Strong Solution Existence for a Class of Degenerate Stochastic Differential Equations ⋆

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Abstract: Existence and uniqueness results for stochastic differential equations (SDEs) under exceptionally weak conditions are well known in the case where the diffusion coefficient is nondegenerate. Here, existence and uniqueness of a strong solution is obtained in the case of degenerate SDEs in a class that is motivated by diffusion representations for solution of Schrödinger initial value problems. In such examples, the dimension of the range of the diffusion coefficient is exactly half that of the state. In addition to the degeneracy, two types of discontinuities and singularities in the drift are allowed, where these are motivated by the structure of the Coulomb potential and the resulting solutions to the dequantized Schrödinger equation. The first type consists of discontinuities that may occur on a possibly high-dimensional manifold (up to codimension one). The second consists of singularities that may occur on a lower-dimensional manifold (up to codimension two).

Keywords: Stochastic control, stochastic processes, stochastic differential equations, optimal control, dynamic programming.

1. INTRODUCTION

Existence and uniqueness results for solutions of stochastic differential equations (SDEs) typically have weaker assumptions on the smoothness of the drift than that which is required in the case of the corresponding ordinary differential equations (ODEs), where more specifically, these ODEs correspond to the case where the diffusion terms in the SDEs are removed. The results with the weakest conditions on the drift have been those where the diffusion coefficient is assumed to be nondegenerate, cf. Albeverio, Kondratiev and Röckner (2003); Krylov and Röckner (2005); Veretennikov (1981); Zvonkin (1974).

In some recent efforts on diffusion representations for solutions of Schrödinger initial value problems (IVPs) Azencott and Doss (1985); Doss (2011); McEneaney and Zhao (2018); McEneaney (2018, 2019), the representation dynamics take the form of complex-valued SDEs. In particular, the SDEs are given as

$$d\xi_t = f(\xi_t)\,dt + \tfrac{1+i}{\sqrt{2}}\sigma\,dB_t,$$

where $B_t \in \mathbb{R}^m$, $\sigma \in \mathbb{R}$, $f : \mathbb{C}^m \to \mathbb{C}^m$, i denotes the imaginary unit, and $\mathbb{C}$ denotes the complex field. Breaking out the real and imaginary parts, one obtains an SDE with $2m$–dimensional state and an $2m \times m$ degenerate diffusion coefficient. Hence, we have an iconic application class for which there were previously no solution existence and uniqueness results. This class motivates the effort here, and a particular example appears in Section 3. We remark here that in the case of that specific example from the problem class, the drift is generated by the gradient of a solution to the dequantized form, cf. Litvinov (2007), of the Schrödinger equation. The particular solution of that motivating example is that which corresponds to the lowest energy electron shell. Moreover, for the general problem class, the $2m$–dimensional real-valued formulation allows for two types of nonsmoothness in the drift. The first consists of discontinuities that may occur on a possibly high-dimensional manifold. The second consists of singularities that may occur on a lower-dimensional manifold. In the our problem class, the discontinuities occur on a manifold of codimension one in $\mathbb{R}^{2m}$, while the singularities occurs on a manifold of codimension two in $\mathbb{R}^{2m}$.

We briefly indicate other recent results on existence and uniqueness for degenerate SDEs, so as to situate the result herein. Kumar (2013) considers degenerate SDEs with non-Lipschitz coefficients and states taking values in the positive orthant, where in the particular case where the coefficients are Lipschitz, both existence and uniqueness

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of a strong solution is obtained. Figalli (2007) employs known results for associated partial differential equations (PDEs), including the Fokker–Planck equation, as an aid in developing results on existence and uniqueness for degenerate SDEs. Raynal also employs known results on associated PDEs to obtain pathwise uniqueness for degenerate SDEs with Hölder drift with exponents greater than 2/3.

In Section 2, the class of SDEs for which the results are obtained is indicated. Section 3 describes the motivational application. In Section 4, mollifiers, indexed by $\delta > 0$, are applied to the drift term in the dynamics, and existence and uniqueness are obtained for the mollified system. Lastly, in Section 5, we take $\delta \downarrow 0$, and obtain the asserted strong solution.

2. THE CLASS OF SDES

We consider SDEs on $[0,T]$ of the form

$$d\eta_t = F(\eta_t,\zeta_t)\,dt + dB_t, \tag{1}$$

$$d\zeta_t = G(\eta_t,\zeta_t)\,dt, \tag{2}$$

$$\eta_0 = y^0 \in \mathbb{R}^m, \quad \zeta_0 = z^0 \in \mathbb{R}^m, \quad l \doteq 2m. \tag{3}$$

In order to describe the problem structure and assumptions, we make some additional definitions. Let $h_0 \in C(\mathbb{R}^m;\mathbb{R})$, $h_1 \in C(\mathbb{R}^m;\mathbb{R})$ and

$$\mathcal{H}_0 \doteq \{(y,z) \in \mathbb{R}^l \,|\, h_0(y) - h_1(z) = 0\}. \tag{4}$$

Note that $\mathcal{H}_0$ will be a set along which the drift may have discontinuities. There will also be a set $\mathcal{G}_0 \subset \mathbb{R}^l$ along which the drift may have singularities. For each $z \in \mathbb{R}^m$, we let $\tilde{\mathcal{G}}_0(z) \subset \mathbb{R}^m$, and define $\mathcal{G}_0 \doteq \{(y,z) \in \mathbb{R}^l \,|\, y \in \tilde{\mathcal{G}}_0(z)\}$, where more specific assumptions on $\tilde{\mathcal{G}}_0$ are given further below. For $\delta > 0$, let $\tilde{\mathcal{G}}_\delta(z) \doteq \{y \in \mathbb{R}^m \,|\, d(y,\tilde{\mathcal{G}}_0(z)) \le \delta|z|\}$ and $\mathcal{G}_\delta \doteq \{(y,z) \in \mathbb{R}^l \,|\, y \in \tilde{\mathcal{G}}_\delta(z)\}$. We assume the following.

(A.1) $F,G \in C^1([\mathcal{G}_0 \cup \mathcal{H}_0]^c)$. For each $\delta > 0$, F and G are bounded on $\mathcal{G}_\delta^c$. For each $\delta > 0$, $\nabla_{(y,z)}F$ and $\nabla_{(y,z)}G$ are bounded on $[\mathcal{G}_\delta \cup \mathcal{H}_0]^c$. $(y^0,z^0) \notin \mathcal{G}_0 \cup \mathcal{H}_0$, and $z^0 \neq 0$.

(A.2) Suppose h_0 is such that for any $[a,b] \subseteq [0,T]$ and any measurable $\nu : [0,T] \to \mathbb{R}^m$, such that each component (say, $[\nu_\cdot]_j$ for $j \le m$) has infinite total variation on $[a,b]$, $h_0(\nu_\cdot)$ has infinite total variation on $[a,b]$.

Let $\bar{\mathcal{L}}$ denote the space of nonsingular $m \times m$ matrices, and let $I_{m\times m} \in \bar{\mathcal{L}}$ denote the identity matrix.

(A.3) Let $\mathcal{I} \doteq [0,1]$, and let $p \in C^1(\mathcal{I}^o;\mathbb{R}^m) \cap C(\mathcal{I};\mathbb{R}^m)$. Let $\bar{e} \in \mathbb{R}^m \setminus \{0\}$. Let $J \in C^2(\mathbb{R}^m \setminus \{0\};\bar{\mathcal{L}})$ be given by $J(z) = (1/|z|)\Gamma(z)$ where $\Gamma : \mathbb{R}^m \setminus \{0\} \to \bar{\mathcal{L}}$ is such that $\Gamma(z)$ is orthonormal for all $z \in \mathbb{R}^m \setminus \{0\}$, and such that $J(z)z = \bar{e}$ for all $z \in \mathbb{R}^m \setminus \{0\}$, $[J(\bar{e})]^{-1} = I_{m\times m}$, and $\frac{dJ}{dz}$ is bounded on $\mathbb{R}^m \setminus B_\delta(0)$ for all $\delta > 0$. Finally, suppose $\mathcal{G}_0 = \{(y,z) \in \mathbb{R}^l \,|\, y \in \tilde{\mathcal{G}}_0(z)\}$ where for all $z \in \mathbb{R}^m$, $\tilde{\mathcal{G}}_0(z) \doteq \{y \in \mathbb{R}^m \,|\, \exists \lambda \in \mathcal{I} \text{ s.t. } y = [J(z)]^{-1}p(\lambda)\}$.

Remark 1. The above structure for $\mathcal{G}_0$, which may at first seem unusual, was chosen for the case where the singular set is defined in terms of η_t relative to ζ_t. A motivational example where these assumptions are satisfied is given in Section 3. In that case, $m = 3$, $\mathcal{H}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y|^2 - |z|^2 = 0\}$ and $\mathcal{G}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y|^2 - |z|^2 = 0,$ and $y^Tz = 0\}$. In that case, one may take $\bar{e}$ to be $(1,0,0)^T$ and $p(\cdot)$ to be a parameterization of the unit circle in the plane perpendicular to $\bar{e}$.

Remark 2. The assumptions may be weakened to allow for a finite number of both discontinuity and singularity manifolds, with no fundamental change in the proofs. For clarity of exposition, we do not include the details.

An additional assumption will appear in Section 4, and it will be the final assumption. That assumption is more easily indicated there, after some additional definitions.

3. MOTIVATION

One motivation for consideration of this large class of SDE problems is the staticization based diffusion representation for the solution of Schrödinger initial value problems (IVPs) McEneaney and Dower (2019); McEneaney and Zhao (2018); McEneaney (2018, 2019). The case of the Coulomb potential was discussed in McEneaney and Dower (2019). For $x \in \mathbb{C} \setminus \{0\}$, define the single-valued logarithm and square-root operations

$$\log_q(x) \doteq \log(r) + i\theta, \quad \sqrt{x} \doteq \exp\left[\tfrac{1}{2}\log_q(x)\right],$$

where $r \in (0,\infty)$ and $\theta \in (-\pi,\pi]$ are such that $x = re^{i\theta}$. We specifically look at the Maslov dequantization (cf. Litvinov (2007)) of the solution of a Schrödinger IVP associated to the lowest energy electron shell (cf. Folland (2008)), which may be extended to complex-valued states as $S^0 : [0,\infty) \times \mathbb{C}^3 \to \mathbb{C}$ given by

$$S^0(t,x) = \tfrac{-c_1^2}{2\bar{m}}t + ic_1\sqrt{x^Tx},$$

where $c_1 = \frac{2\bar{m}C}{(n-1)\hbar} = \frac{\bar{m}C}{\hbar}$, $\bar{m}$ denotes mass, $\hbar$ denotes Planck's constant, $C \doteq q_0q_1/(4\pi\bar{\epsilon}_0)$, q_0 denotes the central charge, q_1 denotes the electron charge and $\bar{\epsilon}_0$ denotes the vacuum permittivity. One may check that $S_t^0(r,x) = \frac{-c_1^2}{2\bar{m}}$ and $S_x^0(r,x) = ic_1x/\sqrt{x^Tx}$, $\Delta S^0(r,x) = 2ic_1/\sqrt{x^Tx}$, and further, that S^0 satisfies the dequantized and time-reversed form of the Schrödinger equation, given by

$$0 = S_t(r,x) + \tfrac{i\hbar}{2\bar{m}}\Delta S(r,x) - \tfrac{1}{2\bar{m}}(S_x(r,x))^T S_x(r,x) - V(x),$$

where $V(x) = -C/\sqrt{x^Tx}$.

The dynamics of the diffusion process generating the solution as the associated stationary value function are given by McEneaney and Dower (2019)

$$d\xi_r = (-1/\bar{m})S_x^0(r,\xi_r)\,dr + \sqrt{\hbar/\bar{m}}\frac{1+i}{\sqrt{2}}\,dB_r,$$

with $\xi_0 = x$. One may separate the three-dimensional complex state, ξ_r, into its real and imaginary parts as $\xi_r = \hat{\eta}_r + i\hat{\zeta}_r$. Similarly, letting $S^0(r,x) = R^0(r,\hat{y},\hat{z}) + iT^0(r,\hat{y},\hat{z})$ with $x = \hat{y} + i\hat{z}$, and employing the Cauchy-Riemann equations, the SDE system becomes

$$d\hat{\eta}_r = (-1/\bar{m})R_y^0(r,\hat{\eta}_r,\hat{\zeta}_r)\,dr + \sqrt{\tfrac{\hbar}{2\bar{m}}}\,dB_r, \quad \hat{\eta}_0 = \hat{y},$$

$$d\hat{\zeta}_r = (1/\bar{m}) R_z^0(r, \hat{\eta}_r, \hat{\zeta}_r)\, dr + \sqrt{\tfrac{\hbar}{2\bar{m}}}\, dB_r, \quad \hat{\zeta}_0 = \hat{z}.$$

Performing the change of coordinates $\eta_r = (1/\sqrt{2})[\hat{\eta}_r + \hat{\zeta}_r]$, $\zeta_r = (1/\sqrt{2})[-\hat{\eta}_r + \hat{\zeta}_r]$ yields

$$d\eta_r = (1/\sqrt{2}\bar{m})[-R_y^0 + R_z^0](r, \tfrac{\eta_r - \zeta_r}{2}, \tfrac{\eta_r + \zeta_r}{2})\, dr + \sqrt{\tfrac{\hbar}{\bar{m}}}\, dB_r,$$
$$d\zeta_r = (1/\sqrt{2}\bar{m})[R_y^0 + R_z^0](r, \tfrac{\eta_r - \zeta_r}{2}, \tfrac{\eta_r + \zeta_r}{2})\, dr,$$

with $\eta_0 = y^0 \doteq (1/\sqrt{2})[\hat{y} + \hat{z}]$ and $\zeta_0 = z^0 \doteq (1/\sqrt{2})[-\hat{y} + \hat{z}]$. Using the specific form of S^0 in this example, this reduces to

$$\begin{aligned} d\eta_r &= F(\eta_r, \zeta_r)\, dr + \sigma\, dB_r & (5)\\ &\doteq \frac{c_1}{\bar{m}\sqrt{\tilde{R}_r}}[\sin(\tilde{\theta}_r)\eta_r - \cos(\tilde{\theta}_r)\zeta_r]\, dr + \sqrt{\tfrac{\hbar}{\bar{m}}}\, dB_r, \\ d\zeta_r &= G(\eta_r, \zeta_r)\, dr & (6)\\ &\doteq \frac{-c_1}{\bar{m}\sqrt{\tilde{R}_r}}[\cos(\tilde{\theta}_r)\eta_r + \sin(\tilde{\theta}_r)\zeta_r]\, dr, \end{aligned}$$

where $\tilde{R}_r \doteq \bar{R}(\eta_r, \zeta_r) \doteq \left[(-2\eta_r^T\zeta_r)^2 + (|\eta_r|^2 - |\zeta_r|^2)^2\right]^{1/2}$, $\cos(2\tilde{\theta}_r) = \frac{-2\eta_r^T\zeta_r}{\tilde{R}_r}$ and $\sin(2\tilde{\theta}_r) = \frac{|\eta_r|^2 - |\zeta_r|^2}{\tilde{R}_r}$ with $\tilde{\theta}_r \in (-\pi/2, \pi/2]$.

In this case, $\mathcal{H}_0$ corresponds to the branch cut induced by $\sqrt{x^T x}$, which is at $|\hat{y}|^2 - |\hat{z}|^2 < 0$, $\hat{y}^T\hat{z} = 0$, or equivalently, at $y^T z > 0$, $|y|^2 - |z|^2 = 0$. That is, $\mathcal{H}_0 = \{(y, z) \in \mathbb{R}^l \,|\, |y| = |z|,\ y^T z > 0\}$. In particular, one may take $h_0(y) = |y|$ and $h_1(z) = |z|$. From this, one may easily verify Assumption $(A.2)$. Also, we see that the singularities occur on

$$\begin{aligned} \mathcal{G}_0 &= \{(y, z) \in \mathbb{R}^l \,|\, \bar{R}(y, z) = 0\} \\ &= \{(y, z) \in \mathbb{R}^l \,|\, y^T z = 0 \ \text{ and } \ |y| = |z|\}. \end{aligned}$$

One easily finds that Assumption $(A.1)$ is satisfied. Lastly, to see that Assumption $(A.3)$ is satisfied, note that one may take $\tilde{\mathcal{G}}_0(z) \doteq \{y \in \mathbb{R}^m \,|\, y^T z = 0 \text{ and } |y| = |z|\}$. Note that if $z = (1, 0, 0)^T$, then $\tilde{\mathcal{G}}_0(z)$ is the unit circle in the (z_2, z_3)–plane. Hence, one may take $p(\lambda) \doteq (0, \cos(2\pi\lambda), \sin(2\pi\lambda))$ and $\bar{e} = (1, 0, 0)^T$. See Figure 3 for a depiction of this over the y-space. Then, for $z \in \mathbb{R}^m \setminus \{0\}$, one may then let

$$\Gamma(z) \doteq \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix}, \quad \text{where } u \doteq \frac{z}{|z|}, \quad \hat{v} \doteq \sum_{k=1}^{2} u \times e_k,$$
$$v = \frac{\hat{v}}{|\hat{v}|}, \quad w \doteq \frac{u \times v}{|u \times v|},$$

and e_k denotes the k^{th} standard basis vector in $\mathbb{R}^3$. One may then easily verify Assumption $(A.3)$.

We remark that another example, again associated to a classical energy shell, is given by

$$S^1(t, x) \doteq \frac{-c_{1,1}^2}{2m}t + ic_{1,1}\sqrt{x^T x} - i\hbar \log_q(x_1),$$

where $c_{1,1} \doteq \frac{mC}{2\hbar}$. In this case there are additional discontinuity and singularity manifolds. In particular, in addition to $\mathcal{H}_0$ and $\mathcal{G}_0$ from the S^0, we also have $\mathcal{H}_{0,1} \doteq \{(y, z) \in \mathbb{R}^l \,|\, z_1 = 0 \text{ and } y_1 < 0\}$ and $\mathcal{G}_{0,1} \doteq \{(y, z) \in \mathbb{R}^l \,|\, y_1 = z_1 = 0\}$.

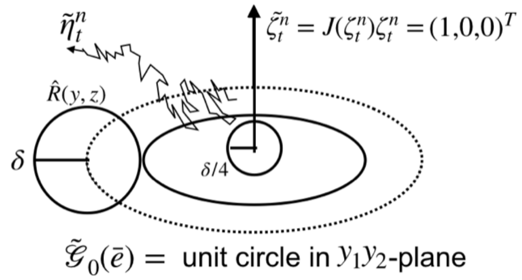

Fig. 1. Pictorial description of $\tilde{\mathcal{G}}_0(z)$.

4. THE $\delta > 0$ PRELIMIT

We smooth the dynamics as follows. For $\delta > 0$, let $g^\delta, \tilde{g}^{\delta/4,\delta} \in C^\infty(\mathbb{R})$ be given by

$$g^\delta(\rho) \doteq \begin{cases} 1 - \exp\left\{\dfrac{1}{\delta^2} + \dfrac{1}{\rho^2 - \delta^2}\right\} & \text{if } |\rho| \in [0, \delta), \\ 1 & \text{if } |\rho| \geq \delta, \end{cases} \quad (7)$$

$$\tilde{g}^{\delta/4,\delta}(\rho) \doteq \begin{cases} 0 & \text{if } |\rho| \in [0, \delta/4], \\ g^{3\delta/4}(|\rho| - \delta/4) & \text{if } |\rho| > \delta/4. \end{cases} \quad (8)$$

We also let

$$\hat{\mathcal{G}}_\delta \doteq \mathcal{G}_\delta \cup [\mathbb{R}^m \times B_\delta(0)].$$

Defining $\hat{R}(y, z) \doteq d\big(y, \tilde{\mathcal{G}}_0(z)\big)/|z|$ for $|z| > 0$, we let

$$F^\delta(y, z) \doteq g^\delta(\hat{R}(y, z))F(y, z),$$
$$G^\delta(y, z) \doteq \tilde{g}^{\delta/4,\delta}(|z|)g^\delta(\hat{R}(y, z))G(y, z)$$

for all $(y, z) \in [\mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}]$. Note that

$$F^\delta = F \ \text{ and } \ G^\delta = G \ \text{ on } \ (\hat{\mathcal{G}}_\delta)^c. \quad (9)$$

Our final assumption is that for each $\delta > 0$,

$F^\delta, G^\delta \in C^1\big(\mathcal{H}_0^c \cap [\mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}]\big)$, F^δ and G^δ are bounded on $[\mathbb{R}^m \times (B_{\delta/4}(0))^c]$, and $\nabla_{(y,z)}F^\delta, \nabla_{(y,z)}G^\delta$ are bounded on $\mathcal{H}_0^c \cap [\mathbb{R}^m \times (B_{\delta/4}(0))^c]$. $\quad (A.4)$

Note that $(A.4)$ holds for the examples given in Section 3, and that it will hold more generally when the dynamics are bounded by the the multiplicative inverse of appropriate polynomial forms.

Consider the system with modified dynamics given in integral form as

$$\eta_t^\delta = y^0 + \int_0^t F^\delta(\eta_r^\delta, \zeta_r^\delta)\, dr + B_t, \quad (10)$$
$$\zeta_t^\delta = z^0 + \int_0^t G^\delta(\eta_r^\delta, \zeta_r^\delta),\, dr \quad (11)$$

for $t \in [0, T]$. We demonstrate existence and uniqueness of a strong solution via application of the Girsanov transform approach to first obtain existence of a weak solution, followed by a demonstration of pathwise uniqueness to then obtain the strong-solution assertion.

Using the infinite total linear variation of a brownian motion versus the finite variation of the ζ^δ process, one obtains the following.

Lemma 3. Suppose η^δ is a brownian motion on probability space $(\Omega, \bar{\mathcal{F}}, \hat{P})$ where Ω, $\bar{\mathcal{F}}$ and $\hat{P}$ denote a sample space, σ-algebra and probability measure, respectively, and with filtration denoted by $\mathcal{F}_\cdot$. Let ζ^δ be continuous and of bounded variation on $[0,T]$. Then, for a.e. $\omega \in \Omega$, $\mu(\{t \in [0,T] \,|\, (\eta^\delta_t, \zeta^\delta_t) \in \mathcal{H}_0\}) = 0$, where μ denotes Lebesgue measure.

Lemma 4. For a.e. $\omega \in \Omega$, There exists absolutely continuous, unique $\zeta^\delta_\cdot(\omega)$ satisfying (11).

The proof follows the standard successive approximations approach, and we do not inlcude the details.

Using the Novikov condition and Girsanov Theorem, one may obtain a weak solution.

Lemma 5. Let $\delta > 0$. There exists a weak solution to (10)–(11).

Applying the pathwise uniqueness approach, one may obtain the strong solution.

Theorem 6. Let $\delta > 0$. There exists a unique strong solution to (10)–(11).

5. TAKING $\delta \downarrow 0$

We obtain the limit result in the case where the dimension satisfies $m \geq 3$. This restriction is related to the form of $\tilde{\mathcal{G}}_0$, which takes the form of a curve in $\mathbb{R}^m$. It is expected that in the case where $\tilde{\mathcal{G}}_0$ is a point, the result would follow for $m \geq 2$.

Fix a probability space, say $(\Omega, \bar{\mathcal{F}}, \bar{P})$, and brownian motion, $B_\cdot$, with filtration $\mathcal{F}_\cdot$ generated by $B_\cdot$. As $(y^0, z^0) \notin \mathcal{G}_0$, there exists $\bar{\delta} > 0$ such that $(y^0, z^0) \notin \mathcal{G}_\delta$ for all $\delta \in [0, \bar{\delta}]$. Let $\delta_n \downarrow 0$ with $\delta_1 \in (0, \bar{\delta})$. Let the corresponding strong solutions of (10)–(11) be denoted by (η^n, ζ^n). Note that $G^{\delta_n}(y,z) = 0$ for all $z \in B_{\delta_n/4}(0)$, and hence

$$|\zeta^n_t| \geq \delta_n/4 \quad \forall\, t \in [0,T],\, \omega \in \Omega,\, n \in I\!N. \tag{12}$$

For $n \in I\!N$, let

$$\mathcal{A}_n \doteq \{\, \omega \in \Omega \,|\, \nexists t \in [0,T] \text{ s.t. } (\eta^n_t, \zeta^n_t) \in \mathcal{G}_{\delta_n} \cup [\mathbb{R}^m \times B_{\delta_n}(0)]\,\}. \tag{13}$$

Recalling that $F^\delta = F$ on $\mathcal{G}^c_\delta$ and $G^\delta = G$ on $\mathcal{G}^c_\delta \cap B_{\delta/4}(0)^c$, we see that

$$(\eta^m, \zeta^m) = (\eta^n, \zeta^n) \quad \forall \omega \in \mathcal{A}_n \text{ and } m \geq n \geq 1. \tag{14}$$

Lastly, let

$$\tilde{\eta}^n_t = J(\zeta^n_t)\eta^n_t, \tag{15}$$

$$\tilde{\zeta}^n_t = J(\zeta^n_t)\zeta^n_t = \bar{e}, \tag{16}$$

for all $t \in [0,T]$.

Lemma 7. For each $\omega \in \Omega$, $(\eta^n_t, \zeta^n_t)(\omega) \in \mathcal{G}_\delta$ if and only if $\eta^n_t(\omega) \in \tilde{\mathcal{G}}_\delta(\zeta^n_t(\omega))$ if and only if $\tilde{\eta}^n_t(\omega) \in \tilde{\mathcal{G}}_{|\zeta^n_t(\omega)|\delta}(\tilde{\zeta}^n_t(\omega))$ if and only if there exists $\lambda^n_t(\omega) \in \mathcal{I}$ such that $|\tilde{\eta}^n_t(\omega) - p(\lambda^n_t(\omega))| \leq \delta$.

Proof. For compactness of the presentation, we suppress the ω arguments. The first assertion is by definition. Noting Assumption $(A.3)$, that assertion is true if and only if $d(\eta^n_t, \tilde{\mathcal{G}}(\zeta^n_t)) \leq \delta|z|$, or equivalently, $\min_{\lambda \in \mathcal{I}} |\eta^n_t - J^{-1}(\zeta^n_t)p(\lambda)| \leq \delta|z|$. Using the orthonormality of $\Gamma(\zeta^n_t)$, one finds that this is true if and only if $\min_{\lambda \in \mathcal{I}} |J(\zeta^n_t)\eta^n_t - p(\lambda)| \leq \delta$, which by (16), is equivalently, $\min_{\lambda \in \mathcal{I}} |\tilde{\eta}^n_t - p(\lambda)| \leq \delta$, which yields the remaining two assertions.

Lemma 8. For each $n \in I\!N$, there exists a probability measure, P_n, mutually absolutely continuous with respect to $\bar{P}$, such that η^n is a brownian motion with respect to P_n.

Proof. By the boundedness of F^{δ_n} and (10), one finds that the Novikov condition is satisfied, and hence the assertion follows from the Girsanov theorem, cf. Karatzas and Shreve (1987).

Let

$$\hat{\mathcal{A}}_n \doteq \{\, \omega \in \Omega \,|\, \nexists t \in [0,T] \text{ s.t. either } \tilde{\eta}^n_t \in \tilde{\mathcal{G}}_{\delta_n}(\bar{e}) \text{ or } \zeta^n_t \in B_{\delta_n}(0)\,\}. \tag{17}$$

Using Lemma 7 and (12), we see that

$$\mathcal{A}_n = \hat{\mathcal{A}}_n. \tag{18}$$

Lemma 9. There exists a probability measure, $\breve{P}_n$, mutually absolutely continuous with respect to P_n, such that

$$d\tilde{\eta}^n_t = J(\zeta^n_t)\, d\breve{\eta}^n_t,$$

where $\breve{\eta}^n_t$ is a brownian motion under $\breve{P}_n$.

Proof. Applying Itô's rule to $\tilde{\eta}^n$, and noting that one has $d\langle[\zeta^n]_k, [\zeta^n]_j\rangle_t \equiv 0$ for all $k, j \in]1, m[$, one sees that

$$\begin{aligned} d\tilde{\eta}^n_t &= \bar{F}^n(\tilde{\eta}^n_t, \zeta^n_t)\, dt + J(\zeta^n_t)\, d\eta^n_t \\ &= J(\zeta^n_t)\Big[(J(\zeta^n_t))^{-1}\bar{F}^n(\tilde{\eta}^n_t, \zeta^n_t)\, dt + d\eta^n_t\Big]. \end{aligned} \tag{19}$$

We examine $\bar{F}^n$. By Assumption $(A.4)$, there exists $M^1_n < \infty$ such that

$$|G^{\delta_n}\big([J(\zeta^n_t)]^{-1}\tilde{\eta}^n_t, \zeta^n_t\big)| \leq M^1_n \quad \forall\, t \in [0,T],\, \omega \in \Omega. \tag{20}$$

Also, by (12) and Assumption $(A.3)$, there exists $M^2_n < \infty$ such that

$$|J(\zeta^n_t)|,\ \Big|\frac{\partial J}{\partial z_j}(\zeta^n_t)\Big| \leq M^2_n \quad \forall\, j \in]1, m[,\, t \in [0,T],\, \omega \in \Omega. \tag{21}$$

Lastly, by (11), (20) and Assumption $(A.3)$ one sees that

$$|(J(\zeta^n_t))^{-1}| = |\zeta^n_t| \leq |z^0| + M^1_n T \doteq M^3_n < \infty \tag{22}$$

for all $t \in [0,T]$ and $\omega \in \Omega$. By (19)–(22), we see that there exists $\bar{M}_n < \infty$ such that

$$|(J(\zeta^n_t))^{-1}\bar{F}^n(\tilde{\eta}^n_t, \zeta^n_t)| \leq \bar{M}_n |\tilde{\eta}^n_t| \quad \forall\, t \in [0,T],\, \omega \in \Omega. \tag{23}$$

For integers $0 \leq k \leq K < \infty$, let $\Delta_K \doteq T/K$ and $t_k \doteq k\Delta_K$. By (23),

$$\begin{aligned} &\mathbb{E}\Big\{\exp\Big[\tfrac{1}{2}\int_{t_k}^{t_{k+1}} |(J(\zeta^n_t))^{-1}\bar{F}^n(\tilde{\eta}^n_t, \zeta^n_t)|^2\, dt\Big]\Big\} \\ &\leq \exp\big[(\bar{M}_n)^2/2\big]\mathbb{E}\Big\{\exp\Big[\tfrac{1}{2}\int_{t_k}^{t_k+\Delta_K} |\tilde{\eta}^n_t|^2\, dt\Big]\Big\} \end{aligned}$$

for all $0 \leq k \leq K < \infty$ and $n \in I\!N$. However, recalling that $\tilde{\eta}^n$ is a brownian motion on measure P_n, this is finite for sufficiently large K. Hence, a weak Novikov condition is satisfied, cf. (Karatzas and Shreve, 1987, Cor. 3.5.14), and we may apply a Girsanov transformation, yielding measure $\breve{P}_n$, mutually absolutely continuous with respect to P_n, given by $d\breve{P}_n \doteq \tilde{\mu}^n_T dP_n$, where

$$\tilde{\mu}^n_T \doteq \exp\Big[-\int_0^T (v^n_t)^T\, d\eta^n_t - \tfrac{1}{2}\int_0^T |v^n_t|^2\, dt\Big],$$

with $v_t^n \doteq (J(\zeta_t^n))^{-1}\bar{F}^n(\tilde{\eta}_t^n, \zeta_t^n)$, and such that under $\breve{P}_n$, the process $\breve{\eta}_t^n \doteq \int_0^t v_r^n\,dr + \eta_t^n$ is a brownian motion. Recalling (19), we have $d\tilde{\eta}_t^n = J(\zeta_t^n)\,d\breve{\eta}_t^n$.

We defne $\{\beta_t^n\}_{t\geq 0}$ and $\{\alpha_s^n\}_{s\geq 0}$ by

$$\beta_t^n \doteq \int_0^{t\wedge T} \frac{dr}{|\zeta_r^n|^2},\ \alpha_s^n \doteq \inf\{t \in [0,\infty)\,|\, \beta_t^n > s\}.$$

We understand the infimum of empty set is ∞.

Lemma 10. There exists a Brownian motion $\{w_s\}_{s\geq 0}$ on an enlarged probability space of $(\Omega, \bar{\mathcal{F}}, \breve{P}_n)$, which we denote by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_n)$, such that $w_s = \tilde{\eta}_{\alpha_s^n}^n$ for $0 \leq s \leq \beta_T^n$. Moreover there exist $0 \leq \underline{\alpha} \leq \bar{\alpha} < \infty$ such that $\alpha_{s+r}^n - \alpha_s^n \in [\,\underline{\alpha} r,\, \bar{\alpha} r\,]$ for all $0 \leq s, r < \infty$.

Proof. The asserted bounds on $\alpha_\cdot^n$ follow from Assumption $(A.3)$, (12) and (22). We extend $\{\tilde{\eta}_t^n\}_{0\leq t\leq T}$ and $\{\mathcal{F}_t\}_{0\leq t\leq T}$ to $[0,\infty)$ by

$$\hat{\eta}_t^n \doteq \tilde{\eta}_{t\wedge T}^n,\ \hat{\mathcal{F}}_t \doteq \mathcal{F}_{t\wedge T},\ t \geq 0.$$

Noting that $\{\hat{\eta}_t^n\}_{t\geq 0}$ is a continuous $\{\hat{\mathcal{F}}_t\}$-martingale and $J(\zeta_t^n)J^T(\zeta_t^n) = |\zeta_t^n|^{-2} I_{m\times m}$ for all $t \in [0,T]$, $\omega \in \Omega$, we have

$$\langle \hat{\eta}^{n,i}, \hat{\eta}^{n,j}\rangle_t = \langle \tilde{\eta}^{n,i}, \tilde{\eta}^{n,j}\rangle_{t\wedge T} = \delta_{ij}\beta_t^n.$$

Thus, by (1.10) Th., Chap.V of Revuz and Yor (1999), there exists a Brownian motion $\{w_s\}_{s\geq 0}$ on an enlarged probability space of $(\Omega, \bar{\mathcal{F}}, \breve{P}_n)$ satisfying $w_s = \tilde{\eta}_{\alpha_s^n}^n$ for $s \in [0, \beta_T^n]$. To clarify the enlargement procedure and the construction of $\{w_s\}$ in the above theorem, let $(\Omega', \mathcal{F}', P')$ be a probability space with a filtration $\{\mathcal{F}_s'\}$ and $\{b_s\}_{s\geq 0}$ be an m-dimensional $\{\mathcal{F}_s'\}$-Brownian motion with $b_0 = 0$. Define $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_n)$ and $\{\tilde{\mathcal{F}}_s\}$ by

$$\tilde{\Omega} \doteq \Omega \times \Omega',\ \tilde{\mathcal{F}} \doteq \bar{\mathcal{F}} \otimes \mathcal{F}',\ \tilde{P}_n \doteq \breve{P}_n \otimes P',$$
$$\tilde{\mathcal{F}}_s \doteq \hat{\mathcal{F}}_{\alpha_s^n} \otimes \mathcal{F}_s'.$$

Then $\{w_s\}_{s\geq 0}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_n)$ is given by

$$w_s(\tilde{\omega}) \doteq \begin{cases} \hat{\eta}_{\alpha_s^n(\omega)}^n(\omega), & 0 \leq s \leq \beta_\infty^n(\omega), \\ \hat{\eta}_\infty^n(\omega) + b_{s-\beta_\infty^n(\omega)}(\omega'), & s > \beta_\infty^n(\omega), \end{cases}$$
$$= \begin{cases} \tilde{\eta}_{\alpha_s^n(\omega)}^n(\omega), & 0 \leq s \leq \beta_T^n(\omega), \\ \tilde{\eta}_T^n(\omega) + b_{s-\beta_T^n(\omega)}(\omega'), & s > \beta_T^n(\omega), \end{cases} \tag{24}$$

where we denote $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega} = \Omega \times \Omega'$.

For $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega}$, we let $\breve{\zeta}_s^n(\tilde{\omega}) \doteq \zeta_s^n(\omega)$ for all $s \in [0,T]$ and $n \in I\!N$. By (24), we note that

$$\begin{aligned}
&\hat{\mathcal{A}}_n \times \Omega' \\
&= \{\tilde{\omega} \in \tilde{\Omega}\,|\, \nexists s \in [0, \beta_T^n(\omega)]\ s.t.\ either\ \ \tilde{\eta}_{\alpha_s^n(\omega)}^n \in \tilde{\mathcal{G}}_{|\breve{\zeta}_s^n(\tilde{\omega})|\delta_n}(\bar{e}) \\
&\qquad\quad or\ \ \breve{\zeta}_s^n(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \\
&= \{\tilde{\omega} \in \tilde{\Omega}\,|\, \nexists s \in [0, \beta_T^n(\omega)]\ s.t.\ either\ \ w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_{|\breve{\zeta}_s^n(\tilde{\omega})|\delta_n}(\bar{e}) \\
&\qquad\quad or\ \ \breve{\zeta}_s^n(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \\
&\supset \{\tilde{\omega} \in \tilde{\Omega}\,|\ \nexists s \in [0,\infty)\ s.t.\ either\ \ w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_{|\breve{\zeta}_s^n(\tilde{\omega})|\delta_n}(\bar{e}) \\
&\qquad\quad or\ \ \breve{\zeta}_s^n(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \\
&\doteq \mathcal{C}_n \quad \forall n \in I\!N.
\end{aligned} \tag{25}$$

For $n \in I\!N$, let

$$\tau_n \doteq \inf\{s \geq 0\,|\, (\eta_s^n, \zeta_s^n)(\omega) \in \mathcal{G}_{\delta_n} \cup [\mathbb{R}^m \times B_{\delta_n}(0)]\},$$

and let

$$(\bar{\eta}_s, \bar{\zeta}_s) = (\eta_s^n, \zeta_s^n) \quad \forall\, s \leq \tau_n,$$

which is

$$= (\eta_s^k, \zeta_s^k) \quad \forall\, k \geq n, \forall s \leq \tau_n. \tag{26}$$

Next, let $\breve{\bar{\zeta}}_s(\tilde{\omega}) \doteq \bar{\zeta}_s(\omega)$ for all $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega}$. Letting

$$\begin{aligned}
\mathcal{C}_n = &\{\tilde{\omega} \in \tilde{\Omega}\,|\ \nexists s \in [0,\infty)\ s.t. \\
&\quad either\ \ w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_{|\breve{\zeta}_s^n(\tilde{\omega})|\delta_n}(\bar{e}) \\
&\quad or\ \ \breve{\bar{\zeta}}_s(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \quad \forall\, n \in I\!N,
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathcal{C}} \doteq &\{\tilde{\omega} \in \tilde{\Omega}\,|\ \ \nexists s \in [0,\infty)\ s.t.\ either\ \ w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_0(\bar{e}) \\
&\quad or\ \ \breve{\bar{\zeta}}_s(\tilde{\omega}) = 0\,\},
\end{aligned}$$

one obtains $\bar{\mathcal{C}} = \lim_{n\to\infty} \mathcal{C}_n$.

Lemma 11. Let $m \geq 3$. $P_n(\bar{\mathcal{C}}) = 1$.

Proof. (sketch) Note that

$$P_n(\{\omega \in \Omega\,|\,\exists s \in [0,\infty)\ s.t.\ \ \breve{\bar{\zeta}}_s = 0\,\}) = 0.$$

One then applies classical results, cf. Doob (2001); Möters and Peres (2010); Port and Stone (1972).

Note that $\hat{\mathcal{A}}_k = \mathcal{A}_k \subseteq \mathcal{A}_n = \hat{\mathcal{A}}_n$ for all $k \leq n$, and let $\bar{\mathcal{A}} \doteq \bigcup_{n\in I\!N} \mathcal{A}_n$.

Lemma 12. Let $m \geq 3$. $\bar{P}(\bar{\mathcal{A}}) = \lim_{n\to\infty} \bar{P}(\mathcal{A}_n) = 1$.

Theorem 13. Let $m \geq 3$. $(\bar{\eta}, \bar{\zeta})$ is a unique strong solution of (1)–(3).

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