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Disjoint cycles and chorded cycles in a graph with given minimum degree

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ABSTRACT

In 1963, Corrádi and Hajnal settled a conjecture of Erdős by showing that every graph on at least $3r$ vertices with minimum degree at least $2r$ contains a collection of r disjoint cycles, and in 2008, Finkel proved that every graph with at least $4s$ vertices and minimum degree at least $3s$ contains a collection of s disjoint chorded cycles. The same year, a generalization of this theorem was conjectured by Bialostocki, Finkel, and Gyárfás: every graph with at least $3r+4s$ vertices and minimum degree at least $2r+3s$ contains a collection of $r+s$ disjoint cycles, s of them chorded. This conjecture was settled and further strengthened by Chiba et al. (2010). In this paper, we characterize all graphs on at least $3r+4s$ vertices with minimum degree at least $2r+3s-1$ that do not contain a collection of $r+s$ disjoint cycles, s of them chorded. In addition, we provide a conjecture regarding the minimum degree threshold for the existence of $r+s$ disjoint cycles, s of them chorded, and we prove an approximate version of this conjecture.

1. Introduction

Most of our notation is standard. All graphs in this paper are simple, unless otherwise noted. If D is a graph that contains a spanning cycle C and $e(D) > e(C)$, then we say that D is a *chorded cycle*, and we call every $e \in E(D) \setminus E(C)$ a *chord* (of C). When we say that two graphs are *disjoint*, we mean that they have no vertices in common. For non-negative integers r and s, we call the pair $(\mathcal{C}, \mathcal{D})$ an (r, s)-*family* if $\mathcal{C}$ and $\mathcal{D}$ are disjoint collections of subgraphs of G, such that $\mathcal{C}$ contains a collection of r disjoint cycles and $\mathcal{D}$ contains a collection of s disjoint chorded cycles. For disjoint graphs G and H, we use $G \vee H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G \cup H) \cup \{xy : x \in G \text{ and } y \in H\}$. We write $\overline{G}$ for the *complement* of G, which is the graph with vertex set $V(G)$ and edge set $\binom{V(G)}{2} \setminus E(G)$. For a graph G, we use $\delta(G)$ for its minimum degree, $\alpha(G)$ for its independence number, and $\alpha'(G)$ for its matching number.

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1.1. Background

In 1963, Corrádi and Hajnal settled a conjecture of Erdős by proving the following theorem.

Theorem 1 (*The Corrádi–Hajnal Theorem [4]*). *Every graph G on at least $3r$ vertices with $\delta(G) \geq 2r$ contains an $(r, 0)$-family (that is, G contains r disjoint cycles).*

A well-known corollary of Theorem 1 is that if $|G| = 3r$ and $\delta(G) \geq 2r$, then G contains r disjoint copies of K_3. This corollary was generalized by Hajnal and Szemerédi in 1970.

Theorem 2 (*The Hajnal–Szemerédi Theorem [8]*). *Every graph G on kr vertices with $\delta(G) \geq (k-1)r$ contains r disjoint copies of K_k.*

With the work in [10], a proof of the as yet unresolved Chen–Lih–Wu Conjecture [2] would provide a characterization of the sharpness examples for the Hajnal–Szemerédi Theorem. The following theorem of Kierstead and Kostochka proves the Chen–Lih–Wu Conjecture when $k \in \{3, 4\}$.

Theorem 3 (*Kierstead and Kostochka [10,11]*). *Let r be a positive integer, $k \in \{3, 4\}$, $m = (k-1)r - 1$, and let G be a graph on $n = kr$ vertices such that $\delta(G) \geq m$. Then G does not contain r disjoint copies of K_k if and only if either*

- $\alpha(G) = n - m$;
- $k = 3$, *r is odd, and* $G \cong \overline{K_r} \vee \overline{K_{r,r}}$; *or*
- $k = 4$, *r is odd, and* $G \cong \overline{H} \vee \overline{K_{r,r}}$ *where H is an r-equitable graph on $2r$ vertices.*[3]

When $k = 2$, a statement analogous to Theorem 3 holds and is a consequence of Tutte's Theorem on matchings. We state a slightly more general consequence of the Berge–Tutte Formula in Corollary 15.

In 1963, Dirac [5] characterized 3-connected multigraphs with no two disjoint cycles. In the same paper, he asked for a characterization of $(2r-1)$-connected multigraphs that do not contain an $(r, 0)$-family. Lovász [15] answered Dirac's question when $r = 2$ by describing all multigraphs that do not contain 2 disjoint cycles. Using Lovász's result and Theorem 3, Kierstead, Kostochka, and Yeager [13] gave the following characterization of the sharpness examples for Theorem 1.

Theorem 4 (*Kierstead, Kostochka, and Yeager, [13]*). *Given $r \geq 2$, let $m = 2r - 1$, and let G be a graph on $n \geq 3r$ vertices with $\delta(G) \geq m$. Then G does not contain an $(r, 0)$-family if and only if either*

- $\alpha(G) = n - m$;
- $n = 3r$, *r is odd, and* $G \cong \overline{K_r} \vee \overline{K_{r,r}}$; *or*
- $r = 2$ *and* $G \cong K_1 \vee C$ *where C is a cycle.*[4]

With Theorem 4, Kierstead, Kostochka, and Yeager [12] were able to completely characterize all $(2r-1)$-connected multigraphs that do not contain r-disjoint cycles, fully answering Dirac's question from 1963.

Theorem 1 has been expanded in other directions. Independently, Enomoto [6] and Wang [18] strengthened Theorem 1 by considering, instead of the minimum degree, the *minimum Ore-degree*, which is the minimum of $d_G(x) + d_G(y)$ over all non-adjacent pairs of distinct vertices x and y. We use $\sigma_2(G)$ to denote the minimum Ore-degree of a graph G, which is sometimes referred to as the *minimum degree-sum*. (When G is complete, we let $\sigma_2(G) = +\infty$.)

Theorem 5 (*Enomoto [6], Wang [18]*). *Every graph G on at least $3r$ vertices with $\sigma_2(G) \geq 4r - 1$ contains an $(r, 0)$-family.*

In 2008, Finkel proved a chorded-cycle analogue of Theorem 1.

Theorem 6 (*Finkel [7]*). *Every graph G on at least $4s$ vertices with $\delta(G) \geq 3s$ contains a $(0, s)$-family (that is, G contains s disjoint chorded cycles).*

The authors [16] recently characterized the sharpness examples of Theorem 6. In addition, they considered the minimum degree-sum instead of the minimum degree.

Theorem 7 (*Molla, Santana, and Yeager [16]*). *Given $s \geq 2$, let $m = 3s - 1$ and let G be a graph on $n \geq 4s$ vertices with $\sigma_2(G) \geq 2m$. Then G does not contain a $(0, s)$-family if and only if either*

(i) $n \geq 2m = 6s - 2$ *and* $G \cong K_{m,n-m}$; *or*
(ii) $n = 2m - 1 = 6s - 3$ *and* $G \cong K_{1,n-m,n-m}$.

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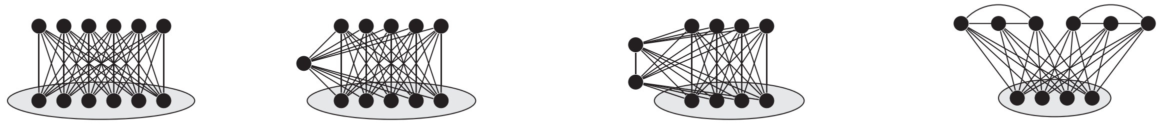
[3] An *r-equitable graph H* is an r-colorable graph on n vertices such that n is divisible by r and every color class of every r-coloring has order exactly n/r. The r-equitable graphs with maximum degree r are characterized in [10]. For odd r, a graph H on $2r$ vertices with $\Delta(H) \leq r$ is r-equitable if and only if $H \cong \overline{K_{r,r}}$; $r = 3$ and H is the graph F_5 from [10]; or $r = 5$ and H is the graph F_1 from [10].

[4] When C is a cycle, $K_1 \vee C$ is often referred to as a *wheel*.



(i) $K_{m,n-m}$ for $n = 2m$ (ii) $K_{1,n-m,n-m}$ (iii) $K_{2m-n} \vee K_{n-m,n-m}$ (iv) $\overline{K_{m/2,m/2}} \vee \overline{K_{n-m}}$

Fig. 1. Sample graphs from Theorem 9 when $r = 2$ and $s = 1$ (and so $m = 6$). In any one of these graphs, it is not possible to find three disjoint cycles, one of which is chorded. Note that each graph has an independent set of $n - m$ vertices circled.

Note that Theorem 7 implies the minimum degree condition in Theorem 6 is not sharp when $4s \leq n \leq 6s - 4$ (see Section 2 for further discussion of this case).

The following theorem of Chiba, Fujita, Gao and Li [3] is an extension of Theorems 1 and 6 and was initially conjectured (in a weaker form) and partially proved by Bialostocki, Finkel, and Gyárfás [1]. It serves as the principal motivation for our main result.

Theorem 8 (*Chiba, Fujita, Gao, and Li, [3]*)**.** *Given non-negative integers r and s with $r + s \geq 1$, let G be a graph on at least $3r + 4s$ vertices with $\sigma_2(G) \geq 4r + 6s - 1$. Then G contains an (r, s)-family.*

Let r and s be non-negative integers. We call a graph G on n vertices *(r, s)-extremal* if $n \geq 3r+4s$ and $\delta(G) \geq 2r+3s-1$, but G does not contain an (r, s)-family. For $r \geq 2$, Theorem 4 characterizes the $(r, 0)$-extremal graphs and, for $s \geq 2$, Theorem 7 characterizes the $(0, s)$-extremal graphs.

The main result of this paper is the following, which together with Theorem 4, characterizes the sharpness examples to Theorem 8 when the minimum degree-sum condition is replaced with the appropriate minimum degree condition. (See Fig. 1.)

Theorem 9. *Given $r \geq 0$ and $s \geq 1$ such that $r + s \geq 2$, let $m = 2r + 3s - 1$ and let G be a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq m$. Then G is an (r, s)-extremal graph if and only if either:*

(i) $n \geq 2m = 4r + 6s - 2$ and $G \cong K_{m,n-m}$;
(ii) $n = 2m - 1 = 4r + 6s - 3$ and $G \cong K_{1,n-m,n-m} = K_{2m-n,n-m,n-m}$;
(iii) $s = 1$, $3r + 4s \leq n \leq 4r + 6s - 4$, and

$$K_{2m-n,n-m,n-m} \subseteq G \subseteq \left(K_{2m-n} \vee K_{n-m,n-m}\right); \text{ or}$$

(iv) $s = 1$, r is even, $n = 3r + 4s$, and $G \cong \overline{K_{m/2,m/2}} \vee \overline{K_{n-m}} = \overline{K_{r+1,r+1}} \vee \overline{K_{n-m}}$.

Since Theorem 7 proves the case when $r = 0$, the remainder of this paper is dedicated to characterizing the (r, s)-extremal graphs when both r and s are positive.

Observe that just as Theorem 7 shows the minimum degree condition in Theorem 6 is not sharp when $4s \leq n \leq 6s-4$, Theorem 9 demonstrates that $2r + 3s$ is not the correct minimum degree threshold for an (r, s)-family when $3r + 4s \leq n \leq 4r + 6s - 4$ and $s \geq 2$.

The remainder of this paper is structured as follows. In Section 2, we present a conjecture on the correct minimum degree thresholds for an (r, s)-family, which if true would be a best possible result, and we prove an approximate version of this conjecture. In Section 3, we present Theorem 14, which is equivalent to Theorem 9, and is the version that we will prove in the remainder of the paper. We also introduce notation that will be used heavily in our proof of this theorem. Section 4 contains all of the lemmas from which we will deduce Theorem 14.

2. The case when $n < 4r + 6s$

In this section we address the minimum degree threshold for an (r, s)-family in a graph on n vertices when n, r, and s are non-negative integers such that $3r + 4s \leq n < 4r + 6s$. In this case, the problem of finding the minimum degree threshold for an (r, s)-family in graphs on n vertices is closely related to the theory of Tiling Turán Numbers introduced by Komlós in [14].

For every nonempty graph H, let $\sigma(H)$ be the size of the smallest color class over all proper $\chi(H)$-colorings of H, and define the *critical chromatic number of H* to be

$$\chi_{cr}(H) = \frac{(\chi(H) - 1)|H|}{|H| - \sigma(H)}.$$

For example, $\chi_{cr}(K_3) = 3$, $\chi_{cr}(K_{1,1,2}) = 8/3$, and $\chi_{cr}(K_{2,2}) = \chi_{cr}(K_{3,3}) = 2$.

For $1 \leq t \leq n/3$, define $G(t, n) = K_{t,\lfloor(n-t)/2\rfloor,\lceil(n-t)/2\rceil}$ and note that

$$\chi_{cr}(G(t, n)) = \frac{2n}{n - t} \qquad \text{and} \qquad \delta(G(t, n)) = \lfloor(n + t)/2\rfloor = \left\lfloor \left(1 - \frac{1}{\chi_{cr}(G(t, n))}\right) n \right\rfloor. \tag{1}$$

Let r_3, r_4, s_4, and s_6 be non-negative integers such that $r = r_3+r_4$ and $s = s_4+s_6$. Call an (r, s)-family an (r_3, r_4, s_4, s_6)-*family* if it contains exactly r_3 triangles, r_4 copies of $K_{2,2}$, s_4 copies of $K_{1,1,2}$, and s_6 copies of $K_{3,3}$. By induction on s_4', we have that $G(s_4', n')$ contains a $(0, 0, s_4', 0)$-family when $n' \geq 4s_4'$. This implies that

$$G(r_3 + s_4, n) \text{ contains an } (r_3, r_4, s_4, s_6)\text{-family when } n \geq 3r_3 + 4r_4 + 4s_4 + 6s_6. \tag{2}$$

Also note that every cycle in $G(t, n)$ on fewer than 4 vertices (i.e. the triangle) has at least one vertex in the part of size t. Similarly, every chorded cycle in $G(t, n)$ on less than 6 vertices must have at least one vertex in the part of size t. Therefore, $G(t, n)$ does not contain an (r, s)-family

$$\text{when } t < s \text{ and } n < 4r + 6s - 2t; \text{ or when } t \geq s \text{ and } n < 4r + 6s - (2s + (t - s)) = 4r + 4s - (t - s). \tag{3}$$

Define

$$r_3 = r_3(r, s, n) = \max\{0, 4r + 4s - n\} \qquad \text{and} \qquad s_4 = s_4(r, s, n) = \min\{s, \lceil (4r + 6s - n)/2 \rceil\}.$$

Because $r_3 \leq r$ and $1 \leq s_4 \leq s$ we have that $r_3 + s_4 \leq r + s \leq n/3$, and we can define $H(r, s, n) = G(r_3 + s_4, n)$ and $H'(r, s, n) = G(r_3 + s_4 - 1, n)$. If we let $r_4 = r - r_3$, and $s_6 = s - s_4$, we have that

$$3r_3 + 4r_4 + 4s_4 + 6s_6 = 4r + 6s - r_3 - 2s_4 \leq n,$$

so (2) implies that $H(r, s, n)$ contains an (r_3, r_4, s_4, s_6)-family, and, hence, an (r, s)-family. To show that $H'(r, s, n)$ does not contain an (r, s)-family we consider two cases. First, when $n \geq 4r+4s$, we have that $r_3 = 0$ and $s_4 - 1 < (4r+6s-n)/2 \leq s_4 \leq s$, so $r_3 + s_4 - 1 < s$ and

$$4r + 6s - 2(r_3 + s_4 - 1) = 4r + 6s - 2(s_4 - 1) > n.$$

Second, when $n < 4r + 4s$ we have that $r_3 = 4r + 4s - n \geq 1$ and $s_4 = s$, so $r_3 + s_4 - 1 \geq s$, and

$$4r + 4s - (r_3 + s_4 - 1 - s) = 4r + 4s - (r_3 - 1) = n + 1 > n.$$

Therefore, in either case, (3) implies that the graph $H'(r, s, n)$ does not contain an (r, s)-family.

For every integer m, we have that $\left\lceil \frac{m}{2} \right\rceil - 1 = \left\lfloor \frac{m-1}{2} \right\rfloor$ and $\left\lfloor \frac{\left\lfloor \frac{m-1}{2} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{m-1}{4} \right\rfloor$, so (with $m = 4r + 6s + n$)

$$\left\lfloor \frac{n + \left\lceil \frac{4r+6s-n}{2} \right\rceil - 1}{2} \right\rfloor = \left\lfloor \frac{\left\lceil \frac{4r+6s+n}{2} \right\rceil - 1}{2} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{4r+6s+n-1}{2} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{4r + 6s + n - 1}{4} \right\rfloor = r + \left\lfloor \frac{6s + n - 1}{4} \right\rfloor.$$

Using this with (1), we have

$$\delta(H'(r, s, n)) = \left\lfloor \frac{n + r_3 + s_4 - 1}{2} \right\rfloor = \begin{cases} \left\lfloor \frac{n+s+(4r+4s-n)-1}{2} \right\rfloor = 2r + \left\lfloor \frac{5s-1}{2} \right\rfloor & \text{if } n < 4r + 4s, \\ \left\lfloor \frac{n+\left\lceil \frac{4r+6s-n}{2} \right\rceil - 1}{2} \right\rfloor = r + \left\lfloor \frac{6s+n-1}{4} \right\rfloor & \text{if } n \geq 4r + 4s. \end{cases}$$

Therefore, we pose the following conjecture, which (if true) would be a tight result.

Conjecture 10. *Suppose that r, s and n are non-negative integers and $n \geq 3r + 4s$. If G is a graph on n vertices such that*

$$\delta(G) \geq \begin{cases} 2r + \frac{5s}{2} & \text{if } 3r + 4s \leq n < 4r + 4s, \\ r + \frac{3s}{2} + \frac{n}{4} & \text{if } 4r + 4s \leq n \leq 4r + 6s - 4, \\ 2r + 3s & \text{if } n \geq 4r + 6s - 3, \end{cases} \tag{4}$$

then G contains an (r, s)-family.

Note that, when $s = 0$, Conjecture 10 is equivalent to the Corrádi–Hajnal Theorem. If either $s = 1$ or both $s \geq 2$ and $n \geq 4r + 6s - 3$, then it is implied by Theorem 8. When $n \leq 4r + 6s - 4$, Theorem 9 implies the conjecture if either $s = 2$ or both $s \geq 3$ and $n \geq 4r + 6s - 7$. Additionally, when $r = 0$ and $n = 4s$, Conjecture 10 is implied by the following theorem of Kawarabayashi [9].

Theorem 11 (*Kawarabayashi [9]*)**.** *Every graph G on $4s$ vertices with $\delta(G) \geq \frac{5}{2}s$ contains s disjoint copies of $K_{1,1,2}$.*

Thus to prove Conjecture 10, it remains to consider the case when $s \geq 3$, $n \leq 4r + 6s - 8$, and either $r \geq 1$ or $n > 4s$.

When n is large and at most $4r + 6s$, an approximate version of Conjecture 10 is implied by the following theorem of Shokoufandeh and Zhao [17] which was originally conjectured by Komlós [14]. (Komlós proved a weaker version of this theorem in [14], which also implies a similar approximate version of Conjecture 10.)

Theorem 12 (*Shokoufandeh and Zhao 2003 [17]*)**.** *For every graph H such that $\chi(G) \geq 3$ there exists $n_0 = n_0(H)$ such that for every $n \geq n_0$ the following holds. If G is an n vertex graph and*

$$\delta(H) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n,$$

then G contains a collection of disjoint copies of H that covers all but at most

$$\frac{5(\chi(H)-2)(|H|-\sigma(H))^2}{\sigma(H)(\chi(H)-1)}$$

vertices.

To highlight the connection between Theorem 12 and Conjecture 10, we define the following function

$$f(x,y) = \begin{cases} 2x + \frac{5y}{2} & \text{if } x+y > \frac{1}{4}, \\ x + \frac{3y}{2} + \frac{1}{4} & \text{if } x+y \leq \frac{1}{4}. \end{cases}$$

Note that $f(r/n, s/n) \cdot n$ is the right-hand-side of (4) whenever $n \leq 4r + 6s - 4$. Also observe that $f(x, y)$ is continuous (since $x + y = \frac{1}{4}$ implies that $2x + \frac{5y}{2} = x + \frac{3y}{2} + \frac{1}{4}$) and, with (1), we have that

$$1 - \frac{1}{\chi_{cr}(H(r,s,n))} = \frac{n + r_3(r,s,n) + s_4(r,s,n)}{2n} = \begin{cases} \frac{n+4r+4s-n+s}{2n} & \text{if } n < 4r+4s \\ \frac{n+0+\lceil(4r+6s-n)/2\rceil}{2n} & \text{if } n \geq 4r+4s \end{cases} \geq \begin{cases} \frac{2r}{n} + \frac{5s}{2n} & \text{if } (r+s)/n > \frac{1}{4} \\ \frac{r}{n} + \frac{3s}{2n} + \frac{1}{4} & \text{if } (r+s)/n \leq \frac{1}{4} \end{cases} = f\left(\frac{r}{n}, \frac{s}{n}\right), \tag{5}$$

with equality holding whenever n is even or $n < 4r + 4s$.

Corollary 13. *For every $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$ the following holds. Let r and s be non-negative integers such that $3r + 4s + \varepsilon n \leq n < 4r + 6s$. If G is an n vertex graph such that*

$$\delta(G) \geq \left(f\left(\frac{r}{n}, \frac{s}{n}\right) + \varepsilon\right) n,$$

then G contains an (r, s)-family.

Proof. Let $C = 28/\varepsilon + 3$, and assume that $n_0 \geq 5C^3$. Also assume that n_0 is greater than the maximum of $n_0(H)$ from Theorem 12 over every 3-chromatic graph H on at most C vertices.

Let

$$h = 2\left\lfloor C \cdot \frac{n - 5C^2}{2n} \right\rfloor, r' = \left\lceil C \cdot \frac{r}{n} \right\rceil, \text{ and } s' = \left\lceil C \cdot \frac{s}{n} \right\rceil. \tag{6}$$

We claim that

$$3r' + 4s' \leq h < 4r' + 6s'. \tag{7}$$

To see that the second inequality in (7) holds, note that, using (6), we have $h \leq C$ and $4r'/h + 6s'/h \geq 4r/n + 6s/n > 1$. To see that, $3r'/h + 4s'/h \leq 1$ and the first inequality in (7) holds, first note that

$$h \geq 2\left(C \cdot \frac{n - 5C^2}{2n} - 1\right) = C - \frac{5C^3}{n} - 2 \geq C - 3 = \frac{28}{\varepsilon}.$$

Then, using the fact that $r/n \leq 1$, we have that

$$\frac{r'}{h} \leq h^{-1} \cdot \left(\frac{Cr}{n} + 1\right) \leq \frac{1}{C-3}\left(\frac{Cr}{n} + 1\right) = \frac{r}{n} + \frac{3}{C-3} \cdot \frac{r}{n} + \frac{1}{C-3} \leq \frac{r}{n} + \frac{4}{C-3} = \frac{r}{n} + \frac{\varepsilon}{7}, \tag{8}$$

and, similarly

$$\frac{s'}{h} \leq \frac{s}{n} + \frac{\varepsilon}{7}. \tag{9}$$

So, with (8) and (9), we have that $3r'/h + 4s'/h \leq 3r/n + 4s/n + \varepsilon \leq 1$.

Now fix H to be the complete tripartite graph $H(r', s', h)$. Note that $|H| = h$ is even, $\chi(H) = 3$ (since we can assume $r + s \geq 1$), and $h \leq C$, so, with (5),

$$f\left(\frac{r'}{h}, \frac{s'}{h}\right) = \left(1 - \frac{1}{\chi_{cr}(H)}\right) \qquad \text{and} \qquad \frac{5(\chi(H)-2)(|H|-\sigma(H))^2}{\sigma(H)(\chi(H)-1)} \leq 5C^2. \tag{10}$$

Also note that (7) implies that

$$H \text{ contains an } (r', s')\text{-family.} \tag{11}$$

Furthermore, with (8), (9), (10) and the definition of $f(x, y)$, we have that

$$\delta(G) \geq \left(f\left(\frac{r}{n}, \frac{s}{n}\right) + \varepsilon\right) n \geq f\left(\frac{r}{n} + \frac{\varepsilon}{7}, \frac{s}{n} + \frac{\varepsilon}{7}\right) n \geq f\left(\frac{r'}{h}, \frac{s'}{h}\right) n = \left(1 - \frac{1}{\chi_{cr}(H)}\right) n.$$

Therefore, by our choice of n, Theorem 12 with (10) implies that G contains a collection of at least $(n - 5C^2)/h$ disjoint copies of H. Note that this implies that G contains an (r, s)-family, because, by (11), every copy of H contains an (r', s')-family and, using (6),

$$\frac{n - 5C^2}{h} \cdot r' \geq (n - 5C^2) \cdot \frac{n}{C(n - 5C^2)} \cdot \frac{Cr}{n} = r,$$

and, similarly, $(n - 5C^2)/h \cdot s' \geq s$. $\square$

This proves an approximate version of Conjecture 10.

3. Preliminaries

In this section we provide additional notation that will be used throughout the remainder of this paper. However, we first begin with a restatement of Theorem 9, which is the version of our main theorem that we will prove in the rest of this paper.

Theorem 14 (*Restatement of Theorem 9*)**.** *Given $r \geq 0$ and $s \geq 1$ such that $r + s \geq 2$, let $m = 2r + 3s - 1$ and let G be a graph on n vertices where $n \geq 3r + 4s$ and $\delta(G) \geq m$. Then G is an (r, s)-extremal graph if and only if there exists a partition $\{A, B\}$ of $V(G)$ such that*

(a) $|A| = m$, $|B| = n - m$ and $\alpha'(G[A]) = \delta(G[A]) = \max\{0, m - |B| = 2m - n\}$;
(b) B is an independent set; and
(c) $s = 1$ when $n \leq 2m - 2 = 4r + 6s - 4$.

To see that Theorem 14 is equivalent to Theorem 9, let r, s, n, m and G be as they are in the statements of Theorems 9 and 14. If $n \geq 2m - 1$, the equivalence of the two Theorems is clear, so assume that $n \leq 2m - 2$. Let $n' = m$ and $m' = \max\{0, 2m - n\} = 2m - n$. If (iii) or (iv) hold, then $s = 1$, and there exists a partition $\{A, B\}$ of $V(G)$ such that B is an independent set, $|B| = n - m$ and $\delta(G[A]) = \alpha'(G[A]) = m'$. Therefore, (a), (b) and (c) of Theorem 14 hold. Now assume that (a), (b) and (c) all hold with the partition $\{A, B\}$. So $|A| = m = n'$ and $\alpha'(G[A]) = \delta(G[A]) = 2m - n = m' = \delta(G) - |B|$. This implies that every vertex in A is adjacent to every vertex in B. Since $s = 1$, we have that

$$m' = 2m - n \leq 4r + 6s - 2 - (3r + 4s) = r < r + 1 = \lfloor n'/2 \rfloor.$$

Therefore, the following well-known corollary to the Berge–Tutte Formula on matchings implies that either (iii) or (iv) of Theorem 9 hold. To see this, first note either (A) or (B) of Corollary 15 must hold for $G[A]$. If (A) holds for $G[A]$, then (iii) holds for G. If (B) holds for $G[A]$, then we must have that $|A| = 2m' + 2$ and that m' is even., which further implies that $2m' + 2 = n' = m = 2r + 3s - 1 = 2r + 2$, so it must be that $m' = r$ which implies that r is even. This further implies that $r = m' = 2m - n = 4r + 6s - 2 - n = 4r + 4 - n$, so $n = 3r + 4 = 3r + 4s$. Therefore, (iv) holds for G.

Although the following corollary is well-known, we provide the proof for completeness.

Corollary 15. *Suppose that G is a graph on n' vertices and let $m' = \delta(G)$. Then $\alpha'(G) \geq \min\{\lfloor n'/2 \rfloor, m'\}$. Furthermore when $m' < \lfloor n'/2 \rfloor$, we have $\alpha'(G) = m'$ if and only if*

(A) $K_{m',n'-m'} \subseteq G \subseteq K_{m'} \vee \overline{K_{n'-m'}}$; or
(B) m' is even and G is isomorphic to $\overline{K_{m'+1,m'+1}}$.

Proof. Note that if (A) or (B) hold in G, then $\alpha'(G) = \delta(G) = m'$.

On the other hand, let G be a graph on n' vertices, $\delta = \delta(G)$ and $\alpha' = \alpha'(G)$. The Berge–Tutte Formula implies that there exists $S \subseteq V(G)$ such that if $\mathcal{O}$ is the set of components of odd order in $G - S$, then

$$|\mathcal{O}| - |S| = n' - 2\alpha'. \tag{12}$$

Note that $n' - 2\alpha' \leq 1$ implies that $\alpha' = \lfloor n'/2 \rfloor$ and we are done. Therefore, assume that

$$|\mathcal{O}| - |S| \geq 2. \tag{13}$$

Let X be the component of smallest order in $\mathcal{O}$. By the minimum degree condition,

$$|S| \geq \delta - (|X| - 1) \tag{14}$$

Using (12) and (14) and the fact that $n' \geq |S| + \sum_{C\in\mathcal{O}} |C|$, we have

$$2\alpha' + |\mathcal{O}| = |S| + n' \geq 2|S| + \sum_{C\in\mathcal{O}} |C| \geq 2(\delta - (|X| - 1)) + |X||\mathcal{O}|, \tag{15}$$

so

$$2(\alpha' - \delta) \geq (|\mathcal{O}| - 2)(|X| - 1). \tag{16}$$

Therefore, with (13), we have that $\alpha' \geq \delta$.

If $\alpha' = \delta$, then, by (13) and (16), either $|\mathcal{O}| = 2$ and $|S| = 0$, or $|X| = 1$. If $|\mathcal{O}| = 2$ and $|S| = 0$, then G consists of two odd components each with minimum degree δ. Since we can assume that $\delta \geq 1$, neither component has a matching of size δ. So, if X and Y are the two components, we have that $|X|, |Y| \geq \delta + 1$ and $\lfloor |X|/2\rfloor + \lfloor |Y|/2\rfloor \leq \alpha' = \delta$. Therefore, $|X| = |Y| = \delta + 1$, δ is even and both X and Y are cliques on $\delta + 1$ vertices. That is, G meets (B). If $|X| = 1$, then (15) holds with equality, which implies that every component in $\mathcal{O}$ consists of a single vertex. Furthermore, (14) implies that $|S| \geq \delta$, and (12) yields $|\mathcal{O}| - |S| = n' - 2\delta$. Because $|S| + |\mathcal{O}| \leq n'$, this implies that $|\mathcal{O}| = n' - \delta$ and $|S| = \delta$. Hence, the union of the vertices in the components in $\mathcal{O}$ is an independent set of size $n' - \delta$, so, by the minimum degree condition, G meets (A). □

3.1. Additional notation

Let G be a graph, $v \in V(G)$, and A and B be two, not necessarily disjoint, subsets of $V(G)$. We let $N_B(v)$ denote $N_G(v)\cap B$, and let both $\|v, B\|$ and $d_B(v)$ denote $|N_B(v)|$. We also let $\|A, B\| = \sum_{v\in A} \|v, B\|$. For every collection of subgraphs $\mathcal{H}$ of G, we let $V(\mathcal{H}) = \bigcup_{H\in\mathcal{H}} V(H)$. If H is a subgraph of G, we often replace $V(H)$ with H in the above notation (e.g., $N_H(v) = N_{V(H)}(v)$, $\|v, H\| = \|v, V(H)\|$, and $\|A, H\| = \|A, V(H)\|$). Similarly, we often replace $V(\mathcal{H})$ with $\mathcal{H}$ when $\mathcal{H}$ is a collection of subsets of G (e.g., $\|A, \mathcal{H}\| = \|A, V(\mathcal{H})\|$).

If $P = v_1 \dots v_m$ is a path, then for $1 \leq i \leq j \leq m$, we let v_iPv_j denote the path $v_i \cdots v_j$. An n-cycle (n-chorded-cycle) is a cycle (chorded-cycle) with n vertices.

Let $(\mathcal{C}, \mathcal{D})$ be an (a, b)-family for some non-negative a and b, and let $\mathcal{U} = \mathcal{C} \cup \mathcal{D}$. We say that $(\mathcal{C}, \mathcal{D})$ *covers* the vertex set $V(\mathcal{U})$. We identify $(\mathcal{C}, \mathcal{D})$ with the quadruple $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ where $R = G[V(G) \setminus V(\mathcal{U})]$ is the graph induced by vertices not covered by $(\mathcal{C}, \mathcal{D})$.

A *leaf* is a vertex that has degree 1, and a *star* is a tree in which all of the vertices except at most one is a leaf. Note that when T is a star and $|T| \geq 3$ vertices, there exists a vertex with degree in T exactly $|T| - 1$. We call this vertex the *center* of the star.

If G is a graph, H is a subgraph of G and $A \subseteq V(G)$, we let $H + A = G[V(H) \cup A]$ and $H - A = G[V(H) \setminus A]$. If $|A|$ is small, we often replace A with the vertices of A in the above notation (e.g., if $A = \{v\}$, we use $H + v = H + A$ and $H - v = H - A$). If F is a subgraph of G, we let $H + F = H + V(F)$ and $H - F = H - V(F)$.

4. Main lemmas

We divide the majority of the proof of Theorem 14 into the following two lemmas. We give the proof of Lemma 16 in Section 4.2 and the proof of Lemma 17 in Section 4.3. In Section 4.1, we prove several structural lemmas that will be used in both Sections 4.2 and 4.3.

Lemma 16. *Let r and s be positive integers, and let G be an (r, s)-extremal graph on n vertices. If G contains an $(r, s-1)$-family that covers at most $n-4$ vertices, then either $n = 4r+6s-2$ and $\alpha(G) = n-(2r+3s-1)$; or G contains an $(r-1, s)$-family that covers at most $n-3$ vertices.*

Lemma 17. *Let r and s be positive integers, and let G be an (r, s)-extremal graph on n vertices. If G contains an $(r-1, s)$-family that covers at most $n-3$ vertices, then $\alpha(G) = n-(2r+3s-1)$. Furthermore, if $n \leq 4r+6s-4$, then $s = 1$.*

Proof of Theorem 14 from Lemmas 16 and 17. First note that by Theorem 7, we can assume throughout that r is positive. Let $m = 2r + 3s - 1$, and let G be a graph on $n \geq 3r + 4s$ vertices such that $\delta(G) \geq m$.

To prove sufficiency, we assume that G contains $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an (r, s)-family and also that both (a) and (b) hold, and we then show that (c) does not hold, by proving that we must have $n \leq 4r + 6s - 4$ and $s \geq 2$. By (a) and (b), there exists a partition $\{A, B\}$ of $V(G)$ such that $|A| = m$ and B is an independent set. By (a), we can assume that $z = \alpha'(G[A]) = \max\{0, 2m - n\}$. Note that when $z > 0$,

$$n = 2m - z = 4r + 6s - 2 - z. \tag{17}$$

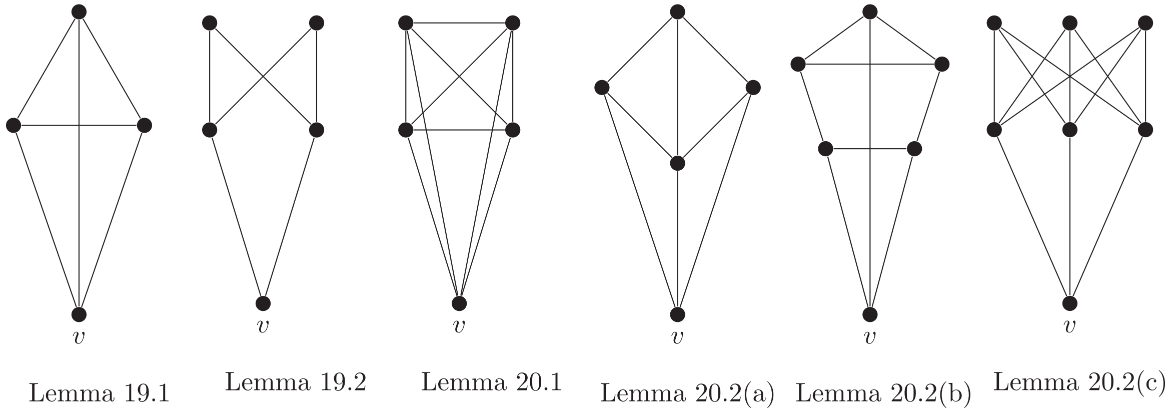


Fig. 2. Configurations of Lemmas 19 and 20.

Since every $C \in \mathcal{C}$ has at least 2 vertices in A, there must exist $D^* \in \mathcal{D}$ such that $|V(D^*) \cap A| \leq 2$, so there exists an edge in $G[A]$; that is, $z \geq 1$. Therefore, with (17), we have $n \leq 4r + 3s - 3$. Note that, because B is independent, D^* must have exactly 2 vertices in both A and B. Because $n \leq 4r + 6s - 3$, we have $|B| \leq 2r + 3s - 2$, so, in $\mathcal{U} - D^*$, there must exist either a chorded cycle with at most 2 vertices in B or a cycle with at most one vertex in B. This implies that $z = \alpha'(G[A]) \geq 2$, so, with (17), $n \leq 4r + 6s - 4$. If $s = 1$, then $|A| = 2r + 2$ and $|B| = 2r + 2 - z$. Therefore, every cycle in $\mathcal{C} = \mathcal{U} - D^*$ has exactly 2 vertices in A, so there are at least z triangles in $\mathcal{C}$ with two vertices in A and one vertex in B. This implies that $\alpha'(G[A]) \geq z + 1$, but $z = \alpha'(G[A])$, a contradiction. Therefore, $s \geq 2$, and this completes the proof of sufficiency.

To prove necessity, we assume that G is (r, s)-extremal. We begin by showing that $\alpha(G) \geq n - (2r + 3s - 1)$ and that $s = 1$ when $n \leq 4r + 6s - 4$. If necessary, add edges to G to form a graph G' that is edge-maximal with respect to being (r, s)-extremal. By the definition of an (r, s)-extremal graph, $n \geq 3r + 4s$, so G' is not a clique and there exist vertices x and y that are not adjacent. Since G' is edge-maximal with respect to being (r, s)-extremal, there exists an (r, s)-family in $G' + xy$. Therefore, in G', there exists either (Case 1) an $(r - 1, s)$-family that covers at most $n - 3$ vertices, or (Case 2) an $(r, s - 1)$-family that covers at most $n - 4$ vertices. If we are in Case 2, then Lemma 16 implies that $n = 4r + 6s - 2$ and $\alpha(G') = n - (2r + 3s - 1)$, or we are in Case 1. When we are in Case 1, we can use Lemma 17 to conclude that $\alpha(G') = n - (2r + 3s - 1)$ and $s = 1$ when $n \leq 4r + 6s - 4$. So in either case we may assume $\alpha(G') = n - (2r + 3s - 1)$, and either $n = 4r + 6s - 2$, or $s = 1$ when $n \leq 4r + 6s - 4$. Because G is a spanning subgraph of G', we have $\alpha(G) = \alpha(G') = n - (2r + 3s - 1)$.

Therefore, there exists a partition $\{A, B\}$ of $V(G)$ such that $|A| = 2r + 3s - 1$ and B is an independent set. To complete the proof, we will show that the matching number of $G[A]$ is $z = \max\{0, m - |B|\}$.

By the minimum degree condition, all possible edges exists between A and B. When $n \geq 4r + 6s - 2$, we have $z = 0$ and $|B| \geq 2r + 3s - 1$. In this case, if A is not an independent set, then we can find an (r, s)-family consisting of one chorded 4-cycle with 2 vertices in both A and B, $(s - 1)$ copies of $K_{3,3}$ and r copies of $K_{2,2}$. This implies that $\alpha'(G[A]) = 0 = z$. If $n = 4r + 6s - 3$, then $|A| = 2r + 3s - 1$, $|B| = 2r + 3s - 2$, and $\delta(G[A]) \geq m - |B| = 1$, so $\alpha'(G[A]) \geq 1$. If $\alpha'(G[A]) \geq 2$, then we can construct an (r, s)-family with one chorded 4-cycle, one triangle, $(r - 1)$ copies of $K_{2,2}$ and $(s - 1)$ copies of $K_{3,3}$, so $\alpha'(G[A]) = z = 1$ when $n = 4r + 6s - 3$. Now assume $3r + 4s \leq n \leq 4r + 6s - 4$ and recall that we have previously shown that, in this case, $s = 1$. We also have that $2 \leq z \leq r + 2(s - 1) = r$, $|A| = 2r + 2$, and $|B| = 2r + 2 - z$. Since $\delta(G[A]) \geq m - |B| = z$ and $z \leq r < |A|/2$, we have that $\alpha'(G[A]) \geq z$. If $\alpha'(G[A]) \geq z + 1$, then we can find an (r, s)-family consisting of one chorded 4-cycle with 2 vertices in A, z triangles with 2 vertices in A, and $(r - z)$ copies of $K_{2,2}$ each with 2 vertices in A. Therefore, $\alpha'(G[A]) = z$, and this completes the proof of necessity. □

4.1. Optimal families

The following definition is critical to the proofs of Lemmas 16 and 17.

Definition 18. Let a and b be non-negative integers and let G be a graph. We say that an (a, b)-family $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ is an *optimal* (a, b)*-family* if, over all (a, b)-families in G, the following conditions hold

(O1) the number of vertices in $\mathcal{U}$ is minimized,
(O2) subject to (O1), the total number of chords in the cycles of $\mathcal{D}$ is maximized, and
(O3) subject to (O1) and (O2), the length of the longest path in R is maximized.
(O4) subject to (O1), (O2) and (O3), the number of vertices in $\mathcal{C}$ is minimized.

Lemma 19. *Let G be a graph, let a and b be non-negative integers, and suppose that $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ is an optimal (a, b)-family. Then, for every $C \in \mathcal{C}$ and $v \in R$, the following holds (see Fig. 2).*

1. *If $\|v, C\| \geq 3$, then $\|v, C\| = 3$ and $C \cong K_3$.*
2. *If $\|v, C\| = 2$, then $|C| \in \{3, 4\}$. Moreover, if $|C| = 4$, then $C + v \cong K_{2,3}$.*

Proof. Observe that, by (O1), we can assume that C is an induced cycle.

We first prove 19.1. Suppose $\|v, C\| \geq 3$ with $c_1, c_2, c_3 \in N_C(v)$ appearing in order on C. If there exists $c \in V(C) \setminus \{c_1, c_2, c_3\}$, then without loss of generality, assume c appears on C after c_2 and before c_3. Let $\tilde{P}$ denote the path on C between c_1 and c_2 that does not contain c. Then $vc_1\tilde{P}c_2v$ is a cycle strictly smaller than C, contradicting (O1). Thus, $N_C(v) = \{c_1, c_2, c_3\} = V(C)$, which proves 19.1.

Suppose $\|v, C\| = 2$ with $N_C(v) = \{c_1, c_2\}$. Let P_1 and P_2 be the two paths between c_1 and c_2 on C. Without loss of generality assume there exist internal vertices c and c' on $c_1P_1c_2$. Then $vc_1P_2c_2v$ is a cycle with fewer vertices than C, contradicting (O1). Thus, $|C| \in \{3, 4\}$, and furthermore, if $|C| = 4$, then $G[C + v] \cong K_{2,3}$. This proves 19.2. □

Lemma 20. *Let G be a graph, let a and b be non-negative integers, and suppose that $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ is an optimal (a, b)-family. Then, for every $D \in \mathcal{D}$ and $v \in R$, the following holds (see Fig. 2).*

1. *If $\|v, D\| \geq 4$, then $\|v, D\| = 4$ and $D \cong K_4$.*
2. *If $\|v, D\| = 3$, then $|D| \in \{4, 5, 6\}$. Moreover:*
   - *(a) if $|D| = 4$ and $D \cong K_{1,1,2}$, then the chord is incident to the non-neighbor of v;*
   - *(b) if $|D| = 5$, then D has a single chord and no vertex in $N_D(v)$ is incident to this chord;*
   - *(c) if $|D| = 6$, then $D \cong K_{3,3}$ and $D + v \cong K_{3,4}$.*

Proof. Observe that, by (O2), we can assume that D is an induced graph. Let $C = d_1d_2 \ldots d_m$ be a spanning cycle in D. The proof is implied by the following two properties:

$$\text{(i) } \|v, D - d - d^+\| \leq 2 \text{ if } d \text{ and } d^+ \text{ are consecutive on } C\text{; and (ii) } \|d, D\| \geq \|v, D - d\| \text{ for all } d \in D. \tag{18}$$

To see (18)(i), note that otherwise we could replace D with a chorded cycle in $D - d - d^+ + v$ and contradict (O1). For (18)(ii), note that $\|v, D - d\| > \|d, D\| \geq 2$, implies that $D - d + v$ contains either a chorded cycle with fewer vertices that D or a chorded cycle with the same number of vertices as D, but more edges than D. Therefore, we either contradict (O1) or (O2).

Note that (18)(i) and $\|v, D\| \geq 4$ imply that $\|v, D\| = |D| = 4$, and then (18)(ii) gives us that $\|d, D\| = 3$ for every $d \in D$. This proves 20.1.

Suppose that $\|v, D\| = 3$ and let $A = N(v)$ and $B = V(D) \setminus A$. Then (18)(i) implies $|B| \leq \|v, D\| = 3$, so $|D| \in \{4, 5, 6\}$. If $|D| = 4$, then (18)(ii) implies that the vertex $d \in B$ is incident to a chord, which proves 20.2(a). If $|D| = 5$, then, by (18)(i), we can assume that $B = \{d_1, d_3\}$, and (18)(ii) implies that both d_1 and d_3 are incident to a chord. Since $|D| = 5$, it must be that D has exactly one chord, otherwise we could replace D with a chorded cycle on 4 vertices, contradicting (O1), so d_1d_3 is an edge and we have proved 20.2(b). If $|D| = 6$, then, by (18)(i), we can assume that $B = \{d_1, d_3, d_5\}$. Because of (O1), we can assume that $D + v$ does not contain a chorded 4-cycle or a chorded 5-cycle and this implies that both A and B are independent. By (18)(ii), every vertex in B must have 3 neighbors in D. These two observations together imply 20.2(c). □

Lemma 21. *Let G be a graph, let a and b be non-negative integers, and suppose that $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ is an optimal (a, b)-family. Let $u, v \in R$ such that $uv \in E(G)$. If $C \in \mathcal{C}$, $|C| = 4$, $\|u, C\| \geq 2$ and $\|v, C\| \geq 1$, then $N_C(u) \cap N_C(v) = \emptyset$. Similarly, if $D \in \mathcal{D}$, $|D| = 6$, $\|u, D\| \geq 3$ and $\|v, D\| \geq 1$, then $N_D(u) \cap N_D(v) = \emptyset$.*

Proof. By Lemma 19, $C \cong K_{2,2}$ and we can let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be the partite sets of C with $N_C(u) = A$. Suppose on the contrary that $va_1 \in E(G)$. Then we can replace C with the smaller cycle uva_1u, contradicting (O1).

Similarly, by Lemma 20, we may assume $D \cong K_{3,3}$ with partite sets $A' = \{a'_1, a'_2, a'_3\}$ and $B' = \{b'_1, b'_2, b'_3\}$, where $N_D(u) = A'$. If $va'_1 \in E(G)$, then we can replace D with the smaller chorded cycle $ua'_2b'_1a'_1vu$, contradicting (O1). □

Lemma 22. *Let G be a graph, let a and b be non-negative integers, and suppose that $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ is an optimal (a, b)-family. If $D \in \mathcal{D}$ and $\|v_1, D\|, \|v_2, D\| \geq 3$ for distinct $v_1, v_2 \in R$, then D is isomorphic to either $K_{1,1,2}$, K_4, or $K_{3,3}$.*

Proof. Note that by 20, we only need to prove that $|D| \neq 5$. For a contradiction, assume $|D| = 5$. Then Lemma 20 implies that $N_D(v_1) = N_D(v_2)$. Furthermore, there are two adjacent vertices $d, d' \in N_D(v_1) = N_D(v_2)$, and $G[\{d, d', v_1, v_2\}]$ contains $K_{1,1,2}$, contradicting (O1). □

Lemma 23. *Let G be a graph, let a and b be non-negative integers, and suppose that $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ is an optimal (a, b)-family and let P be a longest path in R. Suppose that $|R| > |P|$ and let p be an endpoint of P, $v \in R - P$ and $F = \{p, v\}$. Then*

$\|F, C\| \leq 4$ and $\|F, D\| \leq 6$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Furthermore,

(1) if $\|F, C\| = 4$, then either $\|p, C\| = |C| = 3$, or C is isomorphic to K_3 or $K_{2,2}$ and $N_C(p) = N_C(v)$;
(2) if $\|F, D\| = 6$, then either $\|p, D\| = |D| = 4$, or D is isomorphic to K_4 or $K_{3,3}$ and $N_D(p) = N_D(v)$.

Proof. Suppose $\|F, C\| \geq 4$. Assume $|C| = 3$. If we can label $V(C)$ as $\{c_1, c_2, c_3\}$, so that $pc_1, vc_2, vc_3 \in E(G)$, then we can replace C with vc_2c_3v and P with $P + c_1$, contradicting (O3). This implies that $\|F, C\| = 4$, and furthermore, if $\|p, C\| \leq 2$, then $\|p, C\| = 2 = \|v, C\|$ with $N_C(p) = N_C(v)$. This proves 23.(1) when C is a triangle.

Now assume $|C| \geq 4$. By Lemma 19, we can assume that $|C| = 4$. Furthermore, if we let $C = c_1c_2c_3c_4c_1$, we have that $N_C(p), N_C(v) \in \{\{c_1, c_3\}, \{c_2, c_4\}\}$, so if we assume that $N_C(p) \neq N_C(v)$, then we can further assume that $N_C(p) = \{c_1, c_3\}$ and $N_C(v) = \{c_2, c_4\}$. Then we can replace C with $vc_2c_3c_4v$ and P with $P + c_1$, contradicting (O3). This contradiction, completes the proof of 23.(1).

Consider $D \in \mathcal{D}$ and suppose $\|F, D\| \geq 6$. Assume $D \cong K_4$. If we can label $V(D)$ as $\{d_1, d_2, d_3, d_4\}$ so that $d_1, d_2, d_3 \in N_G(v)$ and $d_4 \in N_G(p)$, then we can replace D and P with $D - d_4 + v$ and $P + d_4$, respectively, which violates (O3). Thus, if $D \cong K_4$, then $\|F, D\| = 6$ and either $\|p, D\| = |D| = 4$ or $N_D(p) = N_D(v)$.

So we may assume $D \ncong K_4$. As a result, by Lemmas 20 and 22, $\|\{v, p\}, D\| = 6$, $\|v, D\| = \|p, D\| = 3$, and D is either $K_{1,1,2}$ or $K_{3,3}$. Suppose $D \cong K_{1,1,2}$. Let $d \in N_D(p)$ such that d is not incident to a chord in D. By Lemma 20, we can replace D and P with $D - d + v$ and $P + d$, respectively, which violates (O3). Now suppose $D \cong K_{3,3}$. By Lemma 20, if v and p do not have the same neighborhood, they are adjacent to disjoint sets of vertices, and $D + p$ and $D + v$ both contain $K_{3,4}$. In this case, we extend P using a $d \in N_D(p)$, and replace D with the chorded cycle $D - d + v$. This violates (O3), and completes the proof. □

4.2. Proof of Lemma 16

In this section, we will prove Lemma 16. The following lemma contains most of the argument and is broken up into several claims.

Lemma 24. *Let $r \geq 1$ and $s \geq 1$ and suppose that G is an (r, s)-extremal graph on n vertices that contains an optimal $(r, s-1)$-family $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ that covers at most $n - 4$ vertices, but does not contain any $(r-1, s)$-family that covers at most $n - 3$ vertices. Then*

- *$R \cong K_{2,2}$;*
- *$C \cong K_{2,2}$ and $R + C \cong K_{4,4}$ for every $C \in \mathcal{C}$; and*
- *$D \cong K_{3,3}$ and $R + D \cong K_{5,5}$ for every $D \in \mathcal{D}$.*

In particular, for every $v \in R$, $d_G(v) = 2r + 3s - 1$, and, for every $u \in R$ that is not adjacent to v, we have that $N_G(u) = N_G(v)$.

Proof. In all the following we assume $r \geq 1$ and $s \geq 1$. In addition, G is an (r, s)-extremal graph on n vertices that contains an optimal $(r, s-1)$-family $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ that covers at most $n - 4$ vertices (so $|R| \geq 4$), but does not contain any $(r-1, s)$-family that covers at most $n - 3$ vertices.

Claim 24.1. *The graph R does not contain a chorded cycle, and for every $v \in R$ and for every $C \in \mathcal{C}$, the graph $C + v$ does not contain a chorded-cycle.*

Proof. The graph R cannot contain a chorded cycle as G is (r, s)-extremal. Let $v \in R$. Because there are at least 4 vertices in R, there are at least 3 vertices in $R - v$, so if there was a chorded cycle in $\mathcal{C} + v$, then G would contain an $(r-1, s)$-family on at most $n - 3$ vertices, a contradiction. □

Claim 24.2. *Suppose that P' is a path of maximum length in R and that p is one of its endpoints. If $v \in R - P'$, then $\|\{p, v\}, V(\mathcal{C}) \cup V(R)\| \geq 4r + 4$.*

Proof. Suppose $\|\{p, v\}, V(\mathcal{C}) \cup V(R)\| < 4r + 4$. Then

$$\|\{p, v\}, \mathcal{D}\| \geq 2(2r + 3s - 1) - \| \{p, v\}, V(\mathcal{C}) \cup V(R)| > 6(s - 1).$$

Yet this implies that there exists $D \in \mathcal{D}$ such that $\|\{p, v\}, D\| \geq 7$, which contradicts Lemma 23. □

Claim 24.3. *Suppose that P' is a (not necessarily maximum length) path in R such that $|R| - |P'| \geq 3$. Then $|P', C| \leq 3$ for every $C \in \mathcal{C}$, and, furthermore, if $|P'| \leq 2$, then $\|P', C\| \leq 2$ for every $C \in \mathcal{C}$.*

Proof. Note that we can assume that the graph induced by $V(P') \cup V(C)$ does not contain a chorded cycle, since we could then replace C with such a chorded cycle to produce an $(r-1, s)$-family that covers at most $n - (|R| - |P'|) \leq n - 3$ vertices, a contradiction.

Let $P' = v_1, \ldots, v_\ell$ and suppose that $\|P', C\| \geq 3$ for some $C \in \mathcal{C}$. Let $1 \leq i \leq k \leq \ell$ be such that $k - i$ is maximized subject to $\|v_i, C\| \geq 1$ and $\|v_k, C\| \geq 1$, i.e., v_i and v_k are, respectively, the first and last vertices on P' that have neighbors in C. If v_i and v_k have distinct neighbors on C, then there are two cycles, say C_1 and C_2, in $P' + C$ that both contain the path $v_iP'v_k$ and are such that $V(C) \subseteq V(C_1) \cup V(C_2)$, so since $\|P', C\| = \|v_iP'v_k, C\| \geq 3$, at least one of the two cycles C_1 and C_2 span a chorded cycle. So assume that there exists a unique vertex c in $V(C) \cap (N(v_i) \cup N(v_k))$. Because $\|P', C\| > 1$, this implies that $v_i \neq v_k$ and $\|\{v_i, v_k\}, C\| = 2$. The fact that $\|P', C\| \geq 3$ then implies that there exists a smallest j such that $i < j < k$ and $\|v_j, C\| \geq 1$, i.e., v_i and v_j are the first two vertices on P' that have a neighbor on C and v_k is the last vertex in P that has a neighbor in C. Note that this implies that $|P'| \geq 3$, so we have proved the second part of the statement of the claim. To complete the proof of the first part of the claim, let c' be a neighbor of v_j on C. Since $v_iP'v_k + C$ does not contain a chorded cycle, we have that $c' \neq c$. This implies that there exist two distinct cycles, say C_1' and C_2', that both contain the path $v_jP'v_k$ and such that $V(C) \subseteq V(C_1') \cup V(C_2')$. So, using the same logic as before, since $P' + C$ does not contain a chorded cycle, we have that $\|v_jP'v_k, C\| = 2$. Therefore, v_ic, v_jc' and v_kc are the only edges between P' and C so $\|P', C\| = 3$. □

Claim 24.4. *For every $v \in R$ and every $C \in \mathcal{C}$, we have that $\|v, C\| \leq 2$.*

Proof. This follows from Claim 24.3 applied to the path consisting of the single vertex v. □

Claim 24.5. *If R contains a triangle, then $\|v, C\| \leq 1$ for every $v \in R$ and $C \in \mathcal{C}$.*

Proof. Suppose there exists T a triangle in R. Because we could swap each cycle in $\mathcal{C}$ with this triangle to form a new $(r, s-1)$-family, the condition (O1) implies that every cycle in $\mathcal{C}$ is a triangle. Thus, if $\|v, C\| \geq 2$ for some $v \in R$ and some $C \in \mathcal{C}$, then $C + v$ would form a chorded cycle, contradicting Claim 24.1. □

Claim 24.6. *If P' is a path of maximum length in R, then $|P'| \geq 4$.*

Proof. First suppose $\Delta(R) \leq 1$, which implies that $|P'| \leq 2$. Let p be one of the endpoints of P'. Because $|R| \geq 4$, there exists $v \in V(R - P')$. With Claim 24.4, we have that $\|\{p, v\}, V(\mathcal{C}) \cup V(R)\| \leq 4r + 2$, a contradiction to Claim 24.2.

Therefore, we can assume that there exists a vertex in R with at least 2 neighbors in R. This further implies that if P' is a longest path in R, then $|P'| \geq 3$.

Suppose P' is a maximum length path in R and that $|P'| = 3$. Since $|R| \geq 4$, we have that $|R - P'| \geq 1$, and we can let v be the endpoint of a longest path in $R - P'$. Further suppose that the endpoints of P' are adjacent, so the vertices of P' induce a triangle. Note that this implies that v does not have a neighbor on P', because P' is a longest path in R, so Claim 24.1 implies that $\|v, R\| \leq 2$. With Claim 24.5, we have that $\|\{p, v\}, V(\mathcal{C}) \cup V(R)\| \leq 2r + 4 < 4r + 4$, contradicting Claim 24.2. Now suppose that the vertices in P' do not induce a triangle and let p, q and p' be the vertices on P' in order. Note that, because P' is a longest path in R, if v has a neighbor on P', then it must be q. In this case, v has no neighbors in $R - P'$, because vqp' is also a longest path in R. If v does not have a neighbor on P', then Claim 24.1 implies that $\|v, R\| \leq 2$. In either case, with Claim 24.4, we have that $\|\{p, v\}, V(\mathcal{C}) \cup V(R)\| \leq 4r + 3$, contradicting Claim 24.2. □

Now fix P a maximum length path in R. Let p and p' be the endpoints of P, and let q and q' be the neighbors of p and p' on P, respectively, and let $W = \{p, q, q', p'\}$. By Claim 24.6, we know that $|P| \geq 4$, so $|W| = 4$.

Claim 24.7. *We have that $\|W, C\| \leq 8$ for every $C \in \mathcal{C}$ and $\|W, D\| \leq 12$ for every $D \in \mathcal{D}$.*

Proof. Let $C \in \mathcal{C}$. By Claim 24.4, $\|w, C\| \leq 2$ for every $w \in W$, so $\|W, C\| \leq 8$.

Let $D \in \mathcal{D}$ and assume that $\|W, D\| > 12$. Since one of the four vertices in W sends at least 4 edges to D, Lemma 20 implies that $D \cong K_4$. One of p or p', say p, sends at least 3 edges to D, and since p sends at most 4 edges to D, $\|\{q, q', p'\}, D\| \geq 9$. This further implies that one vertex in D, say d, is adjacent to each of the three vertices $\{q, q', p'\}$. We then have that $D - d + p$ is a chorded cycle and the graph induced by $R - p + d$ contains a chorded cycle, a contradiction to the fact that G is (r, s)-extremal. □

Claim 24.8. *R is isomorphic to $K_{2,2}$.*

Proof. Assume that $|R| \geq 5$. Let P' be the 2-vertex path pq. Since $|R| - |P'| \geq 3$ and $|P'| \leq 2$, Claim 24.3 implies $\|P', C\| \leq 2$ for every $C \in \mathcal{C}$. Since the same holds for the 2-vertex path $q'p'$, we have that $\|W, \mathcal{C}\| \leq 4r$.

With Claim 24.7 and the fact that $r \geq 1$, we have that

$$\|W, R\| \geq 4\delta(G) - \|W, \mathcal{C}\| - \|W, \mathcal{D}\| \geq 4(2r + 3s - 1) - 4r - 12(s - 1) = 4r + 8 \geq 12.$$

Since Claim 24.1 implies that $\|\{p, p'\}, R\| \leq 4$, we can assume that

$$\|\{q, q'\}, R\| \geq 8, \tag{19}$$

so one of q or q', say q has at least 4 neighbors in R. Since, by Claim 24.1, $\|q, P\| \leq 3$, q has a neighbor $v \in R - P$. If $|R| \geq 6$, then we can apply Claim 24.3 to the 3-vertex path pqv and this gives us that $\|\{p, q, v\}, \mathcal{C}\| \leq 3r$. Because both p and v are endpoints of longest path in R, we have that

$$\|\{p, v\}, V(\mathcal{C}) \cup V(R)\| \leq 3r + 4 < 4r + 4,$$

a contradiction to Claim 24.2. Therefore, we can assume that $|R| = 5$, and since $v \in P - R$ and $|P| \geq 4$, we have that $P = pqq'p'$ and $V(R - P) = \{v\}$. By Claim 24.1, q and q' cannot both have 4 neighbors in R, which contradicts (19).

Therefore, we can assume that $|R| = |P| = 4$, so that $W = V(R) = V(P)$. Note that Claim 24.7 implies that

$$\|R, R\| \geq 4(2r + 3s - 1) - 8r - 12(s - 1) = 8. \tag{20}$$

Suppose, one of q or q', say q, is such that $\|q, R\| \geq 3$. Then q, q' and p' induce a triangle. By Claim 24.1, we have exactly 4 edges in R. Also, by Claim 24.5, we have that $\|P, C\| \leq 4$ for every $C \in \mathcal{C}$, so, by Claim 24.7,

$$\|R, V(G)\| = \|R, \mathcal{C}\| + \|R, \mathcal{D}\| + \|R, R\| \leq 4r + 12(s - 1) + 8 < 4(2r + 3s - 1)$$

a contradiction. Therefore, $\|q, R\| = \|q', R\| = 2$ and with Claim 24.1 and (20), we have that $\|p, R\| = \|p', R\| = 2$ as well. This implies that R is a cycle on 4 vertices. □

Claim 24.9. *For every $C \in \mathcal{C}$, C is a $K_{2,2}$ and $R + C$ is a $K_{4,4}$; and, for every $D \in \mathcal{D}$, D is a $K_{3,3}$ and $R + D$ is a $K_{5,5}$.*

Proof. First note that Claim 24.8 implies that R is a $K_{2,2}$. This with Claim 24.7 implies that

$$4\delta(G) \leq \|R, \mathcal{C}\| + \|R, \mathcal{D}\| + \|R, R\| \leq 8r + 12(s - 1) + 8 = 4(2r + 3s - 1) \leq 4\delta(G).$$

Therefore,

$$\|R, C\| = 8 \text{ and } \|R, D\| = 12 \text{ for every } C \in \mathcal{C} \text{ and } D \in \mathcal{D}. \tag{21}$$

Let $C \in \mathcal{C}$. By (21), there exists $v \in R$, such that $\|v, C\| \geq 2$, so Claim 24.1 implies that C is not a triangle. This with Lemma 19 gives us that $C \cong K_{2,2}$ and, with Lemma 21, we have that $R + C \cong K_{4,4}$.

Let $D \in \mathcal{D}$. By (21), there exist at least two vertices in R that send at least 3 edges to D. This with, Lemmas 19 and 22 imply that D is either a $K_{1,1,2}$, K_4 or $K_{3,3}$. If D is a $K_{3,3}$, then $R + C$ is isomorphic to $K_{5,5}$ by Lemmas 20 and 21. When $D \cong K_{1,1,2}$, Lemma 20 implies that $\|v, D\| = 3$ for every $v \in R$ and every vertex in R is adjacent to the two vertices in C that are incident to the chord. Therefore, for every edge $xy \in E(R)$, $D + x + y$ contains a K_4, and we can replace D with this K_4 to create a new $(r, s - 1)$-family that violates (O2), a contradiction.

Now assume that $D \cong K_4$ and let d_1, d_2, d_3 and d_4 be the vertices in D labeled so that $\|d_1, R\| \geq \|d_2, R\| \geq \|d_3, R\| \geq \|d_4, R\|$. Also, let r_1, r_2, r_3 and r_4 be the vertices in R labeled in the order they appear on the cycle and so that $\|r_1, D\| \geq \|r_i, D\|$ for every $i \in \{2, 3, 4\}$. Note that $\|d_1, R\| \geq 3$ and $\|r_1, D\| \geq 3$. This implies that $D - d_1 + r_1$ is a chorded cycle, so since G is (r, s)-extremal, we have that $R - r_1 + d_1$ does not contain a chorded cycle. This implies that $\|d_1, R - r_1\| < 3$, so $\|d_1, R\| = 3$ and d_1r_1 is an edge. By (21), we therefore have $\|d_j, R\| = 3$ for every $j \in \{1, 2, 3, 4\}$. Suppose that there exists $i \in \{1, 2, 3, 4\}$ such that $\|r_i, D\| \in \{2, 3\}$, and let $j \in \{1, 2, 3, 4\}$ so that r_id_j is not an edge. Then $D - d_j + r_i$ is a chorded cycle and, since $\|d_j, R - r_i\| = \|d_j, R\| = 3$, we have that $R - r_i + d_j$ is a chorded cycle, a contradiction. Therefore, every vertex in R sends either 4, 1, or 0 edges to D, and this, with the fact that $\|R, D\| = 12$ implies that one of the four vertices in R sends no edges to D while the remaining three vertices in R each send 4 edges to D. Without loss of generality, we can assume that $\|r_4, D\| = 0$. Then $D' = G[\{d_3, d_4, r_1, r_3, r_4\}]$ has a spanning cycle $r_1d_3d_4r_3r_4r_1$ with two chords: r_1d_4 and r_3d_3. And also $d_1d_2r_2$ is a triangle T. Since $r \geq 1$, there exist $C \in \mathcal{C}$, and, by the first part of this claim, we have that C is isomorphic to $K_{2,2}$. If we replace C with T in $\mathcal{C}$ and D with D' in $\mathcal{D}$, then we have an $(r, s - 1)$-family that violates (O4) a contradiction. □

We now finish the proof of Lemma 24. The first part of the lemma then follows from Claim 24.8 and 24.9. To see the second part, let u and v be a pair of non-adjacent vertices in R. Because $\|v, R\| = 2$, $\|v, C\| = 2$ for every $C \in \mathcal{C}$, and $\|v, D\| = 3$ for every $D \in \mathcal{D}$, we have that $d_G(v) = 2r + 3(s - 1) + 2 = 2r + 3s - 1$. We also have that $N_G(u) = N_G(v)$ because $R + U$ is isomorphic to a complete bipartite graph for every $U \in \mathcal{U}$ and u and v are not adjacent. □

With Lemma 24, we can now prove Lemma 16.

Proof of Lemma 16. Let G be an (r, s)-extremal graph on n vertices that contains an $(r, s - 1)$-family that covers at most $n - 4$ vertices, but does not contain an $(r - 1, s)$-family that covers at most $n - 3$ vertices. By (O1), there exists $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an optimal $(r, s - 1)$-family with $|R| \geq 4$. Note that Lemma 24 implies that $n = 4r + 6(s - 1) + 4 = 4r + 6s - 2$.

Let $v \in R$. From Lemma 24, we have that $d_G(v) = 2r + 3s - 1$. We will show that, for every u that is not adjacent to v, $N_G(u) = N_G(v)$. This implies that $V(G) \setminus N_G(v)$ is an independent set of order $n - (2r + 3s - 1)$, which proves the lemma. To this end, let $u \in V(G) \setminus N_G(v)$. If $u \in R$, then Lemma 24 immediately implies that $N_G(u) = N_G(v)$, so assume that $u \notin R$.

Since $u \notin R$, there exists some $U \in \mathcal{U}$ that contains u. By Lemma 24, we have that U is isomorphic to $K_{t,t}$ and $R + U$ is isomorphic to $K_{t+2,t+2}$ for some $t \in \{2, 3\}$. If we let v' be the vertex in R that is not adjacent to v, then $U' = U - u + v'$ is isomorphic to $K_{t,t}$ and $R' = R - v' + u$ is isomorphic to $K_{2,2}$. Therefore, if we replace U with U', then we have a new optimal $(r, s - 1)$-family $(\mathcal{U}', \mathcal{C}', \mathcal{D}', R')$. Since u and v are not adjacent and are both in R', Lemma 24 applied to $(\mathcal{C}', \mathcal{D}')$ gives us that $N_G(u) = N_G(v)$. This completes the proof of the lemma. □

4.3. Proof of Lemma 17

In this section, we will prove Lemma 17. To do so, we prove a series of lemmas from which we easily deduce Lemma 17. Note that in this section, we assume that G is an (r, s)-extremal graph containing an $(r-1, s)$-family. Thus, in many of the following arguments, we arrive at a contradiction when we can create an additional cycle.

Lemma 25. *Given positive integers r and s, let G be an (r, s)-extremal graph and suppose there exists an optimal $(r-1, s)$-family $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$. Further suppose that there exists $F = \{u_1, v_1, u_2, v_2\} \subseteq V(R)$, a set of four vertices such that there exist disjoint paths P_1 and P_2 in R from u_1 to v_1 and from u_2 to v_2, respectively. Then $\|F, C\| \leq 7$ and $\|F, D\| \leq 10$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

Proof. For $i \in \{1, 2\}$, let $F_i = \{u_i, v_i\}$. Fix $C \in \mathcal{C}$ and suppose $\|F, C\| \geq 8$. By Lemma 19, $|C| \in \{3, 4\}$.

First we consider the case where $|C| = 3$. If a vertex $c \in C$ is adjacent to both vertices of F_i for some $i \in \{1, 2\}$, then cP_ic is a cycle, so $G[(C + F_{3-i}) - c]$ is cycle-free, which implies that $\|C - c, F_{3-i}\| \leq 1$. Label $V(C) = \{c_1, c_2, c_3\}$ so that $\|c_1, F\| \geq \|c_2, F\| \geq \|c_3, F\|$. Then $\|c_1, F\| \geq 3$, so (say) $u_1, v_1 \in N(c_1)$. Now $\|C - c_1, F_2\| \leq 1$. If $\|c_1, F\| = 4 = |F|$, then also $\|C - c_1, F_1\| \leq 1$, and $\|C, F\| = \|c_1, F\| + \|C - c_1, F\| \leq 4 + 2 < 8$, a contradiction, so $\|c_1, F\| = 3$, hence $\|c_2, F\| = 3$ and $\|c_3, F\| \geq 2$. Since $\|C - c_1, F_2\| \leq 1$, it follows that $u_1, v_1 \in N(c_3)$, which implies that $\|C - c_3, F_2\| \leq 1$, a contradiction to $\|c_1, F\| = \|c_2, F\| = 3$.

If $|C| = 4$, let $C = c_1c_2c_3c_4c_1$. By Lemma 19, every vertex in F sends exactly two edges to C and every vertex in F sends exactly one edge to every pair of adjacent vertices in C. Therefore, $\|\{c_1, c_2\}, F_1\| = \|\{c_3, c_4\}, F_2\| = 2$, and both $P_1 + c_1 + c_2$ and $P_2 + c_3 + c_4$ contain a cycle, a contradiction.

Consider $D \in \mathcal{D}$ and suppose $\|F, D\| \geq 11$. By Lemmas 20 and 22, $|D| \in \{4, 6\}$.

First, suppose there exists $d \in D$ such that $\|d, F\| = 4$, then, because $P_1 + d$ and $P_2 + d$ both contain cycles, neither F_1 nor F_2 combined with $D - d$ can contain a chorded cycle. Therefore,

$$\|D, F\| \leq \|d, F\| + \|D - d, F_1\| + \|D - d, F_2\| \leq 4 + 3 + 3 < 11,$$

a contradiction.

Now, suppose there exists a vertex $d \in D$ adjacent to either both vertices of F_1 or both vertices of F_2. Say d is adjacent to both vertices of F_1. By the preceding argument, we can assume that d is not adjacent to both vertices of F_2, i.e. $\|d, F_2\| \leq 1$. Since dP_1d is a cycle, we have $\|D - d, F_2\| \leq 3$, and

$$\|F_2, D\| = \|F_2, D - d\| + \|F_2, d\| \leq 3 + 1 = 4,$$

so $\|F_1, D\| \geq 11 - 4 = 7$. Without loss of generality, we can assume that $\|u_1, D\| \geq 4$, so by Lemma 20, $D \cong K_4$, $N_D(u_1) = V(D)$, and v_1 is adjacent to at least three vertices in D. Then, for every pair of distinct vertices $d', d'' \in D$, we have that $P_1 + d' + d''$ contains a chorded cycle so $G[V(D - d' - d'') \cup V(P_2)]$ is cycle-free. Therefore, $\|F_2, D\| \leq 1$. Now $\|F, D\| = \|F_1, D\| + \|F_2, D\| \leq 8 + 1 < 11$, a contradiction.

By the previous paragraph, there is no $d \in D$ adjacent to both vertices of F_1 or both vertices of F_2. Then every vertex in D has at most two neighbors in F, which implies $\|D, F\| \leq 2|D|$, which further implies that $|D| = 6$. Therefore, by Lemma 20, D is isomorphic to $K_{3,3}$ and we let A and B be the two partite sets of D. By Lemma 20, each vertex in F sends at most 3 edges to D, so we can assume without loss of generality that $\|v_2, D\| \geq 2$ and $\|v, D\| = 3$ for every $v \in F - v_2$, and also that $N_D(u_2) = A$. Then, because no vertex in D is adjacent to both vertices of F_2, we have that $N_D(v_2) \subseteq B$, and there exists $a \in A$ and $b \in N_D(v_2)$, such that $P_2 + a + b$ contains a cycle. Therefore $D - a - b + P_1$ should not contain a chorded cycle. However, by Lemma 20 and the fact that no vertex in D is adjacent to both vertices of F_1, we have that one of u_1 or v_1 has A as its neighborhood in D while the other's neighborhood in D is B. This implies that $D - a - b + u_1 + v_1$ contains $K_{3,3}$, a contradiction. □

Lemma 26. *Given positive integers r and s, let G be an (r, s)-extremal graph. Suppose that there exists $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an optimal $(r-1, s)$-family and $F = \{u_1, u_2, w\}$ a set of three vertices in R. Further suppose that, for $i \in \{1, 2\}$, there exists a path P_i from u_i to w in R that avoids u_{3-i}. Then, for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, we have that $\|F, C\| \leq 7$, $\|F, D\| \leq 9$, and the following statements hold.*

(1) If $\|F, C\| = 6$ and $C \cong K_{2,2}$, then $N_C(u_1) = N_C(u_2) = A$ where A is a partite set of C.
(2) If $\|F, C\| = 7$, then $C \cong K_3$, $N_C(w) = V(C)$, and $N_C(u_1) = N_C(u_2)$.
(3) If $\|F, C\| \geq 6$ and there is a path in R between u_1 and u_2 that avoids w, then $\|F, C\| = 6$ and there either exists $v \in F$ such that $\|v, F\| = 0$ and $N(u) = V(C)$ for every $u \in F - v$; or $N_C(u_1) = N_C(u_2) = N_C(w)$.
(4) If $\|F, D\| = 9$, then $D \cong K_{3,3}$ and $N_D(u_1) = N_D(u_2) = A$ where A is a partite set of D.

Proof. First we will prove (1), so assume $C \cong K_{2,2}$ and let A and B be the two partite sets of C. Because $\|F, C\| = 6$, Lemma 19 implies that for every $x \in F$, we have $N_C(x) \in \{A, B\}$. If $N_C(u_1) \neq N_C(u_2)$, then we can assume without loss of generality that $N_C(u_1) = N_C(w) = A$ and $N_C(u_2) = B$. But then, for $a \in A$, au_1P_1wa and $C - a + u_2$ are both cycles. This contradiction implies (1) and $\|F, C\| \leq 6$ when $C \cong K_{2,2}$.

To prove (2), and (3) we use the following claim.

Claim 26.1. *Let $\{x, y, z\} = F$ and suppose that there exists a path P in R from x to z that avoids y and a path Q from y to z that avoids x. For every $C \in \mathcal{C}$, if $\|F, C\| \geq 6$ and $\|x, C\| = 3$, then $\|F, C\| = 6$, $C \cong K_3$, $N_C(y) \cup N_C(z) = V(C)$, and one of the following two options hold: $\|y, C\| = 1$ and $\|z, C\| = 2$; or $\|y, C\|, \|z, C\| \in \{0, 3\}$.*

Proof. Lemma 19 implies that C is a triangle. If y and z have a common neighbor $c \in C$, then $Q + c$ and $C - c + x$ both contain a cycle, a contradiction. Therefore, $\|F, C\| = 6$ and $N_C(y) \cap N_C(z) = \emptyset$, so $\|\{y, z\}, C\| = 3$ and $N_C(y) \cup N_C(z) = V(C)$. If $N_C(z) = \{c\}$, then $N_C(y) = V(C - c)$, so $xPzcx$ and $C - c + y$ are both cycles, a contradiction. Therefore, either $\|z, C\| = 2$ and $\|y, C\| = 1$; or $\|y, C\|, \|z, C\| \in \{0, 3\}$. □

If we assume that $\|F, C\| \geq 7$, then, by Claim 26.1, neither u_1 nor u_2 can send 3 edges to C, so $\|F, C\| = 7$, $\|w, C\| = 3$, $\|u_1, C\| = \|u_2, C\| = 2$ and $C \cong K_3$. If there exists $c \in N_C(u_2) \setminus N_C(u_1)$, then $C - c + u_1$ and $P_2 + c$ both contain cycles, a contradiction. Therefore, we have proved (2).

To see (3), assume that $\|F, C\| \geq 6$ and there exists a path in R between every pair of vertices in F that avoids the remaining vertex in F. Note that Lemma 19 implies that C is either a $K_{2,2}$ or a triangle. If C is a $K_{2,2}$, then (1) implies that $N_C(u_1) = N_C(u_2) = A$ where A is a partite set of C, but, since there is a path from u_1 to u_2 that avoids w, we can apply (1) with w playing the role of u_2 and conclude that $N_C(w) = A$ as well. Now assume that C is a triangle. Let $\{x, y, z\} = F$ be a labeling of F, so that $\|x, C\| \geq \|y, C\| \geq \|z, C\|$. If $\|x, C\| = 3$, then Claim 26.1, implies that $\|y, C\| = 3$ and $\|z, F\| = 0$. Otherwise, $\|x, C\| = \|y, C\| = \|z, C\| = 2$. If the neighborhoods of x, y are z on C are not all identical, then we can assume that there exists $c \in N_C(x) \cap N_C(y)$ such that $c \notin N_C(z)$. If P is the path from x to y in R that avoids z, then $C - c + z$ is a triangle and $cxPyc$ is a cycle, a contradiction. Therefore, we have proved (3)

To prove (4), fix $D \in \mathcal{D}$ and suppose $\|F, D\| \geq 9$. If D is isomorphic to K_4, then one of u_1 or u_2, say u_1, is such that $\|u_1, D\| \geq 3$. Because $\|\{u_2, w\}, D\| \geq 5$, there exists $d \in D$ that is a neighbor of both u_2 and w, which implies that $P_2 + d$ contains a cycle. Since $D - d + u_1$ contains a chorded cycle, we have a contradiction. Therefore, D is not isomorphic to K_4, and Lemma 20 implies that every vertex in F sends 3 edges to D, and Lemma 22 gives us that $|D| \in \{4, 6\}$. If $|D| = 4$, let $d_1d_2d_3d_4d_1$ be a cycle of D with chord d_1d_3. Note that, by Lemma 20, d_2 and d_4 are both in the neighborhood of every vertex in F. We assume without loss of generality that $\|d_1, F\| \geq \|d_3, F\|$. If $\|d_1, F\| = 2$, then $\|d_3, F\| = 1$, and we can label the vertices in F as x, y, and z so that xd_1, yd_1, and $zd_3 \in E(G)$. Then $d_1xd_2yd_1$ is a cycle with chord d_1d_2 and zd_3d_4z is a triangle, a contradiction. If $\|d_1, F\| = 3$, then $\|d_3, F\| = 0$, $P_1 + d_4$ is a cycle, and $d_1u_2d_2d_3d_1$ is a cycle with chord d_1d_2, which is also a contradiction.

Thus, $|D| = 6$ and by Lemma 20, $D \cong K_{3,3}$. If we label the partite sets of D as $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, then every vertex in F has either A or B as its neighborhood in D. If $N_D(u_1) \neq N_D(u_2)$, then without loss of generality, we can assume that $N_D(u_1) = N_D(w) = A$ and $N_D(u_2) = B$. In this case, $a_1u_1P_1wa_2b_1$ is a cycle with two chords, u_1a_2 and wa_1, and $u_2b_2a_3b_3u_2$ is a cycle, a contradiction. Therefore, we can assume that $N_D(u_1) = N_D(u_2) = A$. □

Lemma 27. *For positive integers r and s, let G be an (r, s)-extremal graph on n vertices. Suppose that G contains $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an optimal $(r - 1, s)$-family. If $|R| \geq 3$, then R is isomorphic to a star.*

Proof. We prove this lemma by proving a series of claims. In each, we assume $|R| \geq 3$.

Claim 27.1. *R is a forest.*

Proof. A cycle in R yields an (r, s)-family, contradicting the fact that G is an (r, s)-extremal graph. □

Claim 27.2. *$\delta(R) \geq 1$.*

Proof. Let $v \in R$ and suppose that $d_R(v) = 0$. Let p be the endpoint of a longest path in $R - v$. By Claim 27.1, we have that $d_R(p) \leq 1$. Let $F = \{v, p\}$. Using Lemma 23, we have that

$$\|F, V(G)\| \leq \|F, \mathcal{U}\| + \|F, R\| \leq 4(r - 1) + 6s + 1 = 4r + 6s - 3 < 2\delta(G)$$

a contradiction. □

Claim 27.3. *If there are two maximal paths in R, then they cannot be vertex-disjoint.*

Proof. Let P and Q be two maximal paths in R that are vertex-disjoint, and let F be the set of endpoints of P and Q. By Claim 27.2, both P and Q contain at least 2 vertices, so $|F| \geq 4$ and furthermore $\|F, R\| = 4$. With Lemma 25, we have that

$$\|F, V(G)\| \leq \|F, \mathcal{U}\| + \|F, R\| \leq 7(r - 1) + 10s + 4 \leq 8r + 12s - 6 < 4\delta(G),$$

a contradiction. □

Claim 27.4. *There is at most one vertex $u \in R$ such that $d_R(u) \geq 2$.*

Proof. By Claims 27.1 and 27.3, we have that R is a tree. Suppose there exists two vertices $u, v \in R$ such that $d_R(u)$ is maximum and $d_R(v) \geq 2$. Let P be the unique path from u to v in R, let H_1 be the component containing u in $R - E(P)$, and let H_2 be the component containing v in $R - E(P)$. Since R is a tree, both H_1 and H_2 are trees.

Assume that $d_R(u) \geq 3$ so that H_1 has two leaves, say x_1 and y_1, neither of which are u. Observe that x_1 and y_1 are leaves in R as well. If H_2 has two leaves, say x_2 and y_2, neither of which are v, then R contains two disjoint maximal paths (one between x_1 and y_1 in H_1, and one between x_2 and y_2 in H_2) contradicting Claim 27.3. Therefore, $d_R(v) = 2$ and H_2 has exactly two leaves, v and another vertex, say z. Just as with x_1 and y_1, z is also a leaf in R.

Observe that the unique path in H_1 between x_1 and y_1, and the unique path in H_2 between v and z are disjoint. Let $F = \{x_1, y_1, v, z\}$, and note that $\|F, R\| = 5$. With Lemma 25, we have

$$\|F, V(G)\| \leq \|F, \mathcal{U}\| + \|F, R\| \leq 7(r-1) + 10s + 5 \leq 8r + 12s - 5 < 4\delta(G),$$

a contradiction.

Therefore, we can assume that $d_R(u) = d_R(v) = 2$, which implies that R is a path. Let p and p' be the endpoints of the path R and let $F = \{p, u, v, p'\}$. Lemma 25 implies that

$$\|F, V(G)\| = \|F, \mathcal{U}\| + \|F, R\| \leq 7(r-1) + 10s + 6 = 8r + 12s - 4 - (r + 2s - 3) \leq 4\delta(G) - (r + 2s - 3),$$

so, we have that $r = s = 1$. Therefore, $\mathcal{C}$ is empty and there is a single chorded cycle $D \in \mathcal{D}$. The fact that $\delta(G) = 4$ implies that $d_D(p) = d_D(p') = 3$ and $d_D(u) = d_D(v) = 2$, so, by Lemma 22, we have $|D| \in \{4, 6\}$.

If $|D| = 4$, then since $\|p, D\| = 3$, Lemma 20 implies that there exists $d \in D$ such that for every $d' \in D - d$, we have that $D - d' + p$ is a chorded cycle. But, because $\|F - p, D - d\| \geq 4$, there exists $d' \in D - d$, such that $\|d', F - p\| \geq 2$. Therefore, $R - p + d'$ contains a cycle, a contradiction.

If $|D| = 6$, then Lemma 20 implies that D is a $K_{3,3}$, and we can let A and B denote the two partite sets of D. Furthermore, we can assume that $N_D(p) = A$. If $N_D(p') = A$, then neither u nor v can also have a neighbor a in A, otherwise $R + a$ is a chorded cycle while $D - a$ contains a 4-cycle. So u and v must have a common neighbor $b \in B$. But then $uPv + b$ is a cycle, while $D - b + p$ is a $K_{3,3}$ a contradiction. Therefore, we can assume that $N_D(p') = B$. Because $\|u, D\| + \|v, D\| = 4$, either $N_D(u) \cup N_D(v)$ intersects both A and B; or $N_D(u) \cap N_D(v) \neq \emptyset$. Therefore, in either case, there exist $a \in A$ and $b \in B$ such that $uPv + a + b$ contains a cycle. But we then have a contradiction because $D - a - b + p + p'$ contains a chorded cycle. □

We now prove Lemma 27. Note that Claims 27.1 and 27.3 imply that R is a tree. Pick u in R such that $d_R(u)$ is maximum. Since $|R| \geq 3$ and R is connected, we have that $d_R(u) \geq 2$. By Claim 27.4, u is the only vertex in the tree R with degree greater than 1. This implies that R is a star. □

Lemma 28. *For positive integers r and s, let G be an (r, s)-extremal graph on n vertices. Suppose that G contains $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an optimal $(r-1, s)$-family. If $|R| \geq 4$, then every $D \in \mathcal{D}$ is isomorphic to $K_{3,3}$.*

Proof. By Lemma 27, R is a star. Let $q \in V(R)$ be the non-leaf vertex in R and let F be a 3-vertex subset of $R - q$. Note that, for every vertex in $v \in F$, there exists a path in R that avoids v that has the two remaining vertices in $F - v$ as its endpoints. Therefore, Lemma 26 implies that $\|F, C\| \leq 6$ for every $C \in \mathcal{C}$ and $\|F, D\| \leq 9$ for every $D \in \mathcal{D}$. Hence, we have that

$$3\delta(G) \leq \|F, V(G)\| \leq 6(r-1) + 9s + 3 = 6r + 9s - 3 \leq 3\delta(G),$$

so, for every $D \in \mathcal{D}$, we have $\|F, D\| = 9$, which, with Lemma 26, implies that D is isomorphic to $K_{3,3}$. □

Lemma 29. *For positive integers r and s, let G be an (r, s)-extremal graph on n vertices. Suppose that G contains an optimal $(r-1, s)$-family $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ that covers at most $n - 3$ vertices. Furthermore, suppose no $D \in \mathcal{D}$ is isomorphic to K_4, and there exists $q \in R$ such that $d_R(q) \geq 2$. Then all of the following hold:*

(i) If $n \leq 4r + 6s - 4$, then $s = 1$.
(ii) For every $p \in R - q$, we have that $d_G(p) = 2r + 3s - 1$.
(iii) For every $p, p' \in R - q$ and every $U \in \mathcal{U}$, we have that $N_U(p') = N_U(p)$.
(iv) For every $p \in R - q$, every $U \in \mathcal{U}$, and for every $u \in V(U) \setminus N_U(p)$, there exist two disjoint subgraphs U' and P' in $U + R$ such that U' is isomorphic to U, neither u nor p are contained in U', and P' is a path on 3 vertices.

Proof. In each of the following claims, assume G is an (r, s)-extremal graph on n vertices with an optimal $(r-1, s)$-family $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ covering at most $n - 3$ vertices (i.e., $|R| \geq 3$). Furthermore, suppose no $D \in \mathcal{D}$ is isomorphic to K_4, and there exists $q \in R$ such that $d_R(q) \geq 2$. Recall that by Lemma 27, R is a star, and for every $p \in R - q$, we have $d_R(p) = 1$.

Claim 29.1. *For every $v \in R$, $C \in \mathcal{C}$, and $D \in \mathcal{D}$, we have $\|v, C\| \leq 2$ and $\|v, D\| \leq 3$. Furthermore, if $v \in R - q$, then equality holds.*

Proof. Since $s \geq 1$, there exists $\hat{D} \in \mathcal{D}$. By assumption, $\hat{D}$ is not isomorphic to K_4. If $|\hat{D}| \geq 5$, then let $\hat{C}$ be a cycle contained in $\hat{D}$ such that $|\hat{C}| \leq |\hat{D}| - 2$. Otherwise, $\hat{D}$ is isomorphic to $K_{1,1,2}$ and we let $\hat{C}$ be a triangle contained in $\hat{D}$. Let v be a vertex in R. If there exists $D \in \mathcal{D}$ such that $\|v, D\| \geq 4$, then Lemma 20 implies that $D \cong K_4$, a contradiction to our assumption that no element of $\mathcal{D}$ is isomorphic to a K_4. If there exists $C \in \mathcal{C}$ such that $\|v, C\| \geq 3$, then Lemma 19 implies that $G[C + v] \cong K_4$, so we can replace C with $\hat{C}$ in $\mathcal{C}$ and $\hat{D}$ with $C + v$ in $\mathcal{D}$ to obtain an $(r - 1, s)$-family that violates either (O1) or (O2), a contradiction. Therefore,

$$\|v, D\| \leq 3 \text{ for all } D \in \mathcal{D} \text{ and } \|v, C\| \leq 2 \text{ for all } C \in \mathcal{C}, \tag{22}$$

and, for every $p \in R - q$,

$$2r + 3s - 1 \leq d_G(p) = \|p, \mathcal{U}\| + \|p, R\| \leq (2(r - 1) + 3s) + 1 = 2r + 3s - 1.$$

Thus, for all $p \in R - q$, $\|p, D\| = 3$ and $\|p, C\| = 2$ for all $D \in \mathcal{D}$ and $C \in \mathcal{C}$. □

Since $d_R(p) = 1$ for all $p \in R - q$, Claim 29.1, implies $d_G(p) = 2(r-1)+3s+1 = 2r+3s-1$. So (ii) holds. Furthermore, Claim 29.1 along with Lemmas 19 and 22, imply that every $C \in \mathcal{C}$ is either a triangle or a $K_{2,2}$ and every $D \in \mathcal{D}$ is either a $K_{1,1,2}$ or $K_{3,3}$.

Claim 29.2. *If there exists a triangle in $\mathcal{C}$, then every $D \in \mathcal{D}$ is isomorphic to $K_{1,1,2}$.*

Proof. Assume that there exists a triangle T in $\mathcal{C}$. For any $v \in R - q$, Claim 29.1 implies that $T + v$ is isomorphic to $K_{1,1,2}$. If there exists $D \in \mathcal{D}$ that is isomorphic to $K_{3,3}$, then we could replace D with $T + v$, and T with a $K_{2,2}$ contained in D, which violates (O1). Since every $D \in \mathcal{D}$ is either $K_{1,1,2}$ or $K_{3,3}$, it must be that every $D \in \mathcal{D}$ is isomorphic to $K_{1,1,2}$. □

Claim 29.3. *If $|R| \geq 4$, then (i), (iii), and (iv) hold.*

Proof. Suppose that $|R| \geq 4$ and let $\{p, p', p''\} = F$ be a 3-vertex set in $R - q$. Note that for any labeling $\{x, y, z\} = F$, the path xqy is a path in R that avoids z. Therefore, Claim 29.1 and Lemma 26 imply that for every $U \in \mathcal{U}$, we have that $N_U(p) = N_U(p')$ and, for every $u \in V(U) \backslash N_U(p)$, the graph $U' = U - u + p'$ is isomorphic to U. Since U' and the 3-vertex path pqp'' are disjoint subgraphs in $U + R$, both hold. To show that (i) also holds, note that, by Lemma 28, $D \cong K_{3,3}$ for every $D \in \mathcal{D}$, so since $s \geq 1$, Claim 29.2 implies that $C \cong K_{2,2}$ for every $C \in \mathcal{C}$. This gives us that $n \geq 4(r-1) + 6s + |R| \geq 4r + 6s$. So n is never at most $4r + 6s - 4$, and (i) holds. □

For the remainder of this proof we assume that $|R| = 3$ and let $\{p, p'\} = V(R - q)$. By Lemma 27, we have that R is the path $P = pqp'$ on 3 vertices, so

$$\|P, R\| = 4. \tag{23}$$

Claim 29.4. *If $U \in \mathcal{U}$ is isomorphic to $K_{t,t}$ for some $t \in \{2, 3\}$, then hold for this U.*

Proof. Suppose $U \in \mathcal{U}$ is isomorphic to $K_{t,t}$ for $t \in \{2, 3\}$. Note that Claim 29.1 implies that each of p and p' send exactly t edges to U, with Lemmas 19 and 20, we also know that both p and p' are adjacent to one of the two partite sets of U. Let A and B be the two partite sets of U labeled so that $N_U(p) = A$. By Claim 29.1, $\|q, \mathcal{U}\| \leq 2(r - 1) + 3s$. So if $t = 2$ (i.e., $U \in \mathcal{C}$), we have;

$$2r + 3s - 1 \leq \|q, U\| + \|q, \mathcal{U} - U\| + \|q, R\| \leq \|q, U\| + (2(r - 2) + 3s) + 2 = (2r + 3s - 1) + (\|q, U\| - 1).$$

Therefore, $\|q, U\| \geq 1$. Similarly, if $t = 3$ (i.e., $U \in \mathcal{C}$), we deduce $\|q, U\| \geq 2$. So always $\|q, U\| \geq t - 1$. Lemma 21 implies that $N_U(q) \subseteq B$, so Lemma 21 again implies that $N_U(p') = A$ which proves (iii).

To prove (iv) in this case, fix $u \in V(U) \setminus N_U(p)$. If $u \in N(q)$, then we let $P' = pqu$ and $U' = U - u + p$. If $u \notin N(q)$, then we fix $a \in A$ and let $P' = uap$ and $U' = U - a - u + q + p'$. In both cases U' is isomorphic to U and P' is a 3-vertex path. Therefore, (iv) holds in this case. □

If every chorded cycle in $\mathcal{D}$ is isomorphic to $K_{3,3}$, then Claim 29.2 implies that every cycle in $\mathcal{C}$ is isomorphic to $K_{2,2}$, so by Claim 29.4, hold for every $U \in \mathcal{U}$. Since we also have that $n = 4(r - 1) + 6s + |R| = 4r + 6s - 1$, (i) also holds and we have proved the lemma. Therefore, we can assume that there exists $D^* \in \mathcal{D}$ that is isomorphic to $K_{1,1,2}$.

Because D^* is not isomorphic to $K_{3,3}$, Lemma 26 implies that $\|P, D^*\| \leq 8$. Therefore, with Claim 29.1 and (23), we have that

$$3\delta(G) \leq \|P, V(\mathcal{U})\| + \|P, R\| \leq 6(r - 1) + 8 + 9(s - 1) + 4 = 3(2r + 3s - 1),$$

so, by the minimum degree condition,

$$\|P, D^*\| = 8, \text{ and } \|P, C\| = 6 \text{ for every } C \in \mathcal{C}. \tag{24}$$

We also have that for every $D \in \mathcal{D} - D^*$, $\|P, D\| = 9$, so Lemma 26 implies that D is isomorphic to $K_{3,3}$. If $n \le 4r + 6s - 4$, then

$$|V(\mathcal{C})| \le n - |D^*| - |V(\mathcal{D} - D^*)| - |R| \le (4r + 6s - 4) - 4 - 6(s - 1) - 3 < 4(r - 1),$$

so there exists a triangle in $\mathcal{C}$. With Claim 29.2, this implies that $s = 1$, so we have proved (i).

Claim 29.5. *If $U \in \mathcal{U}$ is isomorphic to $K_{1,1,2}$, then hold for this U.*

Proof. Let $d_1d_2d_3d_4d_1$ be the spanning cycle of U, labeled so that d_1 and d_3 are adjacent. Note that, by Claim 29.1, $\|p, U\| = \|p', U\| = 3$, so (24) implies that $\|q, U\| = 2$. Lemma 20, implies that both p and p' are adjacent to d_2 and d_4 and exactly one of d_1 or d_3. Without loss of generality we can assume that pd_1 is an edge. If $N_U(p') = \{d_2, d_3, d_4\}$, when we have the two disjoint triangles pd_1d_2 and $p'd_3d_4$. Because q has two neighbors in U and is adjacent to both p and p', q must have at least two neighbors in one of these two triangles, so $U + P$ contains a $(1, 1)$-family, contradicting the fact that G is (r, s)-extremal. Therefore, $N_U(p) = N_U(p') = \{d_1, d_2, d_4\}$ and we have proved (iii) when $U \cong K_{1,1,2}$. Note that since $\|q, U\| = 2$, there exist $d \in \{d_2, d_3, d_4\}$ such that qd is an edge. Since pqd is a path on 3 vertices and $D - d + p'$ is isomorphic to $K_{1,1,2}$, we have that (iv) holds. □

Claim 29.6. *If $U \in \mathcal{U}$ is isomorphic to a triangle, then hold for this U.*

Proof. Suppose U is isomorphic to a triangle. Note that Claim 29.1 and (24) imply that $\|p, U\| = \|p', U\| = \|q, U\| = 2$. If p and p' do not have the same neighborhood in U, then there exists a vertex $u \in N_U(q)$ such that $\|u, \{p, p'\}\| = 1$. Assume without loss of generality $\|u, p\| = 1$ and $\|u, p'\| = 0$. Since uqp and $D - u + p'$ are both triangles, we have contradicted the fact that G is (r, s)-extremal. Therefore, p and p' have the same neighborhood on U and we have proved (iii). Let $u \in V(U)$ that is not adjacent to either p or p'. If q is adjacent to u, then pqu is a path on 3 vertices and $U - u + p'$ is a triangle, and we have (iv). Otherwise, $N_U(p) = N_U(p') = N_U(q) = U - u$. If we let $\{u', u''\} = V(U - u)$, then we have the path $pu'u$ and the triangle $qp'u''$, so we have proved (iv). □

This completes the proof of Lemma 29. □

Lemma 30. *For positive integers r and s, let G be an (r, s)-extremal graph on n vertices. Suppose that G contains an optimal $(r - 1, s)$-family that covers at most $n - 3$ vertices, and that does not contain a chorded cycle isomorphic to K_4. Then $\alpha(G) = n - (2r + 3s - 1)$. Furthermore, if $n \le 4r + 6s - 4$, then $s = 1$.*

Proof. By assumption, there exists $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an optimal $(r - 1, s)$-family such that $|R| \ge 3$ and there does not exists a copy of K_4 in $\mathcal{D}$. By Lemma 27, R is a star. Let q be the center of this star. We have that $d_R(q) = |R| - 1 \ge 2$. Let $p \in R - q$, and let $I = V(G) \setminus N_G(p)$. By Lemma 29(i), we have that $s = 1$ when $n \le 4r + 6s - 4$, and by Lemma 29(ii), we have that $|I| = n - (2r + 3s - 1)$.

To complete the proof of the lemma, we will show that I is an independent set by proving that $N_G(u) = N_G(p)$ for every vertex $u \in I$. To this end, let $u \in I$. If $u \in R$, then Lemma 29(iii) implies that $N_G(u) = N_G(p)$. If $u \notin R$, then there exist $U \in \mathcal{U}$ such that $u \in U$. Lemma 29(iv) implies that there are disjoint subgraphs U' and P' in $U + R$ such that U' is isomorphic to U, neither p nor u is contained in U', and P' is a path on 3 vertices. If $U \in \mathcal{C}$, let $\mathcal{C}' = \mathcal{C} - U + U'$ and $\mathcal{D}' = \mathcal{D}$, and if $U \in \mathcal{D}$, let $\mathcal{D}' = \mathcal{D} - U + U'$ and $\mathcal{C}' = \mathcal{C}$. In either case, we have that $(\mathcal{C}', \mathcal{D}')$ is an optimal $(r - 1, s)$-family, and that $\mathcal{D}'$ does not contain a copy of K_4. Let $\mathcal{U}' = \mathcal{C}' \cup \mathcal{D}'$ and let R' be the graph induced by $V(G) \setminus V(\mathcal{U}')$. By Lemma 27, R' is a star. Let $q' \in R'$ be the center of this star and note that $d_{R'}(q') \ge 2$. Since u and p are both in R' and they are not adjacent, neither u nor p is q'. Applying Lemma 29(iii) and the above argument with $(\mathcal{U}', \mathcal{C}', \mathcal{D}', R')$, q', u, and p playing the roles of $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$, q, p' and p, respectively, implies that $N_G(u) = N_G(p)$. This completes the proof of the lemma. □

Lemma 31. *Let G be a graph and let X and Y be two disjoint cycles in G such that $|X| \ge |Y|$ and $|X| \ge 4$. If $\|X, Y\| > 2|X|$, then there exist two cycles X' and Y' in $G[V(X) \cup V(Y)]$ such that $|X'| + |Y'| < |X| + |Y|$.*

Proof. Fix some orientation for X and, for every $x \in X$, let x^- and x^+ be the vertices that proceed and succeed x, respectively. Make the analogous definitions for the cycle Y.

Assume that

$$\|X, Y\| > 2|X| \ge |X| + 4, \tag{25}$$

and, for every pair of disjoint cycles X' and Y' in the graph induced by $V(X) \cup V(Y)$,

$$|X'| + |Y'| \ge |X| + |Y|. \tag{26}$$

Claim 31.1. *For every $x \in X$ and $y \in Y$, if either*

- $\|x, Y\| \geq 3$ *and* $\|y, X - x\| \geq 2$, *or*
- $\|y, X\| \geq 3$ *and* $\|x, Y - y\| \geq 2$,

then $N_Y(x) = \{y^-, y, y^+\}$ *and* $N_X(y) = \{x^-, x, x^+\}$.

Proof. If $\|x, Y\| \geq 3$ and $\|y, X - x\| \geq 2$, then $X - x + y$ and $Y - y + x$ both contain cycles. Therefore, (26) implies that $N(x) \cap V(Y - y) = \{y^-, y^+\}$ and $N(y) \cap V(X - x) = \{x^-, x^+\}$. Because $\|x, Y\| \geq 3$, we have that $N_Y(x) = \{y^-, y, y^+\}$ which implies that $N_X(y) = \{x^-, x, x^+\}$. The same argument works when $\|y, X\| \geq 3$ and $\|x, Y - y\| \leq 2$. $\square$

By (25) and the fact that $|X| \geq |Y|$, we have that there exists $x_0 \in X$ such that $\|x_0, Y\| \geq 3$ and there exists $y_0 \in Y$ such that $\|y_0, X\| \geq 3$. By Claim 31.1, we then have that $N_X(y_0) = \{x_0^-, x_0, x_0^+\}$ and $N_Y(x_0) = \{y_0^-, y_0, y_0^+\}$. Also, by Claim 31.1, we have that $\|x, Y\| \leq 1$ for every $x \in X \setminus N_X(y_0)$ and $\|y, X\| \leq 1$ for every $y \in Y \setminus N_Y(x_0)$. Therefore, by (25),

$$|X| + 5 \leq \|X, Y\| = \|x_0, Y\| + \|\{x_0^-, x_0^+\}, Y\| + \|X \setminus N_X(y_0), Y\| \leq 3 + \|\{x_0^-, x_0^+\}, Y\| + (|X| - 3),$$

so $\|\{x_0^-, x_0^+\}, Y\| \geq 5$. Similarly, $\|\{y_0^-, y_0^+\}, X\| \geq 5$. We can assume without loss of generality that $\|x_0^+, Y\| \geq 3$ and $\|y_0^+, X\| \geq 3$. Applying Claim 31.1, first with y_0^+ and x_0 playing the roles of y and x, respectively, and then with y_0^+ and x_0^+ playing the roles of y and x, respectively, we have that

$$N_X(y_0^+) = \{x_0^-, x_0, x_0^+\} = \{x_0, x_0^+, (x_0^+)^+\},$$

so $(x_0^+)^+ = x_0^-$, a contradiction to $|X| \geq 4$. $\square$

Lemma 32. *For positive integers r and s, let G be an (r, s)-extremal graph on n vertices. Suppose that G contains $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an optimal $(r-1, s)$-family, $|R| \geq 3$, and there exists $D \in \mathcal{D}$ such that D is isomorphic to K_4. Then $s = 1$, $n = 3r + 4s = 3r + 4$, and $\alpha(G) = n - (2r + 3s - 1) = r + 2$.*

Proof. By Lemma 28 and the fact that G is (r, s)-extremal, we can assume that $|R| = 3$. By Lemma 27, R is a path P on three vertices. Note that $\|P, R\| = 4$.

Claim 32.1. *There exists $C^* \in \mathcal{C}$ such that $\|P, C^*\| \geq 7$.*

Proof. By assumption there exists $D^* \in \mathcal{D}$ such that D^* is isomorphic to K_4. By Lemma 26, $\|P, D\| \leq 9$ for every $D \in \mathcal{D}$. Therefore, because $\|P, R\| = 4$, we have that

$$\|P, \mathcal{C}\| \geq 3\delta(G) - \|P, \mathcal{D}\| - \|P, R\| \geq 3(2r + 3s - 1) - 9(s - 1) - \|P, D^*\| - 4 = 6(r - 1) + 8 - \|P, D^*\|, \tag{27}$$

so if $\|P, D^*\| \leq 7$, there must exist $C^* \in \mathcal{C}$ such that $\|P, C^*\| \geq 7$.

Assume that $\|P, D^*\| \geq 8$ and $\|P, C\| < 7$ for every $C \in \mathcal{C}$. In this case, (27) implies that, we must have $\|P, D\| = 9$ for every $D \in \mathcal{D} - D^*$ and $\|P, C\| = 6$ for every $C \in \mathcal{C}$. Lemma 26, then implies that every vertex in P sends exactly 3 edges to every $D \in \mathcal{D} - D^*$.

If there exists a vertex $v \in P$ such that $\|v, D^*\| \leq 1$, then $\|P - v, D^*\| \geq 7$, so, because the graph $H = D^* + (P - v)$ has at least $\binom{4}{2} + 7 = \binom{6}{2} - 2$ edges, there are two disjoint triangles in H. Then

$$\|v, \mathcal{C}\| \geq \delta(G) - \|v, \mathcal{D} - D^*\| - \|v, D^*\| - \|v, R\| \geq 2r + 3s - 1 - 3(s - 1) - 1 - 2 = 2(r - 1) + 1,$$

so that v sends at least 3 edges to some $C \in \mathcal{C}$. This, with Lemma 19, implies that $C + v$ is isomorphic to K_4. Because we can replace D^* and C with the two disjoint triangles in H and the chorded cycle $C + v$, we have contradicted the fact that G is (r, s)-extremal. Therefore, we can assume that every vertex in P sends at least 2 edges to D^*.

Let $P = pqp'$. Note that, because G is (r, s)-extremal, the graph $P + D^*$ does not contain a $(1, 1)$-family. Since $\|q, D^*\| \geq 2$, one of the two endpoints of P, say p, is such that p and q have a common neighbor in D^*. For every $d \in N_{D^*}(p) \cap N_{D^*}(q)$, we have that $D^* - d + p'$ is not a chorded cycle, so, because $\|p', D^*\| \geq 2$, we have that $d \in N_{D^*}(p')$ and $\|p', D^*\| = 2$. Furthermore, because $p'qd$ is a triangle, we also have that $\|p, D^*\| = 2$, so $\|q, D^*\| = 4$. Now $N_{D^*}(p) \cap N_{D^*}(q) = N_{D^*}(p)$, which implies that $N_{D^*}(p') = N_{D^*}(p)$. Therefore, $G[\{p, p'\} \cup N_{D^*}(p)]$ is a chorded cycle, and $G[(V(D^*) \setminus N_{D^*}(p)) \cup \{q\}]$ is a triangle, a contradiction. $\square$

Claim 32.2. *Every $D \in \mathcal{D}$ is isomorphic to K_4.*

Proof. By Claim 32.1 and Lemma 19, there exists a triangle $C^* \in \mathcal{C}$ such that $\|P, C^*\| \geq 7$. By Lemma 26, we have that $H = G[P + C^*] \cong K_3 \vee \overline{K_3}$. Note that H contains a copy of K_4, which we will denote D^*. Let $D \in \mathcal{D}$. If D has length $t \geq 5$, then the shortest cycle C in D has length at most $t - 2$, so we can replace C^* with C and D with D^* to obtain a new $(r-1, s)$-family that contradicts (O1). Therefore, we can assume that $|D| = t = 4$. If $D \ncong K_4$, then we can replace C^* with a triangle from D and replace D with D^* to obtain a new $(r-1, s)$-family that contradicts (O2). $\square$

Claim 32.3. *Every $C \in \mathcal{C}$ is a triangle.*

Proof. Let C_1 be a longest cycle in $\mathcal{C}$ and assume that C_1 is not a triangle. By (O1), C_1 is an induced cycle so $\|C_1, C_1\| = 2|C_1|$. By Lemma 31, the assumption that C_1 is the longest cycle, and (O1), we have that $\|C_1, C\| \leq 2|C_1|$ for all $C \in \mathcal{C} - C_1$. Therefore, $\|C_1, V(\mathcal{C})\| \leq 2(r-1)|C_1|$

Let $D \in \mathcal{D}$. By Claim 32.2, D is isomorphic to K_4. If there exists $d \in D$ that has 3 neighbors on C_1, then, because C_1 is not a triangle, we can find a chorded cycle D' in $C_1 + d$ on at most $|C_1|$ vertices. Therefore, we could replace D with D' and C_1 with the triangle $D - d$, violating (O1). This implies that $\|C_1, D\| \leq 2|D| \leq 2|C_1|$. Therefore, $\|C_1, \mathcal{D}\| \leq 2s|C_1|$, and, because $s \geq 1$ and $|C_1| \geq 4$,

$$\|C_1, R\| \geq |C_1|(2r + 3s - 1) - 2s|C_1| - 2(r-1)|C_1| = |C_1|(s+1) \geq 8,$$

so, because $|R| = 3$, there exists $v \in R$ such that $\|v, C_1\| \geq 3$. This, with Lemma 19, contradicts the assumption that $|C_1| \geq 4$. □

By Claims 32.2 and 32.3 and the fact that $|R| = 3$, we have that $n = 3r + 4s$. Let $G' = G \vee \overline{K_r}$. Note that $|G'| = 4(r+s)$ and

$$\delta(G') = \min\{\delta(G) + r, n\} \geq 3(r+s) - 1. \tag{28}$$

If G' contains $r + s$ disjoint copies of K_4, then G contains an (r, s)-family; the r vertices in $V(G' - G)$ would each be in a unique copy of K_4. Therefore, G' does not contain $r + s$ disjoint copies of K_4 and, by (28) and Theorem 3, we have that either $\alpha(G') = r + s + 1$, or $r + s$ is odd and G' contains an induced copy of $\overline{K_{r+s,r+s}}$

Assume first that $\alpha(G') = r + s + 1$, and let B be an independent set of G' of order $r + s + 1$. Due to the construction of G', the independence of B, and the fact that $|B| > |G' - G|$, we have $B \subseteq V(G)$. Let $A = V(G) \setminus B$, so $|A| = 2r + 3s - 1$ and, by the minimum degree condition in G and the fact that B is independent, all possible edges exist between A and B in G. Since we have met the conditions of the lemma if $s = 1$, assume that $s \geq 2$. Then

$$\delta(G[A]) \geq 2r + 3s - 1 - |B| = r + 2s - 2 = (2r + 3s - 2)/2 + (s-2)/2 \geq (|A| - 1)/2,$$

so there exists a Hamiltonian path in $G[A]$. Since $|A| = 2r + 3s - 1$, we can partition this Hamiltonian path into a vertex disjoint collection of $r + 1$ edges and $s - 1$ copies of $K_{1,2}$. We can use this collection to find an (r, s)-family in G consisting of r triangles each with one vertex in B and two vertices in A; one chorded 4-cycle with two vertices in B and two vertices in A; and $s - 1$ chorded 4-cycles each with 1 vertex in B and 3 vertices in A, a contradiction.

Now assume that $r + s$ is odd and G' contains an induced copy of $\overline{K_{r+s,r+s}}$. Let A_1 and A_2 be two disjoint cliques each of size $r + s$ such that $\|A_1, A_2\| = 0$. Let $v \in A_1 \cup A_2$. We have that

$$d_{G'}(v) \leq (r + s - 1) + (|G'| - |A_1 \cup A_2|) = (r + s - 1) + (2r + 2s) = 3r + 3s - 1, \tag{29}$$

which implies that $v \in V(G)$, because we are assuming $s \geq 1$, and because every vertex in $V(G') \setminus V(G)$ has exactly $|G| = 3r + 4s$ neighbors in G'. Therefore, if we let $B = V(G) \setminus (A_1 \cup A_2)$, then we have that $|B| = 3r + 4s - 2(r+s) = r + 2s$. Note that, for $i \in \{1, 2\}$,

$$(|A_i| - 1) + |B| = (r + s - 1) + r + 2s = 2r + 3s - 1,$$

so every vertex in $A_1 \cup A_2$ is adjacent to every vertex in B. If $G[B]$ contains an edge, then we can use this edge to create D a $K_{1,1,2}$ with two vertices in B and one vertex in each of A_1 and A_2. Since $r + s$ is odd, there exists a perfect matching M of size $r + s - 1$ in $G[(A_1 \cup A_2) \setminus V(D)]$. Since $|B \setminus V(D)| = r + 2(s-1)$, we can find an $(r, s-1)$ family in $G - D$ by pairing r edges in M with r of the remaining vertices in B to form r triangles, and by pairing $s - 1$ of the edges of M with $s - 1$ pairs of vertices B, a contradiction. Therefore, B is an independent set. By the minimum degree condition, we have that

$$r + 2s = |B| \leq |G| - (2r + 3s - 1) = r + s + 1,$$

so $s = 1$ and $\alpha(G) = |B| = r + 2 = n - (2r + 3s - 1)$. This completes the proof of the Lemma. □

We are now ready to prove Lemma 17.

Proof of Lemma 17. Let G be an (r, s)-extremal graph on n vertices. If there exists an $(r, s-1)$-family that covers at most $n - 3$ vertices, then there exists $(\mathcal{U}, \mathcal{C}, \mathcal{D}, R)$ an optimal $(r, s-1)$-family with $|R| \geq 3$. If there exists a chorded cycle in $\mathcal{D}$ isomorphic to K_4, then Lemma 32 implies that $s = 1$, $n = 3r + 4s$ and $\alpha(G) = n - (2r + 3s - 1)$. Otherwise, Lemma 30 implies that $\alpha(G) = n - (2r + 3s - 1)$, and also that $s = 1$ if $n \leq 4r + 6s - 4$. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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