Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-019-09794-7

On the Determining Wavenumber for the Nonautonomous Subcritical SQG Equation

Alexey Cheskidov[1] · **Mimi Dai**[1]

Received: 22 June 2018 / Revised: 28 August 2019

Abstract
A time-dependent determining wavenumber was introduced in Cheskidov and Dai (Phys D Nonlinear Phenom 376–377:204–215, 2018) to estimate the number of determining modes for the surface quasi-geostrophic equation. In this paper we continue this investigation focusing on the subcritical case and study trajectories inside an absorbing set bounded in L^∞. Utilizing this bound we find a time-independent determining wavenumber that improves the estimate obtained in Cheskidov and Dai (Phys D Nonlinear Phenom 376–377:204–215, 2018). This classical approach is more direct, but it is contingent on the existence of the L^∞ absorbing set.

Keywords Subcritical quasi-geostrophic equation · Determining modes · Global attractor

Mathematics Subject Classification 35Q35 · 37L30

1 Introduction

In this paper we estimate the number of determining modes for the forced subcritical surface quasi-geostrophic (SQG) equation (see [12])

$$\begin{aligned} \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \nu \Lambda^\alpha \theta &= f, \\ u &= R^\perp \theta, \end{aligned} \tag{1.1}$$

The work of A. Cheskidov was partially supported by NSF grants DMS–1108864 and DMS–1517583. The work of M. Dai was partially supported by NSF Grant DMS–1815069.

✉ Mimi Dai
mdai@uic.edu

Alexey Cheskidov
acheskid@uic.edu

[1] Department of Mathematics, Statistics and Computer Science, University of Illinois Chicago, Chicago, IL 60607, USA

Published online: 16 September 2019

where $x \in \mathbb{T}^2 = [0, L]^2$, $1 < \alpha < 2$, $\nu > 0$, $\Lambda = \sqrt{-\Delta}$ is the Zygmund operator, and

$$R^{\perp}\theta = \Lambda^{-1}(-\partial_2\theta, \partial_1\theta).$$

The initial data $\theta(0) \in L^2(\mathbb{T}^2)$ and the force $f \in L^\infty(0, \infty; L^p(\mathbb{T}^2))$ for some $p > 2/\alpha$ are assumed to have zero average.

A time-dependent determining wavenumber $\Lambda(t)$ was introduced in [5] in the case where $\alpha \in (0, 2)$ and the force could be potentially rough. The determining wavenumber was defined based only on the structure of the equation and without any requirements on the regularity of solutions. It was shown that if two solutions coincide below $\Lambda(t)$, the difference between them decay exponentially, even when they are far away from the attractor. Moreover, $\Lambda(t)$ was shown to be uniformly bounded for all the solutions on the global attractor when $\alpha \in [1, 2)$ and $f \in L^p$, $p > 2/\alpha$, in which case the attractor is bounded in L^∞. In this paper we investigate this situation further and present a different, more direct approach in the subcritical case $\alpha \in (1, 2)$. Here we consider solutions that already entered an L^∞ absorbing set and take advantage of the L^∞ bound (which is proportional to the L^p-norm of the force) to define a time-independent determining wavenumber Λ and improve the final estimate for the number of determining modes that we had in [5]. The drawback of this method is that it is less general and works only for regular solutions in the L^∞ absorbing set. For a more complete background on the topic of finite dimensionality of flows, we refer the readers to [5,7,10,11,15–21] and references therein.

For the critical SQG equation ($\alpha = 1$), due to the balance of the nonlinear term and the dissipative term, the global regularity problem was challenging. However it was solved by different authors using different sophiscated methods in [2,9,13,23,24]. For the subcritical SQG equation with $1 < \alpha < 2$, the dissipative term dominates. In this case the global regularity was obtained in [25].

In this paper, we will consider forces $f \in L^\infty(I; L^p(\mathbb{T}^2))$, $p > 2/\alpha$, where $I = (0, \infty)$ or $(-\infty, \infty)$, such that

$$\sup_{t\in I} \|f(t)\|_p \le F,$$

for some fixed $F > 0$. Then $\{\theta \in L^2 : \|\theta\|_\infty \le R_\infty\}$ is an absorbing set in L^2 (see Sect. 3), where

$$R_\infty \sim \lambda_0^{\frac{2}{p}-\alpha} \frac{F}{\nu}. \tag{1.2}$$

Here $\lambda_0 = 1/L$. We prove the following.

Theorem 1.1 *Let $\alpha \in (1, 2)$, $l > \frac{\alpha}{\alpha-1}$, and $Q \in \mathbb{N}$ be such that*

$$\Lambda := \lambda_0 2^Q \ge \left(\frac{Cl^2 R_\infty}{\nu}\right)^{\frac{1}{\alpha-1}},$$

where C is some absolute constant. Let $\theta_1(t)$ be a solution of (1.1) with $f = f_1$ and $\theta_2(t)$ be a solution to (1.1) with $f = f_2$. If

$$\|\theta_1(t)_{\le Q} - \theta_2(t)_{\le Q}\|_{B^0_{l,l}} \to 0, \quad \text{and} \quad \|f_1 - f_2\|_{B^{-\alpha(1-\frac{1}{l})}_{l,l}} \to 0, \quad \text{as} \quad t \to \infty,$$

then

$$\|\theta_1(t) - \theta_2(t)\|^l_{B^0_{l,l}} \to 0 \quad \text{as} \quad t \to \infty.$$

Moreover, if $\theta_1(t)$ and $\theta_2(t)$ are two complete (ancient) solutions of (1.1) with $f \in L^\infty((-\infty,\infty);L^p)$, $p > 2/\alpha$, such that $\theta_1, \theta_2 \in L^\infty((-\infty,\infty);L^2)$ and

$$\theta_1(t)_{\le Q} = \theta_2(t)_{\le Q}, \qquad \forall t < 0, \tag{1.3}$$

then

$$\theta_1(t) = \theta_2(t), \qquad \forall t \in \mathbb{R}.$$

The second part of the theorem concerns solutions on the pullback attractor

$$\mathcal{A}(t) = \{\theta(t) : \theta(\cdot) \text{ is a complete bounded solution, i.e., } \theta \in L^\infty((-\infty,\infty);L^2)\},$$

that describes the long time behavior of solutions as the initial time goes to minus infinity. The fact that $\mathcal{A}(t)$ is indeed a pullback attractor follows, for example, from the general framework [8].

In the particular case of a time independent force $f \in L^p$, all the time slices of $\mathcal{A}(t)$ coincide, and

$$\mathcal{A} = \mathcal{A}(t), \qquad \forall t \in \mathbb{R},$$

is the global attractor. Again, in the subcritical case $\alpha > 1$, it is easy to show that $\mathcal{A}$ is a global attractor by virtue of classical methods, or applying the evolutionary system framework [4] that requires the existence of an absorbing ball, energy inequality, and continuity of trajectories. This method does not require proving the existence of a compact absorbing set, and was used in [6] to show that $\mathcal{A}$ is the global attractor in the critical case $\alpha = 1$ (see also [9] for the existence of the global attractor in H^1). In addition, in the autonomous case $f \in L^p$, the number of determining modes was estimated in [5] using a much more general method applicable to subcritical, critical, and supercritical regimes. Theorem 1.1 provides an improvement to the upper bound on Λ in the subcritical case $\alpha > 1$.

2 Preliminaries

2.1 Notations

We denote by $A \lesssim B$ an estimate of the form $A \le CB$ with some absolute constant C, and by $A \sim B$ an estimate of the form $C_1 B \le A \le C_2 B$ with some absolute constants C_1, C_2. We write $\|\cdot\|_p = \|\cdot\|_{L^p}$, and $(\cdot,\cdot)$ stands for the L^2-inner product.

2.2 Littlewood–Paley Decomposition

We recall briefly the Littlewood–Paley decomposition theory, which is one of the main techniques used in the paper. For a more detailed description on this theory we refer readers to the books [1,22].

Denote $\lambda_q = \frac{2^q}{L}$ for integers q. A nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ is chosen such that

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \le \frac{3}{4} \\ 0, & \text{for } |\xi| \ge 1. \end{cases} \tag{2.1}$$

Let

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$$

and

$$\varphi_q(\xi) = \begin{cases} \varphi(2^{-q}\xi) & \text{for } q \geq 0, \\ \chi(\xi) & \text{for } q = -1. \end{cases}$$

For a tempered distribution vector field u, its Littlewood–Paley projection u_q is defined as follows.

$$h_q := \sum_{k\in\mathbb{Z}^n} \varphi_q(k) e^{i\frac{2\pi k\cdot x}{L}}$$

$$u_q := \Delta_q u = \sum_{k\in\mathbb{Z}^n} \hat{u}_k \varphi_q(k) e^{i\frac{2\pi k\cdot x}{L}} = \frac{1}{L^2}\int_{\mathbb{T}^2} h_q(y) u(x-y)dy, \qquad q \geq -1,$$

where $\hat{u}_k$ is the kth Fourier coefficient of u. Then we have

$$u = \sum_{q=-1}^{\infty} u_q$$

in the distributional sense. We also denote

$$u_{\leq Q} = \sum_{q=-1}^{Q} u_q, \qquad u_{(Q,R]} = \sum_{p=Q+1}^{R} u_p, \qquad \tilde{u}_q = \sum_{|p-q|\leq 1} u_p.$$

The Besov $B^s_{l,l}$-norm is defined as

$$\|u\|_{B^s_{l,l}} = \left(\sum_{q=-1}^{\infty} \lambda_q^s \|u_q\|_l^l\right)^{\frac{l}{l}}.$$

The following inequalities will be frequently used:

Lemma 2.1 (Bernstein's inequality) *Let n be the space dimension and $r \geq s \geq 1$. Then for all tempered distributions u,*

$$\|u_q\|_r \leq \lambda_q^{n(\frac{1}{s}-\frac{1}{r})} \|u_q\|_s.$$

Lemma 2.2 *Assume $2 < l < \infty$ and $0 \leq \alpha \leq 2$. Then*

$$l\int u_q \Lambda^\alpha u_q |u_q|^{l-2} dx \gtrsim \lambda_q^\alpha \|u_q\|_l^l.$$

For a proof of Lemma 2.2, see [3,14].

2.3 Bony's Paraproduct and Commutator

Bony's paraproduct formula will be used to decompose the nonlinear terms. We will use the same version as in [6]:

$$\Delta_q(u\cdot\nabla v) = \sum_{|q-p|\leq 2} \Delta_q(u_{\leq p-2}\cdot\nabla v_p) + \sum_{|q-p|\leq 2} \Delta_q(u_p\cdot\nabla v_{\leq p-2}) + \sum_{p\geq q-2} \Delta_q(\tilde{u}_p\cdot\nabla v_p).$$

Some terms in this decomposition will be estimated using commutators. Let

$$[\Delta_q, u_{\le p-2}\cdot\nabla]v_p := \Delta_q(u_{\le p-2}\cdot\nabla v_p) - u_{\le p-2}\cdot\nabla\Delta_q v_p. \tag{2.2}$$

By definition of Δ_q and Young's inequality,

$$\|[\Delta_q, u_{\le p-2}\cdot\nabla]v_p\|_r \lesssim \|\nabla u_{\le p-2}\|_\infty \|v_p\|_r, \tag{2.3}$$

for any $r > 1$ (see [5] for details).

3 Absorbing Sets

First, we recall the L^∞ estimates from [5].

Lemma 3.1 *Let $\alpha \in (0,2)$ and θ be a solution of (1.1) on $[0,\infty)$ with $\theta(0) \in L^2$ and*

$$\sup_{t>0}\|f(t)\|_p \le F,$$

for some $F \ge 0$ and $p \in (2/\alpha, \infty]$. Then, for every $t > 0$,

$$\|\theta(t)\|_{L^\infty} \lesssim \frac{\|\theta(0)\|_2}{(\nu t)^{\frac{1}{\alpha}}} + \lambda_0^{\frac{2}{p}-\alpha}\frac{F}{\nu}\left(1 + \lambda_0^{\frac{\alpha}{2}-1}(\nu t)^{\frac{1}{2}-\frac{1}{\alpha}}\right). \tag{3.1}$$

Proof Identical to the proof of Lemma 4.2 in [5] provided $\|f\|_p$ is replaced with F. □

Due to the energy equality

$$\|\theta(t)\|_2^2 = \|\theta(t_0)\|_2^2 + \int_{t_0}^{t}\left(-\nu\|\Lambda^{\frac{\alpha}{2}}\theta(\tau)\|_2 + (f(\tau),\theta(\tau))\right)d\tau, \qquad 0 \le t_0 \le t,$$

and the fact that

$$\|\Lambda^{-\frac{\alpha}{2}} f\|_2 \lesssim \lambda_0^{\frac{2}{p}-1-\frac{\alpha}{2}}\|f\|_p \le \lambda_0^{\frac{2}{p}-1-\frac{\alpha}{2}} F,$$

we have

$$\|\theta(t)\|_2^2 \lesssim \|\theta(0)\|_2^2 e^{-\nu(2\pi\lambda_0)^\alpha t} + \frac{\lambda_0^{\frac{4}{p}-2-2\alpha}F^2}{\nu^2}\left(1 - e^{-\nu(2\pi\lambda_0)^\alpha t}\right), \qquad t > 0.$$

which implies the existence of an absorbing ball in L^2. Indeed, for any bounded set $U \subset L^2$ there exists time t_{L^2}, such that

$$\theta(t) \in B_{L^2}, \qquad \forall t \ge t_{L^2},$$

for any solution $\theta(t)$ with $\theta(0) \in U$. Here

$$B_{L^2} = \left\{\theta \in L^2 : \|\theta\|_2 \le R_2\right\}, \qquad R_2 \sim \lambda_0^{\frac{2}{p}-1-\alpha}\frac{F}{\nu}.$$

No consider the following ball in L^∞:

$$B_{L^\infty} = \left\{\theta \in B_{L^2} : \|\theta\|_\infty \le R_\infty\right\}, \qquad R_\infty \sim \lambda_0^{\frac{2}{p}-\alpha}\frac{F}{\nu}.$$

Lemma 3.1 implies that B_{L^∞} is an absorbing set as well, i.e., for any bounded set $U \subset L^2$ there exists time t_{L^∞}, such that

$$\theta(t) \in B_{L^\infty}, \qquad \forall t \ge t_{L^\infty},$$

for any solution $\theta(t)$ with $\theta(0) \in U$. □

4 Proof of the Main Result

First we recall a generalization of Grönwall's lemma from [18].

Lemma 4.1 *Let $\alpha(t)$ be a locally integrable real valued function on $(0,\infty)$, satisfying for some $0 < T < \infty$ the following conditions:*

$$\liminf_{t\to\infty}\int_t^{T+t}\phi(\tau)\,d\tau > 0, \qquad \limsup_{t\to\infty}\int_t^{T+t}\phi^-(\tau)\,d\tau < \infty,$$

where $\phi^- = \max\{-\phi, 0\}$. Let $\psi(t)$ be a measurable real valued function on $(0,\infty)$ such that

$$\psi(t) \to 0, \qquad as \qquad t\to\infty.$$

Suppose $\xi(t)$ is an absolutely continuous non-negative function on $(0,\infty)$ such that

$$\frac{d}{dt}\xi + \phi\xi \le \psi, \qquad a.e.\ on\ (0,\infty).$$

Then

$$\xi(t) \to 0 \qquad as \qquad t\to\infty.$$

Now we are ready to prove the main result.

Proof of Theorem 1.1 Consider two solution θ_1, θ_2 of (1.1) with forces f_1 and f_2. Let t_0 be a time after which the solutions stay in the absorbing set B_{L^∞}:

$$\|\theta_1(t)\|_\infty \le R_\infty, \qquad \|\theta_2(t)\|_\infty \le R_\infty, \qquad t \ge t_0.$$

In what follows we assume that $t \ge t_0$. Denote $u_1 = R^\perp\theta_1$ and $u_2 = R^\perp\theta_2$. Let $f = f_1 - f_2$ and $w = \theta_1 - \theta_2$, which satisfies the equation

$$w_t + u_1\cdot\nabla w + \nu\Lambda^\alpha w + R^\perp w\cdot\nabla\theta_2 = f. \tag{4.1}$$

Projecting Eq. (4.1) onto the q-th shell, multiplying by $l w_q|w_q|^{l-2}$, integrating, adding up for all $q \ge -1$, applying Lemma 2.2, Hölder and Young inequalities, yield

$$\begin{aligned}
&\frac{d}{dt}\|w(t)\|^l_{B^0_{l,l}} + C\nu\|\Lambda^{\alpha/l}w\|^l_{B^0_{l,l}} - \left(\frac{2}{C\nu}\right)^{l-1} l^{l-2}\|\Lambda^{-\alpha\left(1-\frac{1}{l}\right)}f\|^l_{B^0_{l,l}}\\
&\quad\le -l\sum_{q\ge -1}\int_{\mathbb{T}^2}\Delta_q(R^\perp w\cdot\nabla\theta_2)w_q|w_q|^{l-2}\,dx\\
&\qquad - l\sum_{q\ge -1}\int_{\mathbb{T}^2}\Delta_q(u_1\cdot\nabla w)\quad w_q|w_q|^{l-2}\,dx\\
&\quad = I + J,
\end{aligned} \tag{4.2}$$

for some absolute constant C. Using Bony's paraproduct formula, I is decomposed as

$$\begin{aligned} I =& -l \sum_{q\geq -1}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^2}\Delta_q(R^\perp w_{\leq p-2}\cdot\nabla(\theta_2)_p)w_q|w_q|^{l-2}\,dx \\ &- l\sum_{q\geq -1}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^2}\Delta_q(R^\perp w_p\cdot\nabla(\theta_2)_{\leq p-2})w_q|w_q|^{l-2}\,dx \\ &- l\sum_{q\geq -1}\sum_{p\geq q-2}\int_{\mathbb{T}^2}\Delta_q(R^\perp \tilde{w}_p\cdot\nabla(\theta_2)_p)w_q|w_q|^{l-2}\,dx \\ =& \ I_1+I_2+I_3. \end{aligned}$$

Recall that $\Lambda = 2^Q/L$. To estimate I_1 we use Hölder's inequality and split it as follows:

$$\begin{aligned} |I_1| \leq & \ l\sum_{q\geq -1}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^2}\left|\Delta_q(R^\perp w_{\leq p-2}\cdot\nabla(\theta_2)_p)w_q\right||w_q|^{l-2}\,dx \\ \lesssim & \ l\sum_{q>Q}\|w_q\|_l^{l-1}\sum_{|q-p|\leq 2}\lambda_p\|(\theta_2)_p\|_\infty\sum_{Q<p'\leq p-2}\|R^\perp w_{p'}\|_l \\ &+ l\sum_{q>Q}\|w_q\|_l^{l-1}\sum_{|q-p|\leq 2}\lambda_p\|(\theta_2)_p\|_\infty\|R^\perp w_{\leq Q}\|_l \\ &+ l\sum_{q\leq Q}\|w_q\|_l^{l-1}\sum_{|q-p|\leq 2}\lambda_p\|(\theta_2)_p\|_\infty\|R^\perp w_{\leq p-2}\|_l \\ \equiv & \ I_{11}+I_{12}+I_{13}. \end{aligned}$$

Then using Young's inequality, Jensen's inequality and the fact that $\|R^\perp w_q\|_l \lesssim \|w_q\|_l$, we obtain

$$\begin{aligned} |I_{11}| \lesssim & \ R_\infty l\sum_{p>Q-2}\lambda_p\sum_{|q-p|\leq 2}\|w_q\|_l^{l-1}\sum_{Q<p'\leq p-2}\|R^\perp w_{p'}\|_l \\ \lesssim & \ R_\infty l\sum_{p>Q}\lambda_p\|w_p\|_l^{l-1}\sum_{Q<p'\leq p-2}\|R^\perp w_{p'}\|_l \\ \lesssim & \ \Lambda^{1-\alpha+\frac{\alpha}{l}}R_\infty l\sum_{p>Q}\lambda_p^{\frac{\alpha(l-1)}{l}}\|w_p\|_l^{l-1}\sum_{Q<p'\leq p-2}\|R^\perp w_{p'}\|_l \\ \lesssim & \ \Lambda^{1-\alpha}R_\infty l\sum_{p>Q}\lambda_p^{\frac{\alpha(l-1)}{l}}\|w_p\|_l^{l-1}\sum_{Q<p'\leq p-2}\lambda_{p'}^{\frac{\alpha}{l}}\|R^\perp w_{p'}\|_l\lambda_{p'-Q}^{-\frac{\alpha}{l}} \\ \lesssim & \ \Lambda^{1-\alpha}R_\infty l\sum_{q>Q}\lambda_q^\alpha\|w_q\|_l^l, \end{aligned}$$

where we needed $1-\alpha+\frac{\alpha}{l}<0$, i.e., $l>\alpha/(\alpha-1)$. Now we take small enough $\epsilon>0$, such that $1-\alpha+\frac{\alpha}{l}+\epsilon<0$, and use Hölder's inequality, Young's inequality, and Jensen's inequality to infer

$$\begin{aligned} I_{12} \lesssim & \ R_\infty l\sum_{q>Q}\|w_q\|_l^{l-1}\sum_{|q-p|\leq 2}\lambda_p\|R^\perp w_{\leq Q}\|_l \\ \lesssim & \ R_\infty l\sum_{q>Q-2}\lambda_q\|w_q\|_l^{l-1}\|R^\perp w_{\leq Q}\|_l \end{aligned}$$

$$
\begin{aligned}
&= R_\infty l \sum_{q>Q-2} \lambda_q^{1-\alpha+\frac{\alpha}{l}+\epsilon} \lambda_q^{-\epsilon} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \|R^\perp w_{\le Q}\|_l \\
&\lesssim \Lambda^{1-\alpha+\frac{\alpha}{l}+\epsilon} R_\infty l \sum_{q>Q-2} \lambda_q^{-\epsilon} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \|R^\perp w_{\le Q}\|_l \\
&\lesssim \Lambda^{1-\alpha} R_\infty l \left(\sum_{q>Q-2} \lambda_q^{-\epsilon} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \right)^{\frac{l}{l-1}} + \Lambda^{1+\epsilon l} R_\infty l \|R^\perp w_{\le Q}\|_l^l \\
&\lesssim \Lambda^{1-\alpha-\epsilon} R_\infty l \sum_{q>Q-2} \lambda_q^\alpha \|w_q\|_l^l + \Lambda^{1+\epsilon l} R_\infty l \|w_{\le Q}\|_l^l;
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
I_{13} &\lesssim R_\infty l \sum_{q\le Q} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \lambda_p \|R^\perp w_{\le p-2}\|_l \\
&\lesssim R_\infty l \sum_{q\le Q} \lambda_q \|w_q\|_l^{l-1} \|R^\perp w_{\le Q}\|_l \\
&= R_\infty l \sum_{q\le Q} \lambda_q^{1-\alpha+\frac{\alpha}{l}+\epsilon} \lambda_q^{-\epsilon} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \|R^\perp w_{\le Q}\|_l \\
&\lesssim \Lambda^{1-\alpha} R_\infty l \sum_{q\le Q} \lambda_q^{-\epsilon} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \Lambda^{\alpha-1} \|R^\perp w_{\le Q}\|_l \\
&\lesssim \Lambda^{1-\alpha} R_\infty l \left(\sum_{q\le Q} \lambda_q^{-\epsilon} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \right)^{\frac{l}{l-1}} + \Lambda^{(l-1)(\alpha-1)} R_\infty l \|R^\perp w_{\le Q}\|_l^l \\
&\lesssim \Lambda^{1-\alpha} R_\infty l \sum_{q\le Q} \lambda_q^\alpha \|w_q\|_l^l + \Lambda^{(l-1)(\alpha-1)} R_\infty l \|w_{\le Q}\|_l^l.
\end{aligned}
$$

For I_2, splitting the summation and using Hölder's inequality, we obtain

$$
\begin{aligned}
|I_2| &\lesssim l \sum_{q\ge -1} \sum_{|q-p|\le 2} \int_{\mathbb{T}^2} |\Delta_q (R^\perp w_p \cdot \nabla (\theta_2)_{\le p-2}) w_q | |w_q|^{l-2} \, dx \\
&\lesssim l \sum_{q>Q-2} \sum_{|p-q|\le 2} \sum_{p'\le p-2} \lambda_{p'} \|(\theta_2)_{p'}\|_\infty \|R^\perp w_p\|_l \|w_q\|_l^{l-1} \\
&\quad + l \sum_{q\le Q-2} \sum_{|p-q|\le 2} \sum_{p'\le p-2} \lambda_{p'} \|(\theta_2)_{p'}\|_\infty \|R^\perp w_p\|_l \|w_q\|_l^{l-1} \\
&\equiv I_{21} + I_{22}.
\end{aligned}
$$

The first term is estimated as

$$
\begin{aligned}
I_{21} &\lesssim R_\infty l \sum_{p>Q-4} \|w_p\|_l^l \sum_{p'\le p-2} \lambda_{p'} \\
&\lesssim R_\infty l \sum_{p>Q-4} \lambda_p^\alpha \|w_p\|_l^l \sum_{p'\le p-2} \lambda_{p'-p} \lambda_p^{1-\alpha} \\
&\lesssim \Lambda^{1-\alpha} R_\infty l \sum_{p>Q-4} \lambda_p^\alpha \|w_p\|_l^l.
\end{aligned}
$$

For the second term we have

$$\begin{aligned} I_{22} &\lesssim R_\infty l \sum_{q\le Q-2}\sum_{|p-q|\le 2}\sum_{p'\le p-2} \lambda_{p'}\|R^\perp w_p\|_l \|w_q\|_l^{l-1} \\ &\lesssim R_\infty l \sum_{q\le Q}\|w_q\|_l^l \sum_{p'\le Q}\lambda_{p'} \\ &\lesssim \Lambda R_\infty l \sum_{q\le Q}\|w_q\|_l^l. \end{aligned}$$

To estimate I_3, we first integrate by parts and then use Hölder's inequality obtaining

$$\begin{aligned} |I_3| &\le l \sum_{q\ge -1}\sum_{p\ge q-2}\int_{\mathbb{T}^2} |\Delta_q(R^\perp \tilde{w}_p(\theta_2)_p)\nabla(w_q|w_q|^{l-2})|\,dx \\ &\lesssim l^2 \sum_{q\ge -1}\sum_{p\ge q-2}\int_{\mathbb{T}^2} |\Delta_q(R^\perp \tilde{w}_p(\theta_2)_p)\nabla w_q||w_q|^{l-2}\,dx \\ &\lesssim l^2 \sum_{q>Q}\lambda_q\|w_q\|_l^{l-1}\sum_{p\ge q-2}\|R^\perp\tilde{w}_p\|_l\|(\theta_2)_p\|_\infty \\ &\quad + l^2 \sum_{q\le Q}\lambda_q\|w_q\|_l^{l-1}\sum_{p\ge q-2}\|R^\perp\tilde{w}_p\|_l\|(\theta_2)_p\|_\infty \\ &\equiv I_{31}+I_{32}. \end{aligned}$$

For the first term we use Jensen's inequality:

$$\begin{aligned} I_{31} &\lesssim R_\infty l^2 \sum_{p>Q-3}\|R^\perp w_p\|_l \sum_{Q<q\le p+2}\lambda_q\|w_q\|_l^{l-1} \\ &\lesssim R_\infty l^2 \sum_{p>Q-3}\lambda_p^{\frac{\alpha}{l}}\|w_p\|_l \sum_{Q<q\le p+2}\lambda_q^{\frac{\alpha(l-1)}{l}}\|w_q\|_l^{l-1}\lambda_q^{1-\alpha}\lambda_{q-p}^{\frac{\alpha}{l}} \\ &\lesssim \Lambda^{1-\alpha} R_\infty l^2 \sum_{q>Q-3}\lambda_q^\alpha\|w_q\|_l^l. \end{aligned}$$

For the second term, Hölder's inequality, Young's inequality, and Jensen's inequality yield

$$\begin{aligned} I_{32} &\lesssim R_\infty l^2 \sum_{q\le Q}\lambda_q\|w_q\|_l^{l-1}\sum_{p\ge q-2}\|R^\perp\tilde{w}_p\|_l \\ &\lesssim R_\infty l^2 \sum_{q\le Q}\|w_q\|_l^{l-1}\sum_{p\ge q-2}\lambda_p^{\frac{\alpha}{l}}\|R^\perp\tilde{w}_p\|_l\lambda_{p-q}^{-\frac{\alpha}{l}}\lambda_q^{1-\frac{\alpha}{l}} \\ &\lesssim \Lambda^{1-\frac{\alpha}{l}} R_\infty l^2 \sum_{q\le Q}\|w_q\|_l^{l-1}\sum_{p\ge q-2}\lambda_p^{\frac{\alpha}{l}}\|R^\perp\tilde{w}_p\|_l\lambda_{p-q}^{-\frac{\alpha}{l}} \\ &\lesssim \Lambda R_\infty l^2 \sum_{q\le Q}\left(\|w_q\|_l^l + \Lambda^{-\alpha}\left(\sum_{p\ge q-2}\lambda_p^{\frac{\alpha}{l}}\|R^\perp\tilde{w}_p\|_l\lambda_{p-q}^{-\frac{\alpha}{l}}\right)^l\right) \end{aligned}$$

$$\lesssim \Lambda R_\infty l^2 \sum_{q\le Q} \|w_q\|_l^l + \Lambda^{1-\alpha} R_\infty l^2 \sum_{q\le Q} \left(\sum_{p\ge q-2} \lambda_p^{\frac{\alpha}{l}} \|R^\perp \tilde{w}_p\|_l \lambda_{p-q}^{-\frac{\alpha}{l}} \right)^l$$
$$\lesssim \Lambda^{1-\alpha} R_\infty l^2 \sum_{q\ge -1} \lambda_q^\alpha \|w_q\|_l^l + \Lambda R_\infty l^2 \sum_{q\le Q} \|w_q\|_l^l.$$

Therefore, for l such that $1-\alpha+\frac{\alpha}{l} < 0$ we have

$$\begin{aligned} |I| &\lesssim \Lambda^{1-\alpha} R_\infty l^2 \sum_{q\ge -1} \lambda_q^\alpha \|w_q\|_l^l + \left(\Lambda^{(l-1)(\alpha-1)} + \Lambda^{1+\epsilon l} \right) R_\infty l^2 \sum_{q\le Q} \|w_q\|_l^l \\ &\lesssim \Lambda^{1-\alpha} R_\infty l^2 \sum_{q\ge -1} \lambda_q^\alpha \|w_q\|_l^l + \Lambda^{(l-1)(\alpha-1)} R_\infty l^2 \sum_{q\le Q} \|w_q\|_l^l, \end{aligned} \tag{4.3}$$

where ϵ is chosen small enough so that $1-\alpha+\frac{\alpha}{l}+\epsilon < 0$ and hence $(l-1)(\alpha-1) > 1+\epsilon l$.

We now estimate J, where we first apply Bony's paraproduct formula:

$$\begin{aligned} J =& -l \sum_{q\ge -1} \sum_{|q-p|\le 2} \int_{\mathbb{T}^2} \Delta_q((u_1)_{\le p-2} \cdot \nabla w_p) w_q |w_q|^{l-2}\, dx \\ & - l \sum_{q\ge -1} \sum_{|q-p|\le 2} \int_{\mathbb{T}^2} \Delta_q((u_1)_p \cdot \nabla w_{\le p-2}) w_q |w_q|^{l-2}\, dx \\ & - l \sum_{q\ge -1} \sum_{p\ge q-2} \int_{\mathbb{T}^2} \Delta_q((u_1)_p \cdot \nabla \tilde{w}_p) w_q |w_q|^{l-2}\, dx \\ =& \; J_1 + J_2 + J_3. \end{aligned}$$

Observing that $\sum_{|p-q|\le 2} \Delta_q w_p = w_q$, we then decompose J_1 using the commutator notation (2.2):

$$\begin{aligned} J_1 =& -l \sum_{q\ge -1} \sum_{|q-p|\le 2} \int_{\mathbb{T}^2} [\Delta_q, (u_1)_{\le p-2} \cdot \nabla] w_p w_q |w_q|^{l-2}\, dx \\ & - l \sum_{q\ge -1} \int_{\mathbb{T}^2} (u_1)_{\le q-2} \cdot \nabla w_q w_q |w_q|^{l-2}\, dx \\ & - l \sum_{q\ge -1} \sum_{|q-p|\le 2} \int_{\mathbb{T}^2} ((u_1)_{\le p-2} - (u_1)_{\le q-2}) \cdot \nabla \Delta_q w_p w_q |w_q|^{l-2}\, dx \\ =& \; J_{11} + J_{12} + J_{13}. \end{aligned}$$

The term J_{12} vanishes because $\operatorname{div}(u_1)_{\le q-2} = 0$. To estimate J_{11} we will use (2.3),

$$\|[\Delta_q, (u_1)_{\le p-2} \cdot \nabla] w_p\|_l \lesssim \|\nabla (u_1)_{\le p-2}\|_\infty \|w_p\|_l.$$

Then splitting the summation we get

$$\begin{aligned}
|J_{11}| &\le l \sum_{q\ge -1}\sum_{|q-p|\le 2} \|[\Delta_q, (u_1)_{\le p-2}\cdot\nabla]w_p\|_l \|w_q\|_l^{l-1}\\
&\le l \sum_{q> Q-2}\sum_{|q-p|\le 2} \|\nabla (u_1)_{\le p-2}\|_\infty \|w_p\|_l \|w_q\|_l^{l-1}\\
&\quad + l \sum_{q\le Q-2}\sum_{|q-p|\le 2} \|\nabla (u_1)_{\le p-2}\|_\infty \|w_p\|_l \|w_q\|_l^{l-1}\\
&\equiv J_{111} + J_{112}.
\end{aligned}$$

Now note that $\|(u_1)_q\|_\infty \lesssim \|(\theta_1)_q\|_\infty \le R_\infty$. So using Hölder's and Bernstein's inequalities, we obtain

$$\begin{aligned}
J_{111} &\lesssim l \sum_{q> Q-2}\sum_{|q-p|\le 2}\sum_{p'\le q} \lambda_{p'} \|(u_1)_{p'}\|_\infty \|w_p\|_l \|w_q\|_l^{l-1}\\
&\lesssim R_\infty l \sum_{q> Q-2}\sum_{|q-p|\le 2}\sum_{p'\le q} \lambda_{p'} \|w_p\|_l \|w_q\|_l^{l-1}\\
&\lesssim R_\infty l \sum_{q> Q-4}\sum_{p'\le q} \lambda_{p'} \|w_q\|_l^{l}\\
&\lesssim R_\infty l \sum_{q> Q-4} \lambda_q^{\alpha} \|w_q\|_l^{l} \sum_{p'\le q} \lambda_{p'-q}\lambda_q^{1-\alpha}\\
&\lesssim \Lambda^{1-\alpha} R_\infty l \sum_{q> Q-4} \lambda_q^{\alpha} \|w_q\|_l^{l}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_{112} &\lesssim l \sum_{q\le Q-2}\sum_{|q-p|\le 2}\sum_{p'\le p-2} \lambda_{p'} \|(u_1)_{p'}\|_\infty \|w_p\|_l \|w_q\|_l^{l-1}\\
&\lesssim R_\infty l \sum_{q\le Q-2}\sum_{|q-p|\le 2}\sum_{p'\le q} \lambda_{p'} \|w_p\|_l \|w_q\|_l^{l-1}\\
&\lesssim R_\infty l \sum_{q\le Q} \|w_q\|_l^{l} \sum_{p'\le Q-2} \lambda_{p'}\\
&\lesssim \Lambda R_\infty l \sum_{q\le Q} \|w_q\|_l^{l}.
\end{aligned}$$

To estimate J_{13}, we first use Hölder's inequality and split the summation as follows:

$$\begin{aligned}
|J_{13}| &\le l \sum_{q\ge -1}\sum_{|q-p|\le 2} \int_{\mathbb{T}^2} \left|((u_1)_{\le p-2} - (u_1)_{\le q-2})\cdot\nabla\Delta_q w_p\right| |w_q|^{l-1}\,dx\\
&\lesssim l \sum_{q> Q-2}\sum_{|q-p|\le 2}\sum_{q-3\le p'\le q} \int_{\mathbb{T}^2} |(u_1)_{p'}||\nabla\Delta_q w_p||w_q|^{l-1}\,dx\\
&\quad + l \sum_{q\le Q-2}\sum_{|q-p|\le 2}\sum_{q-3\le p'\le q} \int_{\mathbb{T}^2} |(u_1)_{p'}||\nabla\Delta_q w_p||w_q|^{l-1}\,dx\\
&\equiv J_{131} + J_{132}.
\end{aligned}$$

Now Jensen's inequality yields

$$\begin{aligned} J_{131} &\lesssim l \sum_{q>Q-2} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \lambda_p \|w_p\|_l \sum_{q-3\le p'\le q} \|(u_1)_{p'}\|_\infty \\ &\lesssim R_\infty l \sum_{q>Q-2} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \lambda_p \|w_p\|_l \\ &\lesssim R_\infty l \sum_{q>Q-2} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \lambda_p^{\frac{\alpha}{l}} \|w_p\|_l \lambda_{p-q}^{1-\frac{\alpha}{l}} \lambda_q^{1-\alpha} \\ &\lesssim \Lambda^{1-\alpha} R_\infty l \sum_{q>Q-4} \lambda_q^\alpha \|w_q\|_l^l. \end{aligned}$$

And similarly, for the second term,

$$\begin{aligned} J_{132} &\lesssim l \sum_{q\le Q-2} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \lambda_p \|w_p\|_l \sum_{q-3\le p'\le q} \|(u_1)_{p'}\|_\infty \\ &\lesssim R_\infty l \sum_{q\le Q-2} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \lambda_p \|w_p\|_l \\ &\lesssim R_\infty l \sum_{q\le Q} \lambda_q \|w_q\|_l^l \\ &\lesssim \Lambda R_\infty l \sum_{q\le Q} \|w_q\|_l^l. \end{aligned}$$

For J_2 we use Hölder's inequality obtaining

$$\begin{aligned} |J_2| &\le l \sum_{q\ge -1} \sum_{|q-p|\le 2} \int_{\mathbb{T}^2} \left|\Delta_q((u_1)_p \cdot \nabla w_{\le p-2})\right| |w_q|^{l-1}\, dx \\ &\lesssim l \sum_{q>Q} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \|(u_1)_p\|_\infty \sum_{p'\le p-2} \lambda_{p'} \|w_{p'}\|_l \\ &\quad + l \sum_{q\le Q} \|w_q\|_l^{l-1} \sum_{|q-p|\le 2} \|(u_1)_p\|_\infty \sum_{p'\le p-2} \lambda_{p'} \|w_{p'}\|_l \\ &\equiv J_{21} + J_{22}. \end{aligned}$$

Recall that $\|(u_1)_q\|_\infty \lesssim \|(\theta_1)_q\|_\infty \le R_\infty$. Hence we can use Jensen's inequality to deduce that

$$\begin{aligned} J_{21} &\lesssim R_\infty l \sum_{q>Q} \|w_q\|_l^{l-1} \sum_{p'\le q} \lambda_{p'} \|w_{p'}\|_l \\ &\lesssim R_\infty l \sum_{q>Q} \lambda_q^{\frac{\alpha(l-1)}{l}} \|w_q\|_l^{l-1} \sum_{p'\le q} \lambda_{p'}^{\frac{\alpha}{l}} \|w_{p'}\|_l \lambda_{p'-q}^{1-\frac{\alpha}{l}} \lambda_q^{1-\alpha} \\ &\lesssim \Lambda^{1-\alpha} R_\infty l \sum_{q>Q} \lambda_q^\alpha \|w_q\|_l^l, \end{aligned}$$

where we needed $l > \alpha$. While the second term is estimated as

$$\begin{aligned} J_{22} &\lesssim R_\infty l \sum_{q\le Q} \|w_q\|_l^{l-1} \sum_{p'\le q} \lambda_{p'}\|w_{p'}\|_l \\ &\lesssim \Lambda R_\infty l \sum_{q\le Q} \|w_q\|_l^{l-1} \sum_{p'\le Q} \|w_{p'}\|_l \\ &\lesssim \Lambda Q R_\infty l \sum_{q\le Q} \|w_q\|_l^{l}. \end{aligned}$$

Since $\|(u_1)_q\|_r \lesssim \|(\theta_1)_q\|_r$ for any $r \in (1,\infty]$, the term J_3 enjoys the same estimate as I_3. Hence we conclude that

$$|J| \lesssim \Lambda^{1-\alpha} R_\infty l^2 \sum_{q\ge -1} \lambda_q^\alpha \|w_q\|_l^l + \Lambda Q R_\infty l^2 \sum_{q\le Q} \|w_q\|_l^l. \tag{4.4}$$

Thanks to (4.2)–(4.4), inequality (4.2) yields

$$\begin{aligned} \frac{d}{dt}\|w(t)\|_{B^0_{l,l}}^l \le -C\nu\|\Lambda^{\alpha/l} w\|_{B^0_{l,l}}^l + C_1\Lambda^{1-\alpha} R_\infty l^2 \sum_{q>Q} \lambda_q^\alpha \|w_q\|_l^l \\ + \left(\frac{2}{C\nu}\right)^{l-1} l^{l-2} \|\Lambda^{-\alpha\left(1-\frac{1}{l}\right)} f\|_{B^0_{l,l}}^l + C_2 \Lambda^{(l-1)(\alpha-1)} R_\infty l^2 \sum_{q\le Q} \|w_q\|_l^l, \end{aligned}$$

for some absolute constants C, C_1, and C_2. Thus we have

$$\frac{d}{dt}\|w(t)\|_{B^0_{l,l}}^l + \phi \|w(t)\|_{B^0_{l,l}}^l \le \psi(t),$$

where

$$\begin{aligned} \phi &= \tfrac{1}{2}(2\pi\lambda_0)^\alpha C\nu, \\ \psi(t) &= \left(\frac{2}{C\nu}\right)^{l-1} l^{l-2} \|\Lambda^{-\alpha\left(1-\frac{1}{l}\right)} f\|_{B^0_{l,l}}^l + C_2 \Lambda^{(l-1)(\alpha-1)} R_\infty l^2 \sum_{q\le Q} \|w_q\|_l^l, \end{aligned}$$

provided

$$\Lambda = \left(\frac{2C_1 l^2}{C\nu} R_\infty\right)^{\frac{1}{\alpha-1}}.$$

Note that

$$\psi(t) \to 0 \quad \text{as} \quad t \to \infty,$$

due to the assumption of the theorem. Since also $\alpha > 0$, the first part of the theorem follows from Lemma 4.1.

To prove the second part, where $\psi(t) \equiv 0$, we note that

$$\|w(t)\|_{B^0_{l,l}}^l \le \|w(t_0)\|_{B^0_{l,l}}^l e^{-\alpha(t-t_0)}, \qquad t_0 \le t \le 0,$$

thanks to Grönwall's inequality. Since θ_1 and θ_2 are ancient solutions,

$$\theta_1(t), \theta_2(t) \in B_{L^2} \cap B_{L^\infty}, \qquad \forall t \le 0.$$

Hence, we have

$$\begin{aligned}\|w(t)\|_{B^0_{l,l}} &\lesssim \|w(t)\|_\infty^{1-\frac{2}{l}} \|w(t)\|_2^{\frac{2}{l}} \\ &\lesssim R_\infty^{1-\frac{2}{l}} R_2^{\frac{2}{l}},\end{aligned}$$

for all t. Taking the limit as $t_0 \to -\infty$ gives $w(t) = 0$ for all $t \leq 0$, and hence $w \equiv 0$. □

Acknowledgements The authors would wish to express their gratitude to the anonymous referee for the careful review and valuable suggestions which helped to improve the manuscript a lot.

References

1. Bahouri, H., Chemin, J., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehrender Mathematischen Wissenschaften, vol. 343. Springer, Heidelberg (2011)
2. Caffarelli, L.A., Vasseur, A.: Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. Ann. Math. (2) **171**(3), 1903–1930 (2010)
3. Chen, Q., Miao, C., Zhang, Z.: A new Bernstein inequality and the 2D dissipative quasi-geostrophic equation. Commun. Math. Phys. **271**, 821–838 (2007)
4. Cheskidov, A.: Global attractors of evolutionary systems. J. Dyn. Differ. Equ. **21**, 249–268 (2009)
5. Cheskidov, A., Dai, M.: Determining modes for the surface quasi-geostrophic equation. Phys. D Nonlinear Phenom. **376–377**, 204–215 (2018)
6. Cheskidov, A., Dai, M.: The existence of a global attractor for the forced critical surface quasi-geostrophic equation in L^2. J. Math. Fluid Mech. **20**(1), 213–225 (2018)
7. Cheskidov, A., Dai, M., Kavlie, L.: Determining modes for the 3D Navier–Stokes equations. Phys. D Nonlinear Phenom. **374–375**, 1–9 (2018)
8. Cheskidov, A., Kavlie, L.: Pullback attractors for generalized evolutionary systems. Discrete Contin. Dyn. Syst. B **20**, 749–779 (2015)
9. Constantin, P., Zelati, M.C., Vicol, V.: Uniformly attracting limit sets for the critically dissipative SQG equation. Nonlinearity **29**(2), 298 (2016)
10. Constantin, P., Foias, C., Manley, O.P., Temam, R.: Determining modes and fractal dimension of turbulent flows. J. Fluid Mech. **150**, 427–440 (1985)
11. Constantin, P., Foias, C., Temam, R.: On the dimension of the attractors in two-dimensional turbulence. Physica D **30**, 284–296 (1988)
12. Constantin, P., Majda, A.J., Tabak, E.: Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. Nonlinearity **7**(6), 1495–1533 (1994)
13. Constantin, P., Vicol, V.: Nonlinear maximum principles for dissipative linear nonlocal operators and applications. Geom. Funct. Anal. **22**(5), 1289–1321 (2012)
14. Córdoba, A., Córdoba, D.: A maximum principle applied to quasi-geostrophic equations. Commun. Math. Phys. **249**(3), 511–528 (2004)
15. Foias, C., Jolly, M., Kravchenko, R., Titi, E.: A determining form for the 2D Navier–Stokes equations—the Fourier modes case. J. Math. Phys. **53**(11), 115623 (2012)
16. Foias, C., Jolly, M., Kravchenko, R., Titi, E.: A unified approach to determining forms for the 2D Navier–Stokes equations—the general interpolants case. Russ. Math. Surv. **69**, 359 (2014)
17. Foias, C., Manley, O., Rosa, R., Temam, R.: Navier–Stokes Equations and Turbulence. Encyclopedia of Mathematics and Its Applications, vol. 83. Cambridge University Press, Cambridge (2001)
18. Foias, C., Manley, O.P., Temam, R., Tréve, Y.M.: Asymptotic analysis of the Navier–Stokes equations. Physica D **9**(1–2), 157–188 (1983)
19. Foias, C., Prodi, G.: Sur le comportement global des solutions non-stationnaires des équations de Navier–Stokes en dimension 2. Rend. Sem. Mat. Univ. Padova **39**, 1–34 (1967)
20. Foias, C., Temam, R.: Some analytic and geometric properties of the solutions of the Navier–Stokes equations. J. Math. Pures Appl. **58**, 339–368 (1979)
21. Foias, C., Titi, E.S.: Determining nodes, finite difference schemes and inertial manifolds. Nonlinearity **4**, 135–153 (1991)
22. Grafakos, L.: Modern Fourier analysis. Graduate Texts in Mathematics, vol. 250, 2nd edn. Springer, New York (2009)

23. Kiselev, A., Nazarov, F.: A variation on a theme of Caffarelli and Vasseur. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 370(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. (40):58–72, 220 (2009)
24. Kiselev, A., Nazarov, F., Volberg, A.: Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. Invent. Math. **167**(3), 445–453 (2007)
25. Resnick, S.: Dynamical Problems in Nonlinear Advective Partial Differential Equations. Ph.D. Thesis, University of Chicago (1995)