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Uniqueness properties of solutions to the Benjamin-Ono equation and related models

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ABSTRACT

We prove that if u_1, u_2 are real solutions of the Benjamin-Ono equation defined in $(x,t) \in \mathbb{R} \times [0,T]$ which agree in an open set $\Omega \subset \mathbb{R} \times [0,T]$, then $u_1 \equiv u_2$. We extend this uniqueness result to a general class of equations of Benjamin-Ono type in both the initial value problem and the initial periodic boundary value problem. This class of 1-dimensional non-local models includes the intermediate long wave equation. We relate our uniqueness results with those for a water wave problem. Finally, we present a slightly stronger version of our uniqueness results for the Benjamin-Ono equation.

1. Introduction

We consider the initial value problem (IVP) for the Benjamin-Ono (BO) equation

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$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \qquad (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a real-valued function, and $\mathcal{H}$ denotes the Hilbert transform

$$\begin{aligned} \mathcal{H}f(x) &:= \frac{1}{\pi}\,\mathrm{p.v.}\Big(\frac{1}{\mathrm{x}} * \mathrm{f}\Big)(\mathrm{x}) \\ &:= \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int\limits_{|y|>\epsilon}\frac{f(x-y)}{y}dy = (-i\,\mathrm{sgn}(\xi)\widehat{f}(\xi))^{\vee}(x) \end{aligned} \tag{1.2}$$

The BO equation was first deduced by Benjamin [3] and Ono [37] as a model for long internal gravity waves in deep stratified fluids. Later, it was shown to be a completely integrable system (see [2], [6] and references therein). In particular, real solutions of the IVP (1.1) satisfy infinitely many conservation laws, which provide an a priori estimate for the $H^{n/2}$-norm, $n \in \mathbb{Z}^+$.

The problem of finding the minimal regularity measured in the Sobolev scale $H^s(\mathbb{R})$, $s \in \mathbb{R}$, required to guarantee that the IVP (1.1) is locally or globally well-posed (WP) in $H^s(\mathbb{R})$ has been extensively studied, see [1], [13], [38], [21], [18], [41], [5] and [12] where global WP was established in $H^0(\mathbb{R}) = L^2(\mathbb{R})$, (for further details and results regarding the well-posedness of the IVP (1.1) we refer to [31] and to [11] for a different proof of the result in [12]).

We remark that a result established in [36] (see also [22]) implies that no well-posedness result in $H^s(\mathbb{R}), s \in \mathbb{R}$, for the IVP (1.1) can be established by using solely a contraction principle argument.

It was first shown in [13] and [14] that polynomial decay of the data may not be preserved by the (real) solution flow of the BO equation. The results in [13] and [14] which present some unique continuation properties of the BO equation have been extended to fractional order weighted Sobolev spaces and have shown to be optimal in [8] and [9]. More precisely, using the notation

$$Z_{s,r} := H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx),\; \dot{Z}_{s,r} = Z_{s,r} \cap \{f \in L^1(\mathbb{R}) : \widehat{f}(0) = 0\},$$

with $s, r > 0$ one has the results:

(i) [8] The IVP (1.1) is locally WP in $Z_{s,r}$ for $s \geq r \in [1, 5/2)$ and if $u \in C([0,T] : Z_{5/2,2})$ is a solution of (1.1) s.t. $u(\cdot,t_j) \in Z_{5/2,5/2}$, $j = 1,2$ with t_1, $t_2 \in [0,T]$, $t_1 \neq t_2$, then $u \in C([0,T] : \dot{Z}_{5/2,5/2})$.

(ii) [8] The IVP (1.1) is locally WP in $\dot{Z}_{s,r}$ $s \geq r \in [5/2, 7/2)$.

(iii) [8] If $u \in C([0,T] : \dot{Z}_{7/2,3})$ is a solution of (1.1) s.t. $\exists\, t_1$, t_2, $t_3 \in [0,T]$, $t_1 < t_2 < t_3$ with $u(\cdot,t_j) \in Z_{7/2,7/2}$, $j = 1,2,3$, then $u \equiv 0$.

(iv) [9] The IVP (1.1) has solutions $u \in C([0,T] : \dot{Z}_{7/2,3})$, $u \not\equiv 0$, for which $\exists\, t_1$, $t_2, \in [0,T]$, $t_1 < t_2$, with $u(\cdot,t_j) \in Z_{7/2,7/2}$, $j = 1,2$.

Our first main result in this work is the following theorem:

Theorem 1.1. *Let $u_1,\ u_2$ be real solutions to the IVP (1.1) for $(x,t) \in \mathbb{R} \times [0,T]$ such that*

$$u_1,\ u_2 \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-2}(\mathbb{R})),\ \ s > 5/2. \tag{1.3}$$

If there exists an open set $\Omega \subset \mathbb{R} \times [0,T]$ such that

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega, \tag{1.4}$$

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.5}$$

In particular, if u_1 vanishes in Ω, then $u_1 \equiv 0$.

Remark 1.2. (i) Under the same hypotheses, Theorem 1.1 applies to real solutions of the generalized BO equation

$$\partial_t u - \mathcal{H}\partial_x^2 u + \partial_x f(u) = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.6}$$

with $f : \mathbb{R} \to \mathbb{R}$ smooth enough and $f(0) = 0$. In particular, it applies for $f(u) = u^k,\ k = 2, 3, 4, \dots$ for which the well posedness of the associated IVP was considered in [1], [19], [18], [20], [32], [33], [42], [43], see also [27].

(ii) The hypothesis (1.3) guarantees that the solutions satisfy the equation (1.1) pointwise, which will be required in our proof.

(iii) A similar result to that described in Theorem 1.1 for the IVP associated to the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + \partial_x u^k = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R},\ k = 2, 3, \dots., \tag{1.7}$$

was established in [40], and for some evolution equations of Schrödinger type in [15]. In both cases, their proofs are based on appropriate forms of the so called Carleman estimates. Our proof of Theorem 1.1 is elementary and relies on simple properties of the Hilbert transform as a boundary value of analytic functions.

(iv) We observe that the unique continuation in (iii) before the statement of Theorem 1.1 applies to a single solution of the BO equation but not to any two solutions as in Theorem 1.1. This is due to the fact that the argument in the proof there depends upon the whole symmetry structure of the BO equation.

(v) It will be clear from our proof below that the last part of Theorem 1.1, i.e. if $u_1(x,t) = 0$ in Ω, then $u_1 \equiv 0$, can be generalized in the following form:

Corollary 1.3. *Let $u \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-2}(\mathbb{R}))$, $s > 2n + 5/2$ be a real solution of the IVP (1.1). Let $p_n(x)$ be a polynomial of degree at most n with real coefficients. If there exists $\Omega \subset \mathbb{R} \times [0,T]$ such that $u(x,t) = p_n(x)$ for any $(x,t) \in \Omega$, then $u \equiv 0$ and $p_n(x) = 0$.*

(vi) Theorem 1.1 can be seen as a corollary of the following linear result whose proof is exactly the one given below for Theorem 1.1:

Assume that k, $j \in \mathbb{Z}^+ \cup \{0\}$ and that

$$a_m : \mathbb{R} \times [0,T] \to \mathbb{R},\ m = 0, 1, ..., k, \quad \text{and} \quad b : \mathbb{R} \times [0,T] \to \mathbb{R}$$

are continuous functions with $b(\cdot)$ never vanishing on $(x,t) \in \mathbb{R} \times [0,T]$, and consider the IVP

$$\begin{cases} \partial_t w - b(x,t)\,\mathcal{H}\partial_x^j w + \displaystyle\sum_{m=0}^{k} a_m(x,t)\partial_x^m w = 0, \\ w(x,0) = w_0(x). \end{cases} \tag{1.8}$$

Theorem 1.4. *Let*

$$w \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-2}(\mathbb{R})), \quad s > \max\{k; j\} + 1/2,$$

be a real solution to the IVP (1.8). If there exists an open set $\Omega \subset \mathbb{R} \times [0,T]$ such that

$$w(x,t) = 0, \quad (x,t) \in \Omega, \tag{1.9}$$

then,

$$w(x,t) = 0 \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.10}$$

Remark 1.5. (i) In particular, applying Theorem 1.4 to the difference of two real solutions u_1, u_2 of the Burgers-Hilbert (BH) equation (see [4])

$$\partial_t u - \mathcal{H}u + u\partial_x u = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.11}$$

one sees that the result in Theorem 1.1, with $s > 3/2$, holds for the IVP associated to the BH equation (1.11).

(ii) The result of Theorem 1.1 extends to solutions of the initial periodic boundary value problem (IPBVP) associated to the generalized BO equation

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + \partial_x f(u) = 0, \qquad (x,t) \in \mathbb{S}^1 \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.12}$$

with $f(\cdot)$ as in part (i) of this remark. More precisely:

Theorem 1.6. *Let $u_1,\ u_2$ be real solutions of the IPBVP (1.12) in $(x,t) \in \mathbb{S}^1 \times [0,T]$ such that*

$$u_1,\ u_2 \in C([0,T] : H^s(\mathbb{S}^1)) \cap C^1((0,T) : H^{s-2}(\mathbb{S}^1)),\ s > 5/2. \tag{1.13}$$

If there exists an open set $\Omega \subset \mathbb{S}^1 \times [0,T]$ such that

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega, \tag{1.14}$$

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{S}^1 \times [0,T]. \tag{1.15}$$

In particular, if u_1 vanishes in Ω, then $u_1 \equiv 0$.

Remark 1.7. The well-posedness of the initial IPBVP (1.12) has been studied in [28], [29], [30], [10] and [34].

Next, we consider the Intermediate Long Wave (ILW) equation

$$\partial_t u - \mathcal{L}_\delta \partial_x^2 u + \frac{1}{\delta}\partial_x u + u\partial_x u = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.16}$$

where $u = u(x,t)$ is a real-valued function, $\delta > 0$ and

$$\mathcal{L}_\delta f(x) := -\frac{1}{2\delta}\,\mathrm{p.v.} \int \coth\left(\frac{\pi(x-y)}{2\delta}\right) f(y)dy. \tag{1.17}$$

Note that $\mathcal{L}_\delta$ is a multiplier operator with $\partial_x \mathcal{L}_\delta$ having symbol

$$\sigma(\partial_x \mathcal{L}_\delta) = \widehat{\partial_x \mathcal{L}_\delta} = 2\pi\xi \coth(2\pi\delta\xi). \tag{1.18}$$

The ILW equation (1.16) describes long internal gravity waves in a stratified fluid with finite depth represented by the parameter δ, see [25], [16], [17].

Also, the ILW equation has been proven to be complete integrable, see [23] and [24].

In [1] it was proven that solutions of the ILW as $\delta \to \infty$ (deep-water limit) converge to solutions of the BO equation with the same initial data.

Also, in [1] it was shown that if $u_\delta(x,t)$ denotes the solution of the ILW equation (1.16), then

$$v_\delta(x,t) = \frac{3}{\delta}\,u_\delta\big(x, \frac{3}{\delta}t\big) \tag{1.19}$$

converges as $\delta \to 0$ (shallow-water limit) to the solution of the KdV equation, i.e. (1.7) with $k = 2$, with the same initial data.

For further comments on general properties of the ILW equation we refer to the recent survey [39] and references therein.

The well-posedness of the IVP associated to the ILW equation (1.16) was studied in [1] and more recently in [35].

Our next theorem extends the result in Theorem 1.1 to solution of the IVP associated to the ILW (1.16):

Theorem 1.8. *Let $u_1,\ u_2$ be real solutions to (1.16) in $(x,t) \in \mathbb{R} \times [0,T]$ such that*

$$u_1,\ u_2 \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-2}(\mathbb{R})), \quad s > 5/2. \tag{1.20}$$

If there exists an open set $\Omega \subset \mathbb{R} \times [0,T]$ such that

$$u_1(x,t) = u_2(x,t), \qquad (x,t) \in \Omega, \tag{1.21}$$

then,

$$u_1(x,t) = u_2(x,t), \qquad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.22}$$

In particular, if u_1 vanishes in Ω, then $u_1 \equiv 0$.

Remark 1.9. The observations in (i) and (v) in Remark 1.2 and (ii) in Remark 1.5 apply, after some simple modifications, to the ILW equation (1.16).

The two main models considered, namely the BO eq. and the ILW eq., are two classical examples of completely integrable systems. Both of them describe in different asymptotic regimes internal waves propagating in one direction. Another quite well known equation, that is also integrable, is the Korteweg-de Vries equation (KdV) which is related to the propagation of waves in shallow water.

So it seems rather natural to try to understand to what extent the uniqueness properties established here for BO and ILW, and in [40] for the KdV (see also [7]) are still true for general water waves. As we will see below the question for water waves turns out to follow from a classical one concerning uniqueness of harmonic functions.

The setting is as follows. We consider an irrotational fluid under the action of gravity. At a given time t there exists an interface $\partial\Omega_t$ which divides the plane in two connected regions where the fluid has two different densities. In the case of water waves there is no fluid in one of the regions. Let us call $\overrightarrow{u}$ the velocity field in one of the regions Ω_t. We know that in Ω_t

$$\nabla \cdot \overrightarrow{u} = 0, \qquad \text{and} \qquad \nabla \times \overrightarrow{u} = 0.$$

Hence, (assuming that Ω_t is simply connected) there exists a harmonic function ϕ such that

$$\nabla \phi = \overrightarrow{u} \qquad \text{in } \ \Omega_t.$$

Assume that $\partial\Omega_t$ is locally given by a Lipschitz graph $z(\alpha, t) = \big(x_1(\alpha, t), x_2(\alpha, t)\big)$ with α a lagrangian parameter. Then

$$z_t = \overrightarrow{u}(z),$$

for details see for example [26].

Assume also that there exist intervals $B \subset \mathbb{R}$ and $0 \in J \subset \mathbb{R}$ such that $z(\alpha, t) =$ constant on $B \times J$. Then $z_t = \nabla\phi|_{\partial\Omega_t} = 0$ for $(\alpha, t) \in B \times J$.

As a consequence we have,

$$\begin{cases} \Delta\phi & = 0 \quad \text{on} \quad \Omega_0, \\ \nabla\phi|_{z(B\times\{0\})} & = 0. \end{cases}$$

Then $\phi =$ constant and $\overrightarrow{u}(x_1, x_2, 0) = 0$. The proof of this is a consequence of the following well known lemma.

Lemma 1.10. *Let Ω be a connected open set in $R^n, n \geq 2$, given locally by the graph of Lipschitz functions. Let ϕ be a harmonic function in Ω, which is in $H^1_{loc}(\Omega)$, the set of functions in $L^2_{loc}(\Omega)$, with gradient in $L^2_{loc}(\Omega)$. Assume that there is an open boundary ball U, which is contained in a piece of the boundary for which the domain is given by a Lipschitz graph intersected with a cylinder in the graph direction. Assume that the trace of ϕ in U is constant and that the normal derivative is 0 also in U. Then ϕ is constant in Ω.*

For the sake of completeness a sketch of the proof of Lemma 1.10 will be given below.

Finally, we present the following slight improvement of Theorem 1.1 and Theorem 1.6:

Theorem 1.11. *Let $u_1,\ u_2$ be real solutions to (1.1) in $(x, t) \in \mathbb{R} \times [0, T]$ such that*

$$u_1,\ u_2 \in C([0, T] : H^s(\mathbb{R})) \cap C^1((0, T) : H^{s-2}(\mathbb{R})), \quad s > 5/2. \tag{1.23}$$

If there exists an open set $I \subset \mathbb{R},\ 0 \in I$ such that

$$u_1(x, 0) = u_2(x, 0), \qquad x \in I, \tag{1.24}$$

and for each $N \in \mathbb{Z}^+$

$$\int\limits_{|x|\leq R} |\partial_t u_1(x, 0) - \partial_t u_2(x, 0)|^2 dx \leq c_N\, R^N \qquad \text{as} \qquad R \downarrow 0, \tag{1.25}$$

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.26}$$

Theorem 1.12. *Let $u_1,\ u_2$ be real solutions of the IPBVP (1.12) in $(x,t) \in \mathbb{S}^1 \times [0,T] \simeq \mathbb{R}/\mathbb{Z} \times [0,T]$ such that*

$$u_1,\ u_2 \in C([0,T] : H^s(\mathbb{S}^1)) \cap C^1((0,T) : H^{s-2}(\mathbb{S}^1)),\ s > 5/2. \tag{1.27}$$

If there exists an open set $I \subset [-1/2, 1/2]$ with $0 \in I$ such that

$$u_1(x,0) = u_2(x,0), \quad x \in I, \tag{1.28}$$

and for each $N \in \mathbb{Z}^+$

$$\int\limits_{|x|\le R} |\partial_t u_1(x,0) - \partial_t u_2(x,0)|^2 dx \le c_N\, R^N \qquad as \qquad R \downarrow 0, \tag{1.29}$$

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{S}^1 \times [0,T]. \tag{1.30}$$

Remark 1.13. It will be clear from our proof of Theorem 1.11 that a similar argument provides the proof of Theorem 1.12 which will be omitted.

The rest of this paper is organized as follows: section 2 contains some preliminary estimates required for Theorem 1.1 as well as its proof. It also includes the modifications needed to extend the argument in the proof of Theorem 1.1 from the IVP to the IPBVP to prove Theorem 1.6. Section 3 consists of the proof of Theorem 1.8, and section 4 enclose the proof of Theorem 1.11. Finally, Lemma 1.10 is proved in section 5.

2. Proof of Theorem 1.1

To prove Theorem 1.1 we need the following result from complex analysis whose proof follows directly from Schwarz reflection principle:

Proposition 2.1. *Let $I \subseteq \mathbb{R}$ be an open interval, $b \in (0,\infty]$ and*

$$D_b = \{z = x + iy \in \mathbb{C} : 0 < y < b\}, \;\; L = \{x + i0 \in \mathbb{C} : x \in I\}. \tag{2.1}$$

Let $F : D_b \cup L \to \mathbb{C}$ be a continuous function such that $F\big|_{D_b}$ is analytic. If $F\big|_{L} \equiv 0$, then $F \equiv 0$.

As a consequence we have

Corollary 2.2. *Let $f \in H^s(\mathbb{R})$, $s > 1/2$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$ such that*

$$f(x) = \mathcal{H}f(x) = 0, \qquad \forall\, x \in I,$$

then $f \equiv 0$.

Proof. Denoting $U = U(x, y)$ the harmonic extension of f to the upper half-plane D, one sees that its harmonic conjugate $V = V(x, y)$ has boundary value $V(x, 0) = \mathcal{H}f(x)$ with

$$(\widehat{f + i\mathcal{H}f})(\xi) = 2\,\chi_{[0,\infty)}(\xi)\,\widehat{f}(\xi), \qquad \widehat{f} \in L^1(\mathbb{R}). \tag{2.2}$$

Thus, $F := U + iV$ is continuous on $\overline{D}_\infty$ and analytic on D_∞ with $F\big|_L \equiv 0$. Hence, Proposition 2.1 yields the desired result $\square$

Proof of Theorem 1.1 . Defining $w(x, t) = (u_1 - u_2)(x, t)$ one has that

$$\partial_t w - \mathcal{H}\partial_x^2 w + \partial_x u_2\, w + u_1\, \partial_x w = 0, \quad (x, t) \in \mathbb{R} \times [0, T]. \tag{2.3}$$

By hypotheses (1.3) and (1.21) there exist open intervals I, $J \subset \mathbb{R}$ such that

$$\begin{aligned} w(x, t) &= \partial_x w(x, t) \\ &= \partial_t w(x, t) = \partial_x^2 w(x, t) = 0, \qquad (x, t) \in I \times J \subset \Omega. \end{aligned} \tag{2.4}$$

Thus, the equation (2.3) tells us

$$\mathcal{H}\partial_x^2 w(x, t) = 0, \;\; (x, t) \in I \times J \subset \Omega. \tag{2.5}$$

Combining (2.4) and (2.5) and fixing $t^* \in J$ it follows that

$$\partial_x^2 w(x, t^*) = \mathcal{H}\partial_x^2 w(x, t^*) = 0, \;\; x \in I, \tag{2.6}$$

with $\partial_x^2 w(\cdot, t^*)$, $\mathcal{H}\partial_x^2 w(\cdot, t^*) \in H^s(\mathbb{R})$, $s > 1/2$.

Therefore, using Corollary 2.2 one has that $\partial_x^2 w(\cdot, t^*) \equiv 0$ which implies that $w(\cdot, t^*) \equiv 0$ and completes the proof. $\square$

To extend the previous argument to prove Theorem 1.6 we need the following result from complex analysis:

Proposition 2.3. *Let $J \subset [-\pi, \pi]$ be an open non-empty interval and*

$$B_1(0) = \{z = x + iy \in \mathbb{C} : |z| < 1\}, \;\; A = \{z \in \mathbb{C} : |z| = 1, \arg(z) \in J\}.$$

Let $F : B_1(0) \cup A \to \mathbb{C}$ be a continuous function such that $F\big|_{B_1(0)}$ is analytic. If $F\big|_A \equiv 0$, then $F \equiv 0$.

Proof. The proof follows from Proposition 2.1 by considering $F_oT(z)$ where T is a fractional linear transformation mapping the upper half-plane to the unit disk $B_1(0)$. □

3. Proof of Theorem 1.8

First, we shall prove the following result:

Corollary 3.1. *Let $f \in H^s(\mathbb{R})$, $s > 3/2$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$ such that*

$$f(x) = \mathcal{L}_\delta \partial_x f(x) = 0, \qquad \forall x \in I,$$

with $\mathcal{L}_\delta$ as in (1.17), (1.18), then $f \equiv 0$.

Proof. We define

$$F(x) = \partial_x f(x) + i\mathcal{L}_\delta \partial_x f(x), \quad x \in \mathbb{R}, \tag{3.1}$$

and consider its Fourier transform

$$\begin{aligned}
\widehat{F}(\xi) &= (\partial_x f \widehat{+ i\mathcal{L}_\delta} \partial_x f)(\xi) \\
&= 2\pi i\xi(1 + \coth(2\pi\delta\xi))\,\widehat{\mathrm{f}}(\xi) \\
&= 2\pi i\xi\Big(1 + \frac{e^{2\pi\delta\xi} + e^{-2\pi\delta\xi}}{e^{2\pi\delta\xi} - e^{-2\pi\delta\xi}}\Big)\,\widehat{f}(\xi) \\
&= -4\pi i\xi\,\frac{e^{4\pi\delta\xi}}{1 - e^{4\pi\delta\xi}}\,\widehat{f}(\xi)
\end{aligned} \tag{3.2}$$

We observe that by considering $\partial_x f$ with $f \in H^s(\mathbb{R})$, $s > 3/2$, one cancels the singularity of F at $\xi = 0$ introduced by $\coth(\xi)$.

By hypothesis and (3.2) one concludes that $\widehat{F} \in L^1(\mathbb{R})$ and has exponential decay for $\xi < 0$. Hence,

$$F(x) = \int_{-\infty}^{\infty} e^{2\pi i\xi x}\,\widehat{F}(\xi)\,d\xi \tag{3.3}$$

has an analytic extension

$$F(x + iy) = \int_{-\infty}^{\infty} e^{2\pi i\xi(x+iy)}\,\widehat{F}(\xi)\,d\xi \tag{3.4}$$

to the strip

$$D_{2\delta} = \{z = x + iy \in \mathbb{C} \, : \, 0 < y < 2\delta\}$$

with F continuous on

$$\{z = x + iy \, : \, 0 \leq y < 2\delta\}$$

from the hypothesis on f. Now, Proposition 2.1 leads the desired result. $\square$

Proof of Theorem 1.8. Once Corollary 3.1 is available the proof of Theorem 1.8 is similar to that given for Theorem 1.1, therefore it will be omitted. $\square$

4. Proof of Theorem 1.11

To prove Theorem 1.11 we need an auxiliary lemma:

Lemma 4.1. *Let $f \in L^2(\mathbb{R})$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$, $0 \in I$, such that*

$$f(x, 0) = 0, \quad x \in I, \tag{4.1}$$

and for each $N \in \mathbb{Z}^+$

$$\int\limits_{|x| \leq R} |\mathcal{H}f(x)|^2 dx \leq c_N \, R^N \qquad as \qquad R \downarrow 0, \tag{4.2}$$

then,

$$f(x) = 0, \quad x \in \mathbb{R}. \tag{4.3}$$

Proof. Consider the analytic function $F = F(x+iy)$ defined in $\mathbb{R} \times (0, \infty)$ with boundary values

$$F(x + i0) = -\mathcal{H}f(x) + if(x).$$

Since $F\big|_I$ is real we can use Schwarz reflexion principle to find $\widetilde{F}$ analytic in $I \times (-\infty, \infty)$ with $\widetilde{F} = F$ on $I \times [0, \infty)$.

We observe: $\Re \widetilde{F}(x + i0) = \mathcal{H}f(x)$, $x \in I$ with $\mathcal{H}f\big|_I \in C^\infty$, by the support property of f, and by assumption (4.2) $\partial_x^j \mathcal{H}f(0) = 0$, $j \in \mathbb{Z}^+ \cup \{0\}$. Hence

$$\frac{\partial^j}{\partial z^j} \widetilde{F}(0, 0) = 0 \qquad j = 0, 1, 2, \ldots,$$

which completes the proof. $\square$

Proof of Theorem 1.11. Defining $w(x,t) = (u_1 - u_2)(x,t)$ it follows that

$$\partial_t w - \mathcal{H}\partial_x^2 w + \partial_x u_1\, w + u_2\, \partial_x w = 0, \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{4.4}$$

Since $w(x,0) = 0,\ x \in I$, one has that $\partial_x^j w(x,0) = 0,\ x \in I,\ j \in \mathbb{Z}^+ \cup \{0\}$, and using (4.4)

$$\mathcal{H}\partial_x^2 w(x,0) = \partial_t w(x,0)$$

We now apply the hypothesis (4.2) and Lemma 4.1 to conclude that $\partial_x^2 w(x,0) = 0$, $x \in \mathbb{R}$. $\square$

5. Proof of Lemma 1.10

Proof. First observe that the trace on the boundary of Ω is well defined and is locally in $H^{1/2}(\partial\Omega)$. Also, from the harmonicity of ϕ and simple integration by parts, one easily concludes that ϕ has a normal derivative in the weak sense on $\partial\Omega$, which is locally in $H^{-1/2}(\partial\Omega)$.

We can assume that

$$x_n = f(\bar{x}) \qquad \bar{x} = (x_1, \cdots, x_{n-1}) \in \tilde{B},$$

with $\tilde{B}$ a ball in $\mathbb{R}^{n-1}$, and Ω is locally given by $x_n < f(\bar{x})$. Then we extend ϕ as a constant for $\{x_n > f(\bar{x}), \bar{x} \in \tilde{B}\}$. For $\eta \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} \int \nabla\phi\nabla\eta\, d\bar{x}dx_n &= \int\limits_{x_n \geq f} + \int\limits_{x_n \leq f} \\ &= (\partial_n\phi, \eta)|_{x_n = f(\bar{x})} \\ &= 0. \end{aligned}$$

Hence the extended ϕ is weakly harmonic, and therefore harmonic, in a cylinder with basis $\tilde{B}$. So ϕ is constant in a ball inside of Ω, but ϕ is analytic. Hence ϕ is constant. $\square$

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