2020 American Control Conference
Denver, CO, USA, July 1-3, 2020

Conversion of a Class of Stochastic Control Problems to Fundamental-Solution Deterministic Control Problems*

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***Abstract*— A class of nonlinear, stochastic staticization control problems (including minimization problems with smooth, convex, coercive payoffs) driven by diffusion dynamics and constant diffusion coefficient is considered. Using dynamic programming and tools from static duality, a fundamental solution form is obtained where the same solution can be used for a variety of terminal costs without re-solution of the problem. Further, this fundamental solution takes the form of a deterministic control problem rather than a stochastic control problem.**

I. INTRODUCTION

We consider nonlinear optimal stochastic control problems where the finite-dimensional dynamics take the form of stochastic differential equations (SDEs). These problems are typically converted into Hamilton-Jacobi partial differential equation (HJ PDE) problems. In the case of deterministic optimal control problems, the HJ PDEs are first-order equations, while in the stochastic case, they are second-order HJ PDEs. The dimension of the space over which these PDEs are defined is that of the state process of the control problem. Realistic control problems typically have relatively high dimensional state processes (i.e., greater than dimension three), leading to PDEs over high dimensional spaces. The solution of such HJ PDE problems has long been hampered by the curse-of-dimensionality, and we note that this has limited solution of such problems by classical methods to relatively low state-space dimensions on the order of three to five (cf. [3] and the references therein). More recently, the max-plus based curse-of-dimensionality-free methods have demonstrated computational tractability for significantly higher space dimension, and this approach have been quite effective in the case of first-order HJ PDE [6], [13], [14], [15], [16], with the caveat being a curse-of-complexity that grows rapidly with back propagation. Extensions of the max-plus based curse-of-dimensionality-free methods to second-order HJ PDE and stochastic control problems has been less computationally successful [11], [1].

Here, we demonstrate that for certain classes of problems, one may convert the second-order HJ PDE associated to stochastic control problems driven by Brownian motion into a first-order HJ PDE combined with a small integral term. Hence, the rapid curse-of-dimensionality-free methods may be applied. Further, the solutions are obtained as fundamental solutions, which implies that the same solution may be applied to varying terminal costs without complete re-solution of the HJ PDE problem.

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*This work was supported by AFOSR Grants FA9550-18-1-0268 and FA2386-16-1-4066 and NSF Grant DMS-1908918.

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II. DEFINITION OF THE PROBLEM CLASS

We consider a nonlinear stochastic control problem where the SDE dynamics and initial state are given by

$$d\xi_t = f(\xi_t, u_t)\,dt + \mu\,dB_t, \quad \xi_s = x \in \mathbb{R}^n, \tag{1}$$

where the underlying probability space is denoted as $(\Omega, \mathcal{F}_\infty, P)$. Also, $B_\cdot$ denotes an n-dimensional Brownian motion adapted to filtration $\mathcal{F}_t$. We suppose the control, $u_t \in U \subseteq \mathbb{R}^k$ for all t. Fix $T \in (0,\infty)$, and for $s \in [0,T]$, let the control and state-process spaces be given by

$$\mathcal{U}_s \doteq \{u : [s,T]\times\Omega \to \mathbb{R}^n \,|\, u \text{ is } \mathcal{F}_t\text{-adapted, right-contin.} \\ \text{and such that } \mathbb{E}\textstyle\int_s^T |u_t|^m\,dt < \infty\ \forall m \in I\!N\}, \tag{2}$$

$$\mathcal{X}_s \doteq \{\xi : [s,T]\times\Omega \to \mathbb{R}^n \,|\, \xi \text{ is } \mathcal{F}_t\text{-adapted, right-contin.} \\ \text{and such that } \mathbb{E}\sup_{t\in[s,T]} |\xi_t|^m < \infty\ \forall m \in I\!N\}. \tag{3}$$

The payoff will be given by

$$J(s,x,u;z) \doteq \mathbb{E}\Big\{\int_s^T L(\xi_t,u_t)\,dt + \psi(\xi_T;z)\Big\}, \tag{4}$$

$$\psi(x;z) \doteq \tfrac{1}{2}(x-z)^T\bar{M}(x-z) + \bar{\gamma}, \tag{5}$$

where $\bar{M}$ is positive-definite and symmetric, and $z \in \mathbb{R}^n$. For reasons of space, we will not include the case of more general payoffs here. The value function is given by

$$\bar{W}(s,x;z) \doteq \operatorname*{stat}_{u\in\mathcal{U}_s} J(s,x,u;z). \tag{6}$$

We remark that in the case of a convex, coercive, C^1 payoff, stat is equivalent to minimization; that is $\bar{W}(s,x;z) = \min_{u\in\mathcal{U}_s} J(s,x,u;z)$. *Hence, all results obtained for staticization problems hold for such minimization problems.*

III. STATICIZATION DEFINITIONS

Although minimization, maximization and saddle-point generation are more common in control theory, "staticization" has recently proven to be quite useful. We make the following definitions. Let $\mathcal{Z}$ be a real normed vector space with $\mathcal{A} \subseteq \mathcal{Z}$, and suppose $G : \mathcal{A} \to \mathbb{R}$. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{A}} G(u) \doteq \operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\}$ if $\bar{u} \in \mathcal{A}$ and either

$$\limsup_{u\to\bar{u},\,u\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(u)-G(\bar{u})|}{|u-\bar{u}|} = 0, \tag{7}$$

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or there exists $\delta > 0$ such that $\mathcal{A} \cap \mathcal{B}_\delta(\bar{u}) = \{\bar{u}\}$. If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\operatorname*{stat^s}_{u \in \mathcal{A}} G(u) \doteq \big\{ G(\bar{u}) \,\big|\, \bar{u} \in \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \big\}. \quad (8)$$

If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} = \emptyset$, then $\operatorname{stat}^s_{u\in\mathcal{A}} G(u)$ is undefined. We are mainly interested in a single-valued stat operation. In particular, if there exists $a \in \mathbb{R}$ such that $\operatorname{stat}^s_{u\in\mathcal{A}} G(u) = \{a\}$, then $\operatorname{stat}_{u\in\mathcal{A}} G(u) \doteq a$; otherwise, $\operatorname{stat}_{u\in\mathcal{A}} G(u)$ is undefined. The following is immediate.

Lemma 1: Suppose $\mathcal{Z}$ is a Banach space, with open set $\mathcal{A} \subseteq \mathcal{Z}$, and that G is Fréchet differentiable at $\bar{u} \in \mathcal{A}$. Then, $\bar{u} \in \operatorname{argstat}\{G(y) \,|\, y \in \mathcal{A}\}$ if and only if $DG(\bar{u}) = 0$.

IV. Recollection of Results

The first step is the development is that of obtaining the equivalence between the value function and the solution of the associated HJ PDE problem. This equivalence is very standard in the minimization, maximization and minimax cases, and less so in staticization cases that do not correspond to these. We recall a dynamic-programming staticization result here, so as to ground the sequel. In particular, we work under strong conditions so as to avoid excessively technical proofs.

We will let $\mathcal{X} \doteq (0,T) \times \mathbb{R}^n$, $\bar{\mathcal{X}} \doteq (0,T] \times \mathbb{R}^n$, $\mathcal{Y} \doteq (0,T) \times \mathbb{R}^{2n}$ and $\bar{\mathcal{Y}} \doteq (0,T] \times \mathbb{R}^{2n}$. We specifically consider

$$\begin{aligned} 0 &= W_t + \operatorname*{stat}_{v \in U} \left\{ f(x,v)^T W_x + L(x,v) \right\} + \tfrac{1}{2} \operatorname{tr}[A W_{xx}] \\ &\doteq W_t + H_0(x, W_x) + \mathcal{Q}_0(x, W_x) + \tfrac{1}{2} \operatorname{tr}[A W_{xx}], \\ &\doteq W_t + \tilde{H}_0(x, W_x) + \tfrac{1}{2} \operatorname{tr}[A W_{xx}], \quad (t,x,z) \in \mathcal{Y}, \qquad (9) \\ W(T,x;z) &= \psi(x;z), \quad (x,z) \in \mathbb{R}^{2n}, \qquad (10) \end{aligned}$$

where $\mathcal{Q}_0$ is a quadratic function of its arguments, and H_0 contains all remaining non-quadratic terms defining the Hamiltonian (where we note that the diffusion coefficient in (1) is constant). We assume the following.

Assume that for $z \in \mathbb{R}^n$, $W = W(\cdot,\cdot;z) \in C^{1,4}(\mathcal{X}) \cap C_p(\bar{\mathcal{X}})$, and there exists $\bar{C}_0 < \infty$ and $q \in I\!N$ such that $|W_x(s,x)| \le \bar{C}_0(1+|x|^{2q})$ and $|W_{xx}(s,x)| \le \bar{C}_0(1+|x|^{2q})$ for all $(s,x) \in \bar{\mathcal{X}}$. Assume $U = \mathbb{R}^k$; $f, L \in C^3(\mathbb{R}^n \times U)$; $\exists \bar{C}_1 < \infty$ such that $|f_x(x,v)|, |f_v(x,v)| \le \bar{C}_1$, $|f_{xx}(x,v)|, |f_{xv}(x,v)|, |f_{vv}(x,v)| \le \bar{C}_1$ and $|L_{xx}(x,v)|, |L_{xv}(x,v)|, |L_{vv}(x,v)|, \le \bar{C}_1$. $(\mathcal{A}.1)$

Theorem 2: Assume $(\mathcal{A}.1)$. Further, suppose that for each $z \in \mathbb{R}^n$, there exists $\bar{u} \in C(\bar{\mathcal{X}})$ such that $f(x, \bar{u}(t,x))$ is globally Lipschitz on $\mathcal{X}$ and such that $\bar{u}(t,x) \in \operatorname{argstat}_{v\in U}\{f(x,v)^T W_x(t,x) + L(x,v)\}$ for all $(t,x) \in \mathcal{X}$. Then $W = \bar{W}$ on $\bar{\mathcal{Y}}$, and $\bar{u}$ yields payoff $\bar{W}$.

The proof of Theorem 2 is quite similar to that of [10, Th. 4.1], but with a real-valued (rather than complex-valued) system and less specific dynamics and cost. For the benefit of the reader, a partial proof appears in the appendix.

All results to follow are obtained under $(\mathcal{A}.1)$.

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V. Conversion to a Fundamental-Solution Form

We now proceed through several steps that will lead to a fundamental solution form, and then further, to a deterministic-control, fundamental solution form. We remark that the term "fundamental solution form" is being employed here to indicate that modifications of the terminal cost, within a certain class, will not require re-solution of the problem. For $x, p, \alpha, \beta \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$, let

$$\mathcal{Q}(x,p,\alpha,\beta) \doteq \tfrac{c_1}{2}|x-\alpha|^2 + \tfrac{c_2}{2}|p-\beta|^2.$$

We assume that $H_0 \in C^3(\mathbb{R}^{2n})$, that the first, second and third derivatives of H_0 are uniformly bounded, and that H_0 is uniformly Morse in $(x,p) \in \mathbb{R}^{2n}$. $(\mathcal{A}.2)$

Using Assumption $(\mathcal{A}.2)$ and [9] Th. 4, one obtains:

Lemma 3: For $|c_1|, |c_2|$ sufficiently large,

$$\begin{aligned} H_0(x,p) &= \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \big[G_0(\alpha,\beta) + \mathcal{Q}(x,p,\alpha,\beta) \big], \\ G_0(\alpha,\beta) &= \operatorname*{stat}_{(x,p)\in\mathbb{R}^{2n}} \big[H_0(x,p) - \mathcal{Q}(x,p,\alpha,\beta) \big]. \end{aligned}$$

Lemma 4: Let $|c_1|, |c_2|$ be sufficiently large. Then, for each $z \in \mathbb{R}^n$, the value function given by (1)–(6) is the unique, classical solution of

$$\begin{aligned} 0 = W_t &+ \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \left\{ G_0(\alpha,\beta) + \mathcal{Q}(x, W_x, \alpha, \beta) \right\} \\ &+ \mathcal{Q}_0(x, W_x) + \tfrac{1}{2} \operatorname{tr}[A W_{xx}], \quad (t,x) \in \mathcal{X}, \qquad (11) \\ W(T,x;z) &= \psi(x;z), \quad x \in \mathbb{R}^n. \qquad (12) \end{aligned}$$

Now, we let $\mathcal{Q}_0$ take the form

$$\begin{aligned} \mathcal{Q}_0(x,p) = &\tfrac{1}{2}\big[x^T D_{1,1} x + 2 x^T D_{1,2} p + p^T D_{2,2} p \big] \\ &+ d_1^T x + d_2^T p \end{aligned}$$

where $D_{1,1}, D_{2,2}$ are symmetric. Note that for $|c_2|$ sufficiently large, with $\Gamma \doteq -(c_2 \mathcal{I} + D_{2,2})^{-1}$,

$$\begin{aligned} &G_0(\alpha,\beta) + \mathcal{Q}_0(x,p) + \mathcal{Q}(x,p,\alpha,\beta) \qquad (13) \\ &= \operatorname*{stat}_{v\in\mathbb{R}^n} \left\{ [D_{1,2} x + d_2 - c_2 \beta + v]^T p + \mathcal{Q}_1(x,\alpha,\beta,v) \right\}, \\ &\mathcal{Q}_1(x,\alpha,\beta,v) \doteq G_0(\alpha,\beta) + \tfrac{1}{2} x^T D_{1,1} x + \tfrac{c_1}{2}|x-\alpha|^2 \\ &\qquad + \tfrac{c_2}{2}|\beta|^2 + d_1^T x + \tfrac{1}{2} v^T \Gamma v. \end{aligned}$$

VI. Underlying Concepts

The discussion to follow in this section provides the conceptual material related to the main result.

A. Iterated Staticization

Consider the following stationarity control problem.

$$d\xi_t = (D_{1,2}\xi_t + d_2 - c_2 \bar{\beta}_t + u_t)\,dt + \mu\, dB_t, \quad \xi_s = x.$$

where $u_\cdot, \bar{\beta}_\cdot \in \mathcal{U}_{s,T}$. Let the payoff and stationary value be

$$\begin{aligned} J^f(s,x,u,\bar{\alpha},\bar{\beta};z) &\doteq \mathbb{E}\Big\{ \int_s^T \mathcal{Q}_1(\xi_t, \bar{\alpha}^*_{T-t}, \bar{\beta}_t, u_t)\, dt \\ &\qquad + \psi(\xi_T; z) \Big\}, \qquad (14) \\ \hat{W}^f(s,x;z) &\doteq \operatorname*{stat}_{\bar{\alpha},\bar{\beta}\in[\mathcal{O}_{s,T}]^2} \ \operatorname*{stat}_{u_\cdot\in\mathcal{U}_{s,T}} J^f(s,x,u,\bar{\alpha},\bar{\beta};z), \quad (15) \end{aligned}$$

$\mathcal{O}_{s,T} \doteq \{\nu : [s,T] \times \Omega \to \mathbb{R}^n \,|\, \nu \text{ is } \mathcal{F}_t\text{-adapted},$

right-contin. and s.t. $\mathbb{E}\int_s^T |\nu_t|^2\,dt < \infty \}$, (16)

Lemma 5: Let $|c_1|, |c_2|$ be sufficiently large. Then, for each $z \in \mathbb{R}^n$, the value function $\bar{W}$ given by (6) is identical to the value function, $\hat{W}^f$, given by (15).

Proof: Note that J^f has a semi-quadratic form. The result the follows from [8], [17]. ■

Note that the inner staticization of (15), $\operatorname{stat}_{u.\in\mathcal{U}_{s,T}} J^f(s,x,u,\bar{\alpha},\bar{\beta};z)$ is a set of linear-quadratic Gaussian control problems, indexed by the $\bar{\alpha},\bar{\beta}$, and that motivates the following.

B. Relevant Differential Riccati Equations

Consider the dynamics, driven by stochastic processes $\bar{\alpha}_t, \bar{\beta}_t$, given by

$$\dot{\Pi}_t = -\bar{F}_1(\Pi_t) \doteq -\big\{\Pi_t K_1 \Pi_t + K_2^T \Pi_t + \Pi_t K_2 + K_3\big\}, \tag{17}$$

$$\begin{aligned}
\dot{\pi}_t &= -\bar{F}_2(\Pi_t,\pi_t,\bar{\alpha}_t,\bar{\beta}_t) \\
&\doteq -\big\{\Pi_t K_1 \pi_t + \Pi_t K_6 V_t^2 + K_2 \pi_t + V_t^1\big\}, && (18)\\
\dot{\gamma}_t &= -\bar{F}_3(\Pi_t,\pi_t,\bar{\alpha}_t,\bar{\beta}_t) \doteq -\big\{G_0(\bar{\alpha}_t,\bar{\beta}_t) + \tfrac{c_1}{2}|\bar{\alpha}_t|^2 && (19)\\
&\quad + \tfrac{c_2}{2}|\bar{\beta}_t|^2 + \tfrac{1}{2}\pi_t^T K_1 \pi_t + (V_t^2)^T \pi_t + \tfrac{1}{2}\operatorname{tr}(K_4 \Pi_t K_5)\big\},
\end{aligned}$$

$$\Pi_T = \bar{\Pi} \doteq \begin{pmatrix} \bar{M} & -\bar{M} \\ -\bar{M} & \bar{M} \end{pmatrix}, \quad \pi_T = \bar{\pi} \doteq 0, \quad \gamma_T = \bar{\gamma}, \tag{20}$$

$$\begin{aligned}
&K_1 \doteq \begin{pmatrix} (c_2\mathcal{I} + D_{2,2}) & 0 \\ 0 & 0 \end{pmatrix}, \quad K_2 \doteq \begin{pmatrix} D_{1,2}^T & 0 \\ 0 & 0 \end{pmatrix},\\
&K_3 \doteq \begin{pmatrix} (c_1\mathcal{I} + D_{1,1}) & 0 \\ 0 & 0 \end{pmatrix}, \quad K_4 \doteq \begin{pmatrix} A^T \\ 0 \end{pmatrix}^T,\\
&K_5 \doteq \begin{pmatrix} \mathcal{I} \\ 0 \end{pmatrix}, \quad K_6 \doteq \begin{pmatrix} \mathcal{I} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_t^1 \doteq \begin{pmatrix} d_1 - c_1\bar{\alpha}_t \\ 0 \end{pmatrix},\\
&\text{and} \quad V_t^2 \doteq \begin{pmatrix} d_2 - c_2\bar{\beta}_t \\ 0 \end{pmatrix}.
\end{aligned} \tag{21}$$

Note that $\bar{F}_1$ is independent of $\bar{\alpha}.,\bar{\beta}.$. Also note that the dynamics of (17)–(19), although stochastic, are not driven by a Brownian motion; the stochasticity arises only through the presence of $\bar{\alpha},\bar{\beta}$. Let a reverse-time state process be given by $(\hat{\Pi}_t,\hat{\pi}_t,\hat{\gamma}_t) \doteq (\Pi_{T-t},\pi_{T-t},\gamma_{T-t})$ for $t \in [0, T-s]$. The reverse-time dynamics and initial condition are given by

$$\begin{aligned}
&\dot{\hat{\Pi}}_t = \bar{F}_1(\hat{\Pi}_t), \quad \dot{\hat{\pi}}_t = \bar{F}_2(\hat{\Pi}_t,\hat{\pi}_t,\hat{\alpha}_t,\hat{\beta}_t),\\
&\dot{\hat{\gamma}}_t = \bar{F}_3(\hat{\Pi}_t,\hat{\pi}_t,\hat{\alpha}_t,\hat{\beta}_t), \hat{\Pi}_0 = \bar{\Pi}, \;\; \hat{\pi}_0 = \bar{\pi} \doteq 0, \;\; \hat{\gamma}_0 = \bar{\gamma},
\end{aligned}$$

where $\hat{\alpha}_t \doteq \bar{\alpha}_{T-t}$, $\hat{\beta}_t \doteq \bar{\beta}_{T-t}$. Consider the payoff given by

$$\begin{aligned}
\bar{\hat{J}}(\tau,\bar{\Pi},\bar{\pi},\bar{\gamma},\hat{\alpha},\hat{\beta};x,z) &= \tfrac{1}{2}\begin{pmatrix} x \\ z \end{pmatrix}^T \hat{\Pi}_\tau \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} x \\ z \end{pmatrix}^T \mathbb{E}\{\hat{\pi}_\tau\}\\
&\quad + \mathbb{E}\{\hat{\gamma}_\tau\},\\
\bar{\hat{W}}(\tau,\bar{\Pi},\bar{\pi},\bar{\gamma};x,z) &\doteq \operatorname*{stat}_{(\hat{\alpha},\hat{\beta})\in[\mathcal{O}_{0,\tau}]^2} \bar{\hat{J}}(\tau,\bar{\Pi},\bar{\pi},\bar{\gamma},\hat{\alpha},\hat{\beta};x,z).
\end{aligned}$$

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Note that this is a problem formulation where the state has been changed from ξ to $\hat{\Pi},\hat{\pi},\hat{\gamma}$. Aside from the inputs $\tilde{\alpha},\tilde{\beta}$, there is no stochastic input in this problem formulation. This allows one to completely remove the stochastic element from the problem. In that vein, note the following general result.

Theorem 6: Let $-\infty < s < T < \infty$ and $\mathcal{N}_{s,T} \doteq L_2((s,T);\mathbb{R}^n)$. Let $\mathcal{O}_{s,T}$ be as in (16), and supply $\mathcal{O}_{s,T}$ with the inner product $\langle \nu^1,\nu^2\rangle \doteq \mathbb{E}\int_s^T (\nu_t^1)^T \nu_t^2\,dt$. Let $H \in C^2(\mathcal{N}_{s,T};\mathbb{R})$ with uniformly bounded second derivative. Define $\mathcal{H} : \mathcal{O}_{s,T} \to \mathbb{R}$ by $\mathcal{H}(\nu) \doteq \mathbb{E}\{H(\nu)\}$. Then $\operatorname{stat}_{\nu\in\mathcal{O}_{s,T}} \mathcal{H}(\nu)$ exists if and only if $\operatorname{stat}_{\eta\in\mathcal{N}_{s,T}} H(\eta)$ exists, and further, $\operatorname{stat}_{\nu\in\mathcal{O}_{s,T}} \mathcal{H}(\nu) = \operatorname{stat}_{\eta\in\mathcal{N}_{s,T}} H(\eta)$.

VII. The First-Order HJ PDE

We now proceed to the equivalent first-order HJ PDE problem. Let $-\infty < s < T < \infty$. Suppose there exists $\Pi \in C^1\big((s,T);\mathbb{R}^{2n\times 2n}\big) \cap C\big([s,T];\mathbb{R}^{2n\times 2n}\big)$ satisfying

$$\dot{\Pi}_t = -\bar{F}_1(\Pi_t), \quad t \in (s,T), \quad \Pi_s = \bar{\Pi}, \tag{22}$$

where we recall that $\bar{\Pi}$ is given in (20). Let $\mathcal{N}_s \doteq L_2\big((s,T);\mathbb{R}^n\big)$, and let $\tilde{\alpha},\tilde{\beta} \in \mathcal{N}_s$. Let $\pi \in C^1\big((s,T);\mathbb{R}^{2n}\big) \cap C\big([s,T];\mathbb{R}^{2n}\big)$ satisfy (with $\pi_s = \bar{\pi} \doteq 0$)

$$\begin{aligned}
&\dot{\pi}_t = \bar{F}_2(\Pi_t,\pi_t,\tilde{\alpha}_t,\tilde{\beta}_t) = B(t)\pi_t + b(t,\tilde{\alpha}_t,\tilde{\beta}_t), && (23)\\
&B(t) \doteq \Pi_t, K_1 + K_2, \quad b(t,\tilde{\alpha}_t,\tilde{\beta}_t) = \Pi_t K_6 V_t^2 + V_t^1.
\end{aligned}$$

Let the state-transition matrix associated to $B(\cdot)$ be denoted by $\Phi(t,s)$. For $(\tilde{\alpha},\tilde{\beta}) \in \mathcal{N}_s^2$, one has solution $\pi_t = \Phi(t,s)\bar{\pi} + \int_s^t \Phi(t,r) b(r,\tilde{\alpha}_r,\tilde{\beta}_r)\,dr$. Note that there exists $\gamma \in C^1\big((s,T);\mathbb{R}\big) \cap C\big([s,T];\mathbb{R}\big)$ such that

$$\dot{\gamma}_t = \bar{F}_3(\Pi_t,\pi_t,\tilde{\alpha}_t,\tilde{\beta}_t), \quad t \in (s,T), \quad \gamma_s = \bar{\gamma}. \tag{24}$$

For $t \in [s,T]$, let

$$\bar{G}(t;\bar{\Pi},\bar{\gamma},x,z) \doteq \tfrac{1}{2}\begin{pmatrix} x \\ z \end{pmatrix}^T \Pi_t \begin{pmatrix} x \\ z \end{pmatrix} + \bar{\gamma}. \tag{25}$$

Also consider the control problem with payoff and value

$$\begin{aligned}
&\tilde{J}(s,\bar{\pi},\tilde{\alpha},\tilde{\beta};\bar{\Pi},x,z,T) \doteq \begin{pmatrix} x \\ z \end{pmatrix}^T \pi_T + \int_s^T \bar{F}_3(\Pi_t,\pi_t,\tilde{\alpha}_t,\tilde{\beta}_t)\,dt\\
&\tilde{W}(s,\bar{\pi};\bar{\Pi},x,z,T) \doteq \operatorname*{stat}_{(\tilde{\alpha},\tilde{\beta})\in[\mathcal{N}_s]^2} \tilde{J}(s,\bar{\pi},\tilde{\alpha},\tilde{\beta};\bar{\Pi},x,z,T) && (26)
\end{aligned}$$

where we recall that $\bar{\pi} = 0$ here. The HJ PDE problem associated to value $\tilde{W}$ is given by

$$\begin{aligned}
0 = -W_t - \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \big\{&W_\pi \cdot \bar{F}_2(\Pi_t,\pi,\alpha,\beta)\\
&+ \bar{F}_3(\Pi_t,\pi,\alpha,\beta)\big\}, && (27)\\
W(T,\pi;\bar{\Pi},x,z,T) = (x^T,z^T)\pi,& \quad \pi \in \mathbb{R}^{2n}, && (28)
\end{aligned}$$

where we let $\mathcal{Y} \doteq (s,T) \times \mathbb{R}^{2n}$ and $\bar{\mathcal{Y}} \doteq (s,T] \times \mathbb{R}^{2n}$.

Theorem 7: Fix x,z. Let $|c_1|,|c_2|$ be sufficiently large, and suppose there exists a solution to (22) on $[s,T]$. Suppose $W(\cdot,\cdot;\bar{\Pi},x,z,T) \in C^{1,4}(\mathcal{Y}) \cap C_p(\bar{\mathcal{Y}})$ satisfies (27)–(28), and that $W_{\pi\pi}$ is uniformly bounded. Then, $W(\cdot,\cdot;\bar{\Pi},x,z,T) = \tilde{W}(\cdot,\cdot;\bar{\Pi},x,z,T)$ for all $(t,\pi) \in \bar{\mathcal{Y}}$, and there exist unique feedback controls $\alpha^*(t,\pi),\beta^*(t,\pi)$ satisfying (27) such that there exists a unique solution to (23), and that yield the stationary value.

Proof: (*Sketch only.*) It is sufficient to demonstrate that W satisfies the conditions of $(\mathcal{A}.1)$ with $\bar{F}_2$ replacing f, $\bar{F}_3$ replacing L, π replacing x and $(\tilde{\alpha},\tilde{\beta})$ replacing v in the assumption. With a little work, one finds that $\bar{F}_2, \bar{F}_3 \in C^3(\mathbb{R}^{2n\times 2n} \times \mathbb{R}^n \times \mathbb{R}^{2n})$. $W(\cdot,\cdot,\bar{\Pi},x,z) \in C^{1,4}(\mathcal{Y}) \cap C_p(\bar{\mathcal{Y}})$, and that P. and the second derivatives of G_0 are bounded. By those bounds, one easily sees that $|[\bar{F}_2]_\pi|, |[\bar{F}_2]_{(\alpha,\beta)}|, |[\bar{F}_3]_{\pi\pi}|, |[\bar{F}_3]_{\pi(\alpha,\beta)}|$ and $|[\bar{F}_3]_{(\alpha,\beta)(\alpha,\beta)}|$ are bounded, while we note that $|[\bar{F}_2]_{\pi\pi}|, |[\bar{F}_2]_{\pi(\alpha,\beta)}|, |[\bar{F}_2]_{(\alpha,\beta)(\alpha,\beta)}| = 0$.

It remains to verify that the feedback controls are globally Lipschitz. Differentiating, we see that the feedback-form control functions, $\alpha^*(t,\pi), \beta^*(t,\pi)$ achieving the argstat in (27) must satisfy

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \hat{C}^{-1}[G_0]_{\alpha,\beta}(\alpha,\beta) = \hat{\mathcal{I}}\pi + M_t W_\pi, \tag{29}$$
$$\hat{C} \doteq \begin{pmatrix} c_1\mathcal{I}_n & 0\\ 0 & c_2\mathcal{I}_n\end{pmatrix}, \quad [G_0]_{\alpha,\beta}(\alpha,\beta) \doteq \begin{pmatrix}[G_0]_\alpha(\alpha,\beta)\\ [G_0]_\beta(\alpha,\beta)\end{pmatrix},$$
$$\hat{\mathcal{I}} \doteq \begin{pmatrix}0 & 0\\ \mathcal{I}_n & 0\end{pmatrix}, \quad M_t \doteq \begin{pmatrix}\mathcal{I}_n & 0\\ P_t & Q_t\end{pmatrix}.$$

where we see that for $|c_1|, |c_2|$ sufficiently large, the right-hand sides are contractions, and hence these define unique feedback controls, $\alpha^*(t,\pi), \beta^*(t,\pi)$. With this in hand, one may show that the remaining conditions of $(\mathcal{A}.1)$ are satisfied. ■

Although we often find that $\tilde{W}$ quite closely matches $\bar{W}$ of (6), a correction term due to the second derivative of $\tilde{W}$ with respect to x is required for equality. It will be helpful to consider the following HJ PDE problems. For any $(\tilde{\alpha},\tilde{\beta}) \in \mathcal{N}_s^2$, the $k=0$ problem is given by

$$0 = J_t^0 + \left[J_\pi^0 \cdot \bar{F}_2(\Pi_t,\pi,\tilde{\alpha}_t,\tilde{\beta}_t) + \bar{F}_3(\Pi_t,\pi,\tilde{\alpha}_t,\tilde{\beta}_t)\right] \tag{30}$$
$$J^0(T,\pi;\tilde{\alpha},\tilde{\beta},\bar{\Pi},x,z,T) = \begin{pmatrix}x\\ z\end{pmatrix}^T \pi, \quad \pi \in \mathbb{R}^{2n}, \tag{31}$$

and for $k \in]1,n[$, the HJ PDE problems are given by

$$0 = -J_t^k - J_\pi^k \cdot \bar{F}_2(\Pi_t,\pi,\tilde{\alpha}_t,\tilde{\beta}_t), \tag{32}$$
$$J^k(T,\pi;\tilde{\alpha},\tilde{\beta},\bar{\Pi},x,z,T) = \pi_k, \quad \pi \in \mathbb{R}^{2n}, \tag{33}$$

and the $k = n+1$ HJ PDE problem is given by

$$\begin{aligned}0 = -J_t^{n+1} &- \big\{J_\pi^{n+1} \cdot \bar{F}_2(\Pi_t,\pi,\tilde{\alpha}_t,\tilde{\beta}_t)\\ &+ \tfrac{1}{2}\sum_{j,k=1}^{n}\sum_{\ell,m=1}^{2n} A_{j,k} J_{\pi_\ell}^k \left[M_t^T\hat{C}[\hat{C}+G_0''(\tilde{\alpha}_t,\tilde{\beta}_t)]^{-1}\hat{C}M_t\right]_{\ell,m}\\ &\cdot J_{\pi_m}^j\big\},\end{aligned} \tag{34}$$
$$J^{n+1}(T,\pi;\tilde{\alpha},\tilde{\beta},\bar{\Pi},x,z,T) = 0, \quad \pi \in \mathbb{R}^{2n}. \tag{35}$$

Note that the unique solution of (30)–(31) is payoff $\tilde{J}$ of (26). One obtains the HJ PDE problems for $k \in]1,n[$ by formal differentiation of problem (30)–(31) by x_k. The HJ PDE problem corresponding to $k = n+1$ may be obtained by formally differentiating again by x, and taking the appropriate linear combination. For $k \in]0,n+1[$, the J'^k solutions are similarly related, with $\tilde{J}'$ replacing $\tilde{J}$.

Lemma 8: Given $\tilde{\alpha},\tilde{\beta} \in C^1((s,t);\mathbb{R}^{2n})$, there exist unique solutions for HJ PDE problems (30)–(35). These solutions may be obtained by the method of characteristics.

We develop some of the characteristic equations for various HJ PDE problems above. Consider the characteristics associated to (30)–(31), for $(\tilde{\alpha},\tilde{\beta}) \in C^1((s,T);\mathbb{R}^{2n})$. We let q^0, p^0, π^0 correspond to J_t^0, J_π^0, π. One finds

$$\pi^0(t) = \Phi(t,T)\pi - \int_t^T \Phi(t,r)b(r,\tilde{\alpha}_r,\tilde{\beta}_r)\,dr,$$
$$p^0(t) = \Phi^-(t,T)(x^T,z^T)^T - \int_t^T \Phi^-(t,r)b(r,\tilde{\alpha}_r,\tilde{\beta}_r)\,dr,$$

One may also find q^0 and J^0, but that is not needed below.

Next, consider the (32)–(33) HJ PDE problems for $k \in]1,n[$. One finds $\pi_\cdot^k = \pi_\cdot^0$ for all k. Further,

$$p^k(t) = \Phi^-(T,t)\tilde{e}^k \quad \forall t \in [s,T], \quad k \in]1,n[, \tag{36}$$

where $\tilde{e}_j^k = \delta_{k,j}$ and $\delta_{k,j}$ denotes the Kronecker δ function.

Lastly, we turn to the $k = n+1$ case. As before, we find $\pi^{n+1}(t) = \pi^0(t)$ for all $t \in [s,T]$. Here, again $\dot{p}^{n+1} = -B^T(t)p^{n+1}(t)$, but now with $p^{n+1}(T) = 0$, and hence $p^{n+1}(t) = 0$ for all $t \in [s,T]$. Using the characteristic representation for J^k and (36) and Lemma 8, we find

$$J^{n+1}(t,\pi^0(t);\tilde{\alpha},\tilde{\beta}) = J^{n+1}(t;\tilde{\alpha},\tilde{\beta}) = \int_t^T G_1(r,\tilde{\alpha}_r,\tilde{\beta}_r)\,dr$$
$$\begin{aligned}G_1(t,\tilde{\alpha}_t,\tilde{\beta}_t) = \tfrac{1}{2}\sum_{j,k=1}^{n}\sum_{\ell,m=1}^{2n} &A_{j,k}[\Phi^-(T,t)\tilde{e}^k]_\ell \big[M_t^T\hat{C}\\ &\cdot[\hat{C}+G_0''(\tilde{\alpha}_t,\tilde{\beta}_t)]^{-1}\hat{C}M_t\big]_{\ell,m}[\Phi^-(T,t)\tilde{e}^j]_m.\end{aligned}$$

We now move to the relation between J^{n+1} and the correction term needed in the adjustment of $\tilde{W}$.

Lemma 9: Fix $\bar{\Pi},x,z,T$. Suppose $\tilde{W}(t,\pi;\bar{\Pi},x,z,T)$ is twice differentiable in x for all $(t,\pi,x) \in \bar{\mathcal{Y}} \times \mathbb{R}^n$. Let $(\tilde{\alpha},\tilde{\beta}) = (\tilde{\alpha}^*,\tilde{\beta}^*)$, and for each $k \in]1,n[$, let $J^k(\cdot,\cdot;\tilde{\alpha}^*,\tilde{\beta}^*,\bar{\Pi},x,z,T)$ be the corresponding solution of HJ PDE problem (32)–(33). Similarly, let $J^{n+1}(\cdot,\cdot;\tilde{\alpha}^*,\tilde{\beta}^*)$ be the solution of HJ PDE problem (34)–(35), again with $(\tilde{\alpha},\tilde{\beta}) = (\tilde{\alpha}^*,\tilde{\beta}^*)$. Then,

$$J^k(t,\pi;\tilde{\alpha}^*,\tilde{\beta}^*,\bar{\Pi},x,z,T) = \tilde{W}_{x_k}(t,\pi;\bar{\Pi},x,z,T)$$
$$J^{n+1}(t;\tilde{\alpha}^*,\tilde{\beta}^*) = \tfrac{1}{2}\operatorname{tr}[A\tilde{W}_{xx}(t,\pi;\bar{\Pi},x,z,T)].$$

For $(s,\pi) \in (-\infty,T] \times \mathbb{R}^{2n}$ let

$$\begin{aligned}W^f(s,\pi;\bar{\Pi},\bar{\gamma},x,z,T) \doteq &\tilde{W}(s,\pi;\bar{\Pi},x,z,T)\\ &+ \bar{G}(T-s;\bar{\Pi},\bar{\gamma},x,z).\end{aligned} \tag{37}$$

Theorem 10: Fix $\bar{M},z,\bar{\gamma}$ and $\bar{\pi} = 0$. Let $|c_1|,|c_2|$ be sufficiently large, and suppose there exists a solution to (22) on $[s,T]$. Then,

$$W^f(T,\bar{\pi};\bar{\Pi},\bar{\gamma},x,z,T) = \bar{W}(T,x;z,\bar{M},\bar{\gamma},T),$$
$$W_t^f + \operatorname*{stat}_{v\in U}\left\{f(x,v)^T W_x^f + L(x,v)\right\} + \tfrac{1}{2}\operatorname{tr}[AW_{xx}^f]$$
$$= J^{n+1}(t;\tilde{\alpha}^*_{(t,T)},\tilde{\beta}^*_{(t,T)}), \quad \forall (t,\pi,x) \in \mathcal{Y} \times \mathbb{R}^n.$$

Proof: It is easily seen that at $t = T$,

$$W^f(T,\bar{\pi};\bar{\Pi},\bar{\gamma},x,z,T) = \tfrac{1}{2}\begin{pmatrix}x\\ z\end{pmatrix}^T \bar{\Pi}\begin{pmatrix}x\\ z\end{pmatrix} + \bar{\gamma}$$
$$= \psi(x;z) = \bar{W}(T,x;z,\bar{M},\bar{\gamma},T).$$

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By (9) and Lemma 4, one has

$$
\begin{aligned}
&W_t^f + \operatorname*{stat}_{v\in U}\left\{f(x,v)^T W_x^f + L(x,v)\right\} + \tfrac{1}{2}\operatorname{tr}[AW_{xx}^f] \\
&= W_t^f + \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}}\left\{G_0(\alpha,\beta) + \mathcal{Q}(x,W_x^f,\alpha,\beta)\right\} \\
&\quad + \mathcal{Q}_0(x,W_x^f) + \tfrac{1}{2}\operatorname{tr}[AW_{xx}^f],
\end{aligned}
$$

which after direct substitution and cancellation,

$$
= \tfrac{1}{2}\operatorname{tr}[A\tilde{W}_{xx}] = J^{n+1}(t;\tilde{\alpha}^*_{(t,T)},\tilde{\beta}^*_{(t,T)}),
$$

for all $(t,\pi,x)\in\mathcal{Y}\times\mathbb{R}^n$. ■

Using a standard dynamic programming verification proof, one immediately obtains the following.

Corollary 11: Suppose the conditions of Theorem 10.

$$
\text{Then } \bar{W}(s,x;z,\bar{M},\bar{\gamma},T) = W^f(s,\bar{\pi};\bar{\Pi},\bar{\gamma},x,z,T) \quad (38)
$$

$$
+\tfrac{1}{2}\mathbb{E}\left\{\int_s^T\int_t^T G_1(r;\tilde{\alpha}^*_r,\tilde{\beta}^*_r)\,dr\,dt\right\}.
$$

Remark 12: The correction term in (38) may be obtained by integration of ODEs through the use of Ito's rule.

VIII. EXAMPLE

We will discuss a simple, scalar-state, nonlinear stochastic control problem to indicate how the approach may be applied in that arena. A very special form is chosen so that we may perform the bulk of the calculations analytically. In particular, the nonlinearities are already in a stat-duality form, thus reducing the computations. We consider the second-order HJ PDE problem given by

$$
\begin{aligned}
&0 = W_t - \tfrac{1}{2}W_x^2 + \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^2}\left\{\tfrac{c_1}{2}(\alpha-x)^2 + c_3\arctan(\alpha)\right. \\
&\qquad \left. + \tfrac{c_2}{2}(\beta - W_x)^2 + c_4\arctan(\beta)\right\} + \tfrac{\mu^2}{2}W_{xx}, \\
&W(T,x) = \tfrac{1}{2}x^2,
\end{aligned}
$$

where $c_1 = 5/4$, $c_2 = 1/2$, $c_3 = 1$, $c_4 = -1/4$ and $\mu^2 = 2/3$. We also have $s = 0$ and $T = 1/4$. As $z = 0$, one may take $q = 0$ (i.e., $\pi = (\rho,0)^T$) in the deterministic version. Standard finite-element methods are used to compute the solutions. Both the solution to the original, second-order HJ PDE problem and the solution to the equivalent first-order HJ PDE problem were computed over the combined original and dual space, $(x,\rho)\in\mathbb{R}^2$. Standard finite-elements methods were used [3], [4]. In figure 1, the solutions at initial time, $s = 0$ (propagated from $T = 1/4$., computed by both methods, are plotted, where one can see the extreme closeness of the solutions. In figure 2, the difference between the solutions at initial time, $s = 0$ is depicted. In figure 3, the change in the solution from time T to time s is depicted as a reference.

APPENDIX

A. Proof Sketch for Theorem 2

We consider not only generic dynamics (1), but also the stationary dynamics given by

$$
d\bar{\xi}^*_t = f(\bar{\xi}^*_t,\bar{u}^*_t)\,dt + \mu\,dB_t, \quad \bar{\xi}^*_s = x\in\mathbb{R}^n, \quad (39)
$$

$$
\bar{u}(t,\bar{\xi}^*_t)\in\operatorname*{argstat}_{v\in U}\{f(\bar{\xi}^*_t,v)^T W_x(t,\bar{\xi}^*_t) + L(\bar{\xi}^*_t,v)\}. \quad (40)
$$

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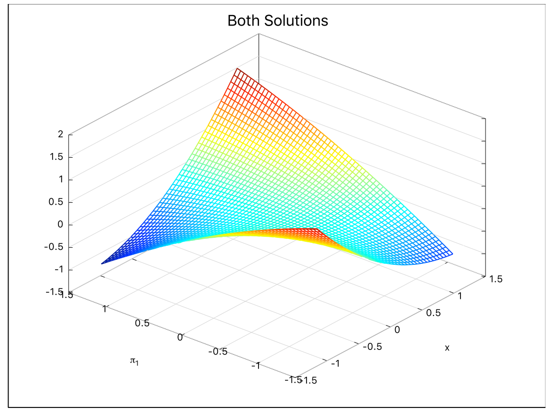

Fig. 1. Both solutions at the initial time

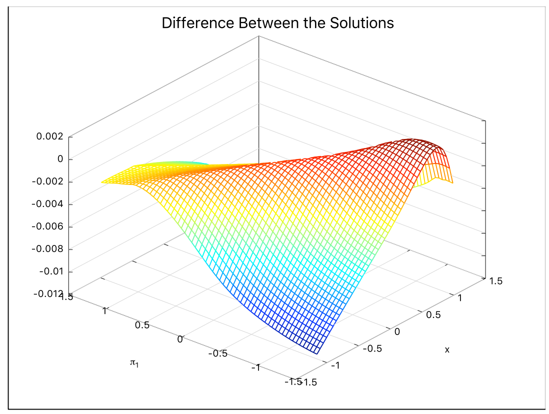

Fig. 2. The difference between the solutions at the initial time

Lemma 13: There exists a unique strong solution of (39)–(40).

Lemma 14: Suppose W satisfies (9)–(10). Let $u\in\mathcal{U}_s$. Let ξ satisfy (1). Let $\bar{\xi}^*,\bar{u}^*$ be given by (39)–(40). Then,

$$
\begin{aligned}
W(s,x) = \mathbb{E}\Big\{\int_s^T &-\big[W_t(t,\bar{\xi}^*_t) + f(\bar{\xi}^*_t,\bar{u}^*_t)^T W_x(t,\bar{\xi}^*_t) \\
&+ \tfrac{1}{2}\operatorname{tr}[AW_{xx}(t,\bar{\xi}^*_t)]\big]\,dt + \psi(\bar{\xi}^*_T,z)\Big\}.
\end{aligned}
$$

Lemma 15: Suppose W satisfies (9)–(10). Let $\bar{u}^*$ be given by (39)–(40). Then, $W(s,x) = J(s,x,\bar{u}^*;z)$.

Lemma 16: Let $\bar{u}^*$ be given by (39)–(40). Then, $\bar{u}^*\in\operatorname{argstat}_{u\in\mathcal{U}_s} J(s,x,u;z)$.

Proof: Let $(s,x)\in\mathcal{X}$. Let $u\in\mathcal{U}_s$, with corresponding trajectory, ξ given by (1), and let $\bar{\xi}^*,\bar{u}^*$ be given by (39)–(40). By (4),

$$
\begin{aligned}
J(s,x,\bar{u}^*;z) - J(s,x,u;z) = \mathbb{E}\Big\{\int_0^T &L(\bar{\xi}^*_t,\bar{u}^*_t) \\
&- L(\xi_t,u_t)\,dt + \psi(\bar{\xi}^*_T,z) - \psi(\xi_T,z)\Big\}. \quad (41)
\end{aligned}
$$

By Lemma 14,

$$
\begin{aligned}
&\mathbb{E}\{\psi(\bar{\xi}^*_T,z) - \psi(\xi_T,z)\} = \mathbb{E}\Big\{\int_s^T\big[W_t(t,\bar{\xi}^*_t) \\
&+ W_x^T(t,\bar{\xi}^*_t)f(\bar{\xi}^*_t,\bar{u}^*_t) + \tfrac{1}{2}\operatorname{tr}[AW_{xx}(t,\bar{\xi}^*_t)]\big] \quad (42) \\
&-\big[W_t(t,\xi_t) + W_x^T(t,\xi_t)f(\xi_t,u_t) + \tfrac{1}{2}\operatorname{tr}[AW_{xx}(t,\xi_t)]\big]\,dt\Big\}.
\end{aligned}
$$

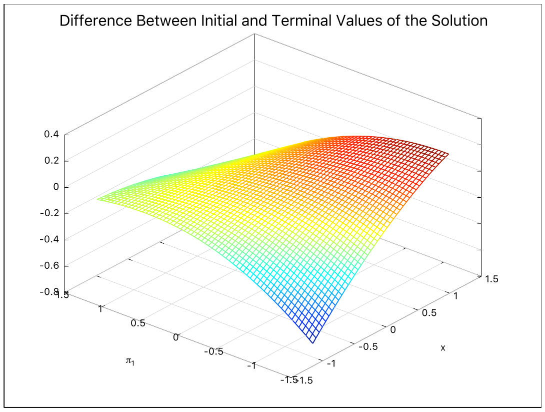

Fig. 3. Change in the solution from the terminal to the initial time

Substituting (42) into (41) yields

$$J(s,x,\bar{u}^*;z) - J(s,x,u;z) = \mathbb{E}\Big\{ \int_0^T \big[L(\bar{\xi}_t^*, \bar{u}_t^*) + W_t(t,\bar{\xi}_t^*)$$
$$+ W_x^T(t,\bar{\xi}_t^*) f(\bar{\xi}_t^*, \bar{u}_t^*) + \tfrac{1}{2}\operatorname{tr}[A W_{xx}(t,\bar{\xi}_t^*)]$$
$$- \big[L(\xi_t,u_t) + W_t(t,\xi_t) + W_x^T(t,\xi_t) f(\xi_t,u_t)$$
$$+ \tfrac{1}{2}\operatorname{tr}[A W_{xx}(t,\xi_t)]\big]\,dt\Big\}$$
$$= \mathbb{E}\Big\{ \int_0^T \big[\hat{H}^0(t,\bar{\xi}_t^*,\bar{u}_t^*) - \hat{H}^0(t,\xi_t,\bar{u}(t,\xi_t))\big]$$
$$+ \big[\hat{H}^0(t,\xi_t,\bar{u}(t,\xi_t)) - \hat{H}^0(t,\xi_t,u_t)\big]\,dt\Big\},$$

which by the choice of $\bar{u}$ and $\bar{u}_t^* = \bar{u}(t,\bar{\xi}_t^*)$ and (9),

$$= \mathbb{E}\Big\{ \int_0^T \big[\hat{H}^0(t,\xi_t,\bar{u}(t,\xi_t)) - \hat{H}^0(t,\xi_t,u_t)\big]\,dt\Big\}. \qquad (43)$$

By using Taylor polynomials, the mean value theorem and a measurable selection theorem [5], there exist progressively measurable $\lambda_\cdot(\cdot), \nu_\cdot(\cdot)$ with $\lambda_t(\omega) \in [0,1]$ for all $(t,\omega) \in [0,T]\times\Omega$ and $\nu_t \doteq \lambda_t \bar{u}(t,\xi_t) + (1-\lambda_t)u_t$, such that

$$|\hat{H}^0(t,\xi_t,\bar{u}(t,\xi_t)) - \hat{H}^0(t,\xi_t,u_t)|$$
$$\le |\hat{H}^0_v(t,\xi_t,\bar{u}(t,\xi_t))|\,|u_t - \bar{u}(t,\xi_t))|$$
$$+ \tfrac{1}{2}|\hat{H}^0_{vv}(t,\xi_t,\nu_t)|\,|u_t - \bar{u}(t,\xi_t))|^2,$$

which by $\bar{u}(t,\xi_t)$ as an element of the argstat,

$$= \tfrac{1}{2}|\hat{H}^0_{vv}(t,\xi_t,\nu_t)|\,|u_t - \bar{u}(t,\xi_t))|^2. \qquad (44)$$

Combining (43), (44) and the assumptions, one finds that there exists $K_1 < \infty$ such that

$$|J(s,x,\bar{u}^*;z) - J(s,x,u;z)|$$
$$\le \frac{K_1}{2}\mathbb{E}\Big\{ \int_0^T (1 + |\bar{\xi}_t^*|^{2q} + \Delta_t^2)\,|u_t - \bar{u}(t,\xi_t))|^2\,dt\Big\},$$

where $\Delta_t \doteq \xi_t - \bar{\xi}_t^*$ and $\delta_t \doteq u_t - \bar{u}_t^*$, and this is

$$\le \frac{K_1}{2}\mathbb{E}\Big\{ \int_0^T (1 + |\bar{\xi}_t^*|^{2q} + \Delta_t^2)[|u_t - \bar{u}_t^*|^2$$
$$+ |\bar{u}_t^* - \bar{u}(t,\xi_t))|^2]\,dt\Big\},$$

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and again using the mean value theorem and a measurable selection theorem, with appropriate $\hat{\nu} \in \mathcal{X}_s$, this is

$$\le K_1 \mathbb{E}\Big\{ \int_0^T (1 + |\bar{\xi}_t^*|^{2q} + \Delta_t^2)[\delta_t^2 + |\bar{u}_x(t,\hat{\nu}_t)|^2 \Delta_t^2]\,dt\Big\}. \qquad (45)$$

Now, using the stat definition of $\bar{u}$ and implicit differentiation, one has $\hat{H}^0_{xv}(t,\hat{\nu}_t,\bar{u}(t,\hat{\nu}_t)) + \hat{H}^0_{vv}(t,\hat{\nu}_t,\bar{u}(t,\hat{\nu}_t))\bar{u}_x(t,\hat{\nu}_t) = 0$. Employing Assumptions $(\mathcal{A}.1)$ and $(\mathcal{A}.2)$, we find that there exists $K_2 < \infty$ such that $|\bar{u}_x(t,\hat{\nu}_t)|^2 \le K_2(1 + |\bar{\xi}_t^*|^{4q} + \Delta_t^{4q})$. Applying this in (45) yields

$$|J(s,x,\bar{u}^*;z) - J(s,x,u;z)| \le \frac{K_1}{2}\mathbb{E}\Big\{ \int_0^T (1 + |\bar{\xi}_t^*|^{2q} + \Delta_t^2)$$
$$\big[\delta_t^2 + K_2(1 + |\bar{\xi}_t^*|^{4q} + \Delta_t^{4q})\Delta_t^2\big]\,dt\Big\}. \quad (46)$$

The remainder of the proof consists of some Hölder estimates as in the proof of [10, Th. 4.1], where we specifically note

$$\mathbb{E}\int_s^T |\Delta_t|^2\,dt = \mathbb{E}\int_s^T \Big|\int_s^t \delta_r\,dr\Big|^2 dt \quad \le (t-s)\|\delta\|^2_{\mathcal{U}_s}.$$

The details are not included, but the reader may refer to [10]. ∎

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