

THE METHOD OF ENERGY CHANNELS FOR NONLINEAR WAVE EQUATIONS

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ABSTRACT. This is a survey of some recent results on the asymptotic behavior of solutions to critical nonlinear wave equations.

In this article we will survey some applications of the method of energy channels, introduced by Duyckaerts–Kenig–Merle [13] to study the long-time behavior of large solutions to nonlinear wave equations. This article is dedicated to Luis Caffarelli, with warm friendship, on the occasion of his 70th birthday.

Nonlinear dispersive equations were introduced in the 19th century to understand models for water waves. Their theory has had a spectacular development in the last 40 years. These equations model phenomena of wave propagation coming from physics and engineering. Some of the areas that give rise to these equations are water waves, optics, lasers, ferromagnetism, particle physics, general relativity, nonlinear elasticity and many others. These equations also have connections to geometric flows arising in Kähler and Minkowski geometries. Nonlinear dispersive equations are time reversible. Typically they have a conserved energy, which gives rise to a Hamiltonian structure. Here are some examples:

- (a) Generalized Korteweg–de Vries equation (water waves in a shallow channel)

$$\begin{cases} \partial_t u - \partial_x^3 u + u^k \partial_x u = 0, & x \in \mathbb{R}, t \in \mathbb{R} \\ u|_{t=0} = u_0. \end{cases}$$

- (b) Nonlinear Schrödinger equations (optics, lasers, ferromagnetism)

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1}u = 0, & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u|_{t=0} = u_0. \end{cases}$$

- (c) Nonlinear wave equations (nonlinear elasticity, toy model for particle physics, general relativity)

$$\begin{cases} \partial_t^2 u - \Delta u \pm |u|^{p-1}u = 0, & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1. \end{cases}$$

2010 *Mathematics Subject Classification*. Primary: 35L70; Secondary: 35L15.

Key words and phrases. Nonlinear wave equations, asymptotic behavior, energy channels, traveling waves, soliton resolution.

*The author is supported in part by NSF Grants DMS-1265249, DMS-1463746 and DMS-1800082.

- (d) (Geometric example) Wave maps (particle physics, σ -models)

Let $u : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{S}^N \subset \mathbb{R}^{N+1}$, where $\mathbb{S}^N$ is the unit sphere with the round metric,

$$\begin{cases} \partial_t^2 u - \Delta u = [|\nabla u|^2 - |\partial_t u|^2]u, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1. \end{cases}$$

This is a geometric equation since it is invariant under isometries of the target.

These equations are called dispersive because their linear part is. Heuristically, a linear dispersive equation spreads out the physical support of a solution over time. For the nonlinear versions, there may be solutions that propagate non-dispersively, like traveling waves, which were first observed in the 19th century. Their essential properties were not understood until much later.

In the late 70's and 80's many properties of nonlinear dispersive equations were discovered, most notably the existence and stability (some times conditionally) of special solutions, such as traveling waves. In the late 80's and early 90's, Kenig–Ponce–Vega introduced the systematic use of the machinery of modern Fourier analysis to study the associated linear problems, which was then applied perturbatively to the corresponding nonlinear problems. The resulting body of techniques (with refinements and extensions by Bourgain, Klainerman–Machedon, Tataru, Tao and many others) proved extremely powerful in many problems. This body of works gave satisfactory theories for the short-time well-posedness with data in Sobolev spaces, and for the global in-time well-posedness for small data. At this time, a notion of “criticality” linked to scaling emerged.

The last 25 years have seen a lot of interest in the study of the long-time behavior of large solutions. Issues like blow-up, global existence, scattering and long-time asymptotic behavior have come to the forefront, especially in critical problems. The study of some of these issues was transformed by Kenig–Merle ([29], [30], [31], [26], [17], etc.) with their introduction of the “concentration–compactness/rigidity theorem” method, which has now become the canonical approach to this problematic. The ultimate goal of the Kenig–Merle method was to attack the problem of asymptotic soliton resolution.

Since the 1970's there has been a widely held belief that “coherent structures” and free radiation describe the long-term asymptotic behavior of generic solutions to nonlinear dispersive equations. This belief has come to be known as the “soliton resolution conjecture”. Roughly speaking, this holds that, asymptotically in time, the evolution of generic solutions decouples as a sum of modulated traveling waves and a free radiation term (which is a dispersive term solving the associated linear equation). For finite time blow-up solutions, a result in the same spirit is expected, depending on the nature of the blow-up. This is a remarkable, beautiful claim, showing a “simplification” in the complex, long-time dynamics of general solutions. The soliton resolution conjecture arose in the 70's and 80's from various numerical experiments (Fermi–Pasta–Ulam, Zabusky–Kruskal) and the integrability theory for the Korteweg–de Vries equation ([20], [21], [43]). The conjecture is quite challenging, even in the completely integrable case. To our knowledge, until recently it was only proved for KdV (power $k = 1$ in (a) above) ([20], [21]), for mKdV (power $k = 2$ in (a) above) ([43]), both of which are completely integrable by the method of inverse scattering, and also for the cubic NLS in one space dimension (power $p = 3$ in (b) above, $N = 1$), ([55], [2]), which is also completely integrable. Very few results exist for equations which are not completely integrable. Weaker theorems are available for some dispersive equations, in cases where traveling waves exist, and in which the

“ground-state” plays an important role as a threshold for the dynamics. All these works imply local versions of the soliton resolution, where at most one soliton (the ground state) appears. We should also mention Tao’s works for L^2 -supercritical, energy-subcritical NLS in high dimensions, where he showed the existence of an attractor, which is compact modulo space translations, up to a dispersive term. This reduces the proof of a weak variant of soliton resolution to a rigidity theorem, showing that any solution with the compactness property (i.e. having a compact trajectory up to the symmetries of the equation) must be a traveling wave, a very difficult problem in itself. It has been solved only in regimes “below” or close to the ground state, except for the case of KdV using complete integrability in work of Martel–Merle and for equation (c) above, $p = (N + 2)/(N - 2)$ (the energy critical wave equation), where the rigidity theorem was proved by Duyckaerts–Kenig–Merle [13] in the radial case, with no smallness assumption, and in the non-radial case, with no smallness assumption, but under an additional non-degeneracy assumption [18]. We refer also to [17], where the importance of this type of solution for general dispersive equations is highlighted.

In the rest of this review we will concentrate the discussion on the energy critical wave equation in the focusing case (− sign in front of the nonlinearity in (c) above). In the defocusing case (+ sign in front of the nonlinearity in (c) above), it was shown in the period 1990–2000, in works of Struwe, Grillakis, Shatah–Struwe, Bahouri–Shatah and Bahouri–Gérard, that all data in the energy space yield global in time solutions which scatter.

The focusing case is very different, since one can have finite-time blow-up, or solutions which exist for all time that do not scatter and traveling wave solutions. The ultimate goal here is to prove soliton resolution for all solutions of the focusing energy critical wave equation which remain bounded in the energy space. I will describe here the progress towards this, obtained in the last 12 years. The hope is that the results that we will discuss will be a model for what to strive for in the study of other nonlinear dispersive equations.

Thus we will be mainly considering the equation

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{4/(N-2)} u = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{cases} \quad (\text{NLW})$$

where $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, and where $u_0 \in \dot{H}^1(\mathbb{R}^N) = \{u_0 : \nabla u_0 \in L^2(\mathbb{R}^N)\}$ and $u_1 \in L^2(\mathbb{R}^N)$. The space $\dot{H}^1 \times L^2(\mathbb{R}^N)$ will be called the energy space. We will restrict ourselves to $3 \leq N \leq 6$, to avoid technical complications due to lack of smoothness of the nonlinearity. In this problem, small data yield solutions which exist for all time and “scatter”, that is to say they exhibit linear behavior for large times, asymptotically. For large data, we have solutions $u \in C(I; \dot{H}^1 \times L^2)$ for small time intervals I , with a maximal interval of existence $(T_-(u), T_+(u))$ and $u \in L^{2(N+1)/(N-2)}(\mathbb{R}^N \times I')$, for each I' compactly contained in $(T_-(u), T_+(u))$. (See for instance [30]). The energy norm is “critical”, since for all $\lambda > 0$, $u_\lambda(x, t) = \lambda^{-(N-2)/2} u(x/\lambda, t/\lambda)$ is also a solution and $\|(u_{0,\lambda}, u_{1,\lambda})\|_{\dot{H}^1 \times L^2} = \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$. The equation is focusing, so there is competition between the linear part and the nonlinearity, and the conserved energy is:

$$\mathcal{E}(u_0, u_1) = \frac{1}{2} \int |\nabla u_0|^2 + |u_1|^2 dx - \frac{N-2}{2N} \int |u_0|^{2N/(N-2)} dx.$$

It is easy to construct solutions that blow-up in finite time, say at $T = 1$, by considering the associated ODE. For instance, when $N = 3$, $u(x, t) = (3/4)^{1/4} (1 -$

$t)^{-1/2}$ is a solution, and using finite speed of propagation, it is then easy to construct solutions with $T_+ = 1$, and $\lim_{t \uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = \infty$. Such solutions are called “type I blow-up” solutions.

(NLW) also admits “type II blow-up” solutions, i.e. solutions which remain bounded in norm. Here the breakdown occurs by “concentration”. The existence of such solutions is a typical feature of energy critical problems. The first examples of such solutions (radial) were constructed for $N = 3$ by Krieger–Schlag–Tataru [38], for $N = 4$ by Hillairet–Raphaël [23] and for $N = 5$ by Jendrej [24]. For (NLW) one expects soliton resolution for solutions that remain bounded in the energy norm, i.e. such that

$$\sup_{0 < t < T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty. \quad (*)$$

Some examples of solutions verifying (*), when $T_+ = \infty$ are “scattering solutions”, that is solutions such that $T_+ = \infty$ and there exists $(u_0^+, u_1^+) \in \dot{H}^1 \times L^2$, with

$$\lim_{t \rightarrow \infty} \|(u(t), \partial_t u(t)) - (S(t)(u_0^+, u_1^+), \partial_t S(t)(u_0^+, u_1^+))\|_{\dot{H}^1 \times L^2} = 0,$$

where $S(t)(u_0^+, u_1^+)$ is the solution of the associated linear wave equation with data (u_0^+, u_1^+) . For instance, as we mentioned before, for (u_0, u_1) small in $\dot{H}^1 \times L^2$, we have a scattering solution. Other examples of solutions of (NLW), with $T_+ = \infty$, and verifying (*) are the stationary solutions, that is the solutions $Q \neq 0$ of the elliptic equation

$$\Delta Q + |Q|^{4/(N-2)}Q = 0 \quad \text{in } \mathbb{R}^N, \quad (\Sigma)$$

with $Q \in \dot{H}^1(\mathbb{R}^N)$. We say that such a solution Q is in Σ . For example,

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-(N-2)/2}$$

is such a solution. Stationary solutions do not scatter, since they do not disperse. W has several important characterizations: up to sign and scaling it is the only radial element in Σ due to work of Pohozaev and Gidas–Ni–Nirenberg. Up to translation and scaling it is also the only non-negative element in Σ (Caffarelli–Gidas–Spruck). The equation (Σ) was extensively studied in connection with the Yamabe problem in differential geometry. There is a continuum of variable sign, non-radial $Q \in \Sigma$ (Ding, del Pino–Musso–Pacard–Pistoia). W also has a variational characterization as the extremal (modulo sign, translation and scaling) in the Sobolev embedding

$$\|f\|_{L^{2N/(N-2)}} \leq C_N \|\nabla f\|_{L^2}.$$

It is also (modulo sign, translation and scaling) the element of Σ of least energy, and there is a positive gap between the energy of W and the energies of the other elements in Σ (up to sign, translation and scaling). Because of this, W is referred to as the “ground state”. In [30], Kenig and Merle established the “ground state” or “threshold” conjecture for (NLW).

Theorem 0.1. *Let u be a solution of (NLW). Assume that $\mathcal{E}(u_0, u_1) < \mathcal{E}(W, 0)$.*

(i) *If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then $T_+ = +\infty$, $T_- = -\infty$, and u scatters in both time directions.*

(ii) *If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$, then $T_+ < +\infty$, $T_- > -\infty$.*

(iii) *The case $\|\nabla u_0\|_{L^2} = \|\nabla W\|_{L^2}$ is vacuous by variational considerations.*

The threshold case $\mathcal{E}(u_0, u_1) = \mathcal{E}(W, 0)$ was completely described by Duyckaerts–Merle.

The proof of Theorem 0.1 was obtained through the “concentration–compactness/rigidity theorem” method, introduced by Kenig–Merle for this purpose (see also [29], [31], [27], etc.). This method has since become the standard tool to understand scattering, below the ground–state threshold, in critical dispersive problems. It was through the work in [30] that Kenig–Merle realized that (NLW) was a favorable non–integrable model in which one could hope to attack the problem of soliton resolution. To understand this further, note that other non–scattering solutions of (NLW), which verify (*) are the traveling wave solutions. These are obtained as Lorentz transforms of $Q \in \Sigma$. Let $\vec{\ell} \in \mathbb{R}^N$, $|\vec{\ell}| < 1$. Then,

$$Q_{\vec{\ell}}(x, t) = Q_{\vec{\ell}}(x - t\vec{\ell}, 0) = Q\left(\left[\frac{-t}{(1 - |\vec{\ell}|^2)^{1/2}} + \frac{1}{|\vec{\ell}|^2} \left(\frac{1}{(1 - |\vec{\ell}|^2)^{1/2}} - 1\right)(\vec{\ell} \cdot x)\right] \vec{\ell} + x\right)$$

is a traveling wave solution of (NLW). Moreover, Duyckaerts–Kenig–Merle showed in [17] that these are all the traveling wave solutions of (NLW) in the energy space. With these traveling wave solutions in hand, it becomes possible to give a rigorous formulation of the soliton resolution conjecture for solutions of (NLW) verifying (*). The first progress in this direction was in the radial case, when $N = 3$, in [15], where it was shown that, if u is a radial solution of (NLW), verifying (*), one can find well–chosen sequences $\{t_n\}$, $t_n \uparrow T_+$, $J \in \mathbb{N}$, signs $i_j = \pm 1$, $1 \leq j \leq J$, scalings $\lambda_{j,n} > 0$ with $0 < \lambda_{1,n} \ll \dots \ll \lambda_{J,n}$ (where $\lambda_{1,n} \ll \lambda_{2,n}$ means that $\lim_{n \rightarrow \infty} \lambda_{1,n}/\lambda_{2,n} = 0$), and a solution of the linear wave equation $v_L(x, t)$, such that

$$(u(t_n), \partial_t u(t_n)) = \sum_{j=1}^J i_j \left(\frac{1}{\lambda_{j,n}^{1/2}} W\left(\frac{x}{\lambda_{j,n}}\right), 0 \right) + (v_L(x, t_n), \partial_t v_L(x, t_n)) + o_n(1),$$

where $o_n(1)$ tends to 0 in $\dot{H}^1 \times L^2$ norm as $n \rightarrow \infty$. Moreover, if $T_+ < \infty$, then $\lambda_{J,n} \ll (T_+ - t_n)$ and if $T_+ = \infty$, then $\lambda_{J,n} \ll t_n$.

Soon after, the same authors showed [16]:

Theorem 0.2. *Let u be a radial solution of (NLW), with $N = 3$. Then one of the following holds (with $\vec{u}(t) = (u(t), \partial_t u(t))$):*

- (i) $T_+ < \infty$ and $\sup_{0 < t < T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} = \infty$ (type I blow–up).
- (ii) $T_+ < \infty$ and $\sup_{0 < t < T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty$ (type II blow–up), and there exist $J \geq 1$, $i_j \in \{\pm 1\}$, $\lambda_j(t) > 0$, for $1 \leq j \leq J$, with $0 < \lambda_1(t) \ll \dots \ll \lambda_J(t) \ll (T_+ - t)$, such that

$$\vec{u}(t) - \vec{v}(t) = \sum_{j=1}^J i_j \left(\frac{1}{\lambda_j(t)^{1/2}} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + o(1) \quad \text{in } \dot{H}^1 \times L^2,$$

where $\vec{u}(t) \rightharpoonup (v_0, v_1)$ as $t \rightarrow T_+$ and v is the solution of (NLW) with $\vec{v}(T_+) = (v_0, v_1)$ which verifies $\text{supp } (\vec{u}(t) - \vec{v}(t)) \subset \{|x| < T_+ - t\}$. (v is the “regular part” of u , it is equivalent to the radiation term since $\vec{v}(t) - \vec{S}_L(t - T_+)(v_0, v_1) \rightarrow 0$ as $t \rightarrow T_+$ in $\dot{H}^1 \times L^2$.)

- (iii) $T_+ = \infty$. Then there exist v_L a solution of the linear wave equation, and $J \geq 0$, and for $1 \leq j \leq J$, $i_j \in \{\pm 1\}$, $\lambda_j(t) > 0$ with $0 < \lambda_1(t) \ll \dots \ll \lambda_J(t) \ll t$, and

$$\vec{u}(t) - \vec{v}_L(t) = \sum_{j=1}^J i_j \left(\frac{1}{\lambda_j(t)^{1/2}} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + o(1) \quad \text{in } \dot{H}^1 \times L^2 \quad \text{as } t \rightarrow \infty.$$

Let us briefly discuss the main idea in the proof of Theorem 0.2. The mechanism for relaxation to a “coherent structure” (a sum of modulated solitons), which has been observed numerically and experimentally, is the radiation of excess energy to spatial infinity. This appears in such diverse settings as the dynamics of gas bubbles in a compressible fluid and the formation of black holes in gravitational collapse. In a series of works with Duyckaerts and Merle ([13], [14], [15], [16], [18], etc.) we have found a way to quantify the ejection of energy, that occurs as we approach the final time T_+ for nonlinear wave equations, through a method that we call “energy channels”. This has allowed us to make significant progress on “soliton resolution”, such as in Theorem 0.2, and other works (sometimes also with other collaborators). The main point in the proof of Theorem 0.2, is a dynamical characterization of W through “energy channels”: if u is a radial solution of (NLW), $N = 3$, which exists for all times and is not 0 or a rescaled $\pm W$, then ([16]) there exist $R > 0$ and $\eta > 0$, such that, for all $t \geq 0$ or all $t \leq 0$, we have

$$\int_{|x| > |t| + R} |\nabla_{x,t} u|^2 \geq \eta. \quad (\dagger)$$

Some of the key tools for proving this are the following “outer energy lower bounds”, valid for radial solutions of the linear wave equation, when $N = 3$. For $r_0 > 0$, let $P_{r_0} = \{(a/r, 0) : a \in \mathbb{R}, r \geq r_0\} \subseteq \dot{H}^1 \times L^2(r > r_0)$. Let $\Pi_{r_0}^\perp$ be the orthogonal projection onto the orthogonal complement of P_{r_0} . Then, for v a radial solution of the linear wave equation, $N = 3$, we have ([13]): for all $t \geq 0$ or all $t \leq 0$,

$$\int_{|x| > |t| + r_0} |\nabla_{x,t} v|^2 dx \geq \frac{1}{2} \|\Pi_{r_0}^\perp(v_0, v_1)\|_{\dot{H}^1 \times L^2(r > r_0)}^2. \quad (1)$$

Note that it is easy to check that

$$\|\Pi_{r_0}^\perp(v_0, v_1)\|_{\dot{H}^1 \times L^2(r > r_0)}^2 = \int_{|x| > r_0} |\partial_r(rv_0)|^2 + v_1^2. \quad (2)$$

Note that (1) and (2) easily give $(\dagger)$ for solutions of the linear wave equation when $N = 3$. The passage to the nonlinear case combines (1), (2) with “elliptic arguments” of iterative type. To see the method of “energy channels” at work, let us give an outline of the proof of Theorem 0.2, (ii), $T_+ = 1$. Take $t_n \uparrow 1$, and consider $\vec{u}(t_n) - \vec{v}(t_n)$, which is supported on $|x| < 1 - t_n$, where v is the regular part of u . Break up $\vec{u}(t_n) - \vec{v}(t_n)$ into a sum of “orthogonal” nonlinear “blocks”. (Technically, nonlinear profiles U_j associated to a Bahouri–Gérard [1] profile decomposition, plus an error w_n , which is a linear solution, tending to 0 in the weaker “dispersive” norm $L_t^5 L_x^{10}$ ($N = 3$).) This can be done through an approximation theorem due to Bahouri–Gérard [1]. If a “block” U_j is not (up to a sign) a rescaled W , by $(\dagger)$, centered at $(0, t_n)$, it will send energy outside the inverted light cone centered at $(0, 1)$, in case $(\dagger)$ holds for $t \geq t_n$ which contradicts that $\text{supp } \vec{u}(t) - \vec{v}(t) \subseteq \{|x| < 1 - t\}$, or close to $|x| = 1$, $t = 0$, for n large (in case $(\dagger)$ holds for $t \leq t_n$), which contradicts the fact that $\vec{u}(0) - \vec{v}(0) \in \dot{H}^1 \times L^2$. Finally one uses (1) to show that the dispersive errors $\vec{w}_n$ tend to 0 in energy norm, by a similar argument.

Let us also give an outline of the earlier argument in [15], which gives the decomposition for well-chosen sequences of times. This is a weaker result, but the argument has proved to be useful in other situations. Let us again concentrate on the analog of (ii) in Theorem 0.2, with $T_+ = 1$. In [15], using (1) and the Bahouri–Gérard decomposition [1], it was shown, for $N = 3$, that no “self-similar blow-up”

is possible for radial solutions verifying $(*)$ above: for any such solution, we have

$$\lim_{t \uparrow 1} \int_{\lambda(1-t) < |x| < (1-t)} |\nabla_{x,t} u(x, t)|^2 dx = 0, \quad \forall 0 < \lambda < 1. \quad (\ddagger)$$

Later, Côte–Kenig–Laurie–Schlag [9] gave a different proof of $(\ddagger)$, which now applies to radial solutions for all N , without using (1) or its analogs (see [25]). The proof was given through an adaptation of the classical argument for showing $(\ddagger)$ for equivariant wave maps due to Christodoulou, Shatah and Tahvildar–Zadeh [4],[5],[46]. Combining $(\ddagger)$ with classical “virial identities” [45], in [15] it was shown that

$$\lim_{t \uparrow 1} \int_t^1 \int_{|x| < 1-t} (\partial_t u(x, s))^2 dx \frac{ds}{1-t} = 0. \quad (\#)$$

This, together with a Tauberian type argument (and hence the need to choose suitable time sequences), and the fact that, in the radial case, the only static solutions are $\pm W$ and their scalings, shows that all nonlinear “blocks” are scalings of $\pm W$. Then, one uses the version of (1) when $r_0 = 0$, and an energy channel argument to show that the dispersive errors go to 0 in energy norm. It turns out that, in the limit as $r_0 \rightarrow 0$, (1) becomes

$$\int_{|x| > |t|} |\nabla_{x,t} v|^2 dx \geq \frac{1}{2} \int |\nabla v_0|^2 + v_1^2 \quad (3)$$

holds for all $t \geq 0$ or all $t \leq 0$. Here v is a solution of the linear wave equation in $\mathbb{R}^N \times \mathbb{R}$, with N odd, and this holds also in the non-radial case. (3) was proved in [14].

Moving forward from the radial case of (NLW), for $N = 3$, we need to take into account a number of facts.

Fact 1. In [10], it is shown that (1) and (3) fail for radial solutions of the linear wave equations, for all even dimensions. However, (3) holds for radial, linear solutions when $N = 4, 8, \dots$, $(v_0, v_1) = (v_0, 0)$ and when $N = 2, 6, 10, \dots$, $(v_0, v_1) = (0, v_1)$.

Fact 2. In [28] it is shown that an analog of (1) holds for radial solutions of the linear wave equation, for all odd N . This is: Let

$$P = \text{span} \left\{ \left(\frac{1}{r^{N-2k_1}}, 0 \right), \left(0, \frac{1}{r^{N-2k_2}} \right) : 1 \leq k_1 \leq \left\lfloor \frac{N+2}{4} \right\rfloor, 1 \leq k_2 \leq \left\lfloor \frac{N}{4} \right\rfloor \right\}.$$

Let $P(R) = P|_{\{|x| > R\}}$. Then, for all $t \geq 0$, or for all $t \leq 0$, if v is a radial solution of the linear wave equation in $\mathbb{R}^N \times \mathbb{R}$,

$$\int_{|x| \geq R+|t|} |\nabla_{x,t} v|^2 dx \geq \frac{1}{2} \|\Pi_{P(R)}^\perp(v_0, v_1)\|_{\dot{H}^1 \times L^2(|x| > R)}^2. \quad (4)$$

Note that when $N = 3$, (4) is precisely (1), and when $N = 5$ P also includes $(0, 1/|x|^3)$. Also as $N \rightarrow \infty$, the dimension of the exceptional subspaces $P(R)$ increases to infinity.

Fact 3. ([11]) (1) and (4) fail in the non-radial case, in all dimensions. Using Fact 2, and $(\ddagger)$, C. Rodriguez [42] was able to extend [15], also for well-chosen sequences of times, for radial solutions of (NLW) in all odd dimensions. Using Fact 1 for $N = 4$, $(\ddagger)$ and $(\#)$ and the fact that the “good” data for (2) in this dimension are $(v_0, 0)$ and $\|\partial_t w_n\|_{L^2} \rightarrow 0$, for the “dispersive error”, in [9] the analog of [15] was obtained for well-chosen sequences of times and radial solutions of (NLW) for $N = 4$.

In [25] Jia–Kenig introduced a new method, based on virial identities, to study soliton resolution, along well-chosen sequences of times, for radial solutions of (NLW) in all dimensions N . This method did not use (1), (3) or (4), but used (†) and (‡), combined with an extra virial identity.

Fact 4. On the difficulty in obtaining (†) for radial solutions of (NLW), in the case $N > 3$. Consider the case $N = 5$. Then, (4) above becomes

$$\int_{|x| \geq R+|t|} |\nabla_{x,t} v|^2 dx \geq \frac{1}{2} \|\Pi_{P(R)}^\perp(v_0, v_1)\|_{H^1 \times L^2(|x| > R)}^2,$$

where $P(R) = \{a(1/r^3, 0) + b(0, 1/r^3) : a, b \in \mathbb{R}\}$ (see [28]). Note that, as in the case $N = 3$, $1/r^3$ is still asymptotic to W . In order to show that the failure of (†) gives a dynamical characterization of W , it is necessary to show that the solution of (NLW) with data $(0, \chi_{\{|x| > R\}}/r^3)$ verifies (*), while for the linear solution, the left-hand side of (4) is 0. This has been a roadblock for a number of years. Recently, in work in progress, Duyckaerts–Kenig–Merle have succeeded in doing this, by a very indirect argument, which has led to a proof of the full soliton resolution for radial solutions of (NLW), when $N = 5$.

We now turn to the non-radial case. Some of the difficulties here are that the set of traveling waves $Q_{\vec{l}}$, Q solving $\Delta Q + |Q|^{4/(N-2)}Q = 0$, is very large and far from being understood and the outer energy lower bounds (1) and (4) fail in the nonradial case, so that a dynamical characterization of traveling waves seems doomed to failure.

Consider now solutions of (NLW) in $\mathbb{R}^N$, $N = 3, 4, 5, 6$, verifying (*). If $T_+ < \infty$ (type II blow-up), we consider the set S of singular points, i.e. the set of points where the solution concentrates energy at the blow-up time. (See [13].) In [13] it is shown that S is non-empty and finite. Moreover, it is shown that $\vec{u}(t) \rightarrow (v_0, v_1)$ weakly in $\dot{H}^1 \times L^2$ as $t \rightarrow T_+$, and that if v solves (NLW), with $\vec{v}(T_+) = (v_0, v_1)$, then $\text{supp} [\vec{u}(t) - \vec{v}(t)] \subset \cup_{k=1}^M \{(x, t) : |x - x_k| < |T_+ - t|\}$, where $S = \{x_1, \dots, x_k\}$. Note that in the radial case $S = \{0\}$. We now describe the result in [11], which gives soliton resolution, along well-chosen sequences of times.

Theorem 0.3 ([11]). *Let u be a solution of (NLW), $N = 3, 4, 5, 6$, verifying (*).*

(i) *Assume $T_+ < \infty$. Fix $x_0 \in S$. Then, there exist $J \geq 1$, $r_0 > 0$, and time sequences $t_n \uparrow T_+$ (well-chosen), scales λ_n^j , with $\lambda_n^j/(T_+ - t_n) \rightarrow 0$ as $n \rightarrow +\infty$, positions $c_n^j \in B_{\beta(T_+ - t_n)}(x_0)$, $\beta \in [0, 1)$, with $\vec{l}_j = \lim_n c_n^j/(T_+ - t_n)$, and traveling waves $Q_{\vec{l}_j}^j$, $Q^j \in \Sigma$, $1 \leq j \leq J$, such that, for $x \in B_{r_0}(x_0)$, we have*

$$\begin{aligned} & \vec{u}(t_n) - \vec{v}(t_n) \\ &= \sum_{j=1}^J \left(\frac{1}{(\lambda_n^j)^{(N-2)/2}} Q_{\vec{l}_j}^j \left(\frac{x - c_n^j}{\lambda_n^j}, 0 \right), \frac{1}{(\lambda_n^j)^{N/2}} \partial_t Q_{\vec{l}_j}^j \left(\frac{x - c_n^j}{\lambda_n^j}, 0 \right) \right) + o_{\dot{H}^1 \times L^2}(1) \end{aligned}$$

as $n \rightarrow \infty$. Moreover,

$$\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|c_n^j - c_n^{j'}|}{\lambda_n^j} \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad \text{for } j \neq j'.$$

(ii) *Assume that $T_+ = \infty$. Then ([19]), there exists v_L solving the linear wave equation such that, for all A ,*

$$\lim_{t \rightarrow \infty} \int_{|x| > t-A} |\nabla_{x,t}(u - v_L)(x)|^2 dx = 0.$$

Moreover, there exists $t_n \rightarrow \infty$ (well-chosen), there exist $J \geq 0$, $\lambda_n^j > 0$, $c_n^j \in B_{\beta t_n}(0) \subset \mathbb{R}^N$, $\beta \in [0, 1)$, $\lim_n c_n^j/t_n = \vec{l}_j$, traveling waves $Q_{\vec{l}_j}^j$, $Q^j \in \Sigma$, such that

$$\begin{aligned} & \bar{u}(t_n) - \bar{v}(t_n) \\ &= \sum_{j=1}^J \left(\frac{1}{(\lambda_n^j)^{(N-2)/2}} Q_{\vec{l}_j}^j \left(\frac{x - c_n^j}{\lambda_n^j}, 0 \right), \frac{1}{(\lambda_n^j)^{N/2}} \partial_t Q_{\vec{l}_j}^j \left(\frac{x - c_n^j}{\lambda_n^j}, 0 \right) \right) + o_{\dot{H}^1 \times L^2}(1) \end{aligned}$$

as $n \rightarrow \infty$, and

$$\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|c_n^j - c_n^{j'}|}{\lambda_n^j} \rightarrow \infty \quad \text{as } n \rightarrow \infty \text{ for } j \neq j', \quad \text{and } \frac{\lambda_n^j}{t_n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Remark 1. The proof of (i), with error $(\varepsilon_n^0, \varepsilon_n^1)$ having corresponding linear solution w_n converging to 0 in $L_t^5 L_x^{10}$ (when $N = 3$) i.e. in the “dispersive” norm, but not necessarily the energy norm, is due earlier to Hao Jia.

Remark 2. The extraction of the radiation term in (ii) is in [19] and it is a delicate result.

Remark 3. The proof of Theorem 0.3, say when $T_+ = \infty$, gives the following additional fact: for any t_n^1, t_n^2 , with $\lim_n t_n^1 = \lim_n t_n^2 = \lim_n t_n^2/t_n^1 = \infty$, after extraction, we can find t_n , with $t_n^1 \leq t_n \leq t_n^2$, so that the decomposition is valid for t_n . Thus, the decomposition holds for many sequences $\{t_n\}$.

The starting point of the proof of Theorem 0.3 (i) is the following Morawetz type estimate (say when $T_- = 0$, $0 \in \mathcal{S}$):

$$\int_{t_1}^{t_2} \int_{|x| < t} \left[\partial_t u + \frac{x}{t} \cdot \nabla u + \left(\frac{N}{2} - 1 \right) \frac{u}{t} \right]^2 dx \frac{dt}{t} \leq C(u) \left(\log \frac{t_1}{t_2} \right)^{1/2}, \quad (5)$$

which is a consequence of preliminary control of the flux. From (5) and Tauberian type arguments one then shows that there exist (many) $t_n \downarrow 0$, such that

$$\sup_{0 < \tau < t_n/16} \frac{1}{\tau} \int_{|t_n - t| < \tau} \int_{|x| < ct} \left[\partial_t u + \frac{x}{t} \cdot \nabla u + \left(\frac{N}{2} - 1 \right) \frac{u}{t} \right]^2 dx dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (6)$$

From this one obtains a preliminary decomposition with an error tending to 0 in the “dispersive” norm. To show that the error goes to 0 in energy, one uses a new “energy channels” argument, valid in the nonradial case, in all dimensions.

Proposition 1 ([11],[12]). Fix $\gamma \in (0, 1)$. There exists $\mu = \mu(\gamma) > 0$ and small such that, if v is a finite energy solution of the linear wave equation in $\mathbb{R}^N \times \mathbb{R}$, $N \geq 1$, with initial data $(v_0, v_1) \in \dot{H}^1 \times L^2$, satisfying

$$\begin{aligned} & \|(v_0, v_1)\|_{\dot{H}^1 \times L^2(B_{1+\mu}^c \cup B_{1-\mu})} + \|\partial_r v_0 + v_1\|_{L^2} \\ & + \left\| \nabla v_0 - \left(\frac{x}{|x|} \cdot \nabla v_0 \right) \frac{x}{|x|} \right\|_{L^2} \leq \mu \|(v_0, v_1)\|_{\dot{H}^1 \times L^2}, \end{aligned}$$

then, for all $t \geq 0$ we have

$$\int_{|x| \geq \gamma + t} |\nabla_{x,t} v|^2 dx \geq \gamma \|(v_0, v_1)\|_{\dot{H}^1 \times L^2}^2. \quad (7)$$

Remark 4. With this method of proof, relying on monotonicity laws giving convergence only after averaging in time (as in (5), (6)) one cannot hope for more than a decomposition for particular (but many, see Remark 3) sequences of times. The

difficulty of obtaining the resolution for all times is illustrated by the harmonic map heat flow, for which the analog of Theorem 0.3 is known, but the soliton resolution for all times need not hold in full generality for general target manifolds, as shown by examples of Topping [54]. For this heat flow, for the case of $\mathbb{S}^2$ target ($\mathbb{S}^2$ being the round sphere in $\mathbb{R}^3$), it is conjectured that the full analog of soliton resolution holds, but this has not been proven yet.

Remark 5. We see Theorem 0.3 as an important step towards the proof of soliton resolution for all times. Indeed, because of it, one is now reduced to the study of the dynamics close to a sum of traveling waves plus a “dispersive” term, rather than the general large data dynamics. The main challenge in order to succeed in this is to show that the collision of two or more traveling waves produces dispersion. Note that in the radial case, when $N = 3$, this is a consequence of (†).

The ideas that we have just outlined are very robust. We will next outline some applications of them to the study of wave maps. This is an extremely well-studied model, a geometric wave equation with applications in physics, to nonlinear sigma models, to scenarios in general relativity and to gauge theories. In the energy critical case, with values in the standard sphere $\mathbb{S}^2$, it takes the form, for $u : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{S}^2$,

$$\partial_t u - \Delta u = (|\nabla u|^2 - |\partial_t u|^2)u,$$

with initial data $\vec{u}(0) = (u_0, u_1)$ verifying the compatibility conditions $|u_0| = 1$ and $u_0 \cdot u_1 = 0$. For simplicity, in the general case when no symmetry is present, as is usual in the subject, we shall assume that $\vec{u}(0)$ is smooth, u_1 is compactly supported and u_0 is a constant u_∞ for large x . Such wave maps are called “classical”. This equation has been studied extensively under symmetry assumptions on the solutions, in the so-called k -equivariant case, for which $u \circ \rho = \rho^k \circ u$, where ρ is a rotation of $\mathbb{R}^2$, which acts on $\mathbb{S}^2$ by rotation about a fixed axis. If we use coordinates (r, ω, t) for $\mathbb{R}^2 \times \mathbb{R}$, and (ψ, θ) as polar coordinates for $\mathbb{S}^2$, then u is given by $(r, \omega, t) \mapsto (\psi(r, t), k\omega)$, and the wave map equation then becomes:

$$\partial_t^2 \psi - \partial_r^2 \psi - \frac{1}{r} \partial_r \psi + k^2 \frac{\sin 2\psi}{2r^2} = 0,$$

which has been studied for example in [4],[5],[46] and [47]. (The radial Yang–Mills system in $\mathbb{R}^4$ can also be put in a similar form.) We consider solutions in the energy space, where

$$\mathcal{E}(\psi, \psi_t) = \int_0^\infty \left[(\partial_r \psi)^2 + (\partial_t \psi)^2 + k^2 \frac{\sin^2 \psi}{r^2} \right] r dr.$$

Note that the finiteness (and conservation) of the energy imply that $\psi(0, t) = m\pi$, $\psi(\infty, t) = l\pi$, $m, l \in \mathbb{Z}$. It has been shown in [41], [39] and [40] that finite time blow-up can occur for k -equivariant wave maps. Following work in the co-rotational case (the 1-equivariant case) in [7],[8],[6], the analog of the result in [15] was obtained in [25], for all k -equivariant wave maps. The static solutions in this case are harmonic maps, which in the k -equivariant setting take the form $Q(r) = 2 \arctan(r^k) + q\pi$, $q \in \mathbb{Z}$. Note that the linearized equation associated to the k -equivariant wave map equation is

$$\partial_t^2 \phi - \partial_r^2 \phi - \frac{1}{r} \partial_r \phi + k^2 \frac{\phi}{r^2} = 0,$$

which under the transformation $u_L = r^{-k}\phi$ verifies the radial $2k + 2$ dimensional wave equation. Hence solutions to the linearized equation preserve the energy

$$\|\vec{\phi}(t)\|_{\mathcal{H}_k \times L^2} = \int_0^\infty \left[(\partial_t \phi)^2 + (\partial_r \phi)^2 + k^2 \frac{\phi^2}{r^2} \right] r dr.$$

Theorem 0.4 ([25]). *Let $\vec{\psi}(t)$ be a finite energy solution of the k -equivariant wave maps equation. Then there exist a sequence of times $t_n \uparrow T_+$, an integer $J \geq 0$, J sequences of scales $0 < r_{J,n} \leq \dots r_{2,n} \leq r_{1,n}$, with $\lim_n r_{j,n}/r_{j-1,n} = 0$, and J harmonic maps $Q_1, \dots, Q_J$ such that $Q_J(0) = \psi_0(0)$, $Q_{j+1}(\infty) = Q_j(0)$, for $j = 1, \dots, J-1$, such that the following holds:*

(i) $T_+ = \infty$. Let $l\pi = \psi_0(\infty)$. Then $Q_1(\infty) = l\pi$, $\lim_n r_{1,n}/t_n = 0$, and there exists a radial finite energy solution $\vec{\phi}_L$ to the linearized equation, such that

$$\vec{\psi}(t_n) = \sum_{j=1}^J \left(Q_j \left(\frac{\cdot}{r_{j,n}} \right) - Q_j(\infty), 0 \right) + (l\pi, 0) + \vec{\phi}_L(t_n) + \vec{b}_n.$$

(ii) $T_+ < \infty$. Denote $l\pi = \lim_{t \rightarrow T_+} \psi(T_+ - t, t)$, which is well-defined. Then, $J \geq 1$, $\lim_n r_{1,n}/(T_+ - t_n) = 0$, and there exists $\vec{\phi} \in \mathcal{H} \times L^2$ of finite energy, such that $Q_1(\infty) = \phi(0) = l\pi$ and

$$\vec{\psi}(t_n) = \sum_{j=1}^J \left(Q_j \left(\frac{\cdot}{r_{j,n}} \right) - Q_j(\infty), 0 \right) + \vec{\phi}_L(t_n) + \vec{b}_n,$$

where $\|\phi\|_{\mathcal{H}}^2 = \int_0^\infty [(\partial_r \phi)^2 + \phi^2/r^2] r dr$, and in both cases $\|\vec{b}_n\|_{\mathcal{H} \times L^2} \rightarrow 0$ as $n \rightarrow \infty$.

The proof in [25] uses a combination of virial identities and Tauberian real variable arguments, bypassing the failure of (3) in the even dimensions $2k + 2$. For the critical wave map equation, without symmetry assumptions, the theory of the local Cauchy problem is quite intricate. The equation is invariant under the scaling $u \mapsto u_\lambda(x, t) = u(\lambda x, \lambda t)$, and the conserved energy is

$$\mathcal{E}(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla_x u|^2 + |\partial_t u|^2] dx.$$

The works [32], [33], [34], [35], [36], [44] established the local well-posedness in subcritical spaces. The small data Cauchy problem in the critical energy space was addressed in the breakthrough works of Tataru [52], [53] and Tao [50], using the null-frame spaces introduced by Tataru and Tao's microlocal gauge. The large data theory has been treated by Sterbenz-Tataru [48], using the diffusive gauge. Sterbenz-Tataru [49] showed that if a wave map blows-up in finite time, or is global in time but does not scatter, then, after modulation, it must converge locally in space time, along a sequence of times, to a harmonic map. This can be seen as a first step towards soliton resolution for a sequence of times. It proves the “ground-state” or “threshold” conjecture: if the energy of the initial data is smaller than the one of the minimal energy harmonic map, the solution exists for all times and scatters. (In the case when the target is the hyperbolic sphere, and hence no harmonic map exists, the corresponding global in time result was also proved by Krieger-Schlag [37] and Tao [51].) Here scattering does not have the usual meaning of the solution approaching asymptotically a solution of the underlying linear equation, due to a non-perturbative portion of the nonlinearity. The meaning of scattering is that of [48], that a certain controlling space-time norm $\mathcal{S}$, introduced in [48], is finite. A characterization of wave maps with finite $\mathcal{S}$ norm is given in [48]. Recently, Grinis

[22] has taken a further step towards establishing soliton resolution along a well-chosen sequence of times. He proved that, along a well-chosen sequence of times, all of the energy concentration strictly inside the light-cone must be in the form of a traveling wave. He showed this by proving that there is no energy in the “neck region” (the region strictly inside the light cone and away from the traveling waves). Here a new difficulty arises, not present in the equivariant case (due to [5], [46]), namely that it is not known in the general case how to prevent energy concentration near the singularity cone. In [12], Duyckaerts, Jia, Kenig and Merle have been able to do this, for the finite time blow-up case, when the energy of the wave map is only slightly bigger than the one of the minimal energy harmonic map for which the energy is 4π , which is achieved for degree 1, 1-equivariant harmonic maps (with respect to an axis of symmetry). Let $\mathcal{M}_1$ be the space of such harmonic maps, and $\mathcal{M}_{\vec{l},1} = \{Q_{\vec{l}} : Q \in \mathcal{M}_1, |\vec{l}| < 1\}$ denote the set of Lorentz transforms of elements in $\mathcal{M}_1$.

Theorem 0.5 ([12]). *Let u be a classical wave map with $\mathcal{E}(\vec{u}) < \mathcal{E}(Q, 0) + \varepsilon_0^2$, $Q \in \mathcal{M}_1$, that blows-up at a finite time T_+ , with the origin being a singular point. Then, there exists $\vec{l} \in \mathbb{R}^2$, $|\vec{l}| \ll 1$, $x(t) \in \mathbb{R}^2$, $\lambda(t) > 0$ with $\lim_{t \rightarrow T_+} x(t)/(T_+ - t) = \vec{l}$, $\lambda(t) = o(T_+ - t)$ and $(v_0, v_1) \in \dot{H}^1 \times L^2 \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$, with $(v_0 - u_\infty, v_1)$ being compactly supported such that*

- (i) $\inf\{\|\vec{u}(t) - (v_0, v_1) - (Q_{\vec{l}}, \partial_t Q_{\vec{l}})\|_{\dot{H}^1 \times L^2} : Q_{\vec{l}} \in \mathcal{M}_{\vec{l},1}\} \rightarrow 0$ as $t \rightarrow T_+$,
- (ii) $\|\vec{u}(t) - (v_0, v_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^2 \setminus B_{\lambda(t)}(x(t)))} \rightarrow 0$ as $t \rightarrow T_+$.

Heuristically, the result says that at the blow-up time, the wave map essentially consists of two parts, one regular part outside the light-cone $\{|x| < T_+ - t\}$, and a traveling wave with small velocity $\vec{l}$ in a small region (relative to the size of the cone) near the point $\vec{l}(T_+ - t)$. Moreover, there is no other energy concentration. The key new idea in the proof of Theorem 0.5 is a new “energy channel” for wave maps.

Proposition 2 ([12]). *Fix $\beta \in (0, 1)$. There exists a small $\delta(\beta) > 0$ and a small $\varepsilon_0(\beta) > 0$, such that if u is a classical wave map with energy $\mathcal{E}(\vec{u}) < \varepsilon_0$, satisfying*

$$\begin{aligned} \|(u_0, u_1)\|_{\dot{H}^1 \times L^2(B_{1+\delta}^c \cup B_{1-\delta})} + \left\| \nabla u_0 - \left(\frac{x}{|x|} \cdot \nabla u_0 \right) \frac{x}{|x|} \right\|_{L^2} \\ + \|\partial_r u_0 + u_1\| \leq \delta \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}, \end{aligned}$$

then, for all $t \geq 0$,

$$\int_{|x| > \beta+t} |\nabla_{x,t} u|^2 dx \geq \beta \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}^2.$$

This is an analog of Proposition 1 (which is valid for solutions of the linear wave equation). The proof of Theorem 0.5 follows, roughly speaking, by using the wave map Morawetz identity [51], [49], Grinis’ result [22] as a substitute for the Bahouri–Gérard decomposition and the energy channels in Proposition 2 to show that there can be no energy concentration on the boundary of the singularity cone. The proof of Proposition 2 uses the linear results in Proposition 1, in dimension 2, a frequency decomposition with a direct argument for low frequencies, and the fact that the outgoing conditions are stable (for most frequencies) with respect to frequency projection. Since the nonlinearity cannot be treated perturbatively ([50]), one has to perform a gauge transform to treat the nonlinearity. The key point is

that, although the gauge transform can change the wave map significantly, it does not change significantly the energy distribution.

Going forward, we would like to prove the analog of Theorem 0.5, without the energy restriction, thus proving soliton resolution for wave maps along well-chosen time sequences. In order to do this one hopes to give an extension of Proposition 2, where one replaces the smallness in energy by smallness in “energy dispersion norm” (see [47],[48]). This seems to require stronger “high frequency perturbation theorems”. We would also like to extend all these results to the scattering situation, where one also needs stronger perturbation theorems. Finally, we believe that the full soliton resolution holds for classical wave maps into $\mathbb{S}^2$. We believe that this should follow from “energy channels for wave maps close to a multisoliton plus radiation” which could be proved exploiting the recent non-degeneracy result for harmonic maps in [3], together with arguments as in [18].

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Received October 2018; revised October 2018.

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