# Cyclic triangle factors in regular tournaments

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**Abstract**

Both Cuckler and Yuster independently conjectured that when $n$ is an odd positive multiple of 3 every regular tournament on $n$ vertices contains a collection of $n/3$ vertex-disjoint copies of the cyclic triangle. Soon after, Keevash & Sudakov proved that if $G$ is an orientation of a graph on $n$ vertices in which every vertex has both indegree and outdegree at least $(1/2 - o(1))n$, then there exists a collection of vertex-disjoint cyclic triangles that covers all but at most 3 vertices. In this paper, we resolve the conjecture of Cuckler and Yuster for sufficiently large $n$.

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## 1 Introduction

Let $H$ and $G$ be graphs or directed graphs. An *$H$-tiling* of $G$ is a collection of vertex-disjoint copies of $H$ in $G$. An $H$-tiling $\mathcal{C}$ *covers* the set $V(\mathcal{C}) = \bigcup_{C\in\mathcal{C}} V(C)$, and is called *perfect* or an *$H$-factor* if it covers $V(G)$.

The celebrated Hajnal-Szemerédi Theorem [7] states that for every positive integer $r$, if $n$ is a positive multiple of $r$ and $G$ is a graph on $n$ vertices such that $\delta(G) \geqslant (1-1/r)n$, then $G$ contains a $K_r$-factor. The case when $r = 3$ is a corollary of an earlier result of Corrádi & Hajnal [3].

In this paper, we consider a similar problem in the context of *oriented graphs*, which are orientations of simple graphs, i.e., oriented graphs are directed graphs in which there is at most one directed edge between every pair of vertices and no loops. A *tournament* is an orientation of a complete graph. For an oriented graph $G$ and $v \in V(G)$, we denote the *out-neighborhood of $v$* and *in-neighborhood of $v$* by $N^+(v)$ and $N^-(v)$, respectively. We let $N(v) = N^+(v) \cup N^-(v)$ be the *neighborhood of $v$*, and we let $d^+(v) = |N^+(v)|$

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and $d^-(v) = |N^-(v)|$ be the *outdegree* and *indegree* of $v$, respectively. The *minimum semidegree of* $G$ is

$$\delta^0(G) = \min_{v \in V(G)} \{\min\{d^+(v), d^-(v)\}\}.$$

An oriented graph $G$ on $n$ vertices is a *regular tournament* if $d^+(v) = d^-(v) = \frac{n-1}{2}$ for every $v \in G$.

A tournament is *transitive* if it contains no directed cycles, and the unique transitive tournament on $r$ vertices is denoted $TT_r$. Up to isomorphism, there are two different tournaments on three vertices: $TT_3$ and the three vertex cycle in which the edges are consistently oriented, which we denote by $C_3$. We call $C_3$ and $TT_3$ the cyclic and transitive triangles, respectively.

There has been some prior work on minimum degree conditions that force an $H$-factor in directed graphs. See [22] and [6] for work on directed graphs, and [1] and [23] for oriented graphs. Also, [21] contains many additional interesting embedding problems for oriented graphs. This paper focuses on the following conjecture that Cuckler and Yuster made independently.

**Conjecture 1** (Cuckler 2008 [5], Yuster 2007 [24])**.** If $n$ is an odd positive multiple of 3, then every regular tournament on $n$ vertices has a cyclic triangle factor.

Keevash & Sudakov then proved the following approximate version of this conjecture.

**Theorem 2** (Keevash & Sudakov 2009 [13])**.** *There exists $c > 0$ and $n_0$ such that for every $n \geqslant n_0$ the following holds. If $G$ is an oriented graph on $n$ vertices and $\delta^0(G) \geqslant (1/2 - c)n$, then there exists a cyclic triangle tiling that covers all but at most 3 vertices.*

A corollary of our main result resolves Conjecture 1 for large tournaments. To see that the resolution of Conjecture 1 is a sharp result, consider the following construction from [13]. For a positive integer $m$, let $G$ be a tournament on $3m$ vertices in which the edges are oriented so that there exists a partition $\{V_1, V_2, V_3\}$ of $V(G)$ such that

- $|V_1| = m - 1$, $|V_2| = m$, and $|V_3| = m + 1$;
- for $i \in [3]$, the oriented graph induced by $V_i$ has minimum semidegree $\lfloor (|V_i| - 1)/2 \rfloor$; and
- no edges are directed from $V_2$ to $V_1$, from $V_3$ to $V_2$, and from $V_1$ to $V_3$.

We have that,

$$\delta^0(G) = \begin{cases} (|V_2| - 2)/2 + |V_1| = \frac{n-4}{2} & \text{if } m \text{ is even} \\ (|V_2| - 1)/2 + |V_1| = \frac{n-3}{2} & \text{if } m \text{ is odd.} \end{cases}$$

To see why $G$ has no cyclic triangle factor, let $\mathcal{C}$ be a cyclic triangle tiling of $G$ and note that, for every $C \in \mathcal{C}$, either $C$ has one vertex in each of $V_1$, $V_2$ and $V_3$, or $V(C) \subseteq V_i$ for some $i \in [3]$. Therefore,

$$|V(\mathcal{C}) \cap V_1| \equiv |V(\mathcal{C}) \cap V_2| \equiv |V(\mathcal{C}) \cap V_3| \pmod 3.$$

Because $|V_1|$, $|V_2|$ and $|V_3|$ are distinct modulo 3, we have that $V(\mathcal{C}) \neq V(G)$. Motivated by this example we make the following definitions.

**Definition 3** (Divisibility barrier and $\boldsymbol{\gamma}$-extremal)**.** Let $G$ be an oriented graph. Call a partition $\mathcal{P}$ of $V(G)$ a *divisibility barrier* if either $\mathcal{P}$ is the trivial partition $\{V(G)\}$ and $|V(G)|$ is not divisible by 3, or if $\mathcal{P}$ has exactly three parts, $V_1$, $V_2$, and $V_3$, such that there are no edges directed from $V_2$ to $V_1$, from $V_3$ to $V_2$, and from $V_1$ to $V_3$; and $|V_1|$, $|V_2|$, and $|V_3|$ are not all equivalent modulo 3.

For $\gamma > 0$, call a partition $\{V_1, V_2, V_3\}$ of $V(G)$ a *$\gamma$-extremal partition* of $G$ if, for every $i \in [3]$,

$$(1/3 - \gamma)\, n \leqslant |V_1|, |V_2|, |V_3| \leqslant (1/3 + \gamma)\, n,$$

and the number of edges directed from $V_2$ to $V_1$, from $V_3$ to $V_2$, and from $V_1$ to $V_3$ are each at most $\gamma n^2$. An oriented graph is called *$\gamma$-extremal* if it contains a $\gamma$-extremal partition.

The following is the main result of this paper.

**Theorem 4.** *There exists $c > 0$ and $n_0$ such that for every $n \geqslant n_0$ and for every oriented graph $G$ on $n$ vertices with $\delta^0(G) \geqslant (1/2 - c)n$ the following holds. $G$ has a cyclic triangle factor if and only if $G$ does not have a divisibility barrier.*

The following corollary resolves Cuckler and Yuster's conjecture for sufficiently large regular tournaments.

**Corollary 5.** *There exists $n_0$ such that when $n$ is a multiple of 3 and $n \geqslant n_0$ the following holds. If $G$ is an oriented graph on $n$ vertices and $\delta^0(G) \geqslant n/2 - 1$, then $G$ has a cyclic triangle factor.*

*Proof.* Assuming that Theorem 4 holds, we only need to show that $G$ does not contain a divisibility barrier. For a contradiction, assume that $\{V_1, V_2, V_3\}$ is a divisibility barrier. Since $n$ is divisible by 3 and $|V_1|$, $|V_2|$, and $|V_3|$ are not all equivalent modulo 3, we have that $|V_1|$, $|V_2|$ and $|V_3|$ are distinct modulo 3. Therefore, there exists a labeling $\{i, j, k\} = [3]$ such that $|V_j| \leqslant |V_k| - 2$. Suppose that $i + 1 \equiv j \pmod 3$. There exists $v \in V_i$ such that $|N^+(v) \cap V_i| < |V_i|/2$, so

$$d^+(v) = \sum_{\ell=1}^{3} |N^+(v) \cap V_\ell| < \frac{|V_i|}{2} + |V_j| \leqslant \frac{|V_i| + |V_j| + |V_k| - 2}{2} = \frac{n}{2} - 1,$$

a contradiction. A similar argument holds when $i - 1 \equiv j \pmod 3$. □

We are not sure how large the constant $c$ can be in Theorem 4, and we do not compute the value of $c$ that our proof implies. An example of Keevash & Sudakov [13], which we will present below, implies that the constant $c$ cannot be larger than $1/18$. This suggests the following problem.

**Problem 6.** What is the smallest $\phi \geqslant 0$ such that there exists $n_0$ such that for every $n \geqslant n_0$ every oriented graph $G$ on $n$ vertices with $\delta^0(G) \geqslant (4/9 + \phi)n$ contains either a divisibility barrier or a cyclic triangle factor?

For $m \geqslant 1$, let $G$ be an oriented graph on $n = 9(m+1)$ vertices and let $\mathcal{P} = \{V_1, V_2, V_3\}$ be a partition of $V(G)$ such that $|V_1| = 3m+1$ and $|V_2| = |V_3| = 3m+4$. Suppose that for every pair $i, j \in [3]$ such that $j \equiv i+1 \pmod 3$, there is a directed edge from every vertex in $V_i$ to every vertex in $V_j$. Further suppose that, for every $i \in [3]$, the vertices of $V_i$ can be cyclically ordered so that, for every $v \in V_i$, the intersection of the out-neighborhood of $v$ and $V_i$ is exactly the $(|V_i| - 1)/3$ vertices that succeed $v$ in this ordering. Note that every cyclic triangle has at least one vertex in $V_1$, so $|\mathcal{C}| \leqslant |V_1| = n/3 - 2$ for every cyclic triangle tiling $\mathcal{C}$, and

$$\delta^0(G) = \frac{|V_2| - 1}{3} + |V_1| = \frac{4n}{9} - 2.$$

Note that $\mathcal{P}$ is not a divisibility barrier, because the three parts of $\mathcal{P}$ all have the same size modulo 3. Additionally, $G$ cannot have a divisibility barrier, because for every partition $\mathcal{P}' \neq \mathcal{P}$, such that $|\mathcal{P}'| \leqslant 3$, there exists a part $U \in \mathcal{P}'$ such that there exist $x, y \in U$ such that $x$ and $y$ are in different parts, say $V_i$ and $V_j$, of $\mathcal{P}$. But then either $\mathcal{P}'$ is not a divisibility barrier or $U$ contains all of the vertices of the third part, say $V_k$, of $\mathcal{P}$. Then, by similar logic, one can argue that either $\mathcal{P}'$ is not a divisibility barrier or $U = V(G)$. Since 3 divides $|V(G)|$, $\mathcal{P}'$ is not a divisibility barrier.

If, for some $\psi > 1/3$ and all sufficiently large $n$, there exists a $C_3$-free oriented graph on $n$ vertices with minimum semidegree $\psi n$, then a similar example would imply that $\phi$ must be strictly greater than 0 in Problem 6. Note that such a $C_3$-free oriented graph would imply that the famous Caccetta-Häggkvist Conjecture [2] is false.[1]

### 1.1 Additional Definitions and Notation

Let $G$ be an oriented graph, $u, v \in V(G)$, and $A, B \subseteq V(G)$. We let $N^+(u)$ be the out-neighborhood of $u$, $N^+(u, B) = N^+(u) \cap B$, $d^+(u, B) = |N^+(u, B)|$, and $e^+(A, B) = \sum_{u \in A} d^+(u, B)$. We define $N^-(u)$, $N^-(u, B)$, $d^-(u, B)$ and $e^-(A, B)$ similarly. If $uv \in E(G)$, we let $d^{+,-}(uv, A) = |N^+(u) \cap N^-(v) \cap A|$ and $d^{-,+}(uv, A)$, $d^{+,+}(uv, A)$, and $d^{-,-}(uv, A)$ are all defined similarly. We also let $d^{\sigma,\tau}(uv) = d^{\sigma,\tau}(uv, V(G))$, for $\sigma, \tau \in \{-, +\}$. We let $E(A)$ be the set of edges in the oriented graph induced by $A$, and let $e(A) = |E(A)|$. We let $\overline{A} = V(G) \setminus A$.

We define $E^+(A, B) = \{uv \in E(G) : u \in A \text{ and } v \in B\}$ and $E^-(A, B) = E^+(B, A)$. We will often write cyclic and transitive triangles $C$ as $abc$ when $V(C) = \{a, b, c\}$. For $V_1, V_2, V_3 \subseteq V(G)$, $\mathrm{cyc}(V_1, V_2, V_3)$ and $\mathrm{trn}(V_1, V_2, V_3)$ count, respectively, the number of cyclic and transitive triangles with vertex set $\{v_a, v_b, v_c\}$ such that $\{a, b, c\} = [3]$ and $v_i \in V_i$ for $i \in [3]$. We abbreviate $\mathrm{cyc}(A, A, A)$ and $\mathrm{trn}(A, A, A)$ as $\mathrm{cyc}(A)$ and $\mathrm{trn}(A)$, respectively. We will often replace $\{v\}$ with $v$ in this notation.

We define the *strong $\beta$-out-neighborhood* of $A$ to be the set of vertices $x \in \overline{A}$ such that $d^-(x, A) \geqslant |A| - \beta n$ and we denote this set by $\mathrm{SN}^+_\beta(A)$. We define the *strong $\beta$-in-neighborhood* $\mathrm{SN}^-_\beta(A)$ of $A$ similarly.

[1] If true, the Cacceta-Häggkvist Conjecture would imply that every oriented graph on $n$ vertices with minimum outdegree at least $n/3$ contains a cyclic triangle.

Throughout the paper, we write $0 < \alpha \ll \beta \ll \gamma$ to mean that we can choose the constants $\alpha, \beta, \gamma$ from right to left. More precisely, there are increasing functions $f$ and $g$ such that, given $\gamma$, whenever we choose $\beta \leqslant f(\gamma)$ and $\alpha \leqslant g(\beta)$, all calculations needed in our proof are valid. Hierarchies of other lengths are defined in the obvious way. For real numbers $x$ and $y$, we write $x = y \pm c$ to mean that $y - c \leqslant x \leqslant y + c$.

In our proof, we use a very small part of the theory and notation developed in [11], [10],[8],[15], and [17]. It is based on the absorbing method of Rödl, Ruciński and Szemerédi [20]. This theory was developed for hypergraphs, so we define, for every oriented graph $G$, the hypergraph $H(G)$ to be the 3-uniform hypergraph in which $xyz$ is in an edge if and only if $xyz$ is a cyclic triangle in $G$. Clearly a cyclic triangle factor in $G$ is equivalent to a perfect matching in $H(G)$.

Let $V$ be a set of order $n$ and let $\mathcal{P} = \{V_1, \dots, V_d\}$ be a partition of $V$. We say that $\mathcal{P}$ is *trivial* if $|\mathcal{P}| = 1$, and, for $\eta > 0$, we call $\mathcal{P}$ an *$\eta$-partition* if $|V_i| \geqslant \eta n$ for every $i \in [d]$. Let $H$ be a $k$-uniform hypergraph with vertex set $V$. We let

$$\delta_1(H) = \min_{v \in V} \{|\{e \in E(H) : v \in e\}|\}.$$

For every subset $U$ of $V$ the *index vector with respect to $\mathcal{P}$*, denoted $\mathbf{i}_{\mathcal{P}}(U)$, is the vector defined by

$$\mathbf{i}_{\mathcal{P}}(U) = (|U \cap V_1|, |U \cap V_2|, \dots, |U \cap V_d|)\,.$$

Let $I_{\mathcal{P}}(H) = \{\mathbf{i}_{\mathcal{P}}(e) : e \in E(H)\}$ be the set of *edge-vectors* and, for $\mu > 0$, let

$$I^{\mu}_{\mathcal{P}}(H) = \{\mathbf{v} \in I_{\mathcal{P}}(H) : \text{ there are at least } \mu n^k \text{ edges } e \text{ in } H \text{ such that } \mathbf{v} = \mathbf{i}_{\mathcal{P}}(e)\},$$

be the set of *$\mu$-robust edge-vectors*.

We call an additive subgroup of $\mathbb{Z}^d$ a *lattice*, and we let $L_{\mathcal{P}}(H)$ and $L^{\mu}_{\mathcal{P}}(H)$ be the lattices generated by $I_{\mathcal{P}}(H)$ and $I^{\mu}_{\mathcal{P}}(H)$, respectively. Clearly if $M$ is a collection of vertex-disjoint edges in $H$, then $\mathbf{i}_{\mathcal{P}}(V(M)) \in L_{\mathcal{P}}(H)$. Therefore, $H$ does not have a perfect matching if $\mathbf{i}_{\mathcal{P}}(V(H)) \notin L_{\mathcal{P}}(H)$. For example, suppose that $G$ is an oriented graph with a divisibility barrier. If $|V(G)|$ is not divisible by 3, then $\mathbf{i}_{\mathcal{P}}(V(G)) \notin L_{\mathcal{P}}(H(G))$ when $\mathcal{P} = \{V(G)\}$. Otherwise, there exists a partition $\mathcal{P} = \{V_1, V_2, V_3\}$ of $V(G)$ such that the entries of $\mathbf{i}_{\mathcal{P}}(V(G))$ are each different modulo 3, and, for every $e \in E(H(G))$, the entries of $\mathbf{i}_{\mathcal{P}}(e)$ are each the same modulo 3, so $\mathbf{i}_{\mathcal{P}}(V(G)) \notin L_{\mathcal{P}}(H(G))$. We let $\mathbf{u}_i \in \mathbb{Z}^d$ be the *$i$th unit vector*, i.e., $\mathbf{u}_i$ is the vector in which the $i$th component is 1 and every other component is 0. A *transferral* is a vector $\mathbf{v}$ in $\mathbb{Z}^d$ such that $\mathbf{v} = \mathbf{u}_i - \mathbf{u}_j$ for distinct $i$ and $j$ in $[d]$. A *2-transferral* $\mathbf{v} \in L^{\mu}_{\mathcal{P}}(H)$ is a transferral such that $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ for $\mathbf{v}_1, \mathbf{v}_2 \in I^{\mu}_{\mathcal{P}}(H)$. We say that $L^{\mu}_{\mathcal{P}}(H)$ is *2-transferral-free* if $L^{\mu}_{\mathcal{P}}(H)$ does not contain a 2-transferral.

Let $x$ and $y$ be vertices in $V$ and let $\beta > 0$. A set $S \subseteq V$ is called an *$(H, x, y)$-linking $(k\ell - 1)$-set* if both $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ have perfect matchings and $|S| = k\ell - 1$. The vertices $x$ and $y$ are *$(H, \beta, \ell)$-reachable* if there are at least $\beta n^{\ell k - 1}$ $(H, x, y)$-linking $(k\ell - 1)$-sets. A set $U \subseteq V$ is *$(H, \beta, \ell)$-closed* if every pair of distinct vertices $x$ and $y$ in $U$ is $(H, \beta, \ell)$-reachable. Call a partition $\mathcal{P} = \{V_1, \dots, V_d\}$ of $V(H)$ a *$(H, \beta, \ell)$-closed partition* if, for every $i \in [d]$, $V_i$ is $(H, \beta, \ell)$-closed. For every vertex $x \in V(H)$, let $\tilde{N}_H(\beta, \ell, x)$ be the set of vertices $y$ such that $x$ and $y$ are $(H, \beta, \ell)$-reachable.

## 2 Proof of Theorem 4

### 2.1 Overview

Our proof follows the stability method, i.e., we divide the proof into two cases depending on whether $G$ is $\gamma$-extremal. The case when $G$ is $\gamma$-extremal is handled in the following lemma, the proof of which we defer until Section 2.5.

**Lemma 7** (Extremal case)**.** *Suppose that $1/n \ll c, \gamma \ll 1$, $n$ is divisible by 3, and $G$ is an oriented graph on $n$ vertices. If $\delta^0(G) \geqslant (1/2 - c)n$ and $G$ is $\gamma$-extremal, then $G$ has a cyclic triangle factor if and only if $G$ does not have a divisibility barrier.*

The proof in the case when $G$ is not $\gamma$-extremal follows the absorbing method, and the following lemma of Lo & Markström [17] serves as our absorbing lemma.

**Lemma 8** (Absorbing Lemma - Lemma 1.1 in [17])**.** *Given*

$$1/n \ll \alpha \ll \eta \ll \beta \ll 1/\ell, 1/k$$

*the following holds. If $H$ is a $k$-uniform hypergraph on $n$ vertices and $V(H)$ is $(H, \beta, \ell)$-closed, then there exists a subset $U$ of $V(H)$ where $|U|$ is at most $\eta n$ and divisible by $k$ such that for every $W \subseteq V(H) \setminus U$ with $|W|$ at most $\alpha n$ and divisible by $k$ there exists a perfect matching of $H[U \cup W]$.*

Lemma 8 and Theorem 2 imply the following lemma.

**Lemma 9.** *Suppose that $1/n \ll c \ll \beta \ll 1/\ell$, $n$ is divisible by 3, and that $G$ is an oriented graph on $n$ vertices. If $\delta^0(G) \geqslant (1/2 - c)n$ and $V(G)$ is $(H(G), \beta, \ell)$-closed, then $G$ has a cyclic triangle factor.*

*Proof.* Introduce constants $\eta$ and $\alpha$ so that

$$1/n \ll \alpha \ll \eta \ll c \ll \beta \ll 1/\ell.$$

Let $U \subseteq V(G)$ be the set guaranteed by Lemma 8 and let $G' = G - U$. Note that

$$\delta^0(G') \geqslant \delta^0(G) - |U| \geqslant (1/2 - c - \eta)n \geqslant (1/2 - 2c)|G'|.$$

Therefore, by Theorem 2, with $2c$ and $G'$ playing the roles of $c$ and $G$, respectively, there exists a cyclic triangle tiling $\mathcal{C}$ of $G'$ such that if $W = V(G') \setminus V(\mathcal{C})$, then $|W| \in \{0, 3\}$. Since $|W| \leqslant \alpha n$ and $|W|$ is divisible by 3, there exists a triangle factor $\mathcal{C}'$ of $G[U \cup W]$. Hence, $\mathcal{C} \cup \mathcal{C}'$ is a triangle factor of $G$. □

Note that, with Lemmas 7 and 9, the following lemma implies Theorem 4.

**Lemma 10.** *Suppose that $1/n \ll c \ll \beta \ll \gamma < 1$ and that $G$ is an oriented graph on $n$ vertices. If $\delta^0(G) \geqslant (1/2 - c)n$, and, for every positive integer $\ell \leqslant 1000$, we have that $V(G)$ is not $(H(G), \beta, \ell)$-closed, then $G$ is $\gamma$-extremal.*

The proof of Lemma 10 relies on the following four lemmas.

**Lemma 11.** *Suppose that $1/n \ll \beta \ll \beta', \mu \ll 1/d, 1/\ell$, $G$ is an oriented graph on $n$ vertices, and that $\mathcal{P} = \{V_1, \dots, V_d\}$ is a partition of $V(G)$. If $\mathcal{P}$ is $(H(G), \beta', \ell)$-closed and there exists a 2-transferral $\mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P}}(H(G))$, then the partition formed by merging $V_i$ and $V_j$ is $(H(G), \beta, 4\ell+1)$-closed.*

**Lemma 12.** *Suppose that $1/n \ll c \ll \beta \ll 1$ and that $G$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. Then there exists $\mathcal{P} = \{V_1, \dots, V_d\}$ a $(H(G), \beta, 8)$-closed partition of $V(G)$ such that $d \leqslant 4$ and $|V_i| > n/9$ for every $i \in [d]$.*

**Lemma 13.** *Suppose that $1/n \ll c \ll \mu \ll \alpha \ll \eta < 1$ and $G$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. If $\mathcal{P}$ is a non-trivial $\eta$-partition of $V(G)$ and $L^{\mu}_{\mathcal{P}}(H(G))$ is 2-transferral-free, then there exists $A \in \mathcal{P}$ such that $\mathrm{cyc}(A, A, \overline{A}) \leqslant \alpha n^3$.*

**Lemma 14.** *Suppose that $1/n \ll c \ll \mu, \alpha \ll \gamma, \eta < 1$ and $G$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. If $\mathcal{P}$ is a non-trivial $\eta$-partition of $V(G)$, $L^{\mu}_{\mathcal{P}}(H(G))$ is 2-transferral-free, and there exists $A \in \mathcal{P}$ such that $\mathrm{cyc}(A, A, \overline{A}) \leqslant \alpha n^3$, then $G$ is $\gamma$-extremal.*

We prove Lemmas 11 and 12 in Section 2.2, Lemma 13 in Section 2.3, and Lemma 14 in Section 2.4.

*Proof of Lemma 10.* Introduce additional constants, $\beta'$, $\mu$ and $\alpha$ so that

$$1/n \ll c \ll \beta \ll \beta' \ll \mu \ll \alpha \ll \gamma < 1.$$

By Lemma 12, with $\beta'$ playing the role of $\beta$, for some $d' \leqslant 4$, there exists $\mathcal{P}' = \{V'_1, \dots, V'_{d'}\}$ a $(H(G), \beta', 8)$-closed 0.1-partition of $V(G)$. If $L^{\mu}_{\mathcal{P}'}(H(G))$ contains a 2-transferral $\mathbf{u}_i - \mathbf{u}_j$ for distinct $i$ and $j$ in $[d']$, then *merge the parts that correspond to the 2-transferral*, i.e., consider the new partition $\mathcal{P}' - V'_i - V'_j + (V'_i \cup V'_j)$. Continue to merge the parts that correspond to 2-transferrals until we have a partition $\mathcal{P}$ such that $L^{\mu}_{\mathcal{P}}(H(G))$ is 2-transferral-free. By Lemma 11, we can assume that $\mathcal{P}$ is an $(H(G), \beta, \ell)$-closed 0.1-partition of $V(G)$ for some $\ell \leqslant 5^3 \cdot 8 = 1000$. If $|\mathcal{P}| = 1$, then $V(G)$ is $(H(G), \beta, \ell)$-closed which contradicts our assumptions, so we can assume that $\mathcal{P}$ is non-trivial. By Lemma 13, there exists $A \in \mathcal{P}$ such that $\mathrm{cyc}(A, A, \overline{A}) \leqslant \alpha n^3$ and by Lemma 14 we have that $G$ is $\gamma$-extremal. □

## 2.2 Proofs of Lemmas 11 and 12

We start this section with a proof of Lemma 11.

*Proof of Lemma 11.* Let $\xi$ be such that $\beta \ll \xi \ll \beta' \ll 1/\ell$. Note that distinct vertices $u_0$ and $v_0$ are $(H(G), \beta, 4\ell+1)$-reachable if there exist at least $\xi n^{12\ell+2}$ ordered $(12\ell+2)$-tuples $\mathcal{T}$ that each can be permuted to form $(12\ell+2)$-tuples $(u_1, \dots, u_{12\ell+2})$ and $(v_1, \dots, v_{12\ell+2})$ such that $u_{3j}u_{3j+1}u_{3j+2}$ and $v_{3j}v_{3j+1}v_{3j+2}$ are both cyclic triangles for every $0 \leqslant j \leqslant 4\ell$.

This is because $\beta < \xi^2/2$; and there are only at most $(12\ell+2)! < 1/\xi$ possible orderings for every such tuple; and only at most

$$2(12\ell+2)n^{12\ell+1} + n \cdot (12\ell+2)^2 \cdot n^{12\ell} < \xi n^{12\ell+2}/2$$

such $(12\ell+2)$-tuples that contain $u_0$ or $v_0$ or that have repeated vertices.

We will first show that if $u, v \in V(G)$ are $(H(G), \beta', \ell)$-reachable, then $u$ and $v$ are also $(H(G), \beta, 4\ell+1)$-reachable. In particular, this will imply that, because $\mathcal{P}$ is $(H(G), \beta', \ell)$-closed, $\mathcal{P}$ is also $(H(G), \beta, 4\ell+1)$-closed. Let $\mathcal{T}$ be the set of $(12\ell+2)$-tuples $(x_1, x_2, \ldots, x_{12\ell+2})$ such that

- $\{x_1, x_2, \ldots, x_{3\ell-1}\}$ is a $(H(G), u, v)$-linking $(3\ell-1)$-set, and
- $G[\{x_{3j}, x_{3j+1}, x_{3j+2}\}]$ is a cyclic triangle for $\ell \leqslant j \leqslant 4\ell$.

Since $u$ and $v$ are $(H(G), \beta', \ell)$-reachable, there are at least $\beta' n^{3\ell-1}$ ways to select the first $3\ell-1$ entries of a tuple in $\mathcal{T}$. For every such selection, there are exactly $(6\operatorname{cyc}(V(G)))^{3\ell+1}$ ways to select the remaining $9\ell+3$ entries of a tuple in $\mathcal{T}$. To see that $6\operatorname{cyc}(V(G)) \geqslant 2(\beta')^2 n^3$, let $V_q$ be the part in $\mathcal{P}$ of largest cardinality. We have that $|V_q| \geqslant n/d \geqslant \beta' n$. Because $V_q$ is $(H(G), \beta', \ell)$-closed, for every $v \in V_q$, the set $\tilde{N}_{H(G)}(\beta', \ell, v)$ is not empty, which implies that $v$ is in at least $\beta' n^2$ cyclic triangles. Therefore, because $6\operatorname{cyc}(V(G))$ is the number of ordered triples $(x, y, z)$ such that $G[\{x, y, z\}]$ is a cyclic triangle in $G$, we have that $6\operatorname{cyc}(V(G)) \geqslant 2(\beta')^2 n^3$. Hence,

$$|\mathcal{T}| \geqslant \beta' n^{3\ell-1} \cdot \left(2(\beta')^2 n^3\right)^{3\ell+1} > \xi n^{12\ell+2},$$

so $u$ and $v$ are $(H(G), \beta, 4\ell+1)$-reachable.

Now we will complete the proof by showing that if $u_0 \in V_i$ and $v_3 \in V_j$, then $u_0$ and $v_3$ are $(H(G), \beta, 4\ell+1)$-reachable. By assumption, there are $A, B \in \mathcal{P}$ such that $\operatorname{cyc}(V_i, A, B)$ and $\operatorname{cyc}(A, B, V_j)$ are both at least $\mu n^3$. Let $\mathcal{T}$ be the set of $(12\ell+2)$-tuples

$$(v_0, v_1, v_2, u_1, u_2, u_3, w_1, \ldots, w_{12\ell-4})$$

that satisfy the following:

- $v_0, v_1, v_2$ is a cyclic triangle with $v_0 \in V_i$, $v_1 \in A$ and $v_2 \in B$;
- $u_1, u_2, u_3$ is a cyclic triangle with $u_1 \in A$ and $u_2 \in B$, and $u_3 \in V_j$; and
- $\{w_{i(3\ell-1)+1}, \ldots, w_{(i+1)(3\ell-1)}\}$ is a $(H(G), u_i, v_i)$-linking $(3\ell-1)$-set for $i \in \{0, 1, 2, 3\}$.

Since $\operatorname{cyc}(V_i, A, B)$ and $\operatorname{cyc}(A, B, V_j)$ are both at least $\mu n^3$ and $V_i$, $V_j$, $A$, $B$ are all $(H(G), \beta', \ell)$-closed, we have that

$$|\mathcal{T}| \geqslant (\mu n^3) \cdot (\mu n^3) \cdot (\beta' n^{3\ell-1})^4 \geqslant \xi n^{12\ell+2},$$

so $u_0$ and $v_3$ are $(H(G), \beta, 4\ell+1)$-reachable. □

To prove Lemma 12, we use the following lemma of Han [8].

**Lemma 15** (Lemma 3.8 in [8]). *Given $1/n \ll \beta \ll \alpha \ll \delta, \delta', 1/k$ the following holds. Let $H$ be a $k$-uniform hypergraph on $n$ vertices. Assume that $\left|\tilde{N}_H(\alpha, 1, v)\right| \geqslant \delta' n$ for every $v \in V(H)$ and $\delta_1(H) \geqslant \delta\binom{n-1}{k-1}$. Then for $d \leqslant \min\{\lfloor 1/\delta \rfloor, \lfloor 1/\delta' \rfloor\}$ there exists an $(H, \beta, 2^{\lfloor 1/\delta \rfloor - 1})$-closed partition $\mathcal{P} = \{V_1, \ldots, V_d\}$ of $V(H)$ such that for every $i \in \{1, \ldots, d\}$ we have that $|V_i| \geqslant (\delta' - \alpha)n$.*

To apply Lemma 15 in our context, we need lower bounds on $\delta_1(H(G))$ and on $\left|\tilde{N}_{H(G)}(\alpha, 1, v)\right|$ for every vertex $v \in V(G)$ when $G$ is an oriented graph with sufficiently high minimum semidegree. The following series of lemmas provide these lower bounds. Note that Lemmas 16, 17, and 18 are also used in other sections.

**Lemma 16.** *Suppose that $c > 0$ and $G = (V, E)$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. Then, for every $A \subseteq V$,*

$$e^+(A, \overline{A}), e^-(A, \overline{A}) = \frac{|A| \cdot |\overline{A}|}{2} \pm c|A|n.$$

*Proof.* We first get a lower bound for $e^+(A, \overline{A})$ as follows,

$$e^+(A, \overline{A}) = \sum_{x \in A} d^+(x) - d^+(x, A) \geqslant$$
$$\left(\sum_{x \in A} d^+(x)\right) - \binom{|A|}{2} \geqslant |A| \left(\frac{1}{2} - c\right) n - \frac{|A|^2}{2} = \frac{|A| \cdot |\overline{A}|}{2} - c|A|n.$$

By a similar computation, we have that $e^-(A, \overline{A}) \geqslant |A||\overline{A}|/2 - c|A|n$. The fact that

$$e^+(A, \overline{A}) + e^-(A, \overline{A}) \leqslant |A| \cdot |\overline{A}|,$$

then implies the upper bounds. □

The following lemma appears in [13]. We provide a proof for completeness.

**Lemma 17.** *Suppose that $c > 0$ and $G = (V, E)$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. Then, $\delta_1(H(G)) = n^2/8 \pm 2cn^2$.*

*Proof.* Let $u \in V$. We will show that $\mathrm{cyc}(u, V, V) = n^2/8 \pm 2cn^2$. Let $m = d^+(u)$ and assume $d^+(u) \leqslant d^-(u)$, so $m \leqslant (n-1)/2$. By the minimum semidegree condition, $|\overline{N(u)}| \leqslant 2cn$, so

$$e^+(N^+(u), \overline{N(u)}) \leqslant m \cdot 2cn.$$

Since we can assume $c < 1/2$,

$$n^2/4 - cn^2 < n^2/4 - c^2n^2 \leqslant m(n-m) \leqslant n^2/4.$$

With Lemma 16 and the fact that $m \leqslant n/2$, we can compute that $\mathrm{cyc}(u, V, V) = e^+(N^+(u), N^-(u))$ is equal to

$$e^+(N^+(u), \overline{N^+(u)}) - e^+(N^+(u), \overline{N(u)}) = \frac{m(n-m)}{2} \pm 3cnm = n^2/8 \pm 2cn^2.$$

Applying a similar argument when $d^-(u) \leqslant d^+(u)$ proves the lemma. □

**Lemma 18.** *Suppose that $c > 0$ and $G = (V, E)$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. Then, for every edge $uv \in E$,*

$$d^{-,+}(uv) - d^{+,-}(uv) = 0 \pm 4cn.$$

*This further implies that, for every pair of disjoint subsets $A$ and $B$ of $V$,*

$$\sum_{uv \in E^+(A,B)} \mathrm{cyc}(u, v, V) \geqslant \frac{e^+(A, B)^2}{2|A|} - cn^3.$$

*Proof.* To prove the first part of the lemma, note that

$$d^{+,-}(xy) = d^+(x) - d^{+,+}(xy) - |N^+(x) \setminus N(y)|,$$

and similarly $d^{-,+}(xy) = d^+(y) - d^{+,+}(xy) - |N^+(y) \setminus N(x)|$. Therefore,

$$d^{-,+}(xy) - d^{+,-}(xy) = (d^+(x) - d^+(y)) + (|N^+(y) \setminus N(x)| - |N^+(x) \setminus N(y)|) = 0 \pm 4cn, \quad (1)$$

by the minimum semidegree condition.

To prove the second part of the lemma, note that if $u \in A$ and $v \in N^+(u, B)$, then

$$d^{+,-}(uv) \geqslant d^-(v, N^+(u, B)).$$

This implies that

$$\sum_{v \in N^+(u,B)} d^{+,-}(uv) \geqslant \sum_{v \in N^+(u,B)} d^-(v, N^+(u, B)) = e(N^+(u, B)) \geqslant d^+(u, B) \left( \frac{d^+(u, B) - 2cn}{2} \right),$$

where the last inequality follows from the fact that $n - 2\delta^0(G) \leqslant 2cn$. This observation, with the first part of the lemma gives us that

$$\sum_{v \in N^+(u,B)} \mathrm{cyc}(u, v, V) = \sum_{v \in N^+(u,B)} d^{-,+}(uv) \geqslant \sum_{v \in N^+(u,B)} \left( d^{+,-}(uv) - 4cn \right) \geqslant d^+(u, B) \left( \frac{d^+(u, B)}{2} - 5cn \right).$$

Letting $m = e^+(A, B)$ and $|A| = a$, we have that, by the convexity of $f(x) = x^2$,

$$\sum_{uv \in E^+(A,B)} \mathrm{cyc}(u, v, V) \geqslant \sum_{u \in A} d^+(u, B) \left( \frac{d^+(u, B)}{2} - 5cn \right) \geqslant \frac{a}{2} \left( \frac{m}{a} \right)^2 - m \cdot 5cn \geqslant \frac{m^2}{2a} - cn^3,$$

where the last inequality follows because, by Lemma 16, $m \leqslant n^2/5$. □

**Lemma 19.** *Suppose $1/n \ll c \ll \alpha \ll 1$ and that $G = (V, E)$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. Then, for every $v \in V$,*

$$\left|\tilde{N}_{H(G)}(\alpha, 1, v)\right| \geqslant \left(\frac{1}{8} - 10\alpha\right) n.$$

*Proof.* Fix $v \in V$ and let $N = \tilde{N}_{H(G)}(\alpha, 1, v)$ and $\overline{N} = V \setminus N$. For every $U \subseteq V$, let $T(U)$ be the set of ordered triples $(x, y, u)$ such that $u \in U$, $xy \in E$ and both $vxy$ and $xyu$ are cyclic triangles. By definition, $u \in \overline{N}$ if and only if $|T(\{u\})| < \alpha n^2$, so

$$|T(N)| = |T(V)| - |T(\overline{N})| \geqslant |T(V)| - \alpha n^3.$$

For every $u \in V$, Lemma 17 implies that $|T(\{u\})| \leqslant (1/8 + 2c)n^2$, so

$$(1/8 + 2c)n^2 \cdot |N| \geqslant |T(N)| \geqslant |T(V)| - \alpha n^3$$

Therefore, to show that $|N| \geqslant (1/8 - 10\alpha)n$ and complete the proof, it suffices to prove that $|T(V)| \geqslant (1/64 - 2c)n^3$.

Let $m = e^+(N^+(v), N^-(v))$. By Lemma 18, we have that

$$|T(V)| = \sum_{xy \in E^+(N^+(v), N^-(v))} \mathrm{cyc}(x, y, V) \geqslant \frac{m^2}{2d^+(v)} - cn^3. \tag{2}$$

Because Lemma 17 implies that $m \geqslant (1/8 - 2c)n^2$, we have that

$$\frac{m^2}{2d^+(v)} \geqslant \frac{mn}{2} \cdot \frac{1/8 - 2c}{1/2 + c} \geqslant (1/16 - c)n^3 \cdot (1/4 - 5c) \geqslant (1/64 - c)n^3,$$

which, with (2), completes the proof of the lemma. □

*Proof of Lemma 12.* Let $\delta = 2/9$, $\delta' = 1/8 - 1/100$ and let $\beta \ll \alpha \ll 1$. By Lemmas 17 and 19, we have that $\delta_1(H(G)) \geqslant (1/8 - 2c)n^2 > \delta\binom{n-1}{2}$ and that $\left|\tilde{N}_{H(G)}(\alpha, 1, v)\right| \geqslant (1/8 - 10\alpha)n \geqslant \delta' n$ for every $v \in V(G)$. Since $\min\{\lfloor 1/\delta \rfloor, \lfloor 1/\delta' \rfloor\} = 4$, Lemma 15 implies that, for some $d \leqslant 4$, there exists $\mathcal{P} = \{V_1, \ldots, V_d\}$ a $(H(G), \beta, 8)$-closed partition of $V(G)$ such that and $|V_i| \geqslant (\delta' - \alpha)n > n/9$ for every $i \in [d]$. □

### 2.3 Proof of Lemma 13

In this section, we prove Lemma 13. In an effort to explain the structure of the proof at a high level, we first informally discuss how to derive a contradiction in the following situation. Suppose that $G$ is a regular tournament on $n$ vertices that has an $\eta$-partition $\mathcal{P}$ of $V(G)$ with three distinct parts $A, B, D \in \mathcal{P}$ such that

(S1) $A$ is the largest part in $\mathcal{P}$,
(S2) $G[A], G[B]$ are transitive tournaments,
(S3) every cyclic triangle with at least two vertices in $A$ has one vertex in $B$, and

(S4) every cyclic triangle with at least two vertices in $B$ has one vertex in $D$.

These conditions are an idealized version of the conditions we will use to produce a contradiction to prove Lemma 13. (Note that (S3) and (S4) together imply (S2).)

Let $C = \overline{A \cup B}$. Since $G[A]$ is a transitive tournament, there exist $x^+, x^- \in A$ such that $d^+(x^+, A) = d^-(x^-, A) = |A| - 1$. This implies that if we let $C^- = N^-(x^+, C)$, then we have that $|C^-|$ is at least

$$\delta^0(G) - |B| = (n-1)/2 - (|A|+|B|)/2 + (|A|-|B|)/2 = |C|/2 + (|A|-|B|)/2 - 1/2. \quad (3)$$

The same lower bound holds for $|C^+|$ when we let $C^+ = N^+(x^-, C)$. Let $v \in C^-$. Note that, because no cyclic triangle contains both $v$ and $x^+$ and has its third vertex in $A$, we have that $N^+(v, A) = A$. Similarly $N^-(v, A) = A$ for every $v \in C^+$. Therefore, $C^+$ and $C^-$ are disjoint, so, with (3) and the fact that $|A| \geqslant |B|$, we have that $|C^+|, |C^-| = |C|/2 \pm 1/2$ and $|B| \geqslant |A| - 1$.

Since $|B| \geqslant |A| - 1$ and $A$ is the largest part in $\mathcal{P}$, we have that $|D| \leqslant |A| \leqslant |B| + 1$. Therefore, using the same logic with $B$ and $D$ playing the roles of $A$ and $C$, respectively, we can find $y^+, y^- \in B$ such that $d^+(y^+, B) = d^-(y^-, B) = |B| - 1$ and the disjoint union of $N^-(y^+, \overline{B \cup D})$ and $N^+(y^-, \overline{B \cup D})$ covers all but at most two vertices of $\overline{B \cup D}$. In particular, because $A \subseteq \overline{B \cup D}$, if we let $A^- = N^-(y^+, A)$ and $A^+ = N^+(y^-, A)$ we have that $A^-$ and $A^+$ are disjoint sets and $|A^- \cup A^+| \geqslant |A| - 2$. Assume that $|A^-| \geqslant |A^+|$, so $|A^-| \geqslant |A|/2 - 1$. (Similar logic gives a contradiction when $|A^+| \geqslant |A^-|$). Because $G[A^-]$ is a transitive tournament, there exists $x \in A^-$ such that $d^+(x, A^-) = |A^-| - 1$.

To get a contradiction, we now argue that the out-degree of $x$ is greater than $(n-1)/2$. Recall that $N^-(v, A) = A$ for every $v \in C^+$. This means that $N^+(x)$ contains $C^+$. Moreover, since $xy^+$ is an edge and there are no triangles with one vertex in $A$ and two vertices in $B$, we have that $N^+(x)$ contains $B$. Therefore, $d^+(x) \geqslant |A^- \setminus x| + |B| + |C^+|$. Recall that $|A^-| \geqslant |A|/2 - 1$ and $|C^+| \geqslant |C|/2 - 1/2$. Therefore, the fact that $\{A, B, C\}$ partitions $V(G)$ implies that $d^+(x) \geqslant n/2 + |B|/2 - 5/2$. Because $\mathcal{P}$ is an $\eta$-partition, we have that $|B| \geqslant \eta n$, so this contradicts the fact that $G$ is a regular tournament. (See Figure 1.)

**Lemma 20.** *Suppose that $1/n \ll \alpha, c \ll \xi, \beta \ll 1$ and that $G$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant (1/2 - c)n$. If $A, B \subseteq V(G)$ are disjoint and $e^+(A, B) \leqslant \alpha n^2$, then $|\operatorname{SN}^-_\beta(A) \cap B| \geqslant |B| - \xi n$ and $|\operatorname{SN}^+_\beta(B) \cap A| \geqslant |A| - \xi n$*

*Proof.* Note that for every $v \in B \setminus \operatorname{SN}^-_\beta(A)$, we have that $d^-(v, A) \geqslant \beta n - (n - 2\delta^0(G)) \geqslant \beta n/2$, and also, for every $v \in A \setminus \operatorname{SN}^+_\beta(B)$, we have that $d^+(v, B) \geqslant \beta n/2$. Therefore, if we let $m$ be the maximum of $|B \setminus \operatorname{SN}^-_\beta(A)|$ and $|A \setminus \operatorname{SN}^+_\beta(B)|$, then $m \cdot \beta n/2 \leqslant e^+(A, B) \leqslant \alpha n^2$, which implies that $m \leqslant \xi n$. □

**Lemma 21.** *Suppose that $1/n \ll c, \alpha \ll \beta < 1$, and that $G = (V, E)$ is an $n$-vertex oriented graph such that $\delta^0(G) \geqslant (1/2 - c)n$. If $A \subseteq V$ and $\operatorname{cyc}(A) \leqslant \alpha n^3$, then, for $\sigma \in \{+, -\}$, there exists $x^\sigma \in A$ such that $d^\sigma(x^\sigma, A) \geqslant |A| - \beta n$.*

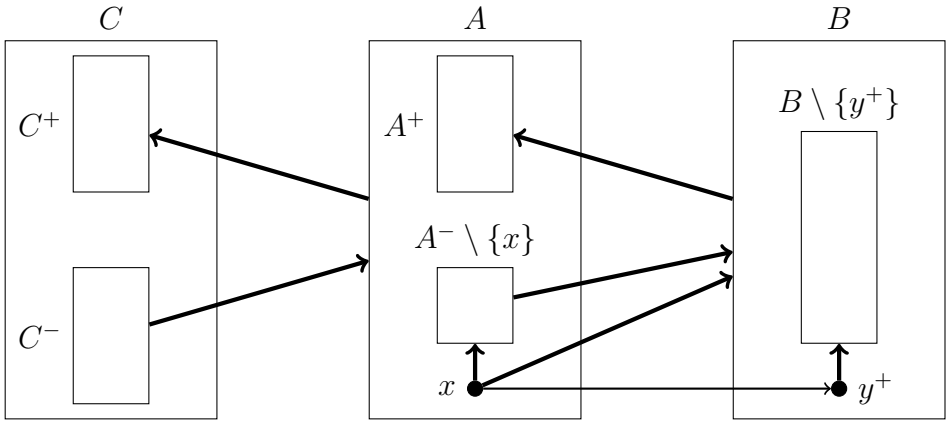


Figure 1: The situation at the end of the sketch of the proof of Lemma 13. The out-neighborhood of the vertex $x$ contains $A^- \setminus \{x\} \cup C^+ \cup B$ which has cardinality larger than $(n-1)/2$, a contradiction to the assumption that $G$ is a regular tournament.

*Proof.* Pick $\xi$ so that

$$1/n \ll c, \alpha \ll \xi \ll \beta \ll 1.$$

Let $X$ be the set of vertices $x$ in $A$ such that $\operatorname{cyc}(x, A, A) \leqslant \alpha^{1/2} n^2$. Since

$$(|A| - |X|) \cdot \alpha^{1/2} n^2 \leqslant 3 \operatorname{cyc}(A) \leqslant 3\alpha n^3$$

we have that

$$|X| \geqslant |A| - 3\alpha^{1/2} n. \tag{4}$$

Pick $x^+ \in X$ so as to maximize $d^+(x^+, X)$, and let $X^+ = N^+(x^+, X)$ and $X^- = N^-(x^+, X)$. Note that, because of the minimum semidegree condition, $|X^-| \geqslant |X| - |X^+| - 2cn$, and that $\alpha^{1/2} n^2 \geqslant \operatorname{cyc}(x^+, A, A) \geqslant e^+(X^+, X^-)$. Therefore, with Lemma 20, we have that

$$|\operatorname{SN}^-_\xi(X^+) \cap X^-| \geqslant |X^-| - \xi n \geqslant |X| - |X^+| - 2cn - \xi n,$$

and, with the minimum semidegree condition, there exists $y \in \operatorname{SN}^-_\xi(X^+) \cap X^-$ such that

$$d^+(y, X^-) \geqslant (|\operatorname{SN}^-_\xi(X^+) \cap X^-| - 2cn)/2 \geqslant (|X| - |X^+| - 2\xi n)/2$$

Because $y \in \operatorname{SN}^-_\xi(X^+)$, we have that

$$d^+(y, X^+) \geqslant |X^+| - \xi n.$$

Therefore, by the selection of $x^+$ we have that,

$$|X^+| \geqslant d^+(y, X^+) + d^+(y, X^-) \geqslant (|X^+| - \xi n) + (|X| - |X^+| - 2\xi n)/2 = (|X| + |X^+| - 4\xi n)/2$$

which, with (4), implies $d^+(x^+, A) \geqslant |X^+| \geqslant |X| - 4\xi n \geqslant |A| - \beta n$. By a similar argument, we can find $x^- \in A$, such that $d^-(x^-, A) \geqslant |A| - \beta n$. □

**Lemma 22.** *Let $1/n \ll \alpha, c \ll \xi \ll \eta < 1$. Let $G = (V, E)$ be an $n$-vertex oriented graph such that $\delta^0(G) \geqslant (1/2 - c)n$. For every partition $\{A, B, C\}$ of $V$ such that $|A| \geqslant \eta n$ and $\mathrm{cyc}(A), \mathrm{cyc}(A, A, C) \leqslant \alpha n^3$ there exist disjoint subsets $C^+$ and $C^-$ of $C$ such that, for $\sigma \in \{+, -\}$,*

$$e^{-\sigma}(A, C^\sigma) \leqslant \xi n^2 \qquad \text{and} \qquad |C^\sigma| \geqslant |C|/2 + (|A| - |B|)/2 - \xi n.$$

*Proof.* We will prove the lemma by showing that, with $\xi/2$ playing the role of $\xi$, there exist subsets $C^+$ and $C^-$ of $C$ that meet the conditions of the lemma except that $C^+ \cap C^-$ might not be empty. This will prove the lemma, because then, by the minimum semidegree condition,

$$(|A| - 2cn)|C^+ \cap C^-| \leqslant e^+(A, C^+ \cap C^-) + e^-(A, C^+ \cap C^-) \leqslant \xi n^2,$$

and this, with the fact that $|A| \geqslant \eta n$, implies that $|C^+ \cap C^-| \leqslant \xi n/2$, which means that the sets $C^+ \setminus C^-$ and $C^- \setminus C^+$ are disjoint sets that meet the conditions of the lemma. Furthermore, we will only prove that such a $C^+$ exists, because the existence of the desired set $C^-$ follows by a similar argument.

Let $X$ be the set of vertices $x$ in $A$ such that $\mathrm{cyc}(x, A, C) \leqslant \alpha^{1/2} n^2$. Because

$$(|A| - |X|) \cdot \alpha^{1/2} n^2 \leqslant 3\,\mathrm{cyc}(A, A, C) \leqslant 3\alpha n^3$$

we have that $|X| \geqslant |A| - 3\alpha^{1/2} n$. Since $\mathrm{cyc}(X, X, X) \leqslant \mathrm{cyc}(A, A, A) \leqslant \alpha n^3$, Lemma 21 implies that there exists $x \in X$ such that

$$d^-(x, A) \geqslant d^-(x, X) \geqslant |X| - \xi n/4 \geqslant |A| - \xi n/3.$$

Let $C^+ = N^+(x, C)$. Because every edge directed from $C^+$ to $N^-(x, A)$ corresponds to a triangle in $\mathrm{cyc}(x, A, C)$, we have that

$$e^-(A, C^+) \leqslant \mathrm{cyc}(x, A, C) + e^-(A \setminus N^-(x, A), C^+) \leqslant \xi n^2/2.$$

We also have that

$$|C^+| \geqslant \delta^0(G) - d^+(x, A) - |B| \geqslant n/2 - |B| - cn - \xi n/3 \geqslant |C|/2 + (|A| - |B|)/2 - \xi n/2.$$

This completes the proof of the lemma. □

*Proof of Lemma 13.* Pick $\beta$, $\alpha'$ and $\xi$ so that

$$1/n \ll c \ll \mu \ll \alpha' \ll \alpha, \xi \ll \beta \ll \eta < 1.$$

For a contradiction, assume that

$$\mathrm{cyc}(V_i, V_i, \overline{V_i}) \geqslant \alpha n^3 \qquad \text{for all } V_i \in \mathcal{P} = \{V_1, \ldots, V_d\}. \tag{5}$$

Because $\mathcal{P}$ is an $\eta$-partition we have that $1/d \geqslant \eta$ which implies that

$$\forall J \subseteq [d], \forall i \in [d], \exists j' \in J \text{ such that } \mathrm{cyc}(V_i, V_i, V_{j'}) \geqslant \eta \cdot c(V_i, V_i, \bigcup_{j \in J} V_j). \tag{6}$$

Let $A = V_x$ be the largest part in $\mathcal{P}$. By (5) and (6) with $J = [d] - x$, there exists $V_y = B \in \mathcal{P} - A$ such that $\mathrm{cyc}(A, A, B) \geqslant \alpha' n^3$. Let $C = V \setminus (A \cup B)$. By (5) and (6) with $J = [d] - y$, we also have $V_z = D \in \mathcal{P} - B$ such that $c(B, B, D) \geqslant \alpha' n^3$. Let $F = V \setminus (B \cup D)$.

Note that $\mathbf{v}_1 = 2\mathbf{u}_x + \mathbf{u}_y \in I^{\mu}_{\mathcal{P}}(H(G))$ and $\mathbf{v}_2 = 2\mathbf{u}_y + \mathbf{u}_z \in I^{\mu}_{\mathcal{P}}(H(G))$. If $A = D$, then $\mathbf{v}_1 - \mathbf{v}_2$ is a 2-transferral in $L^{\mu}_{\mathcal{P}}(H(G))$. Furthermore, if $\mathrm{cyc}(A) \geqslant \alpha' n^3$, then $\mathbf{v}_3 = 3\mathbf{u}_x \in I^{\mu}_{\mathcal{P}}(H(G))$, so $\mathbf{v}_1 - \mathbf{v}_3$ is a 2-transferral in $L^{\mu}_{\mathcal{P}}(H(G))$. Also, if $\mathrm{cyc}(A, A, C) \geqslant \alpha' n^3$, then, by (6) with $J = [d] - x - y$, there exists $w \in [d] - x - y$ such that $\mathbf{v}_4 = 2\mathbf{u}_x + \mathbf{u}_w \in I^{\mu}_{\mathcal{P}}(H(G))$, so $\mathbf{v}_1 - \mathbf{v}_4$ is a 2-transferral in $L^{\mu}_{\mathcal{P}}(H(G))$. With additional similar arguments, the fact that there are no 2-transferrals in $L^{\mu}_{\mathcal{P}}(H(G))$ implies that $A \neq D$ and

$$\mathrm{cyc}(A), \mathrm{cyc}(A, A, C), \mathrm{cyc}(B), \mathrm{cyc}(B, B, F) < \alpha' n^3. \tag{7}$$

By Lemma 22, there exist disjoint subsets $C^+$, $C^-$ of $C$ such that, for $\sigma \in \{-, +\}$,

$$e^{-\sigma}(A, C^{\sigma}) \leqslant \xi n^2, \tag{8}$$

and

$$|C^{\sigma}| \geqslant |C|/2 + (|A| - |B|)/2 - \xi n. \tag{9}$$

By the selection of $A$ we have that $|A| \geqslant |B|$, so (9) implies that

$$|C^{\sigma}| \geqslant |C|/2 - \xi n. \tag{10}$$

Because $C^+$ and $C^-$ are disjoint subsets of $C$, we have that $\min\{|C^+|, |C^-|\} \leqslant |C|/2$, so, with (9),

$$|B| \geqslant |A| - 2\xi n. \tag{11}$$

By Lemma 22, with $B$ and $F$ playing the roles of $A$ and $C$, respectively, there are disjoint subsets $F^+$ and $F^-$ of $F$ such that, for $\sigma \in \{-, +\}$,

$$e^{-\sigma}(B, F^{\sigma}) \leqslant \xi n^2. \tag{12}$$

and

$$|F^{\sigma}| \geqslant |F|/2 + (|B| - |D|)/2 - \xi n.$$

By the selection of $A$ and (11), we have that

$$|B| + 2\xi n \geqslant |A| \geqslant |D|,$$

so $|F^{\sigma}| \geqslant |F|/2 - 2\xi n$, and

$$|F^+| + |F^-| \geqslant |F| - 4\xi n. \tag{13}$$

Note that $A \subseteq F$ and fix $\sigma^* \in \{-, +\}$ so that $|F^{-\sigma^*} \cap A| \geqslant |F^{\sigma^*} \cap A|$. By (13) and the selection of $\sigma^*$, we have that

$$|F^{-\sigma^*} \cap A| \geqslant (|F^+ \cap A| + |F^- \cap A|)/2 \geqslant (|A| - 4\xi n)/2 = |A|/2 - 2\xi n. \tag{14}$$

Note that Lemma 20, with (8), implies that

$$|A \setminus \mathrm{SN}_\beta^{\sigma^*}(C^{\sigma^*})| \leqslant \beta n/2,$$

and Lemma 20, with (12) and (14), implies that

$$|\mathrm{SN}_\beta^{\sigma^*}(B) \cap F^{-\sigma^*} \cap A| \geqslant |A|/2 - \beta n/2.$$

Therefore, if we set $A' = \mathrm{SN}_\beta^{\sigma^*}(B) \cap \mathrm{SN}_\beta^{\sigma^*}(C^{\sigma^*}) \cap A$, then

$$|A'| \geqslant |\mathrm{SN}_\beta^{\sigma^*}(B) \cap F^{-\sigma^*} \cap A| - |A \setminus \mathrm{SN}_\beta^{\sigma^*}(C^{\sigma^*})| \geqslant |A|/2 - \beta n.$$

Since $\mathrm{cyc}(A') \leqslant \mathrm{cyc}(A) \leqslant \alpha n^3$, Lemma 21 implies that there exists $x \in A'$ such that

$$d^{\sigma^*}(x, A') \geqslant |A'| - \beta n \geqslant |A|/2 - 2\beta n.$$

Recall that $x \in \mathrm{SN}_\beta^{\sigma^*}(B) \cap \mathrm{SN}_\beta^{\sigma^*}(C^{\sigma^*})$. This implies that $d^{\sigma^*}(x, B) \geqslant |B| - \beta n$, and, with (10), we have that

$$d^{\sigma^*}(x, C^{\sigma^*}) \geqslant |C^{\sigma^*}| - \beta n \geqslant |C|/2 - 2\beta n.$$

Therefore, with the fact that $|B| \geqslant \eta n$, we have

$$d^{\sigma^*}(x) \geqslant (|A|/2 - 2\beta n) + (|B| - \beta n) + (|C|/2 - 2\beta n) = n/2 + |B|/2 - 5\beta n > (1/2 + c)n,$$

so $d^{-\sigma^*}(x) < (1/2 - c)n$, a contradiction to the minimum semidegree condition. □

### 2.4 Proof of Lemma 14

We need the following lemma for the proof of Lemma 14. Informally, it says that, in an oriented graph $G$ with sufficiently high minimum semidegree, if $A$ is a reasonably large subset of $V(G)$ for which $\mathrm{cyc}(A, A, \overline{A})$ is $o(n^3)$, then essentially half of the vertices in $\overline{A}$ have almost all of $A$ as out-neighbors and half of the vertices in $\overline{A}$ have almost all of $A$ as in-neighbors. From this, we then argue that $|A|$ cannot be much larger than $n/3$.

At a high-level, we use Lemma 23 to prove Lemma 14 in the following way (the actual proof of Lemma 14 appears after the proof of Lemma 23). We start with a non-trivial $\eta$-partition of $G$ such that $L^\mu_{\mathcal{P}}(H(G))$ is 2-transferral-free and such that there exists $A \in \mathcal{P}$ such that $\mathrm{cyc}(A, A, \overline{A})$ is $o(n^3)$. We then use Lemma 23 to get a partition $\{S^+, S^-\}$ of $\overline{A}$ such that $S^+$ and $S^-$ have roughly the same size and such that $e^+(A, S^-)$ and $e^+(S^+, A)$ are both $o(n^2)$. We can then finish the proof by showing that $A$, $S^+$ and $S^-$ each have roughly the same size (which is implied if $|S^+|$ is not much more than $n/3$) and that $e^+(S^-, S^+)$ is $o(n^2)$. To this end, we use the fact that $L^\mu_{\mathcal{P}}(H(G))$ is 2-transferral-free to show that $\mathrm{cyc}(S^+, S^+, \overline{S^+})$ is $o(n^3)$, and then apply Lemma 23 again with $S^+$ now playing the role of $A$.

**Lemma 23.** *Suppose that*

$$1/n \ll c \ll \alpha \ll \beta \ll \xi \ll \eta < 1,$$

*and $G = (V, E)$ is an oriented graph such that $\delta^0(G) \geqslant (1/2 - c)n$. For every $A \subseteq V$ such that $|A| > \eta n$ and $\mathrm{cyc}(A, A, \overline{A}) \leqslant \alpha n^3$, we have that*

- $|\operatorname{SN}^+_\beta(A)|, |\operatorname{SN}^-_\beta(A)| = |\overline{A}|/2 \pm \xi n$, *and*
- $|A| \leqslant (1/3 + \xi)n$.

*Proof.* Let $S^+ = \operatorname{SN}^+_\beta(A)$, $S^- = \operatorname{SN}^-_\beta(A)$ and $S = V \setminus (S^+ \cup S^- \cup A)$, so $\{S, S^+, S^-, A\}$ is a partition of $V$. We will first prove that

$$|S| \leqslant \beta n. \tag{15}$$

To this end, let $x \in S \subseteq \overline{A}$ and then $d^\sigma(x, A) \leqslant |A| - \beta n$ for any $\sigma \in \{+, -\}$. Since $\delta^0(G) \geqslant (1/2 - c)n$, then $x$ must have

$$d^\sigma(x, A) \geqslant |A| - d^{-\sigma}(x, A) - 2cn \geqslant \beta n - 2cn \geqslant \beta n/2.$$

Therefore,

$$\begin{aligned} |S|(\beta n)^2/4 &\leqslant \sum_{x \in S} d^+(x, A) \cdot d^-(x, A) \leqslant \sum_{x \in \overline{A}} d^+(x, A) \cdot d^-(x, A) \\ &= \sum_{xy \in E(A)} d^{-,+}(xy, \overline{A}) + d^{+,-}(xy, \overline{A}) = \operatorname{cyc}(A, A, \overline{A}) + \sum_{xy \in E(A)} d^{+,-}(xy, \overline{A}). \end{aligned} \tag{16}$$

By Lemma 18, we have that

$$\begin{aligned} \sum_{xy \in E(A)} d^{+,-}(xy, \overline{A}) &= \sum_{xy \in E(A)} \big(d^{+,-}(xy) - d^{+,-}(xy, A)\big) \\ &\leqslant \sum_{xy \in E(A)} \big(d^{-,+}(xy) - d^{+,-}(xy, A) + 4cn\big) \\ &= \sum_{xy \in E(A)} \big(d^{-,+}(xy, \overline{A}) + d^{-,+}(xy, A) - d^{+,-}(xy, A) + 4cn\big) \\ &= \operatorname{cyc}(A, A, \overline{A}) + 3\operatorname{cyc}(A) - \operatorname{trn}(A) + 4cn \cdot e(A). \end{aligned} \tag{17}$$

Because $e(A)/|A| = \frac{\sum_{v \in A} d^+(v, A)}{|A|} = \frac{\sum_{v \in A} d(v, A)}{2|A|} \geqslant \frac{|A| - 2cn}{2}$ and $f(x) = \binom{x}{2}$ is convex, we have that

$$\operatorname{trn}(A) = \sum_{v \in A} \binom{d^+(v, A)}{2} \geqslant |A| \binom{e(A)/|A|}{2} \geqslant \frac{|A|(|A| - 2cn - 1)^2}{8} \geqslant \frac{|A|^3}{8} - cn^3$$

Therefore, since $\operatorname{cyc}(A) + \operatorname{trn}(A) \leqslant \binom{|A|}{3}$,

$$3\operatorname{cyc}(A) - \operatorname{trn}(A) \leqslant 3\binom{|A|}{3} - 4\operatorname{trn}(A) \leqslant 4cn^3 \tag{18}$$

Combining (16), (17) and (18), we have that

$$|S|(\beta n)^2/4 \leqslant 2\operatorname{cyc}(A, A, \overline{A}) + 8cn^3 \leqslant 3\alpha n^3,$$

and (15) holds.

For $\sigma \in \{+,-\}$, we have that $e^\sigma(A,\overline{A}) \geqslant (|A| - \beta n)\cdot|S^\sigma|$ and, by (15) and the definition of $S^{-\sigma}$,

$$e^\sigma(A,\overline{A}) \leqslant |A|\cdot|S^\sigma| + (|A|-\beta n)\cdot|S| + \beta n\cdot|S^{-\sigma}| \leqslant |A|\cdot|S^\sigma| + 2\beta n^2,$$

so $|A|\cdot|S^\sigma| = e^\sigma(A,\overline{A}) \pm 2\beta n^2$. With Lemma 16, we have that

$$|A|\cdot|S^+| = e^+(A,\overline{A}) \pm 2\beta n^2 = e^-(A,\overline{A}) \pm 4\beta n^2 = |A|\cdot|S^-| \pm 6\beta n^2.$$

Therefore, because $|A| \geqslant \eta n$, we have that $|S^+| = |S^-| \pm \xi n$. This with (15) implies that

$$|\overline{A}| = n - |A| = |S^+| + |S^-| + |S| = 2|S^\sigma| \pm 2\xi n,$$

which proves the first part of the lemma.

We will prove the second part of the lemma by contradiction, so assume that $|A| > (1/3+\xi)n$. There exists $x \in S^+$ such that $d^+(x,S^+) < |S^+|/2$. Using the fact that $|\overline{A}| < (2/3-\xi)n$, and, by the first part of the lemma, $|S^+| \geqslant |\overline{A}|/2 - \xi n$, we have that

$$d^+(x,\overline{A}) < |S| + |S^-| + |S^+|/2 = |\overline{A}| - |S^+|/2 \leqslant |\overline{A}| - |\overline{A}|/4 + \xi n/2 < n/2 - \xi n/4.$$

Because $x \in S^+$, we have that $d^+(x,A) \leqslant \beta n$, so

$$d^+(x) = d^+(x,A) + d^+(x,\overline{A}) < (1/2-c)n,$$

a contradiction to the minimum semidegree condition. □

*Proof of Lemma 14.* Define $\xi$, $\beta$ and $\omega$ so that

$$1/n \ll c \ll \mu,\alpha \ll \xi \ll \beta \ll \omega \ll \gamma,\eta < 1.$$

By assumption, there exists $\mathcal{P}$ a non-trivial $\eta$-partition of $V(G)$ such that $L^\mu_{\mathcal{P}}(H(G))$ is 2-transferral-free and $A \in \mathcal{P}$ such that $\mathrm{cyc}(A,A,\overline{A}) \leqslant \alpha n^3$. Since $\mathcal{P}$ is an $\eta$-partition, we have that

$$|A| \geqslant \eta n. \tag{19}$$

For (possibly overlapping) subsets $U_1,U_2,U_3,U_4$ of $V$, let $L(U_1,U_2,U_3,U_4)$ be the collection of 4-sets $\{u_1,u_2,u_3,u_4\}$ such that $u_i \in U_i$ for $i \in [4]$ and both $u_1u_2u_3$ and $u_2u_3u_4$ are cyclic triangles. We can assume that

$$|L(A,V,V,\overline{A})| \leqslant \xi n^4, \tag{20}$$

because otherwise, since $|\mathcal{P}| \leqslant 1/\eta$, there would exist $B,C \in \mathcal{P}$ and $D \in \mathcal{P} - A$ such that $|L(A,B,C,D)| \geqslant \eta^3\cdot\xi n^4 \geqslant 3\mu n^4$, which would in turn imply that both $c(A,B,C) \geqslant \mu n^3$ and $c(B,C,D) \geqslant \mu n^3$, which contradicts the fact that $L^\mu_{\mathcal{P}}(H(G))$ is 2-transferral-free.

Define $S^\sigma = \mathrm{SN}^\sigma_\beta(A)$ for $\sigma \in \{+,-\}$, and let $S = V \setminus (S^+ \cup S^- \cup A)$. Lemma 23 implies that

$$|S| \leqslant 2\xi n, \tag{21}$$

and that

$$|A| \leqslant (1/3 + \xi)n. \tag{22}$$

For $\sigma \in \{+, -\}$, suppose that there exists $xyz$ a triangle with $xy \in E(S^\sigma)$ and $z \in S^{-\sigma}$, i.e., $xyz$ is one of the triangles that is counted in $\mathrm{cyc}(S^\sigma, S^\sigma, S^{-\sigma})$. If $\sigma = +$, then for every $w \in N^{-,+}(yz, A)$, we have $\{w, y, z, x\} \in L(A, S^+, S^-, S^+)$, and if $\sigma = -$, then for every $w \in N^{-,+}(zx, A)$, we have $\{w, x, z, y\} \in L(A, S^-, S^+, S^-)$. Therefore, because $d^{-\sigma}(x, A), d^{-\sigma}(y, A), d^{\sigma}(z, A) \geqslant |A| - \beta n$, (20) implies that

$$(|A| - 2\beta n) \cdot c(S^\sigma, S^\sigma, S^{-\sigma}) \leqslant |L(A, S^\sigma, S^{-\sigma}, S^\sigma)| \leqslant \xi n^4.$$

Therefore, with (19), we have

$$\mathrm{cyc}(S^\sigma, S^\sigma, S^{-\sigma}) \leqslant \beta n^3.$$

Furthermore,

$$\mathrm{cyc}(S^\sigma, S^\sigma, A) \leqslant 2\beta n \cdot \binom{|S^\sigma|}{2} \leqslant \beta n^3,$$

and, because (21) implies that

$$\mathrm{cyc}(S^\sigma, S^\sigma, S) \leqslant 2\beta n^3,$$

we can deduce that

$$\mathrm{cyc}(S^\sigma, S^\sigma, \overline{S^\sigma}) = \mathrm{cyc}(S^\sigma, S^\sigma, S^{-\sigma}) + \mathrm{cyc}(S^\sigma, S^\sigma, A) + \mathrm{cyc}(S^\sigma, S^\sigma, S) \leqslant 4\beta n^3.$$

Applying Lemma 23 with $S^\sigma$, $4\beta$ and $\omega$ playing the roles of $A$, $\alpha$ and $\xi$, respectively, implies that

$$|S^+|, |S^-| \leqslant (1/3 + \omega)n. \tag{23}$$

Let $A' = A \cup S$ and note that $\{A', S^+, S^-\}$ is a partition of $V(G)$, and, since (21), (22) and (23) imply that $|A'|, |S^+|, |S^-| \leqslant n/3 + \omega n$, we have

$$|A'|, |S^+|, |S^-| = n/3 \pm 2\omega n. \tag{24}$$

By (21), we have that

$$d^{-\sigma}(v, A') \geqslant |A| - \beta n \geqslant |A'| - \omega n \text{ for every } v \in S^\sigma,$$

which implies that

$$e^\sigma(S^\sigma, A') \leqslant |S^\sigma| \cdot \omega n \leqslant \omega n^2, \tag{25}$$

and, also, with Lemma 16, that

$$e^+(S^+, \overline{S^+}) \geqslant e^+(\overline{S^+}, S^+) - cn^2 \geqslant e^+(A', S^+) - cn^2 \geqslant (|A'| - \omega n)|S^+| - cn^2.$$

Combining this with (24) and (25) gives us that

$$e^+(S^+, S^-) = e^+(S^+, \overline{S^+}) - e^+(S^+, A') \geqslant (|A'| - \omega n)|S^+| - 2\omega n^2 \geqslant |S^-||S^+| - \gamma n^2,$$

so $e^+(S^-, S^+) \leqslant \gamma n^2$, which, with (24) and (25), implies that $G$ is $\gamma$-extremal. □

### 2.5 Proof of Theorem 7 - The Extremal Case

We use the following tripartite Corrádi-Hajnal theorem from [9], although we do not require its full strength. In fact, we only use it for the corollary which follows that could easily be proved directly. The multipartite Hajnal-Szemerédi theorem has been heavily studied by many researchers, for example, see [4, 12, 14, 18, 19].

**Theorem 24** (Johansson 2000 [9])**.** *Let $G = (V, E)$ be a graph. If there exists $\{V_1, V_2, V_3\}$ a partition of $V$ such that $|V_1| = |V_2| = |V_3| = m$ and, for every $i \in [3]$ and $v \in V_i$, $|N(v) \cap V_{i+1}|$, $|N(v) \cap V_{i-1}| \geqslant \frac{2}{3}m + \sqrt{m}$ then $G$ has a triangle factor.*

**Corollary 25.** *Suppose that $0 < 1/n \ll c, \xi \ll 1$, $n$ is a multiple of 3, and $G = (V, E)$ is an oriented graph on $n$ vertices such that $\delta^0(G) \geqslant \left(\frac{1}{2} - c\right) n$. If there exists a partition $\{V_1, V_2, V_3\}$ of $V$ such that $|V_1| \equiv |V_2| \equiv |V_3| \pmod 3$ and, for every $i \in [3]$ and $v \in V_i$, we have $d^+(v, V_{i+1})$, $d^-(v, V_{i-1}) \geqslant \left(\frac{1}{3} - \xi\right) n$, then $G$ has a cyclic triangle factor.*

*Proof.* Assume without loss of generality that $|V_1| \leqslant |V_2| \leqslant |V_3|$. For every $i \in [3]$, the degree condition implies that $|V_i| = (1/3 \pm 2\xi)n$, so

$$\delta^0(G[V_i]) \geqslant (1/2 - c)n - 3\xi n - (1/3 + 2\xi)n \geqslant (1/2 - 3c - 18\xi)|V_i|.$$

Therefore, by Lemma 17, we can greedily find a collection $\mathcal{C}_2$ of $(|V_2| - |V_1|)/3 \leqslant 2\xi n$ vertex-disjoint cyclic-triangles in $G[V_2]$, and a collection $\mathcal{C}_3$ of $(|V_3| - |V_1|)/3 \leqslant 2\xi n$ vertex-disjoint cyclic-triangles in $G[V_3]$. Let $H$ be the oriented graph induced by $V \setminus V(\mathcal{C}_2 \cup \mathcal{C}_3)$. Theorem 24 implies that there exists a triangle factor of the spanning subgraph of the simple graph underlying $H$ that contains the edges that correspond to the edges of $H$ that are directed either from $V_1$ to $V_2$, from $V_2$ to $V_3$, or from $V_3$ to $V_1$. This triangle factor corresponds to a cyclic triangle factor $\mathcal{C}$ of $H$. Then $\mathcal{C} \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is a triangle factor of $G$. □

*Proof of Lemma 7.* If $G$ has a divisibility barrier, then, by the argument presented before Definition 3, $G$ does not contain a cyclic triangle factor. For the other direction, assume that $G$ does not have a divisibility barrier. Since $G$ is $\gamma$-extremal, there exists a partition $\{V_1, V_2, V_3\}$ of $V$ such that for every $i \in [3]$, $|V_i| = (1/3 \pm \gamma)n$ and $e^-(V_i, V_{i+1}) \leqslant \gamma n^2$. Introduce new constants $\alpha$ and $\beta$, so that

$$0 < c, \gamma \ll \alpha \ll \beta \ll 1.$$

Let

$$U_i = \mathrm{SN}^-_\beta(V_{i+1}) \cap \mathrm{SN}^+_\beta(V_{i-1}) \qquad \text{for } i \in [3],$$

and let $U_0 = V \setminus (U_1 \cup U_2 \cup U_3)$. Note that $\{U_0, U_1, U_2, U_3\}$ is a partition of $V$. For every $i \in [3]$ and every $v \in V_i \setminus U_i$, at least one of $d^-(v, V_{i+1})$ or $d^+(v, V_{i-1})$ is at least $\beta n - 2cn \geqslant \beta n/2$, so

$$\frac{1}{2}\sum_{i=1}^{3} |V_i \setminus U_i| \cdot \beta n \leqslant 2e^-(V_1, V_2) + 2e^-(V_2, V_3) + 2e^-(V_3, V_1) \leqslant 6\gamma n^2,$$

and we have that

$$|U_0| \leqslant \sum_{i=1}^{3} |V_i \setminus U_i| \leqslant \alpha n. \tag{26}$$

Suppose that $|U_0| \geqslant 1$. By (26) and Lemma 17, we can greedily find a (possibly empty) collection $\mathcal{C}$ of vertex-disjoint cyclic-triangles such that

$$|V(\mathcal{C})| \leqslant 3(|U_0| - 1) \leqslant 3\alpha n \tag{27}$$

and $|U_0 \cap V(\mathcal{C})| = |U_0| - 1$. Let $u$ be the vertex in $U_0$ that is not covered by $\mathcal{C}$, i.e., $\{u\} = U_0 \setminus V(\mathcal{C})$. Let $G' = G - V(\mathcal{C})$, and, for $i \in [3]$, let $W_i = U_i \setminus V(\mathcal{C})$. Note that, by (26) and (27),

$$|V_i \setminus W_i| \leqslant 4\alpha n \qquad \text{for } i \in [3]. \tag{28}$$

Because $|G'| = |V \setminus V(\mathcal{C})|$ is divisible by 3, we have that $|W_1| + |W_2| + |W_3| \equiv 2 \pmod 3$, and we can assume without loss of generality that $|W_2| \equiv |W_3| \pmod 3$. Fix $x \in \{0, 1, 2\}$ so that

$$|W_1| \equiv x + 2 \text{ and } |W_2| \equiv |W_3| \equiv x \pmod 3.$$

If there exists a triangle $T$ in $G'$ such that $u \in V(T)$ and either $|V(T) \cap W_1| = 2$ or $|V(T) \cap W_2| = |V(T) \cap W_3| = 1$, then we can complete a cyclic triangle factor of $G$ using Corollary 25, so assume the contrary.

Let $W_1^+ = N^+(u, W_1)$, $W_1^- = N^-(u, W_1)$ and $W_1^0 = W_1 \setminus N(u)$. We have that $e^+(W_1^+, W_1^-) = 0$ by assumption, and, by the minimum semidegree condition,

$$|W_1^0| \leqslant 2cn. \tag{29}$$

Assume that $|W_1^-| \leqslant |W_1^+|$. If $W_1^- \neq \varnothing$, then there exists $x \in W_1^-$ such that

$$d^-(x, W_1^-) < |W_1^-|/2 \leqslant |W_1|/4,$$

so, by (28) and (29),

$$d^-(x, W_1^+) \geqslant \delta^0(G) - \big(d^-(x, W_1^- \cap V_1) + |W_1^0 \cap V_1| + |V_1 \setminus W_1| + d^-(x, V_2) + |V_3|\big) > 0,$$

a contradiction. Therefore, we have $W_1^- = \varnothing$. This, with (28) and (29), implies that

$$d^+(u, V_1) \geqslant |W_1 \cap V_1| - |W_1^0| = |V_1| - |V_1 \setminus W_1| - |W_1^0| \geqslant |V_1| - \beta n, \tag{30}$$

and

$$d^-(u, W_3 \cap V_3) \geqslant \delta^0(G) - \big(d^-(u, V_1) + |V_2| + |V_3 \setminus W_3|\big) > (1/6 - 2\beta)n.$$

By the definition of $W_2$, this implies that every vertex in $W_2$ has an out-neighbor in $N^-(u, W_3 \cap V_3)$. Therefore, because $e^+(N^+(u, W_2), N^-(u, W_3)) = 0$ by assumption, we have that $N^+(u, W_2) = \varnothing$, and, by a computation similar to (30)

$$d^-(u, V_2) \geqslant |W_2 \cap V_2| - |W_2 \setminus N(u)| = |V_2| - |V_2 \setminus W_2| - |W_2 \setminus N(u)| \geqslant |V_2| - \beta n,$$

and this with (30) implies that $u \in U_3$, a contradiction to the fact that $u \in U_0$. By a similar argument, if $|W_1^-| \geqslant |W_1^+|$, one can show that $W_1^+ = \varnothing$ and then $u \in U_2$, a contradiction.

Now suppose that $|U_0| = 0$. By Corollary 25, we are done if $|U_1| \equiv |U_2| \equiv |U_3| \pmod 3$, so assume the contrary. If $G$ has no divisibility barriers, then $|V|$ is divisible by 3, and, without loss of generality we can assume that there exists an edge $ab$ such that $a \in U_2$ and $b \in U_1$. Because $|V|$ is divisible by 3, we can fix $x \in \{0, 1, 2\}$ and $y \in \{1, 2\}$ so that

$$|U_1| \equiv x,\ |U_2| \equiv x + y, \text{ and } |U_3| = x - y \pmod 3$$

Note that, by (26),

$$d^+(b, U_1) \geqslant \delta^0(G) - (|V_1 \setminus U_1| + |V_2| + d^+(b, V_3)) \geqslant (1/6 - 2\beta)n,$$

and a similar bound holds for $d^-(a, U_2)$, so there exists a vertex $c \in U_2 \cap N^+(b) \cap N^-(a)$, and a vertex $d \in U_1 \cap N^+(b) \cap N^-(a)$. Hence $T_1 = abc$ and $T_2 = abd$ are cyclic triangles and, for $j \in \{1, 2\}$,

$$|V(T_j) \cap U_1| \equiv j \text{ and } |V(T_j) \cap U_2| \equiv 2j \pmod 3.$$

Therefore, Corollary 25 implies that there exists a cyclic triangle factor $\mathcal{C}$ of $G - T_y$, and $\mathcal{C} \cup \{T_y\}$ is a cyclic triangle factor of $G$. □

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