LOW MODES REGULARITY CRITERION FOR A CHEMOTAXIS-NAVIER-STOKES SYSTEM

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Abstract. In this paper we study the regularity problem of a three dimensional chemotaxis-Navier-Stokes system. A new regularity criterion in terms of only low modes of the oxygen concentration and the fluid velocity is obtained via a wavenumber splitting approach. The result improves certain existing criteria in the literature.

KEY WORDS: Chemotaxis model; Navier-Stokes equations; conditional regularity.
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1. Introduction

We consider the following chemotaxis-Navier-Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \\ u_t + (u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\Phi, \\ \nabla \cdot u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3. \end{cases} \tag{1.1}$$

This coupled system arises from modelling aerobic bacteria, e.g. *Bacillus subtilis*, suspended into sessile drops of water. It describes a scenario in which both the bacteria, whose population density is denoted by $n = n(t,x)$, and oxygen, whose concentration is denoted by $c = c(t,x)$, are transported by the fluid and at the same time diffuse randomly. In addition, the bacteria, which have chemotactic sensitivity $\chi(c)$, tend to swim towards their nutrient oxygen and consume it at a per-capita rate $f(c)$. Meanwhile, since the bacteria are heavier than water, their chemotactic swimming induces buoyant forces which affects the fluid motion. This buoyancy-driven effect is reflected in the third equation in system (1.1), represented by an extra term $n\nabla\Phi$ added to the Navier-Stokes equation. In this extra term, Φ denotes the gravitational potential, whereas the Navier-Stokes equation is conventionally written with $u = u(t,x)$ denoting the fluid velocity, and $P = P(t,x)$ the pressure. In this paper, we study a simple yet prototypical case in which

$$\nabla\Phi \equiv \text{const.}, \quad \chi(c) \equiv \text{const.}, \quad f(c) \equiv c. \tag{1.2}$$

We note that in this case, solutions to system (1.1) satisfy the following scaling property:

$$n_\lambda(t,x) = \lambda^2 n(\lambda^2 t, \lambda x), \quad c_\lambda(t,x) = c(\lambda^2 t, \lambda x), \quad u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

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solve (1.1) with initial data

$$n_{\lambda,0} = \lambda^2 n(\lambda x), \quad c_{\lambda,0} = c(\lambda x), \quad u_{\lambda,0} = \lambda u(\lambda x),$$

if $\big(n(t,x), c(t,x), u(t,x)\big)$ solves (1.1) with initial data $(n_0(x), c_0(x), u_0(x))$. It is obvious that in 3D the Sobolev space $\dot{H}^{-\frac{1}{2}} \times \dot{H}^{\frac{1}{2}} \times \dot{H}^{\frac{3}{2}}$ is scaling invariant (aka critical) for (n, c, u) under the above natural scaling of the system.

Experiments showed that under the chemotaxis-fluid interaction of system (1.1), even almost homogeneous initial bacteria distribution can evolve and exhibit quite intricate spatial patterns (see [10, 31, 25]). In [25] Lorz proved the existence of a local-in-time weak solution to the 3D chemotaxis-Navier-Stokes system on bounded domains. In a recent work by Winkler [34], the existence of global weak solutions was proved under more general assumptions via entropy-energy estimates. We refer readers to the works of Winkler [32, 33, 34], Liu and Lorz [24], Duan, Lorz and Markowich [11], Chae, Kang and Lee [3, 4], Jiang, Wu and Zheng [16] as well as He and Zhang [14] for more details about the well-posedness results for the chemotaxis-Navier-Stokes system.

As of now, the global regularity of the 3D Navier-Stokes equations remains an outstanding unresolved problem, which is a fundamental reason why a mathematical theory for system (1.1) is yet to be completed. A classical result due to Prodi [27], Serrin [30] and Ladyzhenskaya [25] states that if a Leray-Hopf solution u to the 3D Navier-Stokes equations satisfies

$$\|u\|_{L^q(0,T;L^p)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p \le \infty, \tag{1.3}$$

then u is in fact smooth on $[0, T]$. It is also well-known that a smooth solution to the Navier-Stokes equations on $[0, T)$ can be extended beyond time T if

$$\int_0^T \|\nabla \times u\|_{L^\infty} \mathrm{d}t < \infty, \tag{1.4}$$

which is the Beale-Kato-Majda regularity criterion (see [2]). Among a myriad of refined or generalized criteria for the Navier-Stokes equations, we list the ones from [8, 12, 19, 26]. Particularly relevant to this paper is the regularity criterion due to Cheskidov and Shvydkoy (see [8]). It was discovered that by devising the concept of the critical wavenumber 2^Q, the condition in Beale-Kato-Majda regularity criterion can be weakened into

$$\int_0^T \|(\nabla \times u)_{\le Q}\|_{B^0_{\infty,\infty}} \mathrm{d}t < \infty, \tag{1.5}$$

with $B^0_{\infty,\infty}$ being a Besov space and $(\nabla \times u)_{\le Q}$ the low frequency part of $\nabla \times u$ below the wavenumber 2^Q, both of which can be clearly defined within the framework of Littlewood-Paley theory in the upcoming sections. Intuitively speaking, condition (1.5) implies that the integrability of a certain lower frequency part of a solution alone can ensure the regularity of the solution. The idea of separating high and low frequency parts by a wavenumber originates in Kolmogorov's theory of turbulence, which predicts the existence of a critical wavenumber above which the dissipation term is dominant. Recently, this wavenumber splitting mechanism has been applied to various fluid models, e.g., the liquid crystal model with Q-tensor configuration (see [9]), leading to a series of refined regularity criteria.

Concerning the three dimensional chemotaxis-Navier-Stokes system, we are aware of several regularity criteria. In [3], Chae, Kang and Lee obtained local-in-time classical solutions and Prodi-Serrin type regularity criteria. In particular, suppose that

$$\|u\|_{L^q(0,T;L^p)} + \|\nabla c\|_{L^2(0,T;L^\infty)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p \le \infty, \tag{1.6}$$

then the corresponding classical solution can be extended beyond time T. In [4], Chae, Kang and Lee also obtained regularity criteria in terms of the L^p norms of u and n. Jiang, Wu and Zheng, in their recent paper [17], proved that a classical solution to the initial boundary value problem of the Keller-Segel model i.e. the fluid free version of system (1.1), can be extended beyond time T if

$$\begin{aligned} &\|\nabla c\|_{L^2(0,T;L^\infty)} < \infty, \\ \text{or} \quad &\|n\|_{L^q(0,T;L^p)} < \infty, \quad \frac{3}{p} + \frac{2}{q} \le 2, \quad \frac{3}{2} < p \le \infty. \end{aligned}$$

In this paper, we aim to establish a regularity condition only imposed on the low frequency part of the concentration function c and the velocity field u. For a given solution (n, c, u) without knowing its regularity, we shall define the wavenumber $\Lambda_u = 2^{Q_u}$ and $\Lambda_c = 2^{Q_c}$ for u and c as in (3.10), respectively, according to the structure of the equations. Let $u_{\le Q_u}$ and $c_{\le Q_c}$ denote the low modes of the velocity and oxygen concentration below wavenumber Λ_u and Λ_c, respectively. The main result is stated as follows.

Theorem 1.1. *Let $(n(t), c(t), u(t))$ be a weak solution to (1.1) on $[0, T]$ on $\mathbb{R}^3$. Assume that $(n(t), c(t), u(t))$ is regular on $[0, T)$ and*

$$\int_0^T \|\nabla c_{\le Q_c(t)}(t)\|_{L^\infty}^2 + \|u_{\le Q_u(t)}(t)\|_{B^1_{\infty,\infty}} \mathrm{d}t < \infty, \tag{1.7}$$

then $(n(t), c(t), u(t))$ is regular on $[0, T]$.

Remark 1.2. We note that the quantity in (1.7) is invariant with respect to the scaling of system (1.1). It is obvious that the condition on the oxygen concentration c in (1.7) is weaker than that of (1.6). It was also shown that the condition on velocity u in (1.7) is weaker than that of (1.6) (see [7]).

Remark 1.3. The same result as in Theorem 1.1 holds on torus $\mathbb{T}^3$ as well. However, on bounded domain with boundary, such result may not be acheived since the analysis to obtain Theorem 1.1 relies heavily on harmonic analysis techniques and Littlewood-Paley theory on bounded domain is more involved.

The upcoming sections are organized as follows – in Section 2, we shall give concise introductions to the mathematical tools used in this paper and various notions of solutions to system (1.1), then in Section 3 we shall formulate the wavenumbers Λ_u and Λ_c via Littlewood-Paley theory and proceed to prove Theorem 1.1.

2. Preliminaries

2.1. **Notation.** The symbol $A \lesssim B$ denotes an estimate of the form $A \le CB$ with some absolute constant C, and $A \sim B$ denotes an estimate of the form $C_1 B \le A \le C_2 B$ with absolute constants C_1, C_2. The Sobolev norm $\|\cdot\|_{L^p}$ is shortened as $\|\cdot\|_p$ without no confusion. The symbols $W^{k,p}$ and H^s represent the standard Sobolev spaces and L^2-based Sobolev spaces, respectively.

2.2. **Littlewood-Paley decomposition.** The main analysis tools are the frequency localization method and a wavenumber splitting approach based on the Littlewood-Paley theory, which we briefly recall here. For a complete description of the theory and applications, the readers are referred to the books [1] and [13].

We construct a family of smooth functions $\{\varphi_q\}_{q=-1}^{\infty}$ with annular support that forms a dyadic partition of unity in the frequency space, defined as

$$\varphi_q(\xi) = \begin{cases} \varphi(\lambda_q^{-1}\xi) & \text{for } q \geq 0, \\ \chi(\xi) & \text{for } q = -1, \end{cases}$$

where $\lambda_q = 2^q$, $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ and $\chi \in C_0^\infty(\mathbb{R}^d)$ is a nonnegative radial function chosen in a way such that

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4} \\ 0, & \text{for } |\xi| \geq 1. \end{cases}$$

Introducing $\tilde{h} := \mathcal{F}^{-1}\chi$ and $h := \mathcal{F}^{-1}\varphi$, we define the Littlewood-Paley projections for a function $u \in \mathcal{S}'$ as

$$\begin{cases} u_{-1} = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}u) = \displaystyle\int \tilde{h}(y)u(x-y)dy, \\ u_q := \Delta_q u = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1}\xi)\mathcal{F}u) = \lambda_q^d \displaystyle\int h(\lambda_q y)u(x-y)dy, \; q \geq 0. \end{cases}$$

Then the identity

$$u = \sum_{q=-1}^{\infty} u_q$$

holds in the sense of distributions. To simplify the notation, we denote

$$\tilde{u}_q = u_{q-1} + u_q + u_{q+1}, \qquad u_{\leq Q} = \sum_{q=-1}^{Q} u_q, \qquad u_{(P,Q]} = \sum_{q=P+1}^{Q} u_q.$$

We note that

$$\|u\|_{H^s} \sim \left(\sum_{q=-1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{1}{2}},$$

for each $u \in H^s$ and $s \in \mathbb{R}$. Using the Littlewood-Paley projections, we can define the Besov space $B^s_{p,\infty}$ as follows.

Definition 2.1. Let $s \in \mathbb{R}$, and $1 \leq p \leq \infty$. The Besov space $B^s_{p,\infty}$ is the space of tempered distributions u whose Besov norm $\|u\|_{B^s_{p,\infty}} < \infty$, where

$$\|u\|_{B^s_{p,\infty}} := \sup_{q \geq -1} \lambda_q^s \|u_q\|_p.$$

Moreover, we recall Bernstein's inequality, whose proof can be found in [1].

Lemma 2.2. *Let d be the space dimension and $1 \leq s \leq r \leq \infty$. Then for all tempered distributions u,*

$$\|u_q\|_r \lesssim \lambda_q^{d(\frac{1}{s} - \frac{1}{r})} \|u_q\|_s.$$

Throughout the paper, we will also utilize Bony's paraproduct decomposition

$$\Delta_q(u \cdot v) = \sum_{|q-p| \leq 2} \Delta_q(u_{\leq p-2} \cdot v_p) + \sum_{|q-p| \leq 2} \Delta_q(u_p \cdot v_{\leq p-2}) + \sum_{p \geq q-2} \Delta_q(u_p \cdot \tilde{v}_p),$$

as well as the commutator notation

$$[\Delta_q, u_{\le p-2}\cdot\nabla]v_p = \Delta_q(u_{\le p-2}\cdot\nabla v_p) - u_{\le p-2}\cdot\nabla\Delta_q v_p.$$

An estimate for the commutator is given by the following lemma, proven in [7].

Lemma 2.3. *Let $\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r_1}$, we have the estimate*

$$\|[\Delta_q, u_{\le p-2}\cdot\nabla]v_q\|_{r_1} \lesssim \|v_q\|_{r_3}\sum_{p'\le p-2}\lambda_{p'}\|u_{p'}\|_{r_2}.$$

2.3. **Weak solution and regular solution to system (1.1).** From [34], we know that on bounded, smooth and convex domain Ω in three dimension, system (1.1) has a global weak solution (n,c,u) which satisfies the equations in (1.1) in the distributional sense, provided that the initial data (n_0,c_0,u_0) satisfy $n_0>0$, $c_0>0$, and

$$n_0\in L^1\cap L\log L,\quad c_0\in L^\infty,\quad \sqrt{c_0}\in H^1,\quad u_0\in L^2,\quad \nabla\cdot u_0=0. \tag{2.8}$$

Global existence of weak solutions on the full space $\mathbb{R}^d$ with $d=2,3$ was established with initial data satisfying (2.8) in [14]. Adapting the same argument therein on a periodic domain shows that initial data satisfying (2.8) also generate at least one global weak solution (n,c,u).

We highlight the following properties of the weak solution (n,c,u) in particular

$$\begin{aligned}&n\in L^\infty(0,\infty;L^1(\Omega)),\quad c\in L^\infty(0,\infty;L^\infty(\Omega)),\\ &u\in L^\infty_{loc}(0,\infty;L^2(\Omega))\cap L^2_{loc}(0,\infty;H^1_0(\Omega)).\end{aligned}$$

A regular solution of (1.1) is understood in the way that the solution has enough regularity to satisfy the equations of the system in a point-wise manner. Typically, a solution in a space with higher regularity than its critical space can be shown regular via bootstrap arguments. The local-in-time existence of regular solutions to system (1.1) was shown in [3].

2.4. **Parabolic regularity theory.** We consider the heat equation on $\mathbb{R}^d$ with $d\ge 2$

$$u_t-\Delta u=f \tag{2.9}$$

with initial data u_0. We shall see that the solution u turns out to be smoother than the source term f.

Lemma 2.4. *Let u be a solution to (2.9) with $u_0\in H^{\alpha+1}$ and $f\in L^2(0,T;H^\alpha)$ for $\alpha\in\mathbb{R}$. Then we have $u\in L^2(0,T;H^{\alpha+2})\cap H^1(0,T;H^\alpha)$.*

Proof: Projecting equation (2.9) by Δ_q and taking inner product of the resulted equation with $\lambda_q^{2\alpha+4}u_q$ leads

$$\frac{1}{2}\frac{d}{dt}\lambda_q^{2\alpha+4}\|u_q\|_2^2+\lambda_q^{2\alpha+4}\|\nabla u_q\|_2^2=\lambda_q^{2\alpha+4}\int f_qu_q\,\mathrm{d}x.$$

Applying Hölder's and Young's inequalities to the right hand side yields

$$\frac{d}{dt}\lambda_q^{2\alpha+4}\|u_q\|_2^2+\lambda_q^{2\alpha+4}\|\nabla u_q\|_2^2\le 4\lambda_q^{2\alpha+2}\|f_q\|_2^2.$$

As a consequence of Duhamel's formula, summation in q and integration over $[0,T]$, we obtain

$$\begin{aligned}\int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha+4} \|u_q(t)\|_2^2 \,\mathrm{d}t \leq& \int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha+4} \|u_q(0)\|_2^2 e^{-\lambda_q^2 t}\,\mathrm{d}t\\ &+4\int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha+2} \int_0^t e^{-\lambda_q^2(t-s)} \|f_q(s)\|_2^2 \mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

The first integral on the right hand side is handled as

$$\int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha+4} \|u_q(0)\|_2^2 e^{-\lambda_q^2 t}\,\mathrm{d}t \leq \sum_{q\geq -1} \lambda_q^{2\alpha+2} \|u_q(0)\|_2^2 \left(1-e^{-\lambda_q^2 T}\right) \lesssim \|u_0\|_{H^{\alpha+1}}^2.$$

In order to estimate the second integral, we exchange the order of integration to obtain

$$\begin{aligned}&\int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha+2} \int_0^t e^{-\lambda_q^2(t-s)} \|f_q(s)\|_2^2 \mathrm{d}s\,\mathrm{d}t\\ \leq& \int_0^T \int_s^T \sum_{q\geq -1} \lambda_q^{2\alpha+2} e^{-\lambda_q^2(t-s)} \|f_q(s)\|_2^2 \mathrm{d}t\,\mathrm{d}s\\ \leq& \int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha} \|f_q(s)\|_2^2 \left(1-e^{-\lambda_q^2(T-s)}\right)\mathrm{d}s\\ \lesssim& \|f\|_{L^2(0,T;H^\alpha)}^2.\end{aligned}$$

Combining the estimates above, we conclude that $u \in L^2(0,T;H^{\alpha+2})$ for $\alpha \in \mathbb{R}$.

To prove $u \in H^1(0,T;H^\alpha)$, we first project equation (2.9) to the q-th dyadic shell

$$(u_t)_q = \Delta u_q + f_q.$$

It follows

$$\|(u_t)_q\|_2^2 \leq 2\|\Delta u_q\|_2^2 + 2\|f_q\|_2^2.$$

Thus we deduce that

$$\begin{aligned}\int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha} \|(u_t)_q\|_2^2 \,\mathrm{d}t \lesssim& \int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha} \|\Delta u_q\|_2^2 \,\mathrm{d}t + \int_0^T \sum_{q\geq -1} \lambda_q^{2\alpha} \|f_q\|_2^2 \,\mathrm{d}t\\ \lesssim& \|u\|_{L^2(0,T;H^{\alpha+2})}^2 + \|f\|_{L^2(0,T;H^\alpha)}^2.\end{aligned}$$

It is then clear that $u \in H^1(0,T;H^\alpha)$, which completes the proof of the lemma. □

3. Proof of Theorem 1.1

This section is devoted to the proof of the main result. Recall $\lambda_q = 2^q$ for any integer q. We start by introducing the dissipation wavenumber $\Lambda_u(t)$ for u and $\Lambda_c(t)$ for c, respectively

$$\begin{aligned}&\Lambda_u(t) = \min\left\{\lambda_q : \lambda_p^{-1}\|u_p(t)\|_\infty < C_0, \forall p > q, q\in\mathbb{N}\right\},\\ &\Lambda_c(t) = \min\left\{\lambda_q : \lambda_p^{\frac{3}{r}}\|c_p(t)\|_r < C_0, \forall p > q, q\in\mathbb{N}\right\}, \quad r \in \left(3, \frac{3}{1-\varepsilon}\right),\end{aligned} \tag{3.10}$$

where $\varepsilon > 0$ is a fixed arbitrarily small constant, and C_0 is a small constant to be determined later. Through out this section, we use C for various absolute constants which can be different from line to line. In addition, we let $Q_u(t)$ and $Q_c(t)$ be the integers such that

$$\Lambda_u(t) = \lambda_{Q_u(t)} \text{ and } \Lambda_c(t) = \lambda_{Q_c(t)}.$$

The constraint on the low modes is then defined as

$$f(t) := \|\nabla c_{\leq Q_c(t)}(t)\|_{L^\infty}^2 + \|u_{\leq Q_u(t)}(t)\|_{B^1_{\infty,\infty}}.$$

Notice that the wavenumber Λ_u separates the inertial range from the dissipation range where the viscous term Δu dominates; and Λ_c has the analogous property. Precisely, we have

$$\|u(t)_{Q_u}\|_\infty \geq C_0 \Lambda_u(t), \quad \Lambda_c^{\frac{3}{r}}(t)\|c(t)_{Q_c}\|_r \geq C_0;$$

$$\lambda_q \|u(t)_q\|_\infty < C_0, \ \forall q > Q_u; \quad \lambda_q^{\frac{3}{r}}(t)\|c(t)_q\|_r < C_0, \ \forall q > Q_c.$$

The crucial part of the proof is to establish a uniform (in time) bound for each of the unknowns n, u and c in a space with higher regularity than the critical Sobolev space. In fact, it is sufficient to prove that

$$(n, u, c) \in L^\infty(0, T; \dot{H}^{s_1}) \times L^\infty(0, T; \dot{H}^{s_2}) \times L^\infty(0, T; \dot{H}^{s_3})$$

for some $s_1 > -\frac{1}{2}, s_2 > \frac{1}{2}$ and $s_3 > \frac{3}{2}$. Due to the complicated interactions among the three equations in (1.1), the aforementioned goal will be achieved in two steps. The first step is to show that

$$(n, u, c) \in L^\infty(0, T; \dot{H}^{s}) \times L^\infty(0, T; \dot{H}^{s+1}) \times L^\infty(0, T; \dot{H}^{s+1})$$

for some $s \in (-\frac{1}{2}, 0)$. The second step consists of applying bootstrap arguments, the L^p–L^q theory for parabolic equations and a mixed derivative theorem to the equation of oxygen concentration c to improve the regularity of c.

To start, we multiply the equations in (1.1) by $\lambda_q^{2s}\Delta_q^2 n$, $\lambda_q^{2s+2}\Delta_q^2 c$ and $\lambda_q^{2s+2}\Delta_q^2 u$, respectively. Integrating and summing lead to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\lambda_q^{2s}\|n_q\|_2^2 \leq &-\sum_{q\geq -1}\lambda_q^{2s}\|\nabla n_q\|_2^2 - \sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla n)n_q \mathrm{d}x\\ &-\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(\nabla\cdot(n\chi(c)\nabla c))n_q\mathrm{d}x;\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\lambda_q^{2s+2}\|c_q\|_2^2 \leq -\sum_{q\geq -1}\lambda_q^{2s+2}\|\nabla c_q\|_2^2\\ &-\sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla c)c_q\mathrm{d}x - \sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(nf(c))c_q\mathrm{d}x;\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 \leq -\sum_{q\geq -1}\lambda_q^{2s+2}\|\nabla u_q\|_2^2\\ &-\sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla u)u_q\mathrm{d}x - \sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(n\nabla\Phi)u_q\mathrm{d}x.\end{aligned} \tag{3.13}$$

For simplicity we label the terms

$$I := -\sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla u)u_q\mathrm{d}x, \quad II := -\sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla c)c_q\mathrm{d}x,$$

$$III := -\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla n)n_q\mathrm{d}x,$$

$$IV := -\sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(n\nabla\Phi)u_q\mathrm{d}x, \quad V := -\sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(nf(c))c_q\mathrm{d}x,$$

$$VI := -\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(\nabla\cdot(n\chi(c)\nabla c))n_q\mathrm{d}x.$$

3.1. **Estimate for I.** We estimate the term I using the wavenumber splitting method. As we shall see, the commutator reveals certain cancellation within the nonlinear interactions.

Lemma 3.1. *Let $s > -\frac{1}{2}$. We have*

$$|I| \lesssim C_0\sum_{q>-1}\lambda_q^{2s+4}\|u_q\|_2^2 + Q_u f(t)\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2.$$

Proof: Applying Bony's paraproduct decomposition to I leads to

$$\begin{aligned}
I =& -\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_{\leq p-2}\cdot\nabla u_p)u_q\,\mathrm{d}x\\
&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla u_{\leq p-2})u_q\,\mathrm{d}x\\
&-\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{u}_p)u_q\,\mathrm{d}x\\
=:& I_1 + I_2 + I_3.
\end{aligned}$$

Using the fact $\sum_{|q-p|\leq 2}\Delta_q u_p = u_q$ and the commutator notation, we have

$$\begin{aligned}
I_1 =& -\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}[\Delta_q, u_{\leq p-2}\cdot\nabla]u_p u_q\,\mathrm{d}x\\
&-\sum_{q\geq -1}\lambda_q^{2s_3}\int_{\mathbb{R}^3}u_{\leq q-2}\cdot\nabla u_q u_q\,\mathrm{d}x\\
&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}(u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q u_p u_q\,\mathrm{d}x\\
=:& I_{11} + I_{12} + I_{13}.
\end{aligned}$$

Moreover we have $I_{12} = 0$ due to that $\operatorname{div} u_{\leq q-2} = 0$.

We then split I_{11} based on definition of $\Lambda_u(t)$

$$\begin{aligned}
|I_{11}| \le & \sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}|[\Delta_q, u_{\le p-2}\cdot\nabla]u_p u_q|\ \mathrm{d}x\\
\le & \sum_{p\le Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}|[\Delta_q, u_{\le p-2}\cdot\nabla]u_p u_q|\ \mathrm{d}x\\
& + \sum_{p> Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}|[\Delta_q, u_{\le Q_u}\cdot\nabla]u_p u_q|\ \mathrm{d}x\\
& + \sum_{p> Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\left|[\Delta_q, u_{(Q_u, p-2]}\cdot\nabla]u_p u_q\right|\ \mathrm{d}x\\
=: & I_{111}+I_{112}+I_{113}.
\end{aligned}$$

Using (2.3), Hölder's inequality, and definition of $f(t)$, we obtain

$$\begin{aligned}
I_{111} \le & \sum_{1\le p\le Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\|\nabla u_{\le p-2}\|_\infty\|u_p\|_2\|u_q\|_2\\
\lesssim & f(t)\sum_{1\le p\le Q_u+2}\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^{2s+2}\|u_q\|_2\sum_{p'\le p-2}1\\
\lesssim & Q_u f(t)\sum_{1\le p\le Q_u+2}\lambda_p^{s+1}\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^{s+1}\|u_q\|_2\\
\lesssim & Q_u f(t)\sum_{q\ge -1}\lambda_q^{2s+2}\|u_q\|_2^2;
\end{aligned}$$

and similarly

$$\begin{aligned}
I_{112} \le & \sum_{p> Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\|\nabla u_{\le Q_u}\|_\infty\|u_p\|_2\|u_q\|_2\\
\lesssim & Q_u f(t)\sum_{p> Q_u+2}\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^{2s+2}\|u_q\|_2\\
\lesssim & Q_u f(t)\sum_{p> Q_u+2}\lambda_p^{s+1}\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^{s+1}\|u_q\|_2\\
\lesssim & Q_u f(t)\sum_{q> Q_u}\lambda_q^{2s+2}\|u_q\|_2^2.
\end{aligned}$$

We estimate I_{113} with the help of Hölder's inequality and Lemma (2.3)

$$
\begin{aligned}
I_{113} \leq & \sum_{p>Q_u+2} \sum_{|q-p|\leq 2} \lambda_q^{2s+2} \|[\Delta_q, u_{(Q_u,p-2]} \cdot \nabla] u_p\|_2 \|u_q\|_2 \\
\leq & \sum_{p>Q_u+2} \|u_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^{2s+2} \|u_q\|_2 \sum_{Q_u<p'\leq p-2} \lambda_{p'} \|u_{p'}\|_\infty \\
\lesssim & C_0 \sum_{p>Q_u+2} \|u_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^{2s+2} \|u_q\|_2 \sum_{Q_u<p'\leq p-2} \lambda_{p'}^2 \\
\lesssim & C_0 \sum_{p>Q_u+2} \lambda_p^{2s+4} \|u_p\|_2^2 \sum_{Q_u<p'\leq p-2} \lambda_{p'-p}^2 \\
\lesssim & C_0 \sum_{p>Q_u+2} \lambda_p^{2s+4} \|u_p\|_2^2.
\end{aligned}
$$

We split I_{13} according to the definition of $\Lambda_u(t)$

$$
\begin{aligned}
|I_{13}| \leq & \sum_{q\geq -1} \sum_{|q-p|\leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \left| (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla \Delta_q u_p u_q \right| \mathrm{d}x \\
\leq & \sum_{-1\leq q\leq Q_u} \sum_{|q-p|\leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \left| (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla \Delta_q u_p u_q \right| \mathrm{d}x \\
& + \sum_{q>Q_u} \sum_{|q-p|\leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \left| (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla \Delta_q u_p u_q \right| \mathrm{d}x \\
=: & I_{131} + I_{132}.
\end{aligned}
$$

Using Hölder's inequality and definition of $f(t)$ we can bound I_{131}.

$$
\begin{aligned}
|I_{131}| \leq & \sum_{-1\leq q\leq Q_u} \lambda_q^{2s+3} \|u_q\|_\infty \sum_{|q-p|\leq 2} \|u_{\leq p-2} - u_{\leq q-2}\|_2 \|u_p\|_2 \\
\lesssim & f(t) \sum_{-1\leq q\leq Q_u} \lambda_q^{2s+2} \sum_{|q-p|\leq 2} \|u_{\leq p-2} - u_{\leq q-2}\|_2 \|u_p\|_2 \\
\lesssim & f(t) \sum_{-1\leq q\leq Q_u} \lambda_q^{2s+2} \|u_q\|_2^2.
\end{aligned}
$$

And we estimate I_{132} using Hölder's inequality and the definition of $\Lambda_u(t)$,

$$
\begin{aligned}
|I_{132}| \leq & \sum_{q>Q_u} \lambda_q^{2s+3} \|u_q\|_\infty \sum_{|q-p|\leq 2} \|u_{\leq p-2} - u_{\leq q-2}\|_2 \|u_p\|_2 \\
\lesssim & C_0 \sum_{q>Q_u} \lambda_q^{2s+4} \sum_{|q-p|\leq 2} \|u_{\leq p-2} - u_{\leq q-2}\|_2 \|u_p\|_2 \\
\lesssim & C_0 \sum_{q>Q_u} \lambda_q^{2s+4} \|u_q\|_2^2.
\end{aligned}
$$

We omit the detailed estimation of I_2 as it is similar to that of I_{11}. Meanwhile, for I_3 we have for $s > -\frac{1}{2}$

$$\begin{aligned}
|I_3| \leq& \sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}|\Delta_q(u_p\otimes\tilde{u}_p)\nabla u_q|\,\mathrm{d}x\\
\leq& \sum_{q>Q_u}\lambda_q^{2s+3}\|u_q\|_\infty\sum_{p\geq q-2}\|u_p\|_2^2+\sum_{-1\leq q\leq Q_u}\lambda_q^{2s+3}\|u_q\|_\infty\sum_{p\geq q-2}\|u_p\|_2^2\\
\leq& C_0\sum_{q>Q_u}\lambda_q^{2s+4}\sum_{p\geq q-2}\|u_p\|_2^2+f(t)\sum_{-1\leq q\leq Q_u}\lambda_q^{2s+2}\sum_{p\geq q-2}\|u_p\|_2^2\\
\lesssim& C_0\sum_{p>Q_u}\lambda_p^{2s+4}\|u_p\|_2^2\sum_{Q_u<q\leq p+2}\lambda_{q-p}^{2s+4}+f(t)\sum_{p\geq -1}\lambda_p^{2s+2}\|u_p\|_2^2\sum_{q\leq p+2}\lambda_{q-p}^{2s+2}\\
\lesssim& C_0\sum_{q>Q_u}\lambda_q^{2s+4}\|u_q\|_2^2+f(t)\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2.
\end{aligned}$$

We combine the above estimates to conclude that

$$|I| \lesssim C_0\sum_{q>-1}\lambda_q^{2s+4}\|u_q\|_2^2+Q_uf(t)\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2.$$

□

3.2. Estimate for II.

Lemma 3.2. *Let $s > -\frac{1}{2}$. We have*

$$|II| \leq \left(CC_0+\frac{1}{32}\right)\sum_{q>-1}\left(\lambda_q^{2s+4}\|c_q\|_2^2+\lambda_q^{2s+4}\|u_q\|_2^2\right)+CQ_uf(t)\sum_{q\geq -1}\lambda_q^{2s+2}\|c_q\|_2^2.$$

Proof: We start with Bony's paraproduct decomposition for II,

$$\begin{aligned}
II =& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_{\leq p-2}\cdot\nabla c_p)c_q\,\mathrm{d}x\\
&+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla c_{\leq p-2})c_q\,\mathrm{d}x\\
&+\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{c}_p)c_q\,\mathrm{d}x\\
=&II_1+II_2+II_3.
\end{aligned}$$

Using the commutator notation, we rewrite II_1 as

$$\begin{aligned}
II_1 =& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}[\Delta_q, u_{\leq p-2}\cdot\nabla]c_pc_q\,\mathrm{d}x\\
&+\sum_{q\geq -1}\lambda_q^{2s+2}\int_{\mathbb{R}^3}u_{\leq q-2}\cdot\nabla c_qc_q\,\mathrm{d}x\\
&+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}(u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_qc_pc_q\,\mathrm{d}x\\
=:&II_{11}+II_{12}+II_{13}.
\end{aligned}$$

Here, just as in proposition (3.1), we used $\sum_{|q-p|\leq 2}\Delta_pc_q = c_q$ to obtain II_{12}, which vanishes as $\operatorname{div} u_{\leq q-2} = 0$.

One can see that II_{11} enjoys the same estimates that I_{11} does. Splitting the summation by $Q_u(t)$ yields

$$\begin{aligned}|II_{11}| \leq& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}|[\Delta_q, u_{\leq p-2}\cdot\nabla]c_pc_q|\,\mathrm{d}x\\ \leq& \sum_{p\leq Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}|[\Delta_q, u_{\leq p-2}\cdot\nabla]c_pc_q|\,\mathrm{d}x\\ &+\sum_{p>Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}|[\Delta_q, u_{\leq Q_u}\cdot\nabla]c_pc_q|\,\mathrm{d}x\\ &+\sum_{p>Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\left|[\Delta_q, u_{(Q_u,p-2]}\cdot\nabla]c_pc_q\right|\,\mathrm{d}x\\ =:& II_{111}+II_{112}+II_{113}.\end{aligned}$$

We estimate the first two of the above three terms by Hölder's inequality, Lemma (2.3) and the definition of $f(t)$.

$$\begin{aligned}II_{111}\leq& \sum_{1\leq p\leq Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\|\nabla u_{\leq p-2}\|_\infty\|c_p\|_2\|c_q\|_2\\ \lesssim& Q_uf(t)\sum_{1\leq p\leq Q_u+2}\lambda_p^{s+1}\|c_p\|_2\sum_{|q-p|\leq 2}\lambda_q^{s+1}\|c_q\|_2\\ \lesssim& Q_uf(t)\sum_{q\geq -1}\lambda_q^{2s+2}\|c_q\|_2^2,\end{aligned}$$

$$\begin{aligned}II_{112}\leq& \sum_{p>Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\|\nabla u_{\leq Q_u}\|_\infty\|c_p\|_2\|c_q\|_2\\ \lesssim& Q_uf(t)\sum_{p>Q_u+2}\lambda_p^{s+1}\|c_p\|_2\sum_{|q-p|\leq 2}\lambda_q^{s+1}\|c_q\|_2\\ \lesssim& Q_uf(t)\sum_{q>Q_u}\lambda_q^{2s+2}\|c_q\|_2^2.\end{aligned}$$

And with the help of Hölder's inequality, Lemma (2.3) and the definition of $\Lambda_u(t)$, we estimate I_{113} as

$$\begin{aligned}II_{113}\leq& \sum_{p>Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\|[\Delta_q, u_{(Q_u,p-2]}\cdot\nabla]c_p\|_2\|c_q\|_2\\ \leq& \sum_{p>Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\|c_q\|_2\sum_{Q_u<p'\leq p-2}\lambda_{p'}\|u_{p'}\|_\infty\|c_p\|_2\\ \leq& \sum_{p>Q_u+2}\|c_p\|_2\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\|c_q\|_2\sum_{Q_u<p'\leq p-2}\lambda_{p'}\|u_{p'}\|_\infty\\ \lesssim& C_0\sum_{p>Q_u+2}\|c_p\|_2\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\|c_q\|_2\sum_{Q_u<p'\leq p-2}\lambda_{p'}^2\\ \lesssim& C_0\sum_{p>Q_u+2}\lambda_p^{2s+4}\|c_p\|_2^2\sum_{Q_u<p'\leq p-2}\lambda_{p'-p}^2\\ \lesssim& C_0\sum_{p>Q_u+2}\lambda_p^{2s+4}\|c_p\|_2^2.\end{aligned}$$

II_{13} can be estimated in the same fashion as I_{13}. Splitting the sum by the wavenumber $Q_u(t)$, we have

$$\begin{aligned}|II_{13}| \leq& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\left|(u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q c_p c_q\right|\mathrm{d}x\\ \leq& \sum_{-1\leq q\leq Q_u}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\left|(u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q c_p c_q\right|\mathrm{d}x\\ &+\sum_{q>Q_u}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\left|(u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q c_p c_q\right|\mathrm{d}x\\ =:&II_{131}+II_{132},\end{aligned}$$

which, using Hölder's inequality along with the definition of $f(t)$ and $\Lambda_u(t)$, we estimate as

$$\begin{aligned}|II_{131}| \leq& \sum_{-1\leq q\leq Q_u}\lambda_q^{2s+3}\|c_q\|_2\sum_{|q-p|\leq 2}\|u_{\leq p-2}-u_{\leq q-2}\|_\infty\|c_p\|_2\\ \lesssim& f(t)\sum_{-1\leq q\leq Q_u}\lambda_q^{2s+2}\|c_q\|_2\sum_{|q-p|\leq 2}\|c_p\|_2\\ \lesssim& f(t)\sum_{-1\leq q\leq Q_u}\lambda_q^{2s+2}\|c_q\|_2^2,\end{aligned}$$

$$\begin{aligned}|II_{132}| \leq& \sum_{q>Q_u}\lambda_q^{2s+3}\|c_q\|_2\sum_{|q-p|\leq 2}\|u_{\leq p-2}-u_{\leq q-2}\|_\infty\|c_p\|_2\\ \lesssim& C_0\sum_{q>Q_u}\lambda_q^{2s+3}\|c_q\|_2\sum_{|q-p|\leq 2}\lambda_p\|c_p\|_2\\ \lesssim& C_0\sum_{q>Q_u}\lambda_q^{2s+4}\|c_q\|_2^2.\end{aligned}$$

To estimate II_2, we make use of the wavenumber $Q_c(t)$ instead of $Q_u(t)$. Splitting the summation by $Q_c(t)$ yields

$$\begin{aligned}II_2 =& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla c_{\leq p-2})c_q\mathrm{d}x\\ =& \sum_{q\geq -1}\sum_{|q-p|\leq 2, p\leq Q_c+2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla c_{\leq p-2})c_q\mathrm{d}x\\ &+\sum_{q\geq -1}\sum_{|q-p|\leq 2, p>Q_c+2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla c_{\leq Q_c})c_q\mathrm{d}x\\ &+\sum_{q\geq -1}\sum_{|q-p|\leq 2, p>Q_c+2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla c_{(Q_c,p-2]})c_q\mathrm{d}x\\ =:&II_{21}+II_{22}+II_{23}.\end{aligned}$$

It follows from Hölder's and Young's inequalities that

$$
\begin{aligned}
|II_{21}| \leq & \sum_{q\geq -1}\sum_{|q-p|\leq 2, p\leq Q_c+2} \lambda_q^{2s+2}\int_{\mathbb{R}^3}\left|\Delta_q(u_p\cdot\nabla c_{\leq p-2})c_q\right|\mathrm{d}x\\
\leq & \sum_{q\geq -1}\lambda_q^{2s+2}\|c_q\|_2\sum_{|q-p|\leq 2, p\leq Q_c+2}\|u_p\|_2\|\nabla c_{\leq p-2}\|_\infty\\
\leq & \|\nabla c_{\leq Q_c}\|_\infty\sum_{q\geq -1}\lambda_q^{s+1}\|c_q\|_2\sum_{|q-p|\leq 2, p\leq Q_c+2}\lambda_p^{s+2}\|u_p\|_2\lambda_p^{-1}\\
\leq & \frac{1}{32}\sum_{q\geq -1}\lambda_q^{2s+4}\|u_q\|_2^2 + Cf(t)\sum_{q\geq -1}\lambda_q^{2s+2}\|c_q\|_2^2.
\end{aligned}
$$

While the term II_{22} can be treated the same way as II_{21}, the third term is estimated as follows, by utilizing the definition of $\Lambda_c(t)$

$$
\begin{aligned}
|II_{23}| \leq & \sum_{q\geq -1}\sum_{|q-p|\leq 2, p> Q_c+2}\lambda_q^{2s+2}\|u_p\|_2\|c_q\|_2\|\nabla c_{(Q_c,p-2]}\|_\infty\\
\lesssim & \sum_{q>Q_c}\lambda_q^{2s+2}\|u_q\|_2\|c_q\|_2\|\nabla c_{(Q_c,q]}\|_\infty\\
\lesssim & \sum_{q>Q_c}\lambda_q^{2s+2}\|u_q\|_2\|c_q\|_2\sum_{Q_c<p\leq q}\lambda_p^{1+\frac{3}{r}}\|c_p\|_r\\
\lesssim & C_0\sum_{q>Q_c}\lambda_q^{2s+2}\|u_q\|_2\|c_q\|_2\sum_{Q_c<p\leq q}\lambda_p\\
\lesssim & C_0\sum_{q>Q_c}\lambda_q^{s+2}\|u_q\|_2\lambda_q^{s+1}\|c_q\|_2\sum_{Q_c<p\leq q}\lambda_{p-q}\\
\lesssim & C_0\sum_{q>Q_c}\lambda_q^{2s+4}\|u_q\|_2^2 + C_0\sum_{q>Q_c}\lambda_q^{2s+2}\|c_q\|_2^2.
\end{aligned}
$$

As for II_3, we first integrate by parts, then split by the wavenumber $Q_u(t)$,

$$
\begin{aligned}
|II_3| \leq & \left|\sum_{p\geq -1}\lambda_q^{2s+2}\sum_{-1\leq q\leq p+2}\int_{\mathbb{R}^3}\Delta_q(u_p\otimes\tilde{c}_p)\nabla c_q\,\mathrm{d}x\right|\\
\leq & \sum_{p>Q_u}\lambda_q^{2s+2}\sum_{-1\leq q\leq p+2}\int_{\mathbb{R}^3}|\Delta_q(u_p\otimes\tilde{c}_p)\nabla c_q|\,\mathrm{d}x\\
& + \sum_{-1\leq p\leq Q_u}\lambda_q^{2s+2}\sum_{-1\leq q\leq p+2}\int_{\mathbb{R}^3}|\Delta_q(u_p\otimes\tilde{c}_p)\nabla c_q|\,\mathrm{d}x\\
=: & II_{31} + II_{32}.
\end{aligned}
$$

Proceeding with the help of Hölder's, Young's and Jensen's inequalities, we have for $s > -\frac{1}{2}$

$$\begin{aligned}
|II_{31}| \lesssim & \sum_{p>Q_u} \|u_p\|_\infty \|c_p\|_2 \sum_{-1\le q\le p+2} \lambda_q^{2s+3}\|c_q\|_2 \\
\lesssim & C_0 \sum_{p>Q_u} \lambda_p \|c_p\|_2 \sum_{-1\le q\le p+2} \lambda_q^{2s+3}\|c_q\|_2 \\
\lesssim & C_0 \sum_{p>Q_u} \lambda_p^{s+2} \|c_p\|_2 \sum_{-1\le q\le p+2} \lambda_q^{s+2}\|c_q\|_2 \lambda_{q-p}^{s+1} \\
\lesssim & C_0 \sum_{p>Q_u} \left(\lambda_p^{2s+4}\|c_p\|_2^2 + \left(\sum_{-1\le q\le p+2} \lambda_q^{s+2}\|c_q\|_2 \lambda_{q-p}^{s+1} \right)^2 \right) \\
\lesssim & C_0 \sum_{q\ge -1} \lambda_q^{2s+4}\|c_q\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
|II_{32}| \lesssim & \sum_{-1\le p\le Q_u} \|u_p\|_\infty \|c_p\|_2 \sum_{-1\le q\le p+2} \lambda_q^{2s+3}\|c_q\|_2 \\
\lesssim & f(t) \sum_{-1\le p\le Q_u} \lambda_p^{-1} \|c_p\|_2 \sum_{-1\le q\le p+2} \lambda_q^{2s+3}\|c_q\|_2 \\
\lesssim & f(t) \sum_{-1\le p\le Q_u} \lambda_p^{s+1} \|c_p\|_2 \sum_{-1\le q\le p+2} \lambda_q^{s+1}\|c_q\|_2 \lambda_{q-p}^{s+2} \\
\lesssim & f(t) \sum_{q\ge -1} \lambda_q^{2s+2}\|c_q\|_2^2.
\end{aligned}$$

We combine the above estimates to conclude that

$$|II| \lesssim C_0 \sum_{q\ge -1} (\lambda_q^{2s+4}\|c_q\|_2^2 + \lambda_q^{2s+4}\|u_q\|_2^2) + Q_u f(t) \sum_{q\ge -1} \lambda_q^{2s+2}\|c_q\|_2^2.$$

□

3.3. **Estimate for III.**

Lemma 3.3. *Let $s < 0$. We have*

$$|III| \lesssim C_0 \sum_{q\ge -1} \lambda_q^{2s+2}\|n_q\|_2^2 + Q_u f(t) \sum_{q\ge -1} \lambda_q^{2s}\|n_q\|_2^2.$$

Proof: We start with Bony's paraproduct decomposition for both of the terms.

$$\begin{aligned}
III = & \sum_{q\ge -1} \sum_{|q-p|\le 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_{\le p-2}\cdot \nabla n_p) n_q \,\mathrm{d}x \\
& + \sum_{q\ge -1} \sum_{|q-p|\le 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p\cdot \nabla n_{\le p-2}) n_q \,\mathrm{d}x \\
& + \sum_{q\ge -1} \sum_{p\ge q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p\cdot \nabla \tilde{n}_p) n_q \,\mathrm{d}x \\
= & III_1 + III_2 + III_3.
\end{aligned}$$

Since we can estimate III_1 and III_3 in the same manner as II_1 and II_3, respectively, the details of computation are omitted for simplicity. We claim

$$|III_1| + |III_3| \lesssim C_0 \sum_{q\geq -1} \lambda_q^{2s+2}\|n_q\|_2^2 + Q_u f(t) \sum_{q\geq -1} \lambda_q^{2s}\|n_q\|_2^2.$$

We are then left with the estimation of III_2 to complete the conclusion. Splitting the term by the wavenumber $Q_u(t)$, we have

$$\begin{aligned} III_2 = & \sum_{-1\leq p\leq Q_u} \sum_{|q-p|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla n_{\leq p-2}) n_q \mathrm{d}x \\ & + \sum_{p>Q_u} \sum_{|q-p|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla n_{\leq p-2}) n_q \mathrm{d}x \\ =: & III_{21} + III_{22}. \end{aligned}$$

Applying Hölder's, Young's and Jensen's inequalities, we have for $s < 1$

$$\begin{aligned} |III_{21}| \leq & \sum_{-1\leq p\leq Q_u} \|u_p\|_\infty \sum_{|q-p|\leq 2} \lambda_q^{2s}\|n_q\|_2 \sum_{p'\leq p-2} \lambda_{p'}\|n_{p'}\|_2 \\ \leq & \sum_{-1\leq p\leq Q_u} \lambda_p\|u_p\|_\infty \sum_{|q-p|\leq 2} \lambda_q^{2s-1}\|n_q\|_2 \sum_{p'\leq p-2} \lambda_{p'}\|n_{p'}\|_2 \\ \leq & \sum_{-1\leq p\leq Q_u} \lambda_p\|u_p\|_\infty \sum_{|q-p|\leq 2} \lambda_q^{s}\|n_q\|_2 \sum_{p'\leq p-2} \lambda_{p'}^{s}\|n_{p'}\|_2 \lambda_{q-p'}^{s-1} \\ \lesssim & f(t) \sum_{-1\leq p\leq Q_u} \Big(\lambda_p^{2s}\|n_q\|_2^2 + \big(\sum_{p'\leq p-2} \lambda_{p'}^{s}\|n_{p'}\|_2 \lambda_{p-p'}^{s-1} \big)^2 \Big) \\ \lesssim & f(t) \sum_{q\geq -1} \lambda_q^{2s}\|n_q\|_2^2, \end{aligned}$$

and for $s < 0$

$$\begin{aligned} |III_{22}| \leq & \sum_{p>Q_u} \|u_p\|_\infty \sum_{|q-p|\leq 2} \lambda_q^{2s}\|n_q\|_2 \sum_{p'\leq p-2} \lambda_{p'}\|n_{p'}\|_2 \\ \leq & \sum_{p>Q_u} \lambda_p^{-1}\|u_p\|_\infty \sum_{|q-p|\leq 2} \lambda_q^{2s+1}\|n_q\|_2 \sum_{p'\leq p-2} \lambda_{p'}\|n_{p'}\|_2 \\ \leq & C_0 \sum_{q>Q_u-2} \lambda_q^{s+1}\|n_q\|_2 \sum_{p'\leq q} \lambda_{p'}^{s+1}\|n_{p'}\|_2 \lambda_{q-p'}^{s} \\ \lesssim & C_0 \sum_{q\geq -1} \lambda_q^{2s+2}\|n_q\|_2^2. \end{aligned}$$

□

3.4. **Estimate for** IV**.** Here we use Hölder's and Young's inequalities to obtain

$$\begin{aligned} |IV| \leq & \sum_{q\geq -1} \lambda_q^{s+1}\|n_q\|_2 \lambda_q^{s+1}\|u_q\|_2 \\ \leq & \frac{1}{32} \sum_{q\geq -1} \lambda_q^{2s+2}\|n_q\|_2^2 + C \sum_{q\geq -1} \lambda_q^{2s+2}\|u_q\|_2^2 \end{aligned} \tag{3.14}$$

for an absolute constant C.

3.5. **Estimate for V.**

Lemma 3.4. *If $r>3$ and $s<0$, we have*

$$|V| \le \left(\frac{1}{32}+CC_0\right)\sum_{q\ge -1}\left(\lambda_q^{2s+4}\|c_q\|_2^2+\lambda_q^{2s+2}\|n_q\|_2^2\right)$$
$$+C(f(t)+1)\sum_{q\ge -1}\left(\lambda_q^{2s+2}\|c_q\|_2^2+\lambda_q^{2s}\|n_q\|_2^2\right).$$

Proof: Bony's paraproduct decomposition leads to

$$\begin{aligned}
V =&\sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(n_{\le p-2}c_p)c_q\mathrm{d}x\\
&+\sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(n_p c_{\le p-2})c_q\mathrm{d}x\\
&+\sum_{q\ge -1}\sum_{p\ge q-2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(\tilde{n}_p c_p)c_q\mathrm{d}x\\
=:&V_1+V_2+V_3.
\end{aligned}$$

We further split V_1 by the wavenumber $\Lambda_c(t)$

$$\begin{aligned}
V_1 =&\sum_{q\le Q_c}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(n_{\le p-2}c_p)c_q\mathrm{d}x\\
&+\sum_{q> Q_c}\sum_{|q-p|\le 2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(n_{\le p-2}c_p)c_q\mathrm{d}x\\
=:&V_{11}+V_{12}.
\end{aligned}$$

To estimate V_{11}, Hölder's inequality gives

$$\begin{aligned}
|V_{11}|\le&\sum_{q\le Q_c}\lambda_q^{2s+2}\|c_q\|_\infty\sum_{|q-p|\le 2}\|c_p\|_2\sum_{p'\le p-2}\|n_{p'}\|_2\\
\lesssim&\sum_{q\le Q_c}\lambda_q\|c_q\|_\infty\sum_{|q-p|\le 2}\lambda_p^{s+2}\|c_p\|_2\sum_{p'\le p-2}\lambda_{p'}^{s}\|n_{p'}\|_2\lambda_{p-p'}^{s-1}\lambda_{p'}^{-1}.
\end{aligned}$$

Since $s-1<0$ for $s<0$, we apply Young's and Jensen's inequalities to obtain

$$\begin{aligned}
|V_{11}|\le&\frac{1}{32}\sum_{q\le Q_c+2}\lambda_q^{2s+4}\|c_q\|_2^2+C\sum_{q\le Q_c+2}\lambda_q^{2}\|c_q\|_\infty^2\left(\sum_{p'\le q}\lambda_{p'}^{s}\|n_{p'}\|_2\lambda_{q-p'}^{s-1}\right)^2\\
\le&\frac{1}{32}\sum_{q\le Q_c+2}\lambda_q^{2s+4}\|c_q\|_2^2+Cf(t)\sum_{q\le Q_c+2}\sum_{p'\le q}\lambda_{p'}^{2s}\|n_{p'}\|_2^2\lambda_{q-p'}^{s-1}\\
\le&\frac{1}{32}\sum_{q\le Q_c+2}\lambda_q^{2s+4}\|c_q\|_2^2+Cf(t)\sum_{q\le Q_c}\lambda_q^{2s}\|n_q\|_2^2.
\end{aligned}$$

Regarding V_{12}, we have

$$
\begin{aligned}
|V_{12}| \leq & \sum_{q>Q_c} \lambda_q^{2s+2} \|c_q\|_\infty \sum_{|q-p|\leq 2} \|c_p\|_2 \sum_{p'\leq p-2} \|n_{p'}\|_2 \\
\lesssim & \sum_{q>Q_c} \lambda_q^{2s+2+\frac{3}{r}} \|c_q\|_r \sum_{|q-p|\leq 2} \|c_p\|_2 \sum_{p'\leq p-2} \|n_{p'}\|_2 \\
\lesssim & C_0 \sum_{q>Q_c-2} \lambda_q^{2s+2} \|c_q\|_2 \sum_{p'\leq q} \|n_{p'}\|_2 \\
\lesssim & C_0 \sum_{q>Q_c-2} \lambda_q^{s+2} \|c_q\|_2 \sum_{p'\leq q} \lambda_{p'}^{s} \|n_{p'}\|_2 \lambda_{q-p'}^{s}.
\end{aligned}
$$

Again, since $s<0$, we can apply Young's and Jensen's inequalities

$$
\begin{aligned}
|V_{12}| \lesssim & C_0 \sum_{q>Q_c-2} \lambda_q^{2s+4} \|c_q\|_2^2 + \sum_{q>Q_c} \left(\sum_{p'\leq q} \lambda_{p'}^{s} \|n_{p'}\|_2 \lambda_{q-p'}^{s} \right)^2 \\
\lesssim & C_0 \sum_{q>Q_c-2} \lambda_q^{2s+4} \|c_q\|_2^2 + \sum_{q\geq -1} \lambda_{p'}^{2s} \|n_{p'}\|_2^2.
\end{aligned}
$$

We also split V_2 by the wavenumber $\Lambda_c(t)$ as

$$
\begin{aligned}
V_2 = & \sum_{q\geq -1} \sum_{|q-p|\leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(n_p c_{\leq p-2}) c_q \, \mathrm{d}x \\
= & \sum_{q\geq -1} \sum_{|q-p|\leq 2, p\leq Q_c+2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(n_p c_{\leq p-2}) c_q \, \mathrm{d}x \\
& + \sum_{q\geq -1} \sum_{|q-p|\leq 2, p>Q_c+2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(n_p c_{\leq Q_c}) c_q \, \mathrm{d}x \\
& + \sum_{q\geq -1} \sum_{|q-p|\leq 2, p>Q_c+2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(n_p c_{(Q_c, p-2]}) c_q \, \mathrm{d}x \\
=: & V_{21} + V_{22} + V_{23}.
\end{aligned}
$$

Hölder's and Young's inequalities yield the following estimate on V_{21},

$$
\begin{aligned}
|V_{21}| \lesssim & \sum_{q\geq -1} \lambda_q^{2s+2}\|c_q\|_2 \sum_{|q-p|\leq 2}\|n_p\|_2 \sum_{p'\leq p-2\leq Q_c}\|c_{p'}\|_\infty \\
\lesssim & \sum_{q\geq -1} \lambda_q^{s+1}\|c_q\|_2 \sum_{|q-p|\leq 2}\lambda_p^{s+1}\|n_p\|_2 \sum_{p'\leq p-2\leq Q_c}\lambda_{p'}\|c_{p'}\|_\infty \lambda_{p'}^{-1} \\
\leq & \frac{1}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|n_q\|_2^2 + C\sum_{p\geq -1}\lambda_p^{2s+2}\|c_p\|_2^2\left(\sum_{p'\leq p-2\leq Q_c}\lambda_{p'}\|c_{p'}\|_\infty\lambda_{p'}^{-1}\right)^2 \\
\leq & \frac{1}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|n_q\|_2^2 + C\sum_{p\geq -1}\lambda_p^{2s+2}\|c_p\|_2^2\sum_{p'\leq p-2\leq Q_c}\lambda_{p'}^2\|c_{p'}\|_\infty^2\lambda_{p'}^{-1} \\
\leq & \frac{1}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|n_q\|_2^2 + Cf(t)\sum_{p\geq -1}\lambda_p^{2s+2}\|c_p\|_2^2\sum_{p'\leq p-2\leq Q_c}\lambda_{p'}^{-1} \\
\leq & \frac{1}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|n_q\|_2^2 + Cf(t)\sum_{q\geq -1}\lambda_q^{2s+2}\|c_q\|_2^2.
\end{aligned}
$$

Note that V_{22} can be estimated in the same way. Provided $\frac{3}{r}\leq 1$, the last term V_{23} is estimated as follows,

$$
\begin{aligned}
|V_{23}| \leq & \sum_{q\geq -1}\sum_{|q-p|\leq 2, p>Q_c+2}\lambda_q^{2s+2}\|n_p\|_2\|c_q\|_{\frac{2r}{r-2}}\|c_{(Q_c,p-2]}\|_r \\
\lesssim & \sum_{q\geq -1}\lambda_q^{2s+2+\frac{3}{r}}\|n_q\|_2\|c_q\|_2\sum_{q>Q_c, Q_c<p'\leq q}\|c_{p'}\|_r \\
\lesssim & C_0\sum_{q\geq -1}\lambda_q^{2s+2+\frac{3}{r}}\|n_q\|_2\|c_q\|_2\sum_{q>Q_c, Q_c<p'\leq q}\lambda_{p'}^{-\frac{3}{r}} \\
\lesssim & C_0\sum_{q\geq -1}\lambda_q^{s+1}\|n_q\|_2\lambda_q^{s+2}\|c_q\|_2\lambda_q^{\frac{3}{r}-1}\sum_{q>Q_c, Q_c<p'\leq q}\lambda_{p'}^{-\frac{3}{r}} \\
\lesssim & C_0\sum_{q\geq -1}\lambda_q^{s+1}\|n_q\|_2\lambda_q^{s+2}\|c_q\|_2 \\
\lesssim & C_0\sum_{q\geq -1}\lambda_q^{2s+2}\|n_q\|_2^2 + C_0\sum_{q\geq -1}\lambda_q^{2s+4}\|c_q\|_2^2.
\end{aligned}
$$

The term V_3 can be dealt with in a similar way as V_1. We first split the sum,

$$
\begin{aligned}
V_3 = & \sum_{q\leq Q_c}\sum_{p\geq q-2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(\tilde{n}_p c_p)c_q \mathrm{d}x \\
& + \sum_{q<Q_c}\sum_{p\geq q-2}\lambda_q^{2s+2}\int_{\mathbb{R}^3}\Delta_q(\tilde{n}_p c_p)c_q \mathrm{d}x.
\end{aligned}
$$

Without giving details, we claim that

$$
|V_3| \leq \left(\frac{1}{32}+CC_0\right)\sum_{q\geq -1}\lambda_q^{2s+4}\|c_q\|_2^2 + C(f(t)+1)\sum_{q\geq -1}\lambda_q^{2s}\|n_q\|_2^2.
$$

□

3.6. **Estimate for VI.**

Lemma 3.5. *Let $-\frac{1}{2} < s < 0$ and $3 < r < \frac{3}{1+s}$. We have*

$$|VI| \leq \left(\frac{1}{32} + CC_0\right) \sum_{q\geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 + Cf(t) \sum_{q\geq -1} \lambda_q^{2s} \|n_q\|_2^2.$$

Proof: Utilizing Bony's paraproduct, VI can be decomposed as

$$\begin{aligned} VI =& - \sum_{q\geq -1} \sum_{|q-p|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_{\leq p-2} \nabla c_p) \nabla n_q \, \mathrm{d}x \\ &- \sum_{q\geq -1} \sum_{|q-p|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_p \nabla c_{\leq p-2}) \nabla n_q \, \mathrm{d}x \\ &- \sum_{q\geq -1} \sum_{p\geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(\tilde{n}_p \nabla c_p) \nabla n_q \, \mathrm{d}x \\ =:& VI_1 + VI_2 + VI_3. \end{aligned}$$

We continue to decompose VI_1 by Q_c,

$$\begin{aligned} VI_1 =& - \sum_{q\geq -1} \sum_{|q-p|\leq 2, p\leq Q_c} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_{\leq p-2} \nabla c_p) \nabla n_q \, \mathrm{d}x \\ &- \sum_{q\geq -1} \sum_{|q-p|\leq 2, p> Q_c} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_{\leq p-2} \nabla c_p) \nabla n_q \, \mathrm{d}x \\ =:& VI_{11} + VI_{12}. \end{aligned}$$

To estimate VI_{11}, we apply Hölder's inequality first,

$$\begin{aligned} |VI_{11}| \lesssim& \sum_{q\geq -1} \lambda_q^{2s+1} \|n_q\|_2 \sum_{|q-p|\leq 2, p\leq Q_c} \|\nabla c_p\|_\infty \sum_{p'\leq p-2} \|n_{p'}\|_2 \\ \lesssim& \sum_{q\geq -1} \lambda_q^{s+1} \|n_q\|_2 \sum_{|q-p|\leq 2, p\leq Q_c} \|\nabla c_p\|_\infty \sum_{p'\leq p-2} \lambda_{p'}^{s} \|n_{p'}\|_2 \lambda_{p-p'}^{s}. \end{aligned}$$

We then apply Young's and Jensen's inequalities,

$$\begin{aligned} |VI_1| \leq& \frac{1}{32} \sum_{q\geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 + Cf(t) \sum_{q\geq -1} \big(\sum_{p'\leq q} \lambda_{p'}^{s} \|n_{p'}\|_2 \lambda_{q-p'}^{s} \big)^2 \\ \leq& \frac{1}{32} \sum_{q\geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 + Cf(t) \sum_{q\geq -1} \lambda_q^{2s} \|n_q\|_2^2, \end{aligned}$$

where we require $s < 0$ to ensure $\sum_{p'\leq p-2} \lambda_{p-p'}^{s} < \infty$.

Again, applying Hölder's inequality first to VI_{12} yields

$$
\begin{aligned}
|VI_{12}| \leq & \sum_{q\geq -1} \sum_{|q-p|\leq 2, p>Q_c} \lambda_q^{2s} \|\nabla n_q\|_2 \|\nabla c_p\|_r \|n_{\leq p-2}\|_{\frac{2r}{r-2}} \\
\lesssim & \sum_{q\geq -1} \sum_{|q-p|\leq 2, p>Q_c} \lambda_q^{2s+2-\frac{3}{r}} \|n_q\|_2 \lambda_p^{\frac{3}{r}} \|c_p\|_r \sum_{p'\leq p-2} \lambda_{p'}^{\frac{3}{r}} \|n_{p'}\|_2 \\
\lesssim & C_0 \sum_{q\geq -1} \lambda_q^{2s+2-\frac{3}{r}} \|n_q\|_2 \sum_{p'\leq q} \lambda_{p'}^{\frac{3}{r}} \|n_{p'}\|_2 \\
\lesssim & C_0 \sum_{q\geq -1} \lambda_q^{s+1} \|n_q\|_2 \sum_{p'\leq q} \lambda_{p'}^{s+1} \|n_{p'}\|_2 \lambda_{q-p}^{s+1-\frac{3}{r}}.
\end{aligned}
$$

Following Young's and Jensen's inequalities, we obtain for $s+1<\frac{3}{r}$

$$
\begin{aligned}
|VI_{12}| \leq & CC_0 \sum_{q\geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 + CC_0 \sum_{q\geq -1} \left(\sum_{p'\leq q} \lambda_{p'}^{s+1} \|n_{p'}\|_2 \lambda_{q-p}^{s+1-\frac{3}{r}} \right)^2 \\
\leq & CC_0 \sum_{q\geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 + CC_0 \sum_{q\geq -1} \sum_{p'\leq q} \lambda_{p'}^{2s+2} \|n_{p'}\|_2^2 \lambda_{q-p}^{s+1-\frac{3}{r}} \\
\leq & CC_0 \sum_{q\geq -1} \lambda_q^{2s+2} \|n_q\|_2^2.
\end{aligned}
$$

The first step of dealing with VI_2 is to split it by Q_c as well,

$$
\begin{aligned}
VI_2 = & -\sum_{q\geq -1} \sum_{|q-p|\leq 2, p\leq Q_c+2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (n_p \nabla c_{\leq p-2}) \nabla n_q \, dx \\
& - \sum_{q\geq -1} \sum_{|q-p|\leq 2, p>Q_c+2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (n_p \nabla c_{\leq Q_c}) \nabla n_q \, dx \\
& - \sum_{q\geq -1} \sum_{|q-p|\leq 2, p>Q_c+2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (n_p \nabla c_{(Q_c, p-2]}) \nabla n_q \, dx \\
=: & VI_{21} + VI_{22} + VI_{23}.
\end{aligned}
$$

Using Hölder's and Young's inequalities, and the fact that $\lambda_{q-p}^s \sim C$ as p,q are close to each other, we have

$$
\begin{aligned}
|VI_{21}| \leq & \sum_{q\geq -1} \lambda_q^{2s+1} \|n_q\|_2 \sum_{|q-p|\leq 2, p\leq Q_c+2} \|n_p\|_2 \|\nabla c_{\leq p-2}\|_\infty \\
\leq & \sum_{q\geq -1} \lambda_q^{s+1} \|n_q\|_2 \sum_{|q-p|\leq 2, p\leq Q_c+2} \lambda_p^s \|n_p\|_2 \|\nabla c_{\leq p-2}\|_\infty \lambda_{q-p}^s \\
\leq & \frac{1}{32} \sum_{q\geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 + Cf(t) \sum_{q\geq -1} \lambda_q^{2s} \|n_q\|_2^2.
\end{aligned}
$$

We skip the computation for VI_{22} which can be estimated in a likely way as VI_{21}. We proceed to estimate V_{23}, provided $r>3$

$$
\begin{aligned}
|VI_{23}| \leq & \sum_{q\geq -1}\sum_{|q-p|\leq 2, p>Q_c+2} \lambda_q^{2s}\|n_p\|_2\|\nabla c_{(Q_c,p-2]}\|_r\|\nabla n_q\|_{\frac{2r}{r-2}}\\
\lesssim & \sum_{q\geq -1}\lambda_q^{2s+1+\frac{3}{r}}\|n_q\|_2^2\sum_{Q_c<p'\leq q}\lambda_{p'}\|c_{p'}\|_r\\
\lesssim & C_0\sum_{q\geq -1}\lambda_q^{2s+1+\frac{3}{r}}\|n_q\|_2^2\sum_{Q_c<p'\leq q}\lambda_{p'}^{1-\frac{3}{r}}\\
\lesssim & C_0\sum_{q\geq -1}\lambda_q^{2s+2}\|n_q\|_2^2\sum_{Q_c<p'\leq q}\lambda_{p'-q}^{1-\frac{3}{r}}\\
\lesssim & C_0\sum_{q\geq -1}\lambda_q^{2s+2}\|n_q\|_2^2.
\end{aligned}
$$

Finally, we notice that VI_3 can be handled in an analogous way of VI_1; thus the details of computation are omitted. It completes the proof of the lemma.

□

Summing inequalities in Lemma 3.1- Lemma 3.5 and (3.11)-(3.14) produces the following Grönwall type of inequality,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}(\lambda_q^{2s}\|n_q\|_2^2+\lambda_q^{2s+2}\|c_q\|_2^2+\lambda_q^{2s+2}\|u_q\|_2^2)\\
\leq&(-2+CC_0)\sum_{q\geq -1}(\lambda_q^{2s+2}\|n_q\|_2^2+\lambda_q^{2s+4}\|c_q\|_2^2+\lambda_q^{2s+4}\|u_q\|_2^2)\\
&+CQ_uf(t)\sum_{q\geq -1}(\lambda_q^{2s}\|n_q\|_2^2+\lambda_q^{2s+2}\|c_q\|_2^2+\lambda_q^{2s+2}\|u_q\|_2^2).
\end{aligned}
$$

Thus, C_0 can be chosen small enough such that $CC_0<\frac{1}{4}$. On the other hand, combining the definition of $\Lambda_u(t)$ and Bernstein's inequality, one can deduce

$$
1\leq C_0^{-1}\Lambda_u^{-1}\|u_{Q_u}\|_\infty\lesssim C_0^{-1}\Lambda_u^{\frac{1}{2}}\|u_{Q_u}\|_2\lesssim C_0^{-1}\Lambda_u^{-\frac{1}{2}-s}\|u\|_{\dot H^{s+1}},
$$

which implies that for $s>-\frac{1}{2}$,

$$
Q_u=\log\Lambda_u\lesssim 1+\log\|u\|_{\dot H^{s+1}}.
$$

Hence the energy inequality becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}(\lambda_q^{2s}\|n_q\|_2^2+\lambda_q^{2s+2}\|c_q\|_2^2+\lambda_q^{2s+2}\|u_q\|_2^2)\\
\leq&-\sum_{q\geq -1}(\lambda_q^{2s+2}\|n_q\|_2^2+\lambda_q^{2s+4}\|c_q\|_2^2+\lambda_q^{2s+4}\|u_q\|_2^2)\\
&+Cf(t)\left(1+\log\|u\|_{\dot H^{s+1}}\right)\sum_{q\geq -1}(\lambda_q^{2s}\|n_q\|_2^2+\lambda_q^{2s+2}\|c_q\|_2^2+\lambda_q^{2s+2}\|u_q\|_2^2).
\end{aligned}
$$

By the hypothesis (1.7) and Grönwall's inequality, we can conclude that

$$
\begin{aligned}
n&\in L^\infty(0,T;H^s)\cap L^2(0,T;H^{s+1}),\\
c&\in L^\infty(0,T;H^{s+1})\cap L^2(0,T;H^{s+2}),\\
u&\in L^\infty(0,T;H^{s+1})\cap L^2(0,T;H^{s+2}).
\end{aligned}
$$

We consider a particular case $s = -\varepsilon$ for small enough $\varepsilon > 0$. We realize that our solution is regular via bootstrapping arguments. In fact, we have

$$n \in L^\infty(0,T;H^{-\varepsilon}), \quad u \in L^\infty(0,T;H^{1-\varepsilon}), \quad c \in L^\infty(0,T;H^{1-\varepsilon}) \cap L^\infty(0,T;L^\infty).$$

By scaling, it is known that $H^{-\frac{1}{2}}$, $H^{\frac{1}{2}}$, and $H^{\frac{3}{2}}$ are critical for n, u and c, respectively. For small enough $\varepsilon > 0$, $H^{-\varepsilon}$ is subcritical for n; so is $H^{1-\varepsilon}$ for u. Thus, it suffices to bootstrap the equation of c to obtain higher regularity for c. We recall

$$c_t - \Delta c = -u \cdot \nabla c - nc.$$

Since $u \in L^\infty(0,T;H^{1-\varepsilon})$ and $\nabla c \in L^2(0,T;H^{1-\varepsilon})$, by Sobolev embedding $H^{1-\varepsilon} \hookrightarrow L^{\frac{6}{1+2\varepsilon}}$, we have $u \cdot \nabla c \in L^2(0,T;L^{\frac{3}{1+2\varepsilon}})$. Similarly, the fact of $c \in L^\infty(0,T;H^{1-\varepsilon})$ and $n \in L^2(0,T;H^{1-\varepsilon})$ implies $nc \in L^2(0,T;L^{\frac{3}{1+2\varepsilon}})$. Then the standard maximal regularity theory of heat equation yields

$$c \in H^1(0,T;L^{\frac{3}{1+2\varepsilon}}) \cap L^2(0,T;W^{2,\frac{3}{1+2\varepsilon}}).$$

We need to bootstrap one more time. Now we have $\nabla c \in L^2(0,T;W^{1,\frac{3}{1+2\varepsilon}})$ which, along with $u \in L^\infty(0,T;H^{1-\varepsilon})$, implies $u \cdot \nabla c \in L^2(0,T;L^{\frac{6}{1+6\varepsilon}})$. On the other hand, we have $nc \in L^2(0,T;L^{\frac{6}{1+2\varepsilon}})$ from the estimate $n \in L^2(0,T;H^{1-\varepsilon})$ and the maximal principle $c \in L^\infty(0,T;L^\infty)$. Again, the maximal regularity theory of heat equation produces that

$$c \in H^1(0,T;L^{\frac{6}{1+6\varepsilon}}) \cap L^2(0,T;W^{2,\frac{6}{1+6\varepsilon}}).$$

As a consequence of the mixed derivative theorem (see [28]), we have

$$c \in W^{1-\theta,2}(0,T;W^{2\theta,\frac{6}{1+6\varepsilon}})$$

for any $\theta \in [0,1]$. In fact, if we take $\theta \in (\frac{1+6\varepsilon}{4}, \frac{1}{2})$, Sobolev embedding theorem shows that

$$c \in W^{1-\theta,2}(0,T;W^{2\theta,\frac{6}{1+6\varepsilon}}) \hookrightarrow L^\infty(0,T;H^{\frac{3}{2}+\varepsilon_0})$$

for an small enough constant $\varepsilon_0 > 0$. Notice that $H^{\frac{3}{2}}$ is critical for c. Thus we can stop the bootstrapping for c equation. Regarding the density function n, although the obtained estimates are in subcritical space already, we would like to further improve the estimates to reach spaces with even higher regularity (i.e. Sobolev spaces with positive smoothness index). Indeed, we look at the n equation again,

$$n_t - \Delta n = -u \cdot \nabla n - \nabla \cdot (n \nabla c).$$

Due to the fact $\nabla c \in L^\infty(0,T;H^{\frac{1}{2}+\varepsilon_0})$ and $n \in L^2(0,T;H^{1-\varepsilon})$, Sobolev embedding theorem yields $n\nabla c \in L^2(0,T;L^2)$ and hence $\nabla \cdot (n\nabla c) \in L^2(0,T;H^{-1})$. While $u \in L^\infty(0,T;H^{1-\varepsilon})$ and $n \in L^2(0,T;H^{1-\varepsilon})$ together imply $un \in L^2(0,T;L^{\frac{3}{1+2\varepsilon}})$ which is embedded in $L^2(0,T;L^2)$. Thus $u \cdot \nabla n = \nabla \cdot (un) \in L^2(0,T;H^{-1})$. Applying Lemma 2.4 with $\alpha = -1$ to the n equation, we claim

$$n \in H^1(0,T;H^{-1}) \cap L^2(0,T;H^1).$$

Summarizing the analysis above gives us

$$\begin{aligned} &n \in L^\infty(0,T;H^{-\varepsilon}) \cap L^2(0,T;H^1) \cap H^1(0,T;H^{-1}), \\ &u \in L^\infty(0,T;H^{1-\varepsilon}) \cap L^2(0,T;H^{2-\varepsilon}), \\ &c \in L^\infty(0,T;H^{\frac{3}{2}+\varepsilon_0}) \cap L^2(0,T;W^{2,\frac{6}{1+6\varepsilon}}). \end{aligned}$$

Since each of the three functions n, u and c is in higher regularity space than its critical Sobolev space, further bootstrapping procedures for parabolic equations and standard argument of extending regularity can be applied to infer that the solution (u, n, c) is regular up to time T.

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