensors we have $\lambda = 2n$.

7.5. **Hardy Inequalities.** We adapt the Hardy inequalities found in the work of Andersson-Blue [1] and the earlier work of Blue-Soffer [5], utilizing them to prove Morawetz estimates as in these earlier works. In addition, owing to the non-positivity of our potentials in higher dimensions, we rely upon such inequalities in proving uniform boundedness estimates.

Lemma 10. *Suppose A and W are smooth functions on $[s_0, s_1]$, with A non-negative. Further, assume that the ODE*

$$-\frac{d}{ds}\left(A\frac{dg}{ds}\right) + Wg = 0 \tag{100}$$

has a smooth, positive solution g on $[s_0, s_1]$. Given any smooth function f, as long as

$$f^2 A\frac{d}{ds}(\log g)$$

vanishes at s_0 and s_1, we have the estimate

$$\int_{s_0}^{s_1}\left[A\left(\frac{df}{ds}\right)^2 + Wf^2\right] ds \geq 0. \tag{101}$$

Proof. Let $h = f/g$. Then

$$\begin{aligned}
&\int_{s_0}^{s_1}\left[A\left(\frac{df}{ds}\right)^2 + Wf^2\right] ds \\
&= \int_{s_0}^{s_1}\left[Ag^2\left(\frac{dh}{ds}\right)^2 + Ah^2\left(\frac{dg}{ds}\right)^2 + 2Agh\frac{dg}{ds}\frac{dh}{ds} + Wg^2h^2\right] ds \\
&= \int_{s_0}^{s_1}\left[Ag^2\left(\frac{dh}{ds}\right)^2 + Ah^2\left(\frac{dg}{ds}\right)^2 + Wg^2h^2 + \frac{d}{ds}\left(Ah^2 g\frac{dg}{ds}\right)\right. \\
&\quad \left. - gh^2\frac{d}{ds}\left(A\frac{dg}{ds}\right) - Ah^2\left(\frac{dg}{ds}\right)^2\right] ds \\
&= \int_{s_0}^{s_1} Ag^2\left(\frac{dh}{ds}\right)^2 ds + \left(Ah^2 g\frac{dg}{ds}\right)\Big|_{s_0}^{s_1} \\
&= \int_{s_0}^{s_1} Ag^2\left(\frac{dh}{ds}\right)^2 ds \geq 0.
\end{aligned}$$

□

Often we will assume that $s_0 = 2M$ and $s_1 = \infty$, with A vanishing at this boundary and f compactly supported. In applying the estimate, the primary difficulty lies in finding a positive solution to the associated ODE. To this end, we transform the ODE into a hypergeometric form, from which a well-known positive solution can be constructed.

7.6. **Hypergeometric Differential Equations.** We consider the hypergeometric ODE

$$(1-z)z\frac{d^2\tilde{g}}{dz^2} + \Big(c - (a+b+1)z\Big)\frac{d\tilde{g}}{dz} - ab\tilde{g} = 0, \tag{102}$$

with $z < 1$ and $0 < b < c$. With such constraints, the hypergeometric function $F(a, b; c; z)$, defined by

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{103}$$

is a solution of (102). We can easily divide out the terms involving the Gamma function to obtain a positive solution of (102).

7.7. The T-Energy.

7.7.1. *Definition and Monotonicity.* We define the T-current

$$J_a^T[\Psi] := T_{ab}[\Psi]T^b, \tag{104}$$

and the T-energy

$$\begin{aligned} E_\Psi^T(\Sigma_\tau) &:= \int_{\Sigma_\tau} J_a^T[\Psi] n^a_{\Sigma_\tau} dVol_{\Sigma_\tau} \\ &= \int_{\Sigma_\tau} T_{ab}[\Psi] n^a_{\Sigma_\tau} T^b dVol_{\Sigma_\tau}. \end{aligned} \tag{105}$$

Applying the static Killing multiplier T over the spacetime region bounded by the decay foliation hypersurfaces Σ_{τ_1} and Σ_{τ_2} , where $0 \leq \tau_1 < \tau_2$, we have

$$\begin{aligned} &E_\Psi^T(\Sigma_{\tau_2}) + \int_{\mathcal{H}^+(\tau_1,\tau_2)} J_a^T[\Psi] n^a_{\mathcal{H}^+} dVol_{\mathcal{H}^+} \\ &+ \int_{\mathcal{I}^+(\tau_1,\tau_2)} J_a^T[\Psi] \underline{L}^a (r^n du dVol_{S^n}) = E_\Psi^T(\Sigma_{\tau_1}), \end{aligned}$$

with the terms of the density

$$\text{div} J^T[\Psi] = (\nabla^a T_{ab}[\Psi])T^b + T_{ab}[\Psi]\nabla^{(a}T^{b)}$$

vanishing according to (92) and Killing condition on T,

$$({}^T\pi)^{ab} := \mathcal{L}_T g_M^{ab} = 2\nabla^{(a}T^{b)} = 0.$$

Non-negativity of the boundary integral along the event horizon follows from T being null tangential, which makes the V term have no contribution in $J_a^T[\Psi]n^a_{\mathcal{H}^+}$. Non-negativity of the boundary integral along future null infinity follows from the asymptotic property of the potential V and the spectral property (see 7.4) in each portion, which can be verified individually. Hence we have

$$\begin{aligned} &\int_{\mathcal{H}^+(\tau_1,\tau_2)} J_a^T[\Psi] n^a_{\mathcal{H}^+} dVol_{\mathcal{H}^+} \geq 0, \\ &\int_{\mathcal{I}^+(\tau_1,\tau_2)} J_a^T[\Psi] \underline{L}^a (r^n du dVol_{S^n}) \geq 0. \end{aligned} \tag{106}$$

Together, the two imply monotonicity of the T-energy:

$$E_\Psi^T(\Sigma_{\tau_2}) \leq E_\Psi^T(\Sigma_{\tau_1}). \tag{107}$$

Before proceeding, we define the virtual T-energy:

$$\check{E}^T_\Psi(\Sigma_\tau) := \int_{\Sigma_\tau} \check{T}_{ab}[\Psi] n^a_{\Sigma_\tau} T^b. \tag{108}$$

7.7.2. *The Adapted Hardy Estimate.* The natural stress-energy tensors under consideration often fail to satisfy pointwise positive energy conditions; as a consequence, the T-energies above are not obviously positive-definite. To address this issue, we rely upon the Poincaré inequality (99) and an adapted Hardy estimate, in the spirit of Lemma 10.

To begin, we regard $R = r$ as a function on the hypersurface Σ_τ, and consider the coordinate system (τ, R, x^A). Written in this form, the integrand of the virtual T-energy (108) takes the form

$$\begin{aligned}&\check{T}_{ab}[\Psi] n^a_{\Sigma_\tau} T^b dVol_{\Sigma_\tau}\\ =&\frac{1}{2}\left((1-\mu)^{-1}\cosh^{-2} x|\nabla\!\!\!/_\tau \Psi|^2 + (1-\mu)|\nabla\!\!\!/_R \Psi|^2 + R^{-2}|\mathring{\nabla\!\!\!/}\Psi|^2\right) R^n dRdVol_{S^n},\end{aligned}$$

where $x(r)$ is specified by

$$(1-\mu)^{-1/2}\cosh x = -\left\langle n_{\Sigma_\tau}, T\right\rangle.$$

In particular, the coefficients of $|\nabla\!\!\!/_R\Psi|^2$ and $|\mathring{\nabla\!\!\!/}\Psi|^2$ don't depend on the defining function of Σ_τ. Incorporating the potential V, the T-energy (105) has the form

$$\begin{aligned}E^T_\Psi(\Sigma_\tau) = &\frac{1}{2}\int_{S^n}\int_{r_h}^\infty \left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + V|\Psi|^2\right] R^n dRdVol_{S^n}\\ &+\frac{1}{2}\int_{S^n}\int_{r_h}^\infty \left[\cosh^{-2} x(1-\mu)^{-1}|\nabla\!\!\!/_\tau\Psi|^2 + R^{-2}|\mathring{\nabla\!\!\!/}\Psi|^2\right] R^n dRdVol_{S^n}.\end{aligned}$$

With the change of variable $s = R^{n-1}$, the radial coefficient naturally takes the form $A = s(s-2M)$, in the notation of Lemma 10. Further choosing $s_0 = 2M$ and $s_1 = \infty$, the following Hardy estimate holds for a large class of potentials V:

Lemma 11. *Let $f(s)$ be a function defined on $[2M, \infty)$ and $E, F \geq 0$ be two nonnegative numbers with $|2F - E| \leq 1$ and $E > 1$. Then*

$$\int_{2M}^\infty \left[A\left(\frac{df}{ds}\right)^2 + V(E,F)f^2\right] ds \geq 0, \tag{109}$$

where

$$\begin{aligned}A &= s(s-2M),\\ V(E,F) &= \frac{1}{4}(E^2-1) - \frac{2MF^2}{s}.\end{aligned}$$

Proof. We first assume $f(s)$ has compact support in $[2M,\infty)$. From Lemma 10, the estimate follows from the existence a positive function $g(s)$ defined on $[2M,\infty)$ and satisfying the equation

$$-\frac{d}{ds}\left(A\frac{dg}{ds}\right)+V(E,F)g=0. \tag{110}$$

One can obtain an explicit positive solution of (110) by using hypergeometric functions. Letting $\alpha=\pm F$ and $\tilde{g}=s^{-\alpha}g$, the equation (110) becomes

$$-s(s-2M)\frac{d^2\tilde{g}}{ds^2}+\Big(-2(s-2M)\alpha-(2s-2M)\Big)\frac{d\tilde{g}}{ds}-\frac{1}{4}\Big((2\alpha+1)^2-E^2\Big)\tilde{g}(z)=0.$$

Performing the change of variable $z=1-\frac{s}{2M}$, we have

$$(1-z)z\frac{d^2\tilde{g}}{dz^2}+\Big(1-(2\alpha+2)z\Big)\frac{d\tilde{g}}{dz}-\frac{1}{4}\Big((2\alpha+1)^2-E^2\Big)\tilde{g}(z)=0. \tag{111}$$

Comparing (111) to hypergeometric ODE (102), we have a hypergeometric equation with $z\le 0$, $c=1$, and $\{a,b\}=\frac{1}{2}\Big(1+2\alpha\pm E\Big)=\frac{1}{2}\Big(1\pm 2F\pm E\Big)$. The associated hypergeometric function $F(a,b;c;z)$ has integral representation (103) assuming

$$\begin{aligned}0<\frac{1}{2}\Big(1+2F+E\Big)<1 \text{ or } 0<\frac{1}{2}\Big(1-2F+E\Big)<1\\ \text{or } 0<\frac{1}{2}\Big(1+2F-E\Big)<1 \text{ or } 0<\frac{1}{2}\Big(1-2F-E\Big)<1.\end{aligned}$$

In particular, when $E,F\ge 0$, the above condition is equivalent to $|2F-E|<1$.

For general $f(s)$ assume without loss of generality that

$$\int_{2M}^{\infty}\left[s^2\left(\frac{df}{ds}\right)^2+f^2\right]ds<\infty,$$

as the left hand side of (109) is infinity otherwise. Then approximating such $f(s)$ with compactly supported functions yields the result.

□

7.7.3. *The T-Energy Comparison.* We briefly describe a typical application of the Hardy estimate above. First, we borrow from the angular term using the Poincaré inequality (99), and find the underestimate

$$\begin{aligned}
E^T_\Psi(\Sigma_\tau) &= \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + V|\Psi|^2\right] R^n dr dVol_{S^n} \\
&+ \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\left[\cosh^{-2}x(1-\mu)^{-1}|\nabla\!\!\!/_\tau\Psi|^2 + R^{-2}|\mathring{\nabla\!\!\!/}\Psi|^2\right] R^n dr dVol_{S^n} \\
&\geq \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + \left(V+\lambda(1-\delta)R^{-2}\right)|\Psi|^2\right] R^n dr dVol_{S^n} \\
&+ \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\left[\cosh^{-2}x(1-\mu)^{-1}|\nabla\!\!\!/_\tau\Psi|^2 + \delta R^{-2}|\mathring{\nabla\!\!\!/}\Psi|^2\right] R^n dr dVol_{S^n},
\end{aligned}$$

where $\delta > 0$ is a small residual angular coefficient.

Isolating the term

$$\int_{r_h}^{\infty}\left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + \left(V+\lambda(1-\delta)R^{-2}\right)|\Psi|^2\right] R^n dr,$$

we perform the change of variables $s = R^{n-1}$ and determine a $V(E,F)$ in Lemma 11 underestimating the s-dependent potential associated with $\left(V+(\lambda-\delta)R^{-2}\right)$ pointwise on the exterior region. The lemma then implies

$$\int_{r_h}^{\infty}\left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + \left(V+(\lambda-\delta)R^{-2}\right)|\Psi|^2\right] R^n dr \geq 0,$$

and we can choose $\epsilon(\delta) > 0$ such that

$$\begin{aligned}
&E^T_\Psi(\Sigma_\tau) \\
&\geq \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + \left(V+\lambda(1-\delta)R^{-2}\right)|\Psi|^2\right] R^n dr dVol_{S^n} \\
&+ \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\left[\cosh^{-2}x(1-\mu)^{-1}|\nabla\!\!\!/_\tau\Psi|^2 + \delta R^{-2}|\mathring{\nabla\!\!\!/}\Psi|^2\right] R^n dr dVol_{S^n} \\
&\geq \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\epsilon\left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + \left(V+\lambda(1-\delta)R^{-2}\right)|\Psi|^2\right] R^n dr dVol_{S^n} \\
&+ \frac{1}{2}\int_{S^n}\int_{r_h}^{\infty}\left[\cosh^{-2}x(1-\mu)^{-1}|\nabla\!\!\!/_\tau\Psi|^2 + \delta R^{-2}|\mathring{\nabla\!\!\!/}\Psi|^2\right] R^n dr dVol_{S^n} \\
&\gtrsim \check{E}^T_\Psi(\Sigma_\tau),
\end{aligned}$$

with $\epsilon(\delta)$ chosen small enough to absorb possible negative contributions from $\left(V+\lambda(1-\delta)R^{-2}\right)$ into the δ-small residual angular term, again via application of (99).

We remark that, owing to the characteristics of the potential V and the Poincaré inequality (99), we have the other direction in the comparison

$$\check{E}^T_\Psi(\Sigma_\tau) \gtrsim E^T_\Psi(\Sigma_\tau).$$

In what follows, we assume the T-energy comparison

$$\check{E}^T_\Psi(\Sigma_\tau) \approx E^T_\Psi(\Sigma_\tau). \tag{112}$$

We remark that the comparison is trivial in the event that V is non-negative, as in four spacetime dimensions. On the other hand, if V lacks non-negativity, the inequality is not obvious.

Owing to the spacetime Stokes' theorem, the comparison (112) implies

$$\begin{aligned}\int_{\mathcal{H}^+(\tau_1,\tau_2)} J^T_a[\Psi] n^a_{\mathcal{H}^+} dVol_{\mathcal{H}^+} &\lesssim \check{E}^T_\Psi(\Sigma_{\tau_1}),\\ \int_{\mathcal{I}^+(\tau_1,\tau_2)} J^T_a[\Psi] \underline{L}^a (r^n du dVol_{S^n}) &\lesssim \check{E}^T_\Psi(\Sigma_{\tau_1}).\end{aligned} \tag{113}$$

7.8. **The N-Energy.** We describe the red-shift vector field, introduced in Dafermos-Rodnianski [10] in four spacetime dimensions and generalized to the higher-dimensional setting by Schlue [30].

For convenience, we calculate in the ingoing Eddington-Finkelstein coordinates

$$\begin{aligned}\bar{v} &= t + r_*,\\ R &= r.\end{aligned}$$

Let Y be a smooth vector field, specified by

$$\begin{aligned}Y|_{r=r_h} &= -\frac{\partial}{\partial R},\\ \mathcal{L}_T Y &= 0,\\ \mathcal{L}_{\partial_{x^\alpha}} Y &= 0,\\ (\nabla_Y Y)_{r=r_h} &= -\sigma(Y+T),\end{aligned}$$

where σ is a positive number to be determined. At the event horizon $R = r_h$ we compute

$$\begin{aligned}\check{T}_{ab}[\Psi]\nabla^{(a}Y^{b)} &= \sigma|\nabla\!\!\!/_{\bar{v}}\Psi|^2 + \kappa_n|\nabla\!\!\!/_R\Psi|^2 + \frac{n}{R}\nabla\!\!\!/_{\bar{v}}\Psi\cdot\nabla\!\!\!/_R\Psi\\ &\quad+\frac{1}{R^3}\left(\frac{\sigma R}{2}+\frac{n}{2}-1\right)|\mathring{\nabla\!\!\!/}\Psi|^2\\ &\gtrsim\left(\sigma|\nabla\!\!\!/_{\bar{v}}\Psi|^2+\kappa_n|\nabla\!\!\!/_R\Psi|^2+\frac{\sigma}{R^2}|\mathring{\nabla\!\!\!/}\Psi|^2\right).\end{aligned}$$

Here $\kappa_n = \frac{(n-1)M}{r_h^n}$ is the surface gravity. Hence

$$\begin{aligned}&\nabla^a(\check{T}_{ab}[\Psi]Y^b) = (\nabla^a\check{T}_{ab}[\Psi])Y^b + \check{T}_{ab}\nabla^{(a}Y^{b)}\\ &\gtrsim\left(\sigma|\nabla\!\!\!/_{\bar{v}}\Psi|^2+\kappa_n|\nabla\!\!\!/_R\Psi|^2+\frac{\sigma}{R^2}|\mathring{\nabla\!\!\!/}\Psi|^2\right)+V\nabla_Y\Psi\cdot\Psi\\ &\gtrsim\left(\sigma|\nabla\!\!\!/_{\bar{v}}\Psi|^2+\kappa_n|\nabla\!\!\!/_R\Psi|^2+\frac{\sigma}{R^2}|\mathring{\nabla\!\!\!/}\Psi|^2\right)\end{aligned} \tag{114}$$

using the Cauchy-Schwarz and Poincaré inequalities and a choice of large σ. Note that such a choice can be made for any bounded radial potential V.

We extend Y to the exterior region such that Y is causal and $Y = 0$ for $r \geq r_1$. By continuity, the coercive estimate (114) is satisfied in $r_h \leq r \leq r_0$ for some $r_0 < r_1$.

We define the strictly timelike red-shift multiplier $N := T+Y$, in addition to the current

$$J^N_a[\Psi] := T_{ab}[\Psi]T^b + \check{T}_{ab}[\Psi]Y^b, \tag{115}$$

and energies

$$\begin{aligned} E^N_\Psi(\Sigma_\tau) &:= \int_{\Sigma_\tau} J^N_a[\Psi] n^a_{\Sigma_\tau} dVol_{\Sigma_\tau}, \\ \check{E}^N_\Psi(\Sigma_\tau) &:= \int_{\Sigma_\tau} \check{T}_{ab}[\Psi] n^a_{\Sigma_\tau} N^b dVol_{\Sigma_\tau}. \end{aligned} \tag{116}$$

Lemma 12. *Assume the T-energy comparison* (112) *holds. Defining the red-shift multiplier N as above, we have the pointwise density estimate*

$$\begin{aligned} \mathrm{div} J^N[\Psi] = \nabla^a \left(\check{T}_{ab}[\Psi]Y^b\right) &\gtrsim \check{T}_{ab}[\Psi] n^a_{\Sigma_\tau} N^b \text{ in } r \leq r_0 \\ &\gtrsim -\check{T}_{ab}[\Psi] n^a_{\Sigma_\tau} N^b \text{ in } r \in [r_0, r_1], \end{aligned}$$

in addition to the energy comparison

$$E^N_\Psi(\Sigma_\tau) \approx \check{E}^N_\Psi(\Sigma_\tau)$$

and the boundary estimates

$$\begin{aligned} &\int_{\mathcal{H}^+(0,\tau)} J^N_a[\Psi] n_{\mathcal{H}^+} dVol_{\mathcal{H}^+} \geq 0, \\ &\int_{\mathcal{I}^+(0,\tau)} J^N_a[\Psi] \underline{L}^a (r^n du dVol_{S^n}) \geq 0. \end{aligned}$$

With Lemma 12 we can follow the argument in Dafermos-Rodnianski [10] to obtain uniform boundedness of the non-degenerate N-energy.

Theorem 13. *Suppose Ψ is a solution of the Regge-Wheeler type equation* (87) *as in Definition 9. Further, assume that Ψ satisfies the T-energy comparison* (112). *Then Ψ satisfies the uniform boundedness estimate*

$$\check{E}^N_\Psi(\Sigma_\tau) \lesssim \check{E}^N_\Psi(\Sigma_0), \tag{117}$$

with $0 \leq \tau$. In addition, we have the boundary estimates

$$\begin{aligned} &\int_{\mathcal{H}^+(0,\tau)} \check{T}_{ab}[\Psi] n^a_{\mathcal{H}^+} T^b \; dVol_{\mathcal{H}^+} \lesssim \check{E}^N_\Psi(\Sigma_0), \\ &\int_{\mathcal{I}^+(0,\tau)} T_{ab}[\Psi] \underline{L}^a T^b \;\; (r^n du dVol_{S^n}) \lesssim \check{E}^N_\Psi(\Sigma_0). \end{aligned} \tag{118}$$

7.9. **The Morawetz Estimate.** We assume the existence of a Morawetz estimate, first adapted to curved four-dimensional spacetime backgrounds by Blue-Soffer [4] and Dafermos-Rodnianski [10] and later to the higher-dimensional setting by Schlue [30]. The requirements of such a multiplier are described in the following assumption:

Assumption 14. There exists a current $J^{\mathfrak{M}}_a[\Psi]$, with density $K^{\mathfrak{M}}[\Psi] = \mathrm{div} J^{\mathfrak{M}}[\Psi]$, satisfying the bulk estimates

in $r \geq r_0$,

$$\int_{S^n} K^{\mathfrak{M}}[\Psi] dVol_{S^n} \gtrsim \int_{S^n} \frac{1}{r^n} \left(\left(1 - \frac{(n+1)M}{r^{n-1}}\right)^2 \left(|\nabla\!\!\!/_t \Psi|^2 + |\mathring{\nabla\!\!\!/} \Psi|^2 \right) + |\nabla\!\!\!/_r \Psi|^2 \right) dVol_{S^n},$$

in $r \leq r_0$,

$$\int_{S^n} K^{\mathfrak{M}}[\Psi] dVol_{S^n} \gtrsim \int_{S^n} \check{T}_{ab}[\Psi] N^a N^b dVol_{S^n},$$

and the boundary estimates

$$\int_{S^n} |J^{\mathfrak{M}}_a[\Psi] n^a_{\Sigma_\tau}| dVol_{S^n} \lesssim \int_{S^n} \check{T}_{ab}[\Psi] n^a_{\Sigma_\tau} N^b dVol_{S^n},$$

$$\int_{S^n} \left(J^{\mathfrak{M}}_a[\Psi] n^a_{\mathcal{H}^+} \right)_- dVol_{S^n} \lesssim \int_{S^n} \check{T}_{ab}[\Psi] n^a_{\mathcal{H}^+} T^b dVol_{S^n},$$

$$\int_{S^n} |J^{\mathfrak{M}}_a[\Psi] \underline{L}^a| dVol_{S^n} \lesssim \int_{S^n} \check{T}_{ab}[\Psi] \underline{L}^a T^b dVol_{S^n},$$

where $\left(J^{\mathfrak{M}}_a[\Psi] n^a_{\mathcal{H}^+} \right)_-$ is the negative part of $\left(J^{\mathfrak{M}}_a[\Psi] n^a_{\mathcal{H}^+} \right)$.

Applying the divergence theorem and Theorem 13, we have the following result, essential in proving uniform decay:

Theorem 15. *Assuming the estimates of Assumption 14, along with the conclusions of Theorem 13 following from the T-energy comparison* (112)*, we have the further density estimate*

$$\int_0^\infty \int_{\Sigma_\tau \cap \{r \leq R_1\}} K^{\mathfrak{M}}[\Psi] dVol_{\Sigma_\tau} d\tau \lesssim \check{E}^N_\Psi(\Sigma_0). \tag{119}$$

Proof. By the divergence theorem

$$\begin{aligned} &\int_{\Sigma_\tau} J^{\mathfrak{M}}_a[\Psi] n^a_{\Sigma_\tau} dVol_{\Sigma_\tau} + \int_{\mathcal{H}^+(0,\tau)} J^{\mathfrak{M}}_a[\Psi] n^a_{\mathcal{H}^+} dVol_{\mathcal{H}^+} \\ &+ \int_{\mathcal{I}^+(0,\tau)} J^{\mathfrak{M}}_a[\Psi] \underline{L}^a \left(r^n du dVol_{S^n} \right) + \int_{D(0,\tau)} K^{\mathfrak{M}}[\Psi] dVol \\ &= \int_{\Sigma_\tau} J^{\mathfrak{M}}_a[\Psi] n^a_{\Sigma_\tau} dVol_{\Sigma_\tau} \end{aligned}$$

The boundary terms are controlled by the right hand side of (119) by Assumption 14 and Theorem 13. The result follows from the volume comparison $dVol \approx dVol_{\Sigma_\tau} d\tau$. ∎

Briefly we discuss the construction of candidate Morawetz currents and the calculation of their densities, deferring details on the verification of the estimates of Assumption 14 to the cases at hand. We denote $\frac{\partial f}{\partial r_*}$ by f', and consider the vector field $X = f(r)\partial_{r_*}$. Letting $\omega^X = f' + \frac{n(1-\mu)}{r} f$ and $\beta(r)$ be a radial function. We introduce the currents

$$\begin{aligned} J_a^X[\Psi] &:= T_{ab}[\Psi]X^b, \\ J_a^{X,\omega^X}[\Psi] &:= J_a^X[\Psi] + \frac{1}{4}\omega^X \nabla_a |\Psi|^2 - \frac{1}{4}\nabla_a \omega |\Psi|^2, \\ J_a^{X,\omega^X,\beta}[\Psi] &:= J_a^{X,\omega^X}[\Psi] + f'\beta|\Psi|^2 dr_*, \end{aligned} \tag{120}$$

and calculate the density:

$$\begin{aligned} &\operatorname{div}\left(J^{X,\omega^X,\beta}[\Psi]\right) = \\ =&\frac{f'}{1-\mu}|\nabla\!\!\!/_{r_*}\Psi + \beta\Psi|^2 + \frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla\!\!\!/}\Psi|^2 \\ &+\mathring{W}(f,\beta)|\Psi|^2, \end{aligned} \tag{121}$$

where

$$\mathring{W}(f,\beta) = \left(-\frac{1}{4}\Box\omega^X - \frac{1}{2}V'f + \frac{1}{2}\mu_r V f\right) + \left(\frac{f''\beta}{1-\mu} + f'\left(\frac{\beta'}{1-\mu} + \frac{n\beta}{r} - \frac{\beta^2}{1-\mu}\right)\right).$$

7.10. **The r^p-Hierarchy.** We adapt the r^p-hierarchy appearing in the work of Schlue [30], generalizing the earlier ideas of Dafermos-Rodnianski [11]. See also the work of Moschidis [27] on the r^p-hierarchy in general asymptotically flat spacetimes.

We define

$$\Phi := r^{n/2}\Psi \tag{122}$$

and the current

$$J_b[\Phi] := \frac{1}{r^n(1-\mu)}T_{ab}[\Phi]L^a - \frac{1}{8r^{n+2}(1-\mu)}\left((n^2-2n) + \frac{2n^2M}{r^{n-1}}\right)|\Phi|^2 L_b, \tag{123}$$

from which we obtain the following r^p density estimate:

Proposition 16. *Suppose the potential V has the form* (88)

$$V = \frac{a_0}{r^2} + \frac{b_0 M}{r^{n+1}} + O_1\left(\frac{M^2}{r^{2n}}\right), \ a_0 > -\frac{n^2-2n}{4}.$$

Then for $1 \le p \le 2$, R_1 large, and an appropriate choice of k,

$$\nabla^b\left(\frac{r^p J_b[\Phi]}{(1-\mu)^k}\right) \gtrsim \left(r^{p-n-1}|\nabla\!\!\!/_L\Phi|^2 + (2-p)r^{p-n-3}|\mathring{\nabla\!\!\!/}\Phi|^2\right) \quad \text{as} \quad r \ge R_1.$$

Proof. Except for the presence of the potential V, the construction is the same as the one in [30]. By direct calculation

$$\begin{aligned}\nabla^a(r^pJ_a[\Phi]) =&\frac{r^{p-n-1}}{(1-\mu)^2}\left(\frac{p}{2}(1-\mu)-\frac{(n-1)M}{r^{n-1}}\right)|\bar{\nabla}_L\Phi|^2+\left(1-\frac{p}{2}\right)r^{p-n-3}|\mathring{\bar{\nabla}}\Phi|^2\\&-\frac{1}{2}r^{p-n}\left(\frac{1}{1-\mu}\nabla_L(V+W)-\frac{p}{r}(V+W)\right)|\Phi|^2,\end{aligned}$$

where

$$W=\frac{1}{r^2}\left[\frac{n^2-2n}{4}+\frac{n^2M}{2r^{n-1}}\right].$$

From the assumption on V,

$$V+W=\frac{\bar{a}_0}{r^2}+\frac{\bar{b}_0M}{r^{n+1}}+O_1\left(\frac{M^2}{r^{2n}}\right),$$

where $\bar{a}_0=a_0+\frac{n^2-2n}{4}>0$ and $\bar{b}_0=b_0+\frac{n^2}{2}$, the coefficient of $|\Phi|^2$ has the form

$$r^{p-n-3}\left(\bar{a}_0\left(1-\frac{p}{2}\right)+\frac{\bar{b}_0}{2}(n+1-p)\frac{M}{r^{n-1}}+O\left(\frac{M^2}{r^{2n-2}}\right)\right).$$

The leading positive term $\bar{a}_0\left(1-\frac{p}{2}\right)$ vanishes as $p=2$. In order to make the coefficient positive, we consider

$$\begin{aligned}&\nabla^a\left(\frac{r^pJ_a[\Phi]}{(1-\mu)^{k-1}}\right)\\&=\frac{r^{p-n-1}}{(1-\mu)^{k+1}}\left[\frac{p}{2}(1-\mu)-\frac{k(n-1)M}{r^{n-1}}\right]|\bar{\nabla}_L\Phi|^2\\&+\frac{r^{p-3-n}}{(1-\mu)^{k-1}}\left[\left(1-\frac{p}{2}\right)+(k-1)(n-1)(1-\mu)^{-1}\frac{M}{r^{n-1}}\right]|\mathring{\bar{\nabla}}\Phi|^2\\&+\frac{r^{p-n-3}}{(1-\mu)^{k-1}}\left[\bar{a}_0\left(1-\frac{p}{2}\right)+\left(\frac{\bar{b}_0}{2}(n+1-p)+\bar{a}_0(k-1)(n-1)\right)\frac{M}{r^{n-1}}+O\left(\frac{M^2}{r^{2n-2}}\right)\right]|\Phi|^2.\end{aligned}$$

By taking k large such that $\frac{\bar{b}_0}{2}(n+1-p)+\bar{a}_0(k-1)(n-1)>0$, the coefficient of $|\Phi|^2$ above is non-negative for $r\geq R_1$ sufficiently large. Rechoosing R_1 as necessary, comparison of the coefficient of $|\bar{\nabla}_L\Phi|^2$ above with that of the right-hand side in the proposition holds, and the proposition follows. □

We remark that this lemma will be applied to the Regge-Wheeler type equations (129), (144), and (169). Since the potentials $V_\ell^{(+)}$ in (169) satisfy $V_\ell^{(+)}\to V^{(+)}$ as $\ell\to\infty$, we can choose R_1 and k large which hold for all the equations (129), (144), and (169).

Defining

$$\begin{aligned} E^p_\Psi(\Sigma_\tau) &:= \int_{\Sigma_\tau\cap\{r\geq R_1\}} \frac{r^{p-n}}{(1-\mu)^k}\frac{|\nabla\!\!\!/_L\Phi|^2}{(1-\mu)} r^n dv dVol_{S^n}, \\ A^p_\Psi(\Sigma_\tau) &:= \int_{\Sigma_\tau\cap\{r\geq R_1\}} r^{p-n}|r^{-1}\mathring{\nabla\!\!\!/}\Phi|^2 r^n dv dVol_{S^n}, \end{aligned} \tag{124}$$

with Φ specified in terms of Ψ by (122), the Morawetz estimate of Assumption 14 and the r^p density estimate of Proposition 16 give the r^p-estimate:

Theorem 17. *Suppose Ψ is a solution of the Regge-Wheeler type equation* (87) *as in Definition 9 with potential of the form* (88)*. Further, assume that Ψ satisfies the T-energy comparison* (112) *and the Morawetz estimate of Assumption 14, with choices of R_1 made such that Proposition 16 holds. For $1 \leq p \leq 2$ and $\tau_1 < \tau_2$, we have the r^p-estimate:*

$$\begin{aligned} &E^p_\Psi(\Sigma_{\tau_2}) + \frac{1}{C}\int_{\tau_1}^{\tau_2}\Big(E^{p-1}_\Psi(\Sigma_\tau) + (2-p)A^{p-1}_\Psi(\Sigma_\tau)\Big)d\tau \\ &\leq E^p_\Psi(\Sigma_{\tau_1}) + C\check{E}^N_\Psi(\Sigma_{\tau_1}). \end{aligned} \tag{125}$$

Proof. Let $J^p_a[\Phi] = \eta(r)\frac{r^pJ_a[\Phi]}{(1-\mu)^k}$, where η is a radial cut-off function with $\chi = 1$ for $r \geq R_1$ and $\chi = 0$ for $r \leq R_1 - 1$, with k and R_1 chosen to ensure that Proposition 16 holds. Let $J^{\mathfrak{M}}_a[\Psi]$ be a current satisfying the Morawetz estimates in Assumption 14. Using Proposition 16 and the coercivity of $K^{\mathfrak{M}}[\Psi] := \mathrm{div}J^{\mathfrak{M}}[\Psi]$ in $R_1 - 1 \leq r \leq R_1$, we have for $C >> 1$

$$\begin{aligned} &\mathrm{div}(J^p[\Phi] + CJ^{\mathfrak{M}}[\Psi]) \geq 0 \ \ as\ r \leq R_1, \\ &\mathrm{div}(J^p[\Phi] + CJ^{\mathfrak{M}}[\Psi]) \gtrsim r^{p-n-1}|\nabla\!\!\!/_v\Psi|^2 + (2-p)r^{p-n-3}|\mathring{\nabla\!\!\!/}\Psi|^2 \ \ as\ r \geq R_1. \end{aligned}$$

Owing to the T-energy comparison (112), implying Theorem 13, we have estimates on many of the boundary terms in terms of the initial N-energy. The result then follows from the divergence theorem. □

In addition, we have the lemma:

Lemma 18.

$$E^0_\Psi(\Sigma_\tau) + A^0_\Psi(\Sigma_\tau) \gtrsim \int_{\Sigma_\tau\cap\{r\geq R_1\}} T_{ab}[\Psi]N^a n^b_{\Sigma_\tau} dVol_{\Sigma_\tau}$$

Proof. The integrand of E_0 is $|\nabla\!\!\!/_L(r^{n/2}\Psi)|^2 \approx |r^{n/2}\nabla\!\!\!/_L\Psi + \frac{n}{2}r^{n/2-1}\Psi|^2$. Borrowing some of the $|\Psi|^2$ term from A_0 by applying a Poincaré inequality yields the estimate. □

7.11. **Uniform Decay Estimates.** We introduce the notation

$$\begin{aligned} E^2_{\Psi,\mathcal{L}_K\Psi}(\Sigma_\tau) &:= E^2_\Psi(\Sigma_\tau) + E^2_{\mathcal{L}_T\Psi}(\Sigma_\tau) + E^2_{\mathcal{L}_\Omega\Psi}(\Sigma_\tau), \\ E^N_{\Psi,\mathcal{L}_K\Psi}(\Sigma_\tau) &:= E^N_\Psi(\Sigma_\tau) + E^N_{\mathcal{L}_T\Psi}(\Sigma_\tau) + E^N_{\mathcal{L}_\Omega\Psi}(\Sigma_\tau), \end{aligned} \tag{126}$$

with $K := \{T, \Omega_i\}$ ranging over the background Killing fields and $\Omega := \{\Omega_i\}$ ranging over the angular Killing fields.

Assuming the T-energy comparison (112) and the Morawetz estimate of Assumption 14, we have the following theorem on uniform decay.

Theorem 19. *Suppose Ψ is a solution of the Regge-Wheeler type equation* (87) *as in Definition 9. Further, assume that Ψ satisfies the T-energy comparison* (112) *and the Morawetz estimate of Assumption 14, with choices of k and R_1 made such that Proposition 16 holds. Then Ψ satisfies the uniform decay estimate*

$$\check{E}^N_\Psi(\Sigma_\tau) \lesssim \frac{I_\Psi(\Sigma_0)}{\tau^2}, \tag{127}$$

where

$$I_\Psi(\Sigma_0) := E^2_{\Psi,\mathcal{L}_K\Psi}(\Sigma_0) + E^N_{\Psi,\mathcal{L}_K\Psi,\mathcal{L}^2_K\Psi}(\Sigma_0). \tag{128}$$

Proof. The proof follows the argument in Dafermos-Rodnianski [11], so we only describe main steps and refer readers to [11] for more detail.

By applying Theorem 17 with $p = 2$ and using the mean value theorem, there exists a sequence $2^k \le \bar{\tau}_k \le 2^{k+1}$ such that

$$E^1_\Psi(\Sigma_{\bar{\tau}_k}) \le \frac{C}{\bar{\tau}_k}\left(E^2_\Psi(\Sigma_0) + \check{E}^N_\Psi(\Sigma_0)\right).$$

Through Theorem 17 with $p = 1$, Lemma 18, and the Morawetz estimate (119), we have for any $\tau_2 \ge \tau_1$,

$$E^1_\Psi(\Sigma_{\tau_2}) + \frac{1}{C}\int_{\tau_1}^{\tau_2} E^N_\Psi(\Sigma_\tau)d\tau \le E^1_\Psi(\Sigma_{\tau_1}) + CE^N_{\Psi,\mathcal{L}_T\Psi,\mathcal{L}_\Omega\Psi}(\Sigma_{\tau_1}).$$

Apply this inequality with $\tau_1 = \bar{\tau}_k$ and $\tau_2 = \bar{\tau}_{k+2}$ and the mean value theorem, we obtain another sequence $\tilde{\tau}_k \approx 2^k$ such that

$$E^N_\Psi(\Sigma_{\tilde{\tau}_k}) \lesssim \frac{1}{\tilde{\tau}_k}(E^1_\Psi(\Sigma_{\bar{\tau}_k}) + E^N_{\Psi,\mathcal{L}_T\Psi,\mathcal{L}_\Omega\Psi}(\Sigma_{\bar{\tau}_k})).$$

Combining with the τ^{-1} decay of E^1_Ψ discussed above yields the desired τ^{-2} decay of E^N_Ψ.

∎

8. The Two-Tensor Portion

8.1. **The Linearized Einstein Equations.** There is just a single wave-type equation involving the two-tensor portion $\hat{h}$, a gauge-invariant quantity as remarked following Definition 7. Namely, from (69) we have

$$\begin{aligned}&-r^2\tilde{\Box}\left(r^{-2}\hat{h}_{\alpha\beta}\right)-nr^{-1}r^A\tilde{\nabla}_A\hat{h}_{\alpha\beta}-r^{-2}\mathring{\Delta}\hat{h}_{\alpha\beta}\\&+2r^{-2}\left(n+r^Ar_A-r\tilde{\Box}r\right)\hat{h}_{\alpha\beta}=0.\end{aligned}$$

8.2. **Master Equation for the Two-Tensor Portion.** Expanding the first term in the equation above and noting

$$-r^2\tilde{\Box}\left(r^{-2}\right)=2r^{-1}\tilde{\Box}r-6r^{-2}r^Ar_A,$$

we deduce

$$\not{\Box}_{\mathcal{L}(-2)}\hat{h}_{\alpha\beta}=2r^{-2}\left(n+(1-n)r^Ar_A-r\tilde{\Box}r\right)\hat{h}_{\alpha\beta},$$

or

$$\not{\Box}_{\mathcal{L}(-2)}\hat{h}_{\alpha\beta}=U\hat{h}_{\alpha\beta}, \tag{129}$$

with

$$U:=2r^{-2}. \tag{130}$$

In deriving (129), we have made use of (28) and (51). The equation is analogous to that first discovered in Kodama-Ishibashi-Seto [22].

8.3. **Analysis of the Master Equation.** In contrast with the Regge-Wheeler type equations to be considered later in this work, the potential in (129) is non-negative throughout the exterior region. As a consequence, the natural stress-energy tensor (89) satisfies a positive energy condition, dramatically simplifying the analysis.

8.3.1. *The T-Energy Comparison.* Owing to the positive energy condition, the T-energy comparison (112) is easily satisfied. This, in turn, implies the uniform boundedness result of Theorem 13.

8.3.2. *The Morawetz Estimate.* The calculation of the weighted density of the current $J_a^{X,\omega^X}[\hat{h}]$ (121) with $\beta\equiv 0$ agrees with that of the two-tensor wave equation, up to the presence of the potential terms

$$\begin{aligned}&\left[\frac{1}{2}\partial_r\mu fU-\frac{1}{2}f(1-\mu)\partial_rU\right]|\hat{h}|^2\\&=\frac{2f}{r^3}\left(1-\frac{n+1}{2}\mu\right)|\hat{h}|^2,\end{aligned}$$

with coefficient proportional to that of the angular gradient. Choosing a function f increasing on the exterior and vanishing at the photon sphere, the extra terms above are manifestly non-negative. Such a choice of f appears in the work of Schlue [30]. Then we can define $J_a^{\mathfrak{M}}[\hat{h}]$ as

$$J_a^{\mathfrak{M}}[\hat{h}]:=J_a^{X,\omega^X}[\hat{h}]+\frac{1}{4}(\nabla_ag)|\hat{h}|^2-\frac{1}{4}\nabla_ag|\hat{h}|^2+\epsilon_3\check{T}_{ab}[\hat{h}]Y^b, \tag{131}$$

with suitable choices of $g(r)$ and $\epsilon_3 > 0$ to obtain the density estimate in Assumption 14. See subsection 9.3.2 for details. In addition, the function f constructed in [30] is uniformly bounded on the exterior region, ensuring that the boundary terms associated with the $\mathfrak{M}$-current are dominated by those of the N-current as required in Assumption 14.

8.4. **Uniform Boundedness and Decay of the Master Quantity.** With the T-comparison and the Morawetz estimates in place, we have uniform boundedness and decay of solutions to the Regge-Wheeler equation (129) in all spacetime dimensions:

Theorem 20. *Let δg be a smooth, symmetric two-tensor on a Schwarzschild-Tangherlini spacetime, satisfying the linearized Einstein equation* (2)*. There exists a gauge-invariant master quantity $\hat{h}_{\alpha\beta}$ in the two-tensor portion h_3 of δg satisfying the Regge-Wheeler type equation* (129)*. As a solution of* (129)*, $\hat{h}_{\alpha\beta}$ satisfies the uniform boundedness estimate*

$$\check{E}^N_{\hat{h}}(\Sigma_\tau) \lesssim \check{E}^N_{\hat{h}}(\Sigma_0), \tag{132}$$

and the uniform decay estimate

$$\check{E}^N_{\hat{h}}(\Sigma_\tau) \lesssim \frac{I_{\hat{h}}(\Sigma_0)}{\tau^2}, \tag{133}$$

where

$$I_{\hat{h}}(\Sigma_0) := E^2_{\hat{h},\mathcal{L}_K\hat{h}}(\Sigma_0) + E^N_{\hat{h},\mathcal{L}_K\hat{h},\mathcal{L}_K^2\hat{h}}(\Sigma_0) \tag{134}$$

and $\tau \geq 0$.

We emphasize that the relevant constants in the comparisons depend only upon the orbit sphere dimension n and the mass $M > 0$.

9. The Co-Vector Portion

9.1. **The Linearized Einstein Equations.** The cross-term and the traceless portion of the pure angular term of the linearized Ricci tensor above yield the co-vector equations

$$\begin{aligned}
&-\tilde{\Box}\hat{h}_{A\alpha} - r^{-2}\mathring{\Delta}\hat{h}_{A\alpha} + (2-n)r^{-1}r^B\tilde{\nabla}_B\hat{h}_{A\alpha} \\
&+ (n-1)r^{-2}\hat{h}_{A\alpha} + \tilde{\nabla}^B\tilde{\nabla}_A\hat{h}_{B\alpha} - 2r^{-1}\left(\tilde{\nabla}_A\tilde{\nabla}^B r\right)\hat{h}_{B\alpha} \\
&+ 2(1-n)r^{-2}r_A r^B\hat{h}_{B\alpha} - 2r^{-1}r_A\tilde{\nabla}^B\hat{h}_{B\alpha} + nr^{-1}r^B\tilde{\nabla}_A\hat{h}_{B\alpha} \\
&+ (n-1)\tilde{\nabla}_A\left(r^{-2}\hat{h}_\alpha\right) + \tilde{\nabla}_A\left(r^{-2}\mathring{\Delta}\hat{h}_\alpha\right) = 0,
\end{aligned} \tag{135}$$

$$\begin{aligned}
&\tilde{\nabla}^A\hat{h}_{A\beta} + (n-2)r^{-1}r^A\hat{h}_{A\beta} - r^2\tilde{\Box}\left(r^{-2}\hat{h}_\beta\right) \\
&- nr^{-1}r^A\tilde{\nabla}_A\hat{h}_\beta + 2(n-1)r^{-2}\hat{h}_\beta + 2r^{-2}r^A r_A\hat{h}_\beta \\
&- 2r^{-1}\left(\tilde{\Box}r\right)\hat{h}_\beta = 0.
\end{aligned} \tag{136}$$

We define the connection-level co-vector quantities

$$P_\alpha := r^3 \epsilon^{AB} \tilde{\nabla}_B \left(r^{-2} \hat{h}_{A\alpha} \right), \tag{137}$$

$$Q_{\alpha\beta A} := \mathring{\nabla}_\beta \hat{h}_{A\alpha} + \mathring{\nabla}_\alpha \hat{h}_{A\beta} - r^2 \tilde{\nabla}_A \left(r^{-2} \left(\mathring{\nabla}_\beta \hat{h}_\alpha + \mathring{\nabla}_\alpha \hat{h}_\beta \right) \right). \tag{138}$$

The two quantities are gauge-invariant in view of (72) and Definition 7. The quantity $Q_{\alpha\beta A}$ is the higher-dimensional analog of a gauge-invariant quantity appearing in the authors' earlier work [16]; contraction with r^A yields the gauge-invariant quantity appearing in Kodama-Ishibashi-Seto [22], generalizing the quantity of Regge-Wheeler [29] in four spacetime dimensions. The quantity P_α is a higher dimensional generalization of the Cunningham-Moncrief-Price quantity [8] in four spacetime dimensions, which we introduce in this paper. The co-vector equations above can be rewritten in terms of these gauge-invariant quantities as

$$-r^{-n} \epsilon_{AB} \tilde{\nabla}^B \left(r^{n-1} P_\alpha \right) - r^{-2} \mathring{\nabla}^\beta Q_{\alpha\beta A} = 0, \tag{139}$$

$$r^{2-n} \tilde{\nabla}^A \left(r^{n-2} Q_{\alpha\beta A} \right) = 0. \tag{140}$$

In addition, the two are related by

$$\epsilon^{AB} \tilde{\nabla}_B \left(r^{-2} Q_{\alpha\beta A} \right) - r^{-3} \left(\mathring{\nabla}_\alpha P_\beta + \mathring{\nabla}_\beta P_\alpha \right) = 0. \tag{141}$$

9.2. **Master Equations for the Co-Vector Portion.**

9.2.1. *Decoupling of P_α.* The decoupling of P_α proceeds as follows. Applying the operator $\epsilon^{AB} \tilde{\nabla}_B$ to (139) and rewriting the result with (141), we find

$$\epsilon^{AB} \tilde{\nabla}_B \left(r^{-n} \epsilon_{AC} \tilde{\nabla}^C \left(r^{n-1} P_\alpha \right) \right) + \epsilon^{AB} \tilde{\nabla}_B \left(r^{-2} \mathring{\nabla}^\beta Q_{\alpha\beta A} \right) = 0,$$

$$g^{BC} \left(\tilde{\nabla}_B \left(r^{-n} \tilde{\nabla}_C \left(r^{n-1} P_\alpha \right) \right) \right) + r^{-3} \mathring{\Delta} P_\alpha + (n-1) r^{-3} P_\alpha = 0.$$

Expanding the first term and multiplying through by r, we have

$$\tilde{\Box} P_\alpha + (n-2) r^{-1} r^B \tilde{\nabla}_B P_\alpha + r^{-2} \mathring{\Delta} P_\alpha$$
$$+ \left((n-1) r^{-2} - 2r^{-2} (n-1) r^B r_B + (n-1) r^{-1} \left(\tilde{\Box} r \right) \right) P_\alpha = 0.$$

Noting the formula for the spin-1 d'Alembertian (52), along with the background formulae (28), the equation reduces to

$$\not{\Box}_{\mathcal{L}(-1)} P_\alpha = W P_\alpha, \tag{142}$$

where

$$W := \frac{1}{r^2} - \frac{2Mn^2}{r^{n+1}}. \tag{143}$$

We remark that P_α is the higher-dimensional analog of the Cunningham-Moncrief-Price quantity [8] in four spacetime dimensions.

9.2.2. *Decoupling of* $Q_{\alpha\beta A}r^A$. Multiplying (141) by r^{n+2}, and applying the operator $\epsilon_{AB}\tilde{\nabla}^B$ to the result, we find

$$\begin{aligned}&\epsilon_{AB}\tilde{\nabla}^B\left(r^{n+2}\epsilon^{CD}\tilde{\nabla}_D\left(r^{-2}Q_{\alpha\beta C}\right)\right)\\&-\epsilon_{AB}\tilde{\nabla}^B\left(r^{n-1}\mathring{\nabla}_\alpha P_\beta\right)-\epsilon_{AB}\tilde{\nabla}^B\left(r^{n-1}\mathring{\nabla}_\beta P_\alpha\right)=0,\end{aligned}$$

or, applying (139),

$$\begin{aligned}&r^{-n}\epsilon_{AB}\epsilon^{CD}\tilde{\nabla}^B\left(r^{n+2}\tilde{\nabla}_D\left(r^{-2}Q_{\alpha\beta C}\right)\right)\\&+r^{-2}\mathring{\nabla}_\alpha\mathring{\nabla}^\gamma Q_{\beta\gamma A}+r^{-2}\mathring{\nabla}_\beta\mathring{\nabla}^\gamma Q_{\alpha\gamma A}=0.\end{aligned}$$

The first term above can be expanded by appealing to the relation

$$\epsilon_{AB}\epsilon^{CD}P^B_{DC}=P^B_{BA}-P^B_{AB},$$

valid for tensors on the two-dimensional quotient space. Applying this result, and contracting the equation with r^A, we find

$$\begin{aligned}&\left(\tilde{\Box}Q_{\alpha\beta A}\right)r^A-2r^{-1}r^Ar^B\tilde{\nabla}_BQ_{\alpha\beta A}-r^{-1}\left(\tilde{\Box}r\right)Q_{\alpha\beta A}r^A\\&+2r^{-1}(r^Br_B)\tilde{\nabla}^BQ_{\alpha\beta B}-r^A\left(\tilde{\nabla}^B\tilde{\nabla}_AQ_{\alpha\beta B}\right)\\&+r^{-2}\mathring{\nabla}_\alpha\mathring{\nabla}^\gamma\left(Q_{\beta\gamma A}r^A\right)+r^{-2}\mathring{\nabla}_\beta\mathring{\nabla}^\gamma\left(Q_{\alpha\gamma A}r^A\right)=0.\end{aligned}$$

Commuting the covariant derivative, and applying (140), we rewrite the term

$$\begin{aligned}&r^A\left(\tilde{\nabla}^B\tilde{\nabla}_AQ_{\alpha\beta B}\right)\\&=(n-2)r^{-2}r^Br_BQ_{\alpha\beta A}r^A-(n-2)r^{-1}r^A\left(\tilde{\nabla}_A\tilde{\nabla}^Cr\right)Q_{\alpha\beta C}\\&-(n-2)r^{-1}r^Ar^C\tilde{\nabla}_CQ_{\alpha\beta A}+\tilde{K}Q_{\alpha\beta A}r^A.\end{aligned}$$

With this, and application of (140) to the preceding divergence term, our equation takes the form

$$\begin{aligned}&\left(\tilde{\Box}Q_{\alpha\beta A}\right)r^A+(n-4)r^{-1}r^Ar^B\tilde{\nabla}_BQ_{\alpha\beta B}-r^{-1}\left(\tilde{\Box}r\right)Q_{\alpha\beta A}r^A\\&-2(n-2)r^{-2}r^Br_BQ_{\alpha\beta A}r^A-\tilde{K}Q_{\alpha\beta A}r^A-(n-2)r^{-2}r^Br_BQ_{\alpha\beta A}r^A\\&+(n-2)r^{-1}r^A\left(\tilde{\nabla}_A\tilde{\nabla}^Br\right)Q_{\alpha\beta B}\\&+r^{-2}\left(\mathring{\nabla}_\alpha\mathring{\nabla}^\gamma Q_{\beta\gamma A}r^A\right)+r^{-2}\left(\mathring{\nabla}_\beta\mathring{\nabla}^\gamma Q_{\alpha\gamma A}r^A\right)=0.\end{aligned}$$

Comparing this expression with the spin-2 d'Alembertian (51) applied to $Q_{\alpha\beta A}r^A$,

$$\begin{aligned}&\not\Box_{\mathcal{L}(-2)}\left(Q_{\alpha\beta A}r^A\right) = \left(\tilde{\Box}Q_{\alpha\beta A}\right)r^A + \left(\tilde{\Box}r^A\right)Q_{\alpha\beta A}\\ &+ 2\left(\tilde{\nabla}^A\tilde{\nabla}^A r\right)\tilde{\nabla}_B Q_{\alpha\beta A} + (n-4)r^{-1}r^A\left(\tilde{\nabla}_A\tilde{\nabla}^B r\right)Q_{\alpha\beta B}\\ &+ (n-4)r^{-1}r^A r^B\tilde{\nabla}_B Q_{\alpha\beta B} + r^{-2}\mathring{\Delta}\left(Q_{\alpha\beta A}r^A\right)\\ &+ (6-2n)r^{-2}r^B r_B Q_{\alpha\beta A}r^A - 2r^{-1}\left(\tilde{\Box}r\right)Q_{\alpha\beta A}r^A,\end{aligned}$$

a lengthy reduction, using also the background calculations (28) and the commutation relation

$$\mathring{\nabla}_\alpha\mathring{\nabla}^\gamma Q_{\beta\gamma A} + \mathring{\nabla}_\beta\mathring{\nabla}^\gamma Q_{\alpha\gamma A} - \mathring{\Delta}Q_{\alpha\beta A} = -2Q_{\alpha\beta A},$$

yields the equation

$$\not\Box_{\mathcal{L}(-2)}\left(Q_{\alpha\beta A}r^A\right) = V^{(-)}\left(Q_{\alpha\beta A}r^A\right), \tag{144}$$

with

$$V^{(-)} := \frac{n+2}{r^2} - \frac{2Mn^2}{r^{n+1}}. \tag{145}$$

The quantity and associated equation are analogous to those first discovered in Kodama-Ishibashi-Seto [22], although their derivation is quite different in that work.

Subsequently, we employ the shorthand

$$Q^{(-)}_{\alpha\beta} := Q_{\alpha\beta A}r^A. \tag{146}$$

9.2.3. *Spin-Raising of P_α.* We denote by $\mathcal{D}$ the symmetrized gradient operation, and consider the quantity

$$S_{\alpha\beta} := r\left(\mathcal{D}P\right)_{\alpha\beta} := r\left(\mathring{\nabla}_\alpha P_\beta + \mathring{\nabla}_\beta P_\alpha\right). \tag{147}$$

Expanding with definition (51), we find

$$\begin{aligned}\not\Box_{\mathcal{L}(-2)}S_{\alpha\beta} &= \tilde{\Box}S_{\alpha\beta} + (n-4)r^{-1}r^A\tilde{\nabla}_A S_{\alpha\beta}\\ &+ r^{-2}\mathring{\Delta}S_{\alpha\beta} + (6-2n)r^{-2}r^A r_A S_{\alpha\beta} - 2r^{-1}\left(\tilde{\Box}r\right)S_{\alpha\beta}\\ &= r\left(\mathcal{D}\not\Box_{\mathcal{L}(-1)}P\right)_{\alpha\beta} + r^{-2}(n+1)S_{\alpha\beta}\\ &= \left(W + (n+1)r^{-2}\right)S_{\alpha\beta},\end{aligned}$$

where we have used

$$\mathring{\Delta}\mathcal{D}P = \mathcal{D}\mathring{\Delta}P + (n+1)\mathcal{D}P,$$

applied the definition (52), and used the wave equation (142). That is, the spin-raised quantity $S_{\alpha\beta}$ satisfies

$$\not\Box_{\mathcal{L}(-2)}S_{\alpha\beta} = V^{(-)}S_{\alpha\beta}, \tag{148}$$

with $V^{(-)}$ defined by (145).

9.3. **Analysis of the Master Equation.** We analyze solutions Ψ of the Regge-Wheeler type equation (144), including both $S_{\alpha\beta}$ and $Q^{(-)}_{\alpha\beta}$.

9.3.1. *The T-Energy Comparison.* The T-energy has the form

$$\begin{aligned} E^T_\Psi(\Sigma_\tau) &= \frac{1}{2}\int_{S^n}\int_{r_h}^\infty \left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + V^{(-)}|\Psi|^2\right] R^n dr dVol_{S^n} \\ &+ \frac{1}{2}\int_{S^n}\int_{r_h}^\infty \left[\cosh^{-2}x(1-\mu)^{-1}|\nabla\!\!\!/_\tau\Phi|^2 + R^{-2}|\mathring{\nabla\!\!\!/}\Phi|^2\right] R^n dr dVol_{S^n}. \end{aligned}$$

Performing the change of variables $s = R^{n-1}$, we evaluate the quantity

$$\int_{r_h}^\infty \left[s(s-2M)\left(\frac{df}{ds}\right)^2 + \left(\frac{n+2}{(n-1)^2} - \frac{2Mn^2}{(n-1)^2 s}\right) f^2\right] ds.$$

Choosing $F = \frac{n}{n-1}$ and $E = 2F - 1 = \frac{n+1}{n-1}$, and noting $\frac{n+2}{(n-1)^2} \geq \frac{1}{4}(E^2-1)$, Lemma 11 implies positivity of the expression above.

Choosing $\epsilon(n) > 0$ small, we find the lower bound

$$\begin{aligned} E^T_\Psi(\Sigma_\tau) &\geq \frac{1}{2}\int_{S^n}\int_{r_h}^\infty \epsilon(n)\left[(1-\mu)|\nabla\!\!\!/_R\Psi|^2 + V^{(-)}|\Psi|^2\right] R^n dr dVol_{S^n} \\ &+ \frac{1}{2}\int_{S^n}\int_{r_h}^\infty \left[\cosh^{-2}x(1-\mu)^{-1}|\nabla\!\!\!/_\tau\Phi|^2 + R^{-2}|\mathring{\nabla\!\!\!/}\Phi|^2\right] R^n dr dVol_{S^n} \\ &\gtrsim \check{E}^T_\Psi(\Sigma_\tau), \end{aligned}$$

with the reversed comparison in (112) following trivially. We conclude that Ψ satisfies the uniform boundedness estimate of Theorem 13 in all spacetime dimensions.

9.3.2. *The Morawetz Estimate.* Borrowing from the angular term via the Poincaré inequality (99), with least eigenvalue $\lambda = n$, and applying (121), we find

$$\int_{S^n} \operatorname{div}\left(J^{X,\omega^X}[\Psi] + \frac{f'}{1-\mu}\beta|\Psi|^2\partial_{r_*}\right) \geq \int_{S^n}\left(\frac{f'}{1-\mu}|\nabla\!\!\!/_{r_*}\Psi + \beta\Psi|^2 + W|\Psi|^2\right),$$

where

$$\begin{aligned} W(f,\beta) &= \left(-\frac{1}{4}\Box\omega^X - \frac{1}{2}V^{(-)'}f + \frac{1}{2}\mu_r V^{(-)}f\right) \\ &+ \left(\frac{f''\beta}{1-\mu} + f'\left(\frac{\beta'}{1-\mu} + \frac{n\beta}{r} - \frac{\beta^2}{1-\mu}\right)\right) \\ &+ \frac{nf}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right). \end{aligned}$$

For $n = 3$, choosing

$$\begin{aligned} f &= \left(1 - \frac{(n+1)M}{r^{n-1}}\right)\left(1 - \frac{M}{r^{n-1}} + \frac{4M^2}{5r^{2n-2}}\right) \\ &= \left(1 - \frac{4M}{r^2}\right)\left(1 - \frac{M}{r^2} + \frac{4M^2}{5r^4}\right), \\ r_+ \cdot \beta &= \frac{1}{2}, \end{aligned}$$

gives $f' \geq 0$ and $W(f, \beta) > 0$. For $n = 4$, choices of

$$\begin{aligned} f &= \left(1 - \frac{(n+1)M}{r^{n-1}}\right)\left(1 - \frac{2M}{r^{n-1}} + \frac{8M^2}{5r^{2n-2}}\right) \\ &= \left(1 - \frac{5M}{r^3}\right)\left(1 - \frac{2M}{r^3} + \frac{8M^2}{5r^6}\right), \\ r_+ \cdot \beta &= 1 + \frac{M}{r^3}, \end{aligned}$$

do the same. Furthermore, $W \approx r^{-3}$ near infinity.

Let $J[\Psi] = J^{X,\omega}[\Psi] + \frac{f'}{1-\mu}\beta|\Psi|^2 \frac{\partial}{\partial r_*}$. For any $\epsilon_0 > 0$ we have

$$\begin{aligned} \mathrm{div} J[\Psi] &= \frac{f'}{1-\mu}|\nabla\!\!\!/_{r_*}\Psi + \beta\Psi|^2 + \frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla\!\!\!/}\Psi|^2 + \mathring{W}|\Psi|^2 \\ &= \frac{\epsilon_0 f'}{1-\mu}|\nabla\!\!\!/_{r_*}\Psi|^2 + \frac{f'}{1-\mu}|(1-\epsilon_0)^{1/2}\nabla\!\!\!/_{r_*}\Psi + (1-\epsilon_0)^{-1/2}\beta\Psi|^2 \\ &+ \frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla\!\!\!/}\Psi|^2 + \mathring{W}|\Psi|^2 - \frac{\epsilon_0}{1-\epsilon_0}\frac{\beta^2 f'}{1-\mu}|\Psi|^2 \\ &\geq \frac{\epsilon_0 f'}{1-\mu}|\nabla\!\!\!/_{r_*}\Psi|^2 + \frac{f'}{1-\mu}|(1-\epsilon_0)^{1/2}\nabla\!\!\!/_{r_*}\Psi + (1-\epsilon_0)^{-1/2}\beta\Psi|^2 \\ &+ \frac{1}{2}\left(\frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla\!\!\!/}\Psi|^2 + \mathring{W}|\Psi|^2\right) \\ &+ \left(\frac{1}{2}W - \frac{\epsilon_0}{1-\epsilon_0}\frac{\beta^2 f'}{1-\mu}\right)|\Psi|^2. \end{aligned}$$

For $r \geq r_0$, $W \gtrsim r^{-3}$ and $\frac{\beta f'}{1-\mu} \approx r^{-n}$; by choosing ϵ_0 small enough, the last term is positive in this region. To control the angular derivative, notice that

$$\begin{aligned} &\frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla\!\!\!/}\Psi|^2 + \mathring{W}|\Psi|^2 \\ &\geq \epsilon_1 \frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla\!\!\!/}\Psi|^2 + W|\Psi|^2 - \epsilon_1 \frac{nf}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\Psi|^2. \end{aligned}$$

Again, $W \gtrsim r^{-3}$ for $r \geq r_0/2$ and $\frac{nf}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right) \approx r^{-3}$. By choosing ϵ_1 small, we have as $r \geq r_0/2$

$$\begin{aligned}&\frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla}\Psi|^2 + \mathring{W}|\Psi|^2\\ &\geq \epsilon_1 \frac{f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla}\Psi|^2 + \frac{1}{2}W|\Psi|^2.\end{aligned}$$

Hence for $r \geq r_0$,

$$\text{div}J[\Psi] \geq \frac{\epsilon_0 f'}{1-\mu}|\nabla\!\!\!/_{r_*}\Psi|^2 + \frac{\epsilon_1 f}{2r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)|\mathring{\nabla}\Psi|^2 + \frac{1}{4}W|\Psi|^2.$$

To obtain a $|\nabla\!\!\!/_t\Psi|^2$ term, let χ be a cut-off function with $\chi = 0$ as $r \leq r_0$ and $\chi = 1$ as $r \geq (r_0 + r_1)/2$. Define

$$g(r) := \epsilon_2 \frac{\chi}{r^n}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)^2,$$

where $\epsilon_2 > 0$ is a constant to be determined. Denoting

$$\tilde{J}_a[\Psi] := J_a[\Psi] + \frac{1}{4}(\nabla_a g)|\Psi|^2 - \frac{1}{4}g\nabla_a|\Psi|^2,$$

we calculate

$$\begin{aligned}\text{div}\tilde{J}[\Psi] &= \text{div}J[\Psi] + \frac{1}{4}\Box g|\Psi|^2 - \frac{1}{2}g(\nabla\Psi)^2 - \frac{1}{2}gV^{(-)}|\Psi|^2\\ &= \text{div}J[\Psi] + \frac{1}{2}g\left(\frac{1}{1-\mu}|\nabla\!\!\!/_t\Psi|^2 - \frac{1}{1-\mu}|\nabla\!\!\!/_{r_*}\Psi|^2 - \frac{1}{r^2}|\mathring{\nabla}\Psi|^2\right)\\ &+ \left(\frac{1}{4}\Box g - \frac{1}{2}gV^{(-)}\right)|\Psi|^2.\end{aligned}$$

Together with previous estimate, for $r \geq r_0$

$$\begin{aligned}\text{div}\tilde{J}[\Psi] \geq &\frac{g}{2(1-\mu)}|\nabla\!\!\!/_t\Psi|^2 + \left(\frac{\epsilon_0 f'}{1-\mu} - \frac{g}{2(1-\mu)}\right)|\nabla\!\!\!/_{r_*}\Psi|^2\\ &+ \left(\frac{\epsilon_1 f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right) - \frac{g}{2r^2}\right)|\mathring{\nabla}\Psi|^2\\ &+ \left(\frac{1}{4}W + \frac{1}{4}\Box g - \frac{1}{2}gV^{(-)}\right)|\Psi|^2.\end{aligned}$$

For large radii, $f' \approx g \approx r^{-n}$. By choosing ϵ_2 small, the coefficient of $|\nabla\!\!\!/_{r_*}\Psi|^2$ is greater than $\frac{\epsilon_0 f'}{2(1-\mu)}$. Both $\frac{\epsilon_1 f}{r^2}\left(1 - \frac{(n+1)M}{r^{n-1}}\right)$ and g vanish quadratically at the photon sphere, and the former, comparable to r^{-3}, decays no slower than the latter, comparable to r^{-n}. Hence the coefficient of $|\mathring{\nabla}\Psi|^2$ can also be made positive with ϵ_2 small. The term $\frac{1}{4}\Box g - \frac{1}{2}gV^{(-)}$ behaves like r^{-n-2} toward infinity; yet again, positivity of the coefficient for $|\Psi|^2$ is ensured by a choice of small ϵ_2. With such a choice, we have for $r \geq r_0$

$$\mathrm{div}\tilde{J}[\Psi] \gtrsim \frac{1}{r^n}\left(\left(1-\frac{(n+1)M}{r^{n-1}}\right)^2\left(|\nabla\!\!\!/_t\Psi|^2+|\mathring{\nabla\!\!\!/}\Psi|^2\right)+|\nabla\!\!\!/_r\Psi|^2\right).$$

In $r \leq r_0$, $\mathrm{div}\tilde{J}[\Psi] = \mathrm{div}J[\Psi] \geq 0$. By adding $\epsilon_3\check{T}_{ab}[\Psi]Y^b$ into $\tilde{J}_a[\Psi]$, we obtain a Morawetz current

$$\begin{aligned} J_a^{\mathfrak{M}}[\Psi] &:= \tilde{J}_a[\Psi] + \epsilon_3\check{T}_{ab}[\Psi]Y^b \\ &= T_{ab}[\Psi]X^b - \frac{1}{2}V^{(-)}|\Psi|^2X_a + \frac{1}{4}(\omega^X - g)\nabla_a|\Psi|^2 \\ &\quad - \frac{1}{4}\nabla_a(\omega^X-g)|\Psi|^2 + f'\beta|\Psi|^2 dR_* + \epsilon_3\check{T}_{ab}[\Psi]Y^b. \end{aligned} \tag{149}$$

By choosing ϵ_3 small enough, $K^{\mathfrak{M}}[\Psi] := \mathrm{div}J^{\mathfrak{M}}[\Psi]$ has the desired positivity. To estimate $J_a^{\mathfrak{M}}[\Psi]n^a_{\Sigma_\tau}$, note that X and Y are regular vector fields, such that

$$|(T_{ab}[\Psi]X^b - \frac{1}{2}V^{(-)}|\Psi|^2X_a + \epsilon_3\check{T}_{ab}[\Psi]Y^b)n^a_{\Sigma_\tau}| \lesssim T_{ab}[\Psi]n^a_{\Sigma_\tau}N^b.$$

From $\omega^X - g \approx r^{-1}$ for large radii, the third and the fourth terms are controlled. The estimate for the fifth term follows from regularity of $\frac{f'}{1-\mu} = \frac{\partial f}{\partial r}$ at the horizon in addition to its comparability with r^{-n} for large radii. The estimate for $J_a^{\mathfrak{M}}[\Psi]\underline{L}^a$ is similar. For $J_a^{\mathfrak{M}}[\Psi]n^a_{\mathcal{H}^+}$, we note that on the horizon $\frac{\partial}{\partial r_*}$ and $n^a_{\mathcal{H}^+}$ are proportional to T and that $\omega^X - g = 0$. Except for $\epsilon_3\check{T}_{ab}[\Psi]Y^bn^a_{\mathcal{H}^+}$, satisfying $\check{T}_{ab}[\Psi]Y^bn^a_{\mathcal{H}^+} \geq 0$, the terms in $J_a^{\mathfrak{M}}[\Psi]n^a_{\mathcal{H}^+}$ are bounded by $\check{T}_{ab}[\Psi]T^bn^a_{\mathcal{H}^+}$, from which Assumption 14 follows.

We remark that the construction of a current $\tilde{J}_a[\Psi]$ with positivity in $r \leq r_0$, an essential feature in the construction above, is impossible for $n \geq 5$, i.e. in spacetime dimension seven and above. From direct computation,

$$4W(f,\beta)|_{r=r_+} = -\left(\frac{n-1}{r_+} - 2\beta(r_+)\right)^2\frac{df}{dr}(r_+) + \frac{(n-4)(n^2-1)}{r_+^3}f(r_+).$$

Under the requirement that $\frac{df}{dr} \geq 0$ and $f(r_P) = 0$, one must have $f(r_+) \leq 0$. Therefore as $n \geq 5$, $W(f,\beta)|_{r=r_+}$ is non-positive and equals zero only if $f \equiv 0$ in $r \in [r_+, r_P]$. In this situation, the Morawetz current cannot be used to absorb error in the red-shift estimate. In this regime, a more refined analysis is needed to form a Morawetz current satisfying the necessary density estimate. This difficulty is the reason for our restriction to dimensions six and below in our decay estimates.

9.4. **Uniform Boundedness and Decay of the Master Quantities.** With the T-comparison, we have uniform boundedness of solutions to the Regge-Wheeler equation (142) in all spacetime dimensions; in addition, the

Morawetz estimate for $n \leq 4$ gives uniform decay of solutions in six and fewer spacetime dimensions.

Theorem 21. *Let δg be a smooth, symmetric two-tensor on a Schwarzschild-Tangherlini spacetime, satisfying the linearized Einstein equation* (2)*. There exists a gauge-invariant master quantities $Q^{(-)}_{\alpha\beta}$ and $S_{\alpha\beta}$ in the co-vector portion h_2 of δg satisfying the Regge-Wheeler type equation* (142)*. As solutions of* (142)*, $Q^{(-)}_{\alpha\beta}$ and $S_{\alpha\beta}$ satisfy the uniform boundedness estimate*

$$\begin{aligned} \check{E}^N_{Q^{(-)}}(\Sigma_\tau) &\lesssim \check{E}^N_{Q^{(-)}}(\Sigma_0), \\ \check{E}^N_S(\Sigma_\tau) &\lesssim \check{E}^N_S(\Sigma_0), \end{aligned} \tag{150}$$

in all spacetime dimensions. In six and fewer spacetime dimensions, $Q^{(-)}_{\alpha\beta}$ and $S_{\alpha\beta}$ satisfy the uniform decay estimate

$$\begin{aligned} \check{E}^N_{Q^{(-)}}(\Sigma_\tau) &\lesssim \frac{I_{Q^{(-)}}(\Sigma_0)}{\tau^2}, \\ \check{E}^N_S(\Sigma_\tau) &\lesssim \frac{I_S(\Sigma_0)}{\tau^2}, \end{aligned} \tag{151}$$

where

$$I_{Q^{(-)}}(\Sigma_0) := E^2_{Q^{(-)},\mathcal{L}_K Q^{(-)}}(\Sigma_0) + E^N_{Q^{(-)},\mathcal{L}_K Q^{(-)},\mathcal{L}^2_K Q^{(-)}}(\Sigma_0), \tag{152}$$

and similarly for S. Here we assume $\tau \geq 0$.

We emphasize that the relevant constants in the comparisons depend only upon the orbit sphere dimension n and the mass $M > 0$.

10. The Scalar Portion

10.1. **The Linearized Einstein Equations.** In this subsection we reduce the linearized vacuum Einstein equations for the scalar solution, following Kodama-Ishibashi [21]. Subsequently, we modify the gauge-invariant master quantity appearing in their work, generalizing the Moncrief-Zerilli quantity [26, 32], and argue that our modified quantity satisfies a Regge-Wheeler equation. We remark that such an argument is absent in [21].

The authors consider the linearized Einstein tensor

$$\delta E_{ab} dx^a dx^b = \delta E_{AB} dx^A dx^B + 2\delta E_{A\alpha} dx^A dx^\alpha + \delta E_{\alpha\beta} dx^\alpha dx^\beta, \tag{153}$$

admitting the same Hodge decomposition as the linearized Ricci tensor above. Define

$$e_A := H_A - \frac{1}{2} r^2 \tilde{\nabla}_A \left(r^{-2} H_2\right), \tag{154}$$

$$\tilde{h}_{AB} := h_{AB} - \tilde{\nabla}_A e_B - \tilde{\nabla}_B e_A, \tag{155}$$

$$\tilde{H} := \frac{1}{2r^2}\left(H - \frac{1}{n}\mathring{\Delta} H_2 - 2r r^A e_A\right), \tag{156}$$

gauge-invariant quantities per (71) and Definition 7. The linearized vacuum Einstein equations for the scalar portion can be expressed in terms of $\tilde{h}_{AB}$ and $\tilde{H}$. Following [21], we further rescale the quantities as

$$\begin{aligned} \delta\check{E}_{ab} &:= r^{n-2}\delta E_{ab}, \\ \check{h}_{AB} &:= r^{n-2}\tilde{h}_{AB}, \\ \check{H} &:= r^{n-2}\tilde{H}. \end{aligned} \tag{157}$$

As noted in the same work, owing to the Bianchi identities, it suffices to consider the rescaled linearized Einstein tensor components

$$\delta\check{E}_{A\alpha},\ \delta\check{E}_{\alpha\beta} - \frac{1}{n}\left(\mathring{\sigma}^{\gamma\delta}\delta\check{E}_{\gamma\delta}\right)\mathring{\sigma}_{\alpha\beta},\ \delta\check{E}_{tr},\ \delta\check{E}_{rr}.$$

With the aim of rewriting these remaining linearized equations, we define the further gauge-invariants from $\check{h}_{AB}$ and $\check{H}$

$$\begin{aligned} X &= \check{h}^t_t - 2\check{H}, \\ Y &= \check{h}^r_r - 2\check{H}, \\ Z &= \check{h}^r_t. \end{aligned} \tag{158}$$

Using the algebraic relation provided by the traceless equation

$$\delta\check{E}_{\alpha\beta} - \frac{1}{n}\left(\mathring{\sigma}^{\gamma\delta}\delta\check{E}_{\gamma\delta}\right)\mathring{\sigma}_{\alpha\beta} = 0,$$

we can invert these relations and find

$$\begin{aligned} \check{h}^t_t &= \frac{(n-1)X - Y}{n}, \\ \check{h}^r_r &= \frac{-X + (n-1)Y}{n}, \\ \check{h}^r_t &= Z, \\ \check{H} &= -\frac{X+Y}{2n}. \end{aligned} \tag{159}$$

We rewrite the linearized Einstein equations using scalar spherical harmonic expansion with indices $\ell \geq 2$ and $m_s(n,\ell) \in \{1, \dots, d_s(n,\ell)\}$, with the dimension of the eigenspaces $d_s(n,\ell)$ given by (31), and the inversion (159). First, the cross-term equations $\delta\check{E}_{A\alpha} = 0$ imply

$$\begin{aligned} \partial_r Z_{\ell m_s(n,\ell)} &= -\partial_t X_{\ell m_s(n,\ell)}, \\ \partial_r Y_{\ell m_s(n,\ell)} &= \frac{\partial_r f}{2f}\left(X_{\ell m_s(n,\ell)} - Y_{\ell m_s(n,\ell)}\right) + \frac{1}{f^2}\partial_t Z_{\ell m_s(n,\ell)}, \end{aligned} \tag{160}$$

where we adopt the notation $f := 1 - \mu$ of [21].

Substituting (160), the quotient equation $\delta \check{E}^r_r = 0$ can be rewritten as

$$\begin{aligned}\partial_r X_{\ell m_s(n,\ell)} &= \frac{2}{\partial_r f}\Bigg(\frac{n-1}{r^2}(f-1) + \frac{(n+2)\partial_r f}{2r} + \frac{\partial_r^2 f}{n} \\ &- \left(\frac{\partial_r f}{2} + \frac{nf}{r}\right)\frac{\partial_r f}{2f}\Bigg) X_{\ell m_s(n,\ell)} + \frac{2}{\partial_r f}\Bigg(\frac{1-f}{r^2} - \frac{(3n-2)}{2r}\partial_r f \\ &- \frac{(n-1)}{n}\partial_r^2 f + \frac{\ell(\ell+n-1)-n}{r^2} + \left(\frac{\partial_r f}{2} + \frac{nf}{r}\right)\frac{\partial_r f}{2f}\Bigg) Y_{\ell m_s(n,\ell)} \\ &+ \frac{2}{\partial_r f}\left(\frac{2n}{rf} - \frac{1}{f^2}\left(\frac{\partial_r f}{2} + \frac{nf}{r}\right)\right)\partial_t Z_{\ell m_s(n,\ell)} \\ &+ \frac{2}{f\partial_r f}\left(\partial_t^2 X_{\ell m_s(n,\ell)} + \partial_t^2 Y_{\ell m_s(n,\ell)}\right).\end{aligned} \tag{161}$$

Finally, the quotient equation $\delta \check{E}^r_t = 0$ has the form

$$\begin{aligned}&\left(\frac{\ell(\ell+n-1)}{r^2} - \partial_r^2 f - \frac{n\partial_r f}{r}\right) Z_{\ell m_s(n,\ell)} \\ &+ f\left(\partial_t\partial_r X_{\ell m_s(n,\ell)} + \partial_t\partial_r Y_{\ell m_s(n,\ell)}\right) \\ &- \left(\frac{(n-2)}{r}f + \frac{\partial_r f}{2}\right)\partial_t X_{\ell m_s(n,\ell)} + \left(\frac{2f}{r} - \frac{\partial_r f}{2}\right)\partial_t Y_{\ell m_s(n,\ell)} = 0.\end{aligned} \tag{162}$$

10.2. **Master Equation for the Scalar Portion.** Let us define the further gauge-invariant quantity

$$\begin{aligned}\tilde{Z}_{\ell m_s(n,\ell)} := \frac{1}{\left(\frac{\ell(\ell+n-1)}{r^2} - \partial_r^2 f - \frac{n\partial_r f}{r}\right)}&\Bigg(f\left(\partial_r X_{\ell m_s(n,\ell)} + \partial_r Y_{\ell m_s(n,\ell)}\right) \\ &- \left(\frac{(n-2)}{r}f + \frac{\partial_r f}{2}\right) X_{\ell m_s(n,\ell)} + \left(\frac{2f}{r} - \frac{\partial_r f}{2}\right) Y_{\ell m_s(n,\ell)}\Bigg),\end{aligned} \tag{163}$$

in addition to

$$\Phi_{\ell m_s(n,\ell)} = \frac{n\tilde{Z}_{\ell m_s(n,\ell)} - r(X_{\ell m_s(n,\ell)} + Y_{\ell m_s(n,\ell)})}{r^{n/2-1}(\ell(\ell+n-1) - n + \frac{1}{2}n(n+1)\mu)}, \tag{164}$$

modifying those definitions provided in [21]. Substituting (161) and (160) into $\tilde{Z}_{\ell m_s(n,\ell)}$, we regard $\Phi_{\ell m_s(n,\ell)}$ as an expression in $X_{\ell m_s(n,\ell)}$, $Y_{\ell m_s(n,\ell)}$, $\partial_t Z_{\ell m_s(n,\ell)}$, $\partial_t^2 X_{\ell m_s(n,\ell)}$, $\partial_t^2 Y_{\ell m_s(n,\ell)}$.

As described below, $\Phi_{\ell m_s(n,\ell)}$ satisfies the following scalar wave equation:

$$\tilde{\Box}\Phi_{\ell m_s(n,\ell)} = \tilde{V}_\ell^{(+)}\Phi_{\ell m_s(n,\ell)}, \tag{165}$$

where

$$\tilde{V}_\ell^{(+)} := \frac{Q_\ell}{16r^2\left(q + n(n+1)\mu/2\right)^2}, \tag{166}$$

with

$$q := \ell(\ell + n - 1) - n$$

and

$$\begin{aligned} Q_\ell := n^4(n+1)^2\mu^3 + n(n+1)\Big[4(2n^2 - 3n + 4)q \\ + n(n-2)(n-4)(n+1)\Big]\mu^2 - 12n\Big[(n-4)q \\ + n(n+1)(n-2)\Big]q\mu + 16q^3 + 4n(n+2)q^2. \end{aligned} \tag{167}$$

Note that the equation reduces to the well-known Zerilli equation in the case $n = 2$.

Suppressing the angular harmonic dependence where possible, the equation in standard Schwarzschild coordinates is tantamount to

$$\partial_r\left(f\partial_r\Phi_\ell\right) = \tilde{V}_\ell^{(+)}\Phi_\ell + \frac{1}{f}\partial_t^2\Phi_\ell.$$

Again, we regard Φ_ℓ as an expression in $X, Y, \partial_t Z, \partial_t^2 X, \partial_t^2 Y$, and apply the linearized Einstein equations (160, 161, 162) to the left-hand side of the equation above as follows. Differentiating in r, we replace radial derivatives on $X, Y, \partial_t Z$ using (161) and (160), respectively. Differentiation of the terms $\partial_t^2 X$ and $\partial_t^2 Y$ results in mixed partials, with the same coefficient on each quantity; we replace these terms using (162). After this first differentiation and substitution, we are again left with an expression in $X, Y, \partial_t Z, \partial_t^2 X, \partial_t^2 Y$. Multiplying by f and differentiating the product in r, we again replace radial derivatives on $X, Y, \partial_t Z$ using (161) and (160). The situation for the mixed partial terms is more subtle. Regarding the mixed term obtained via radial derivatives on $\partial_t^2 X$, we first substitute (161) for an appropriate portion in order to match the $\partial_t^4 X + \partial_t^4 Y$ term in the right-hand side of the equation above. For the remainder of the term, we add and subtract appropriate radial derivatives on $\partial_t^2 Y$ and apply (162) to the matching mixed partial terms in X and Y. Finally, the residual radial derivatives on $\partial_t^2 Y$ are handled via substitution with (160). The resulting quantity, an expression in $X, Y, \partial_t Z, \partial_t^2 X, \partial_t^2 Y$ and their second time derivatives, is equal to the right-hand side of the equation above, as can be verified by direct calculation. We remark that verification of the Regge-Wheeler equation for Φ_ℓ is absent in [21].

10.3. **The Spin-Raised Equation.** We spin-raise the equation by associating $\Phi_{\ell m_s(n,\ell)}$ with a symmetric traceless two-tensor $Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}$, specified by

$$Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta} := \left(r^k\Phi_{\ell m_s(n,\ell)}\right)Y^{\ell m_s(n,\ell)}_{\alpha\beta}dx^\alpha dx^\beta, \tag{168}$$

where

$$k := \frac{4-n}{2}$$

and $Y^{\ell m_s(n,\ell)}_{\alpha\beta}$ are the tensor spherical harmonics (35).

We calculate

$$
\begin{aligned}
\not\Box_{\mathcal{L}(-2)} Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta} &= \Big[r^k \tilde{\Box} \Phi_{\ell m_s(n,\ell)} + \tilde{\Box}(r^k) \Phi_{\ell m_s(n,\ell)} \\
&+ (n-4) k r^{-2} r^A r_A r^k \Phi_{\ell m_s(n,\ell)} \Big] Y^{\ell m_s(n,\ell)}_{\alpha\beta} \\
&+ r^{-2} r^k \Phi_{\ell m_s(n,\ell)} \left(2n - \ell(\ell+n-1)\right) Y^{\ell m_s(n,\ell)}_{\alpha\beta} \\
&+ (6-2n) r^{-2} r^A r_A Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta} - 2r^{-1} \left(\tilde{\Box} r\right) Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta} \\
&= \Big[\tilde{V}^{(+)}_\ell + k(k-1) r^B r_B r^{-2} + k r^{-1} \tilde{\Box} r \\
&+ (n-4) r^{-2} r^B r_B + \left(2n - \ell(\ell+n-1)\right) r^{-2} \\
&+ (6-2n) r^{-2} r^B r_B - 2r^{-1} \left(\tilde{\Box} r\right) \Big] Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta},
\end{aligned}
$$

where we have used

$$
\tilde{\Box}(r^k) = k(k-1) r^{k-2} r^A r_A + k r^{k-1} \tilde{\Box} r.
$$

Simplifying, we obtain the master equation

$$
\not\Box_{\mathcal{L}(-2)} Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta} = V^{(+)}_\ell Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}, \tag{169}
$$

where

$$
V^{(+)}_\ell = V_{1,n\ell} + V_{2,n\ell} + V_{3,n\ell}
$$

is given by

$$
\begin{aligned}
r^2 V_{1,n\ell} &= -(\ell(\ell+n-1) - 2n), \\
r^2 V_{2,n\ell} &= \left(\frac{n^2 - 10n + 16}{4} - \frac{3n^2 - 12n + 16}{4} \left(\frac{2M}{r^{n-1}} \right) \right),
\end{aligned} \tag{170}
$$

and

$$
\begin{aligned}
4r^2 V_{3,n\ell} = \frac{1}{D^2_{n\ell}} \Bigg[& (\ell-1)^2 (n+\ell)^2 n(n+2) + 4(\ell-1)^3 (n+\ell)^3 \\
& - \Big(6(\ell-1)(n+\ell)(n-2) n^2 (n+1) M \\
& + 6(\ell-1)^2 (n+\ell)^2 (n-4) n M \Big) r^{-n+1} \\
& + \Big(4(\ell-1)(n+\ell) n(n+1)(2n^2 - 3n + 4) M^2 \\
& + (n-4)(n-2) n^2 (n+1)^2 M^2 \Big) r^{-2n+2} \\
& + 2n^4 (n+1)^2 M^3 r^{-3n+3} \Bigg],
\end{aligned} \tag{171}
$$

with

$$
D_{n\ell} = n(n+1) M r^{-n+1} + (\ell-1)(n+\ell). \tag{172}
$$

Note that the potentials converge as

$$\lim_{\ell\to\infty} V_\ell^{(+)} = V^{(+)} := \frac{1}{2r^2}\left((n^2-2n+8) - (5n^2-10n+8)\frac{2M}{r^{n-1}}\right). \quad (173)$$

10.4. **Analysis of the Master Equation.** In this subsection we show for any dimension $n \geq 3$, the T-energy comparison (112) holds for the scalar portion. In dimensions five and six, we further obtain Morawetz estimates.

Lemma 22. *For any $n \geq 3$ and and $\ell \geq 2$, $Q^{(+)}_{\ell m_s(n\ell)}$ satisfies the T-energy comparison* (112)*. For $n = 5, 6$ and any $\ell \geq 2$, $Q^{(+)}_{\ell m_s(n\ell)}$ satisfies a Morawetz estimate as in Assumption 14.*

This will be proved in the following two subsubsections.

10.4.1. *The T-Energy Comparison.* The T-energy comparison for the Zerilli potential is more involved than the earlier equations. The argument naturally splits into two regimes, $n \geq 4$ and $n \leq 3$.

Turning to the higher dimensional analysis, with $n \geq 4$, we make use of the following inequalities:

For α, β, we have

$$(\alpha+\beta)^2 \geq 4\alpha\beta,$$

and for x, y and $\lambda_1, \lambda_2 \geq 0$,

$$\frac{\lambda_1 x^2 + \lambda_2 y^2}{(x+y)^2} \geq \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}.$$

Moreover, we let $s = r^{n-1}$.

Grouping terms in $V^3_{n\ell}$,

$$\begin{aligned}4r^2V^3_{n\ell} = \frac{1}{D^2_{n\ell}}\Bigg[&\underbrace{(\ell-1)^2(n+\ell)^2n(n+2)}_{\text{I}} + \underbrace{4(\ell-1)^3(n+\ell)^3}_{\text{II}} + 2n^4(n+1)^2M^3r^{-3n+3}\\
&+\Big(\underbrace{(n-4)(n-2)n^2(n+1)^2M^2}_{\text{I}} + \underbrace{4(\ell-1)(n+\ell)n(n+1)(2n^2-3n+4)M^2}_{\text{II}}\Big)r^{-2n+2}\\
&-\Big(\underbrace{6(\ell-1)(n+\ell)(n-2)n^2(n+1)M}_{\text{III}} + 6(\ell-1)^2(n+\ell)^2(n-4)nM\Big)r^{-n+1}\Bigg]\end{aligned}$$

we estimate each of the pieces as follows.

For I and II, we apply the second inequality to deduce

$$\begin{aligned}\text{I}/D^2_{n\ell} &= \frac{\overbrace{(\ell-1)^2(n+\ell)^2}^{x^2}\overbrace{n(n+2)}^{\lambda_1} + \overbrace{n^2(n+1)^2M^2s^{-2}}^{y^2}\overbrace{(n-4)(n-2)}^{\lambda_2}}{((\ell-1)(n+\ell)+n(n+1)Ms^{-1})^2}\\
&\geq \frac{n(n+2)(n-4)(n-2)}{n(n+2)+(n-4)(n-2)} = \frac{n(n+2)(n-4)(n-2)}{2(n^2-2n+4)}.\end{aligned}$$

and

$$\begin{aligned}\mathrm{II}/D_{n\ell}^2 &= \frac{(n+\ell)^2(\ell-1)^2(4(n+\ell)(\ell-1)) + n^2(n+1)^2M^2s^{-2}(\frac{4(n+\ell)(\ell-1)(2n^2-3n+4)}{n(n+1)})}{((\ell-1)(n+\ell)+n(n+1)Ms^{-1})^2} \\ &\geq 4(\ell-1)(n+\ell)\frac{2n^2-3n+4}{3n^2-2n+4}.\end{aligned}$$

For III, we apply the second inequality:

$$\begin{aligned}-\mathrm{III}/D_{n\ell}^2 &= -\frac{6(\ell-1)(n+\ell)(n-2)n^2(n+1)Ms^{-1}}{\Big(n(n+1)Ms^{-1}+(\ell-1)(n+\ell)\Big)^2} \\ &\geq -\frac{6(\ell-1)(n+\ell)(n-2)n^2(n+1)Ms^{-1}}{4(\ell-1)(n+\ell)n(n+1)Ms^{-1}} \\ &= -\frac{3}{2}n(n-2).\end{aligned}$$

After this first round of estimates, we have

$$\begin{aligned}r^2V_{n\ell}^3 \geq{}& \frac{n(n+2)(n-2)(n-4)}{8(n^2-2n+4)} + (\ell-1)(n+\ell)\frac{2n^2-3n+4}{3n^2-2n+4} - \frac{3}{8}n(n-2) \\ &+ \frac{1}{4D_{n\ell}^2}\left[2n^4(n+1)^2M^3s^{-3} - 6(\ell-1)^2(n+\ell)^2(n-4)nMs^{-1}\right].\end{aligned}$$

Defining the quadratic polynomial

$$P(n) = an^2 + bn + c,$$

with a, b, c as yet unchosen, we rewrite the estimate above as

$$\begin{aligned}r^2V_{n\ell}^3 \geq{}& \frac{n(n+2)(n-2)(n-4)}{8(n^2-2n+4)} + (\ell-1)(n+\ell)\frac{2n^2-3n+4}{3n^2-2n+4} - \frac{3}{8}n(n-2) \\ &+ \frac{1}{4D_{n\ell}^2}\Big[\underbrace{2n^4(n+1)^2M^3s^{-3} + P(n)(\ell-1)^2(n-\ell)^2Ms^{-1}}_{\mathrm{IV}} \\ &\underbrace{-P(n)(\ell-1)^2(n+\ell)^2Ms^{-1} - 6(\ell-1)^2(n+\ell)^2(n-4)nMs^{-1}}_{\mathrm{V}}\Big].\end{aligned}$$

Using the second inequality, we estimate

$$\begin{aligned}\mathrm{IV}/D_{n\ell}^2 &= Ms^{-1}\frac{(\ell-1)^2(n-\ell)^2P(n) + n^2(n+1)^2M^2s^{-2}(2n^2)}{((\ell-1)(n+\ell)+n(n+1)Ms^{-1})^2} \\ &\geq Ms^{-1}\frac{2n^2P(n)}{2n^2+P(n)} = \left(\frac{n^2P(n)}{2n^2+P(n)}\right)\frac{2M}{s},\end{aligned}$$

where we have assumed that $P(n)$ is non-negative.

Next, we control V with the first inequality:

$$\begin{aligned}
\mathrm{V}/D_{n\ell}^2 &= \frac{(\ell-1)^2(n+\ell)^2(-6n^2+24n-P(n))Ms^{-1}}{((\ell-1)(n+\ell)+n(n+1)Ms^{-1})^2} \\
&\geq (\ell-1)(n+\ell)\frac{-6n^2+24n-P(n)}{4n(n+1)}.
\end{aligned}$$

After this second round of estimates, we have

$$\begin{aligned}
r^2V_{n\ell}^3 \geq{}& \frac{n(n+2)(n-2)(n-4)}{8(n^2-2n+4)} - \frac{3}{8}n(n-2) + \frac{1}{4}\left(\frac{n^2P(n)}{2n^2+P(n)}\right)\frac{2M}{s} \\
&+ (\ell-1)(n+\ell)\left(\frac{2n^2-3n+4}{3n^2-2n+4} + \frac{-6n^2+24n-P(n)}{16n(n+1)}\right) \\
\geq{}& \frac{n(n+2)(n-2)(n-4)}{8(n^2-2n+4)} - \frac{3}{8}n(n-2) + \frac{1}{4}\left(\frac{n^2P(n)}{2n^2+P(n)}\right)\frac{2M}{s} \\
&+ (n+2)\left(\frac{2n^2-3n+4}{3n^2-2n+4} + \frac{-6n^2+24n-P(n)}{16n(n+1)}\right),
\end{aligned}$$

assuming that

$$\frac{2n^2-3n+4}{3n^2-2n+4} + \frac{-6n^2+24n-P(n)}{16n(n+1)} \geq 0.$$

As $V_{n\ell}^1$ can be accounted for by borrowing from the angular gradient, it remains to consider

$$\begin{aligned}
&\left[r^2V_{n\ell}^2 + r^2V_{n\ell}^3\right]\frac{1}{(n-1)^2} \\
&\geq \left[\frac{n^2-10n+16}{4} + \frac{n(n+2)(n-2)(n-4)}{8(n^2-2n+4)} - \frac{3}{8}n(n-2)\right. \\
&\left. + (n+2)\left(\frac{2n^2-3n+4}{3n^2-2n+4} + \frac{-6n^2+24n-P(n)}{16n(n+1)}\right)\right]\frac{1}{(n-1)^2} \\
&- \left[\frac{3n^2-12n+16}{4} - \frac{1}{4}\frac{n^2P(n)}{2n^2+P(n)}\right]\frac{1}{(n-1)^2}\frac{2M}{s}.
\end{aligned}$$

Setting $F_n^2 = \left[\frac{3n^2-12n+16}{4} - \frac{1}{4}\frac{n^2P(n)}{2n^2+P(n)}\right]\frac{1}{(n-1)^2}$ and $E_n = 2F_n - 1$, we must check that

$$\begin{aligned}
&\left[\frac{n^2-10n+16}{4} + \frac{n(n+2)(n-2)(n-4)}{8(n^2-2n+4)} - \frac{3}{8}n(n-2)\right. \\
&\left. + (n+2)\left(\frac{2n^2-3n+4}{3n^2-2n+4} + \frac{-6n^2+24n-P(n)}{16n(n+1)}\right)\right]\frac{1}{(n-1)^2} \geq \frac{1}{4}(E_n^2-1).
\end{aligned}$$

Choosing $P(n) = 2n^2 - 5n + 5$, fulfilling both of the conditions on $P(n)$ above, we obtain positivity for $n \geq 7$ and $n = 4$. We remark that we can always perturb the estimate slightly and retain $\epsilon(n+\ell)^3(\ell-1)^3$ in II; phrased another way, we can keep a small piece of the angular gradient, as necessary to demonstrating positivity of the T-energy.

To extend the estimate to all $n \geq 4$, we need a refinement of the second inequality above. Namely, given $\lambda_1, \lambda_2 > 0$ and $x, y > 0$, with some $\alpha > 0$ such that $0 < y < \alpha < \frac{\lambda_1}{\lambda_2}x$, we have

$$\frac{\lambda_1 x^2 + \lambda_2 y^2}{(x+y)^2} \geq (1+\epsilon)\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2},$$

where

$$\epsilon = \frac{(\lambda_1 x - \lambda_2\alpha)^2}{\lambda_1\lambda_2(x+\alpha)^2}.$$

Specializing to I, with $x = (\ell-1)(n+\ell), y = n(n+1)Ms^{-1}, \lambda_1 = n(n+2), \lambda_2 = (n-4)(n-2)$, we note that $Ms^{-1} < \frac{1}{2}$, such that $0 < y < \frac{1}{2}n(n+1) < \frac{1}{2}n(n+\ell)(\ell-1)$. The relation

$$y < \alpha = \frac{1}{2}n(n+\ell)(\ell-1) < \frac{n(n+2)}{(n-4)(n-2)}(n+\ell)(\ell-1) = \frac{\lambda_1}{\lambda_2}x$$

holds for $5 \leq n \leq 7$, and we obtain the improvement

$$\begin{aligned} r^2 V^3_{n\ell} \geq{}& (1+\epsilon)\frac{n(n+2)(n-2)(n-4)}{8(n^2-2n+4)} - \frac{3}{8}n(n-2) + \frac{1}{4}\left(\frac{n^2P(n)}{2n^2+P(n)}\right)\frac{2M}{s} \\ &+ (n+2)\left(\frac{2n^2-3n+4}{3n^2-2n+4} + \frac{-6n^2+24n-P(n)}{16n(n+1)}\right), \end{aligned}$$

with

$$\epsilon = \frac{n(n^2-8n+4)^2}{(n-4)(n-2)(n+2)^3}.$$

Using the same choices of $P(n), F_n$ and E_n as above, the result extends to include $n = 5$ and $n = 6$.

Finally, we consider the low dimensions $n = 2$ and $n = 3$. Here we group

$$\begin{aligned} 4r^2V^3_{n\ell} = \frac{1}{D^2_{n\ell}}\Bigg[&(\ell-1)^2(n+\ell)^2 n(n+2) + \underbrace{4(\ell-1)^3(n+\ell)^3}_{\text{II}} + 2n^4(n+1)^2M^3r^{-3n+3} \\ &+ \Big(\underbrace{(n-4)(n-2)n^2(n+1)^2M^2}_{\text{I}} + \underbrace{4(\ell-1)(n+\ell)n(n+1)(2n^2-3n+4)M^2}_{\text{II}}\Big)r^{-2n+2} \\ &- \Big(\underbrace{6(\ell-1)(n+\ell)(n-2)n^2(n+1)M}_{\text{III}} + \underbrace{6(\ell-1)^2(n+\ell)^2(n-4)nM}_{\text{I}}\Big)r^{-n+1}\Bigg]. \end{aligned}$$

The terms II and III are handled just as before. The terms in I change sign for these low dimensions, and we estimate

$$\begin{aligned}\mathrm{I} &= 6(\ell-1)^2(n+\ell)^2(4-n)nMs^{-1} - (n-2)n^2(n+1)^2(4-n)M^2s^{-2}\\ &\geq 12(n+2)^2(4-n)nM^2s^{-2} - (n-2)n^2(n+1)^2(4-n)M^2s^{-2}\\ &\geq n(4-n)\left(12(n+2)^2 - n(n-2)(n+1)^2\right)M^2s^{-2}.\end{aligned}$$

In this way, we obtain

$$r^2V^3_{n\ell} \geq (\ell-1)(n+\ell)\frac{2n^2-3n+4}{3n^2-2n+4} - \frac{3}{8}n(n-2)$$
$$+\frac{1}{4D^2_{n\ell}}\left[\underbrace{(\ell-1)^2(n+\ell)^2n(n+2) + n(4-n)\left(12(n+2)^2 - n(n-2)(n+1)^2\right)M^2s^{-2}}_{\mathrm{IV}}\right].$$

Applying the second inequality, we find

$$\mathrm{IV}/D^2_{n\ell} \geq \frac{n(n+2)(4-n)(48+50n+15n^2-n^4)}{2(n^5-5n^3+6n^2+76n+96)}.$$

With this new lower bound, and the usual choices of F_n and E_n, the result extends to $n=2$ and $n=3$.

10.4.2. *The Morawetz Estimate.* Borrowing from the angular term using (99), we have

$$\begin{aligned}&\int_{S^n}\mathrm{div}\left(J^{X,\omega^X}[Q^{(+)}_{\ell m_s(n,\ell)}] + \frac{f'}{1-\mu}\beta|Q^{(+)}_{\ell m_s(n,\ell)}|^2\partial_{r_*}\right)\\ &\geq \int_{S^n}\left[\frac{f'}{1-\mu}|\nabla\!\!\!/_{r_*}Q^{(+)}_{\ell m_s(n,\ell)} + \beta Q^{(+)}_{\ell m_s(n,\ell)}|^2 + W|Q^{(+)}_{\ell m_s(n,\ell)}|^2\right],\end{aligned}$$

with

$$\begin{aligned}W(f,\beta) = &\left(-\frac{1}{4}\Box\omega^X - \frac{1}{2}V^{(+)'}_{\ell}f + \frac{1}{2}\mu_rV^{(+)}_{\ell}f\right)\\ &+\left(\frac{f''\beta}{1-\mu} + f'\left(\frac{\beta'}{1-\mu} + \frac{n\beta}{r} - \frac{\beta^2}{1-\mu}\right)\right)\\ &+\frac{(\ell(\ell+n-1)-2n)f}{r^3}\left(1 - \frac{(n+1)M}{r^{n-1}}\right).\end{aligned}$$

For $n = 3$, we make the same choice as for the Regge-Wheeler equation (142):

$$\begin{aligned} f &= \left(1 - \frac{(n+1)M}{r^{n-1}}\right)\left(1 - \frac{M}{r^{n-1}} + \frac{4M^2}{5r^{2n-2}}\right) \\ &= \left(1 - \frac{4M}{r^2}\right)\left(1 - \frac{M}{r^2} + \frac{4M^2}{5r^4}\right), \\ r_+ \cdot \beta &= \frac{1}{2}. \end{aligned}$$

For $n = 4$ and $\ell \geq 3$, we choose

$$\begin{aligned} f &= \left(1 - \frac{(n+1)M}{r^{n-1}}\right)\left(1 - \frac{2M}{r^{n-1}} + \frac{6M^2}{5r^{2n-2}}\right) \\ &= \left(1 - \frac{5M}{r^3}\right)\left(1 - \frac{2M}{r^3} + \frac{6M^2}{5r^6}\right), \\ r_+ \cdot \beta &= 1 + \frac{M}{r^3}. \end{aligned}$$

For $n = 4$ and $\ell = 2$, we take

$$\begin{aligned} f &= \left(1 - \frac{(n+1)M}{r^{n-1}}\right)\left(1 - \frac{M}{5r^{n-1}} + \frac{2M^2}{5r^{2n-2}}\right) \\ &= \left(1 - \frac{5M}{r^3}\right)\left(1 - \frac{M}{5r^3} + \frac{2M^2}{5r^6}\right), \\ r_+ \cdot \beta &= \frac{1}{2}\left(2 + \frac{M}{5r^3}\right)^3 \Bigg/ \left(2 + \frac{M}{5r_P^3}\right)^3 \text{ for } r \leq r_P, \\ &= \frac{1}{2}\left(\frac{1}{2} + \frac{2}{5}\left(\frac{2M}{r^3}\right)^{1/6}\right) \Bigg/ \left(\frac{1}{2} + \frac{2}{5}\left(\frac{2M}{r_P^3}\right)^{1/6}\right) \text{ for } r \geq r_P. \end{aligned}$$

Note that $(2M)^{1/3}\beta = \frac{1}{2}$ on the photon sphere $r = r_P$, so that the divergence theorem still applies. With these choices, $f' > 0$ and $W(f, \beta) > 0$, and Assumption 14 can be proved in the same way as in Regge-Wheeler case.

10.5. **Uniform Boundedness and Decay of the Master Quantity.** The estimates for the $Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}$ in the previous subsection are uniform in the angular mode numbers ℓ and m, owing to convergence of the potentials $V^{(+)}_\ell$ to the limiting potential $V^{(+)}$ (173). As the relevant energies involve $L^2(S^n)$-terms integrated over the orbit spheres, there is no difficulty in summing the estimates on the angular modes $Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}$ to obtain estimates on a total object $Q^{(+)}_{\alpha\beta}$, defined as the $L^2(S^n)$-sum

$$Q^{(+)}_{\alpha\beta} := \sum_{\ell \geq 2} \sum_{m_s(n,\ell)=1}^{d_s(n,\ell)} Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}. \tag{174}$$

Following this reasoning, we have the following estimates for the gauge-invariant master quantity $Q^{(+)}_{\alpha\beta}$:

Theorem 23. *Let δg be a smooth, symmetric two-tensor on a Schwarzschild-Tangherlini spacetime, satisfying the linearized Einstein equation* (2)*. There exists a gauge-invariant master quantity $Q^{(+)}_{\alpha\beta}$ in the scalar portion h_1 of δg with harmonics $Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}$ satisfying the Regge-Wheeler type equations* (169)*. Summing estimates for the $Q^{(+)}_{\ell m_s(n,\ell)}$ terms, $Q^{(+)}$ satisfies the uniform boundedness estimate*

$$\check{E}^N_{Q^{(+)}}(\Sigma_\tau) \lesssim \check{E}^N_{Q^{(+)}}(\Sigma_0), \tag{175}$$

in all spacetime dimensions. In six and fewer spacetime dimensions, $Q^{(+)}$ satisfies the uniform decay estimate

$$\check{E}^N_{Q^{(+)}}(\Sigma_\tau) \lesssim \frac{I_{Q^{(+)}}(\Sigma_0)}{\tau^2}, \tag{176}$$

where

$$I_{Q^{(+)}}(\Sigma_0) := E^2_{Q^{(+)},\mathcal{L}_K Q^{(+)}}(\Sigma_0) + E^N_{Q^{(+)},\mathcal{L}_K Q^{(+)},\mathcal{L}_K^2 Q^{(+)}}(\Sigma_0) \tag{177}$$

and $\tau \geq 0$.

We emphasize that the relevant constants in the comparisons depend only upon the orbit sphere dimension n and the mass $M > 0$.

11. Proof of Main Theorem

Theorem 24. *Let δg be a smooth, symmetric two-tensor on a Schwarzschild-Tangherlini spacetime, satisfying the linearized Einstein equation* (2)*. Performing a spacetime Hodge decomposition of δg, each of the portions of δg contains gauge-invariant master quantities satisfying decoupled Regge-Wheeler type wave equations. In particular, the two-tensor portion $h_3 = \hat{h}_{\alpha\beta}$ satisfies the equation* (129)*, the co-vector portion h_2 has quantities $Q^{(-)}_{\alpha\beta}$* (146) *and $S_{\alpha\beta}$* (147) *satisfying the equation* (144)*, and the scalar portion h_1 has quantities $Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}$* (174) *satisfying the equations* (169)*.*

As solutions of Regge-Wheeler type equations, the master quantities satisfy the uniform boundedness estimates

$$\begin{aligned} \check{E}^N_{\hat{h}}(\Sigma_\tau) &\lesssim \check{E}^N_{\hat{h}}(\Sigma_0), \\ \check{E}^N_{Q^{(-)}}(\Sigma_\tau) &\lesssim \check{E}^N_{Q^{(-)}}(\Sigma_0), \\ \check{E}^N_{S}(\Sigma_\tau) &\lesssim \check{E}^N_{S}(\Sigma_0), \\ \check{E}^N_{Q^{(+)}_{\ell m_s(n,\ell)}}(\Sigma_\tau) &\lesssim \check{E}^N_{Q^{(+)}_{\ell m_s(n,\ell)}}(\Sigma_0), \end{aligned} \tag{178}$$

in all spacetime dimensions. In six and fewer spacetime dimensions, the master quantities satisfy the uniform decay estimate

$$\begin{aligned} \check{E}^N_{\hat{h}}(\Sigma_\tau) &\lesssim \frac{I_{\hat{h}}(\Sigma_0)}{\tau^2}, \\ \check{E}^N_{Q^{(-)}}(\Sigma_\tau) &\lesssim \frac{I_{Q^{(-)}}(\Sigma_0)}{\tau^2}, \\ \check{E}^N_{S}(\Sigma_\tau) &\lesssim \frac{I_{S}(\Sigma_0)}{\tau^2}, \\ \check{E}^N_{Q^{(+)}_{\ell m_s(n,\ell)}}(\Sigma_\tau) &\lesssim \frac{I_{Q^{(+)}_{\ell m_s(n,\ell)}}(\Sigma_0)}{\tau^2}, \end{aligned} \tag{179}$$

where

$$I_\Psi(\Sigma_0) := E^2_{\Psi,\mathcal{L}_K\Psi}(\Sigma_0) + E^N_{\Psi,\mathcal{L}_K\Psi,\mathcal{L}_K^2\Psi}(\Sigma_0) \tag{180}$$

and $\tau \geq 0$ for the decay foliation Σ_τ of Subsection 7.3.

Owing to uniformity of the estimates for the $Q^{(+)}_{\ell m_s(n,\ell),\alpha\beta}$ in the angular mode numbers ℓ and $m_s(n,\ell)$, we can concisely encode these estimates by considering their $L^2(S^n)$-sum $Q^{(+)}_{\alpha\beta}$ (174), which satisfies

$$\begin{aligned} \check{E}^N_{Q^{(+)}}(\Sigma_\tau) &\lesssim \check{E}^N_{Q^{(+)}}(\Sigma_0), \\ \check{E}^N_{Q^{(+)}}(\Sigma_\tau) &\lesssim \frac{I_{Q^{(+)}}(\Sigma_0)}{\tau^2}. \end{aligned} \tag{181}$$

We remark that further pointwise uniform boundedness and uniform decay estimates can be derived from those above by means of commutation with the angular Killing fields and application of Sobolev estimates on the orbit spheres.

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Pei-Ken Hung, Department of Mathematics, Massachusetts Institute of Technology, USA
Email address: pkhung@mit.edu

Jordan Keller, Black Hole Initiative, Harvard University, USA
Email address: jordan_keller@fas.harvard.edu

Mu-Tao Wang, Department of Mathematics, Columbia University, USA
Email address: mtwang@math.columbia.edu