

# A NOTE ON THE BILINEAR BOGOLYUBOV THEOREM: TRANSVERSE AND BILINEAR SETS

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ABSTRACT. A set  $P \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  is called *bilinear* when it is the zero set of a family of linear and bilinear forms, and *transverse* when it is stable under vertical and horizontal sums. A theorem of the first author provides a generalization of Bogolyubov's theorem to the bilinear setting. Roughly speaking, it implies that any dense transverse set  $P \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  contains a large bilinear set. In this paper, we elucidate the extent to which a transverse set is forced to be (and not only contain) a bilinear set.

## 1. INTRODUCTION

A simple exercise shows that any nonempty subset  $A \subset \mathbb{F}_p^n$  that is closed under addition is a linear subspace, that is, the zero set of a family of linear forms. Indeed, denoting as usual

$$A \pm A = \{a \pm b : (a, b) \in A^2\},$$

this amounts to the claim that  $A + A = A \neq \emptyset$  if and only if  $A$  is a subspace (and analogously for  $A - A$ ). Considering a large amount of summands, one will eventually get  $\text{span}(A)$ , the linear subspace generated by  $A$ . This may require an unbounded number of summands as the dimension  $n$  or the prime  $p$  tends to infinity.

The following classical result states that a bounded number of summands already suffices to produce a rather large subspace of  $\text{span}(A)$ .

**Theorem 1.1** (Bogolyubov). *Let  $A \subset \mathbb{F}_p^n$  be a subset of density  $\alpha > 0$ , that is,  $|A| = \alpha p^n$ . Then  $2A - 2A$  contains a vector space of codimension  $c(\alpha) = O(\alpha^{-2})$ .*

Bogolyubov's original paper [2] deals with  $\mathbb{Z}/N\mathbb{Z}$ , but the ideas translate to finite  $\mathbb{F}_p$ -vector spaces. Note that if  $A$  is a vector space, its codimension is  $\log_p \alpha^{-1}$ . As a consequence,  $c(\alpha) \geq \log_p \alpha^{-1}$ . Sanders [6] improved the bound in the statement to a nearly optimal  $c(\alpha) = O(\log^4 \alpha^{-1})$ . Recently, bilinear versions of this result by the first author and Lê [1] and, independently, Gowers and Milićević [3] have appeared. Let us now state this bilinear Bogolyubov theorem. We need to introduce a piece of useful notation (cf. [1]).

Given a set  $A \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  we define the vertical sum or difference as

$$A \stackrel{V}{\pm} A := \{(x, y_1 \pm y_2) : (x, y_1), (x, y_2) \in A\}.$$

The set  $A \stackrel{H}{\pm} A$  is defined analogously but fixing the second coordinate. Then we define  $\phi_V$  as the operation

$$A \mapsto (A \stackrel{V}{+} A) \stackrel{V}{-} (A \stackrel{V}{+} A)$$

and  $\phi_H$  similarly. The theorem proved in [1] is the following.

**Theorem 1.2** (Bienvenu and Lê, [1]). *Let  $\delta > 0$ , then there is  $c(\delta) > 0$  such that the following holds. For any  $A \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  of density  $\delta$ , there exists  $W_1, W_2 \subseteq \mathbb{F}_p^n$  subspaces of codimension  $r_1$  and  $r_2$  respectively and bilinear forms  $Q_1, \dots, Q_{r_3}$  on  $W_1 \times W_2$  such that  $\phi_H \phi_V \phi_H(A)$  contains*

$$\{(x, y) \in W_1 \times W_2 : Q_1(x, y) = \dots = Q_{r_3}(x, y) = 0\} \quad (1)$$

where  $\max\{r_1, r_2, r_3\} \leq c(\delta)$ .

The poor bound obtained in [1] and [3] was improved very recently by Hosseini and Lovett [4] to the nearly optimal  $c(\delta) = O(\log^{O(1)} \delta^{-1})$ , at the cost of replacing  $\phi_H \phi_V \phi_H$  by a slightly longer sequence of operations.

We call a set  $A \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  *transverse* if it satisfies  $A \stackrel{V}{+} A = A \stackrel{H}{+} A = A$ . In connection with the result above the following natural problem arose: characterise transverse sets. Examples of transverse sets are what we call *bilinear* sets, that is, zero sets of linear and bilinear forms as in (1). It is tantalizing to suspect that they are the only possible examples. Theorem 1.2 only shows that any transverse set  $A$  of density  $\alpha$  contains a bilinear subset defined by  $c(\alpha)$  linear and bilinear forms.

In this paper, we find transverse, non bilinear sets  $A \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  for any  $(p, n)$  except  $p = 2$  and  $n = 2$  where it is possible to list all transverse sets and check that they are bilinear. In this direction, we provide an explicit counterexample for  $p = 3$  and  $n = 2$  and a non-constructive argument in general.

**Proposition 1.3.** *Let  $P \subset \mathbb{F}_3^2 \times \mathbb{F}_3^2$  be the set of  $((x_1, x_2), (y_1, y_2))$  satisfying*

$$\begin{cases} x_1 y_1^2 + x_2 y_2^2 = 0 \\ x_1^2 y_1 + x_2^2 y_2 = 0 \end{cases} \quad (2)$$

*is transverse but not bilinear.*

Nevertheless, we show that transversity together with an extra largeness hypothesis implies bilinearity for small characteristics. For any transverse set  $P \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$ , let  $P_x = \{y \in \mathbb{F}_p^n : (x, y) \in P\}$  be the vertical *fiber* above  $x \in \mathbb{F}_p^n$ . Notice that a non-empty fiber is a subspace.

**Theorem 1.4.** *Let  $P \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  be a transverse set such that  $P_x$  contains a hyperplane for any  $x$ . Then it is bilinear provided that the prime  $p = 2$  or  $3$ .*

We end the paper providing non constructive counterexamples.

**Theorem 1.5.** *Let  $p$  be a prime and  $n$  a positive integer.*

- (i) *For any prime  $p \geq 5$  and dimension  $n \geq 2$ , there exists a transverse, non-bilinear set  $P \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  for which  $P_{x \cdot}$  contains a hyperplane for any  $x$ .*
- (ii) *For all but finitely many primes  $p$  and dimensions  $n$ , we can find transverse, non-bilinear sets  $P \subset \mathbb{F}_p^n \times \mathbb{F}_p^n$  where  $P_{x \cdot}$  is a space of dimension 1 for any  $x$ .*

The paper is organized as follows. In Section 2 we study the explicit algebraic counterexample. In Section 3 we provide a qualitative classification of transverse sets  $P$  for which  $P_{x \cdot}$  contains a hyperplane; this entails a proof for Theorem 1.4 and the basis for the proof Theorem 1.5, which can be finally found in Section 4.

## 2. PROOF OF PROPOSITION 1.3

Consider  $P \subset \mathbb{F}_3^2 \times \mathbb{F}_3^2$  to be the set defined by the system (2). We want to show that we cannot have

$$P = \{(x, y) \in W_1 \times W_2 : Q_1(x, y) = \dots = Q_{r_3}(x, y) = 0\}$$

for any subspaces  $W_1, W_2$  and any bilinear forms  $Q_1, \dots, Q_{r_3}$  so by contradiction suppose that it is the case.

The set  $P$  is easy to describe: indeed, if  $(x, y) \in P$ , then either  $x_1 y_1 = x_2 y_2 = 0$  or  $x_1 y_1 x_2 y_2 \neq 0$ . Let

$$P_0 = \{(x_1, x_2, y_1, y_2) \in \mathbb{F}_3^2 \times \mathbb{F}_3^2 : x_1 y_1 = 0 \text{ and } x_2 y_2 = 0\}$$

and

$$P_1 = \{(x_1, x_2, y_1, y_2) \in \mathbb{F}_3^2 \times \mathbb{F}_3^2 : x_1 + x_2 = 0 \text{ and } y_1 + y_2 = 0\}$$

which is a subset of  $P$  and contains the set of points where  $x_1 y_1 x_2 y_2 \neq 0$  since  $z^2 \equiv 1 \pmod{3}$  provided  $z \not\equiv 0 \pmod{3}$ . Therefore  $P = P_0 \cup P_1$ .

Let us check that this set satisfies both conditions  $P \stackrel{V}{+} P = P$  and  $P \stackrel{H}{+} P = P$ . By symmetry it is enough to check that  $P \stackrel{H}{+} P = P$ . The cases where the points  $(x_1, x_2, y_1, y_2), (x'_1, x'_2, y_1, y_2)$  are both in  $P_0$  or  $P_1$  are easily verified and if one is in  $P_0$  and the other in  $P_1$  then  $(x_1 + x'_1)y_1^2 + (x_2 + x'_2)y_2^2 = 0$  by the first equation in (2) and

$$(x_1 + x'_1)^2 y_1 + (x_2 + x'_2)^2 y_2 = 2(x_1 x'_1 y_1 + x_2 x'_2 y_2) = 0$$

using the fact that either  $(x_1, x_2, y_1, y_2)$  or  $(x'_1, x'_2, y_1, y_2)$  is in  $P_0$ .

The fact that  $P_1 \subset P$  shows that  $W_1, W_2$  are at least one dimensional but this is not enough. Indeed, suppose they are one dimensional, then  $W_1$  and  $W_2$  should be precisely

$\{(x_1, x_2) : x_1 + x_2 = 0\}$  and  $\{(y_1, y_2) : y_1 + y_2 = 0\}$  but, for example,  $(1, 0, 0, 0) \notin W_1 \times \mathbb{F}_3^2$  and  $(0, 0, 1, 0) \notin \mathbb{F}_3^2 \times W_2$  and they belong to  $P$ . As a consequence  $W_1 = W_2 = \mathbb{F}_3^2$ . Let us show that no bilinear form other than the trivial one can vanish on this  $P$ . Suppose

$$xQy = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

for all  $(x, y) \in P$  or, alternatively,

$$xQy = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2 = 0.$$

On  $P_0 \subset P$ , this equation boils down to

$$a_{12}x_1y_2 + a_{21}x_2y_1 = 0$$

but now  $(0, 1, 1, 0), (1, 0, 0, 1) \in P_0$  imply  $a_{12} = a_{21} = 0$ . On the other hand  $(1, 2, 1, 2) \in P_1$  imply  $a_{11} + a_{22} = 0$ . This implies that if  $P$  is a bilinear set then it must be the zero set of  $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  (or equivalently,  $-Q$ ). But this is impossible because  $(x, y) = (1, 1, 1, 1) \notin P$  and yet  $xQy = 0$ . So the only option left is that  $P = \mathbb{F}_3^2 \times \mathbb{F}_3^2$  and this is not the case either. As an aside, note that  $\dim P_x.$  is not constant on  $\mathbb{F}_p^2 \setminus \{0\}$ , so this example is different from the generic ones mentioned in Theorem 1.5.

### 3. PROOF OF PROPOSITION 1.4

In this section, we prove Theorem 1.4 Let  $V_1$  and  $V_2$  be two  $\mathbb{F}_p$ -vector spaces, and we slightly generalise the above discussion to transverse sets of  $V_1 \times V_2$ . Let  $P \subset V_1 \times V_2$  be a set. Write  $P_{x.} = \{y \in V_2 : (x, y) \in P\}$  and  $P_{.y} = \{x \in V_1 : (x, y) \in P\}$  for the vertical and horizontal fibers, respectively, borrowing the notation from [3]. We now characterise transversity by some rigidity property of the map  $x \mapsto P_{x.}$

**Lemma 3.1.** *A set  $P \subset V_1 \times V_2$  is transverse if, and only if, the map  $x \mapsto P_{x.}$  satisfies the following properties.*

- (i) *For any  $x$ , the set  $P_{x.}$  is the empty set or a subspace and  $P_{x.} \subset P_0..$*
- (ii) *For any  $x \neq 0$ , the set  $P_{x.}$  depends only on the class  $[x] \in P(V_1) = V_1^*/\mathbb{F}_p^*$  of  $x$  in the projective space.*
- (iii) *If  $[z]$  is on the projective line spanned by  $[x]$  and  $[y]$ , we have  $P_{z.} \supset P_{x.} \cap P_{y.}.$*

*Proof.* Let  $P \subset V_1 \times V_2$  be transverse. Let  $x \in V_1$ . Because  $P \stackrel{V}{+} P$ , we find that  $P_{x.} + P_{x.} = P_{x.}$ , so  $P_{x.}$  is empty or a subspace. Similarly  $P_{.y}$  is empty or a subspace. Let  $y \in P_{x.}$ . Then  $x \in P_{.y}$  which implies  $0 \in P_{.y}$ , hence  $y \in P_{0.}$ , which proves the first point. Further,  $(\lambda x, y) \in P$  for any  $\lambda \neq 0$  as well, thus  $y \in P_{\lambda x.}$ ; this shows the second point. To prove the third point, suppose without loss of generality that  $z = x + \lambda y$  for some  $\lambda \in \mathbb{F}_p$ .

Let  $w \in P_x \cap P_y$ . Thus both  $x$  and  $y$  belong to the subspace  $P_w$ , so that  $z \in P_w$  too, which means that  $w \in P_z$ , concluding the proof.

We now prove the converse. Let a set  $P \subset V_1 \times V_2$  satisfy the three properties. The first point means that  $P + V = P$ . The horizontal stability follows from the second and third points.  $\square$

We will need another lemma. Recall the notation  $\mathbb{P}(V) = V^*/\mathbb{F}^*$  for the projective space of an  $\mathbb{F}$ -vector space  $V$ . We will often omit the distinction between  $x \in V$  and its class  $[x] \in \mathbb{P}(V)$ . It will be convenient to use the language of projective geometry, of which we assume some basic facts, such as the fact that any two (projective) lines of a (projective) plane intersect.

**Lemma 3.2.** *Suppose that  $\xi : \mathbb{P}(V_1) \rightarrow \mathbb{P}(V_2)$  has the property that for any  $x, y, z$  in  $V_1$  such that  $z \in \text{span}(x, y)$ , we have  $\xi(z) \in \text{span}(\xi(x), \xi(y))$ . Then  $\xi$  is either constant or injective.*

*Proof.* First we deal with the case where  $\mathbb{P}(V_1)$  is a projective line (i.e.  $\dim V_1 = 2$ ). Suppose  $\xi$  is not injective, thus there exists two non-collinear vectors  $x$  and  $y$  of  $V_1$  such that  $\xi(x) = \xi(y)$ . Now  $(x, y)$  is a basis of  $V_1$ , so for any  $z \in \mathbb{P}(V_1)$ , by the defining property of  $\xi$ , we have  $\xi(z) = \xi(x) = \xi(y)$ . So  $\xi$  is constant.

Now suppose  $\dim V_1 \geq 3$ . We already know that  $\xi$  is either injective or constant on any projective line. Assume that overall  $\xi$  is neither injective nor constant. This means that there exist two distinct points  $x, y$  such that  $\xi(x) = \xi(y)$ , and a third point  $z$  satisfying  $\xi(z) \neq \xi(x)$ . This implies that  $x, y, z$  are not (projectively) aligned, so they span a projective plane. The reader may now wish to follow the proof on Figure 1. Take a point  $w \notin \{y, z\}$  on the line  $(yz)$  spanned by  $y$  and  $z$ . Because  $\xi$  is a bijection on both lines  $(yz)$  and  $(xz)$ , and the image of both lines under  $\xi$  being the same namely  $(\xi(y)\xi(z))$ , we can find  $w' \notin \{x, z\}$  on  $(xz)$  such that  $\xi(w) = \xi(w') \neq \xi(x)$ . Now consider the intersection  $u = (ww') \cap (xy)$  in the projective plane  $\text{span}(x, y, z)$ . Then we have  $\xi(u) = \xi(x) = \xi(y) \neq \xi(w)$ , so that on the line  $(ww')$  the map  $\xi$  is neither constant nor injective, a contradiction.  $\square$

Finally, we recall the fundamental theorem [5, Théorème 7] of projective geometry.

**Theorem 3.3.** *Suppose that  $\xi : \mathbb{P}(V_1) \rightarrow \mathbb{P}(V_2)$  is injective and has the property that for any  $x, y, z$  in  $V_1$  such that  $z \in \text{span}(x, y)$ , we have  $\xi(z) \in \text{span}(\xi(x), \xi(y))$  (i.e. it maps points on a line to points on a line). Further, suppose that  $\dim V_1 \geq 3$ . Then  $\xi$  is a projective map, that is, there exists a linear injection  $f : V_1 \rightarrow V_2$  such that  $\xi([x]) = [f(x)]$  for any  $x \in V_1$ .*

Here we require the field  $\mathbb{F}_p$  to be a prime field; on a non prime finite field  $\mathbb{F}_q$ , we would need to incorporate Frobenius field automorphisms.

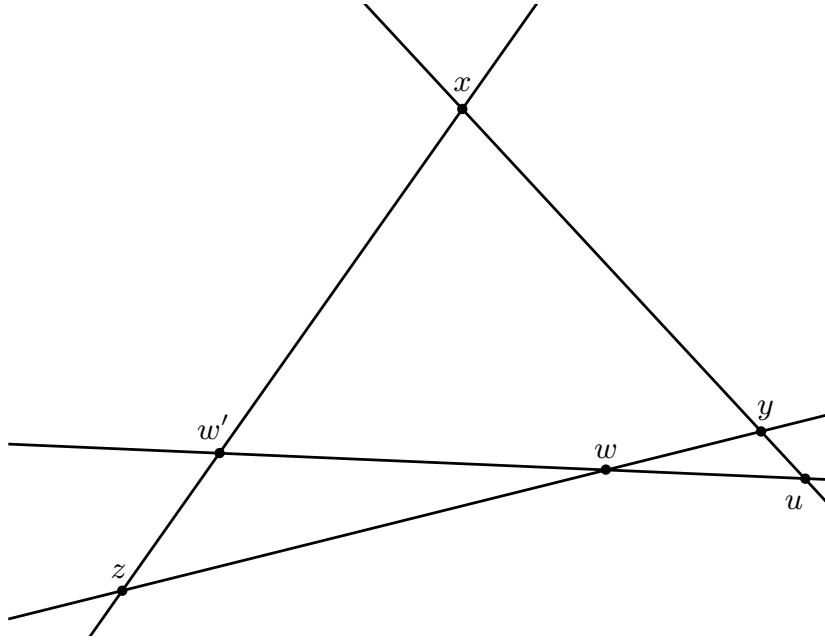


FIGURE 1. Proof of Lemma 3.2.

Note that the result holds even if  $\dim V_1 = 2$  in the case where  $p = 2$  or  $3$ . Indeed, the number of bijections between two projective lines is  $(p+1)!$ . On the other hand, since there are  $(p^2 - 1)(p^2 - p)$  linear bijections between any two given planes, the number of projective bijections is  $(p^2 - 1)(p^2 - p)/(p - 1) = (p+1)p(p-1)$ . These two numbers are equal when  $p \in \{2, 3\}$  which forces any bijection to be projective.

Now we state this section's main result.

**Proposition 3.4.** *Let  $P \subset V_1 \times V_2$  be a transverse set. Suppose that  $\text{codim}_{V_2} P_x \leq 1$  for any  $x \in V_1$ . Then one of the three alternatives holds.*

- (i) *There exist a subset  $W \subset V_1$  which is empty or a subspace, and a hyperplane  $H \leq V_2$ , such that  $P = W \times V_2 \cup V_1 \times H$ .*
- (ii) *There exists a bilinear form  $b$  on  $V_1 \times V_2$  such that  $P = \{(x, y) \in V_1 \times V_2 : b(x, y) = 0\}$ .*
- (iii) *We have  $p \geq 5$  and the minimal codimension of a subspace  $W \leq V_1$  such that  $W \times V_2 \subset P$  is exactly 2.*

Observe that this implies Theorem 1.4, since the first two alternatives correspond to bilinear sets. This is obvious for the second one. For the first one, if  $W$  is empty, it is clear; otherwise, let  $a_1, \dots, a_k$  be linearly independent linear forms such that  $W$  is the intersection of their kernels, and  $\ell$  be a linear form that defines  $H$ . Then

$$P = \{(x, y) \in V_1 \times V_2 : a_1(x)\ell(y) = \dots = a_k(x)\ell(y) = 0\}.$$

One can check that one can not write  $P$  as in (1) with  $W_1$  and  $W_2$  other than  $V_1$  and  $V_2$  and with  $r_3 \neq k$ , and  $k$  may tend to infinity with  $\dim V_1$ , while the density is bounded

below by  $1/p$ , but this is not a contradiction with Theorem 1.2, since  $P$  contains (but may not be equal to) the Cartesian product  $V_1 \times H$ . As for the last alternative, Theorem 1.5 (ii) indicates that it is not necessarily a bilinear set.

*Proof.* Without loss of generality suppose that  $P_0 = V_2$ . Indeed, otherwise  $P_0$  is a hyperplane  $H$  and Lemma 3.1 (i) shows that  $P = V_1 \times H$ . Let  $(x, y) \mapsto x \cdot y$  be a bilinear form of full rank on  $V_1 \times V_2$ . For  $\phi \in V_2$  let  $\phi^\perp = \{y \in V_2 : x \cdot \phi = 0\}$ . The hypothesis allows us to write  $P_x = \xi(x)^\perp$  for some vector  $\xi(x) \in V_2$  that is defined uniquely up to homothety. The proof consists in deriving rigidity properties for  $\xi$  which will eventually make it linear or constant.

With this new notation, the assumption just made implies that  $\xi(0) = 0$ . Further, the second point of Lemma 3.1 means that  $\xi(x)$  depends only on  $[x]$  for  $x \neq 0$  and the third point of that lemma yields that whenever  $[z]$  is on the projective line spanned by  $x$  and  $y$ , we have  $\xi(z) \in \text{span}(\xi(x), \xi(y))$ . Using Lemma 3.1 (iii), one can see that the set

$$W := \{x \in V_1 : P_x = V_2\}$$

is a vector subspace. If  $W = V_1$ , we have  $P = V_1 \times V_2$  so the first alternative holds. Otherwise  $W \neq V_1$ . Let  $V'_1 = V_1/W$  and observe that for any given  $x - y = w \in W$ , we have  $\xi(x) \in \text{span}(\xi(y), \xi(w)) = \text{span}(\xi(y))$ , that is,  $\xi(x) = \xi(y)$  up to homothety, so that  $\xi$  descends to a map  $\xi' : \mathbb{P}(V_1/W) \rightarrow \mathbb{P}(V_2)$ . Thus  $\xi'$  is a map  $\mathbb{P}(V'_1) \rightarrow \mathbb{P}(V_2)$  that maps aligned points to aligned points. If  $\text{codim } W = 1$ , it follows that  $[\xi(x)]$  is a nonzero constant vector  $\phi$  for  $x \in V \setminus W$  so the first alternative is true with  $H = \phi^\perp$ . In the following we assume that  $\text{codim } W \geq 2$ .

By construction  $\xi'$  satisfies the hypothesis of Lemma 3.2, therefore it should be either constant or injective. If  $\xi'$  is constant on  $\mathbb{P}(V'_1)$ , we can take  $\xi(x)$  to be a nonzero constant vector  $\phi \in V_2$  for all  $x \in W^\perp$ , while  $\xi(x) = 0$  on  $W$ . Let  $H$  denote the subspace orthogonal to  $\phi$ . Then  $P = W \times V_2 \cup V_1 \times H$ , which is the first alternative. We suppose now that  $\xi'$  is injective. If  $\dim V'_1 = 2$  and  $p \geq 5$ , the third alternative is true. Now suppose that  $\dim V'_1 \geq 3$  or that  $\dim V'_1 = 2$  and  $p \in \{2, 3\}$ . Theorem 3.3 and the remark following it imply that  $\xi$  comes from an injective linear map  $V'_1 \rightarrow V_2$ , which we extend to a linear map  $f : V_1 \rightarrow V_2$  with kernel  $W$ . In the particular case  $p \in \{2, 3\}$  this proves proposition 1.4. Then  $P$  is the zero set of the bilinear form  $(x, y) \mapsto f(x) \cdot y$ , which concludes the proof of Proposition 3.4.  $\square$

#### 4. PROOF OF PROPOSITION 1.5

First we introduce a new notation and a characterisation of bilinear sets. For a set  $P \subset V_1 \times V_2$  satisfying  $P_0 = V_2$  and  $P_0 = V_1$ , let  $\text{Ann}(P)$  be the subspace of the space  $\mathcal{B}(V_1, V_2)$  of bilinear forms on  $V_1 \times V_2$  that consist of the forms that vanish on  $P$ . For a set  $M \subset \mathcal{B}(V_1, V_2)$ , let  $\text{Orth}(M)$  be the (bilinear) subset  $V_1 \times V_2$  where all the forms of

$x$	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)	(1, 1, 0)	(1, 0, 1)	(0, 1, 1)	(1, 1, 1)
$\sigma(x)$	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)	(1, 1, 0)	(0, 1, 1)	(1, 1, 1)	(1, 0, 1)

FIGURE 2. Table defining the permutation  $\sigma$ .

$M$  vanish simultaneously. Thus in general  $P \subset \text{Orth}(\text{Ann}(P))$ , while the equality holds if and only if  $P$  is a bilinear set.

Now we prove Theorem 1.5 (i), that is, we show that some transverse sets satisfying the third alternative of Proposition 3.4 are not bilinear. Let  $W$  be a subspace of codimension 2 in  $V_1$ . Let  $V'_1 = V_1/W$  and  $\xi' : \mathbb{P}(V'_1) \rightarrow \mathbb{P}(V_2)$  be a non-projective bijection onto a projective line; as observed after Theorem 3.3, this is possible when  $p \geq 5$  since there are  $(p+1)!$  bijection between any two projective lines but only  $(p+1)p(p-1)$  projective maps between them. Extend naturally  $\xi'$  to a map  $\xi : V_1 \rightarrow V_2$  that induces  $\xi'$  by projection and let  $P = \bigcup_{x \in V_1} \{x\} \times \xi(x)^\perp$ . Thanks to the characterization from Lemma 3.1, we see that  $P$  is transverse.

Let  $b \in \text{Ann}(P)$ , one can write  $b(x, y) = f(x) \cdot y$  where  $f$  is a linear map  $V_1 \rightarrow V_2$  vanishing on  $W$ ; thus it induces a linear map  $f' : V'_1 \rightarrow V_2$  satisfying  $f'(x) \in \text{span}(\xi'(x))$  for  $x \in V'_1 \setminus \{0\}$ . Recall that  $W$  has codimension two and therefore  $f'$  has either rank 2, 1 or 0 respectively. In the first case  $f'$  does not vanish on  $V'_1 \setminus \{0\}$  and we get  $\xi'(x) = [f'(x)]$  for any  $x \neq 0$ . As a consequence  $\xi'$  is projective, which is false. The second possibility can be ruled out too. Indeed, in this case the image of  $f'$  is a line  $\ell$ , i.e. a vector space of dimension one. As a consequence  $\xi'([x])$  will have the same constant value for any  $x \in V'_1 \setminus \ker f'$  which contradicts the fact that it is injective by construction. The only possibility left is  $f' = 0$ . This proves that  $\text{Ann}(P) = \{0\}$  and so  $\text{Orth}(\text{Ann}(P)) = V_1 \times V_2 \neq P$ , which means that  $P$  is not bilinear, concluding the proof of Theorem 1.5 (i).

We now show Theorem 1.5 (ii). Here we think of  $V_1$  and  $V_2$  as two  $n$ -dimensional  $\mathbb{F}_p$ -vector spaces. Recall the characterisation of transverse sets obtained in Lemma 3.1. In particular, if  $P_x \cap P_y = \{0\}$  for any  $[x] \neq [y]$ , the third property of that Lemma 3.1 is vacuous. As a consequence the characterisation of transverse sets it provides is easier to satisfy. One can achieve this, for instance, by taking a bijection  $\sigma : \mathbb{P}(V_1) \rightarrow \mathbb{P}(V_2)$  and letting  $P$  be the transverse set

$$P_\sigma = \{0\} \times V_2 \cup \bigcup_{x \in \mathbb{P}(V_1)} \text{span}(x) \times \text{span}(\sigma(x))$$

where  $\text{span}$  denotes the linear span in  $V_1$  or  $V_2$ .

With the assistance of a computer, it is possible to find  $\sigma$  such that  $P_\sigma \neq \text{Orth}(\text{Ann}(P_\sigma))$  for small  $p$  and  $n$ . For instance, for  $p = 2$  and  $n = 3$  one can let  $\sigma$  be the permutation of  $\mathbb{P}(\mathbb{F}_2^3) = \mathbb{F}_2^3 \setminus \{(0, 0, 0)\}$  defined in Figure 2. The above characterisation implies that  $P_\sigma$  is not a bilinear set. Indeed, we find that

$$\text{Ann}(P) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \right\}$$

so that  $\text{Orth}(\text{Ann}(P))$  contains  $((1, 0, 0), (0, 1, 0))$ , an element which does not belong to  $P$ , so  $P$  is not bilinear.

For general  $p$  and  $n$ , the following non-constructive counting argument shows that there exists a permutation  $\sigma$  such that  $P_\sigma$  is not bilinear. On the one hand, the number of points in a projective space can be bounded from below, i.e.

$$|\mathbb{P}(V_1)| = \frac{p^n - 1}{p - 1} \geq p^{n-1}.$$

Thus there are at least

$$p^{n-1}! \geq \left( \frac{p^{n-1}}{e} \right)^{p^{n-1}}$$

transverse sets  $P_\sigma$ , where we used the inequality  $e^m \geq m^m/(m!)$  valid for any positive integer  $m$ . On the other hand, the number of subspaces  $M$  of  $\mathcal{B}(V_1, V_2)$  can be bounded from above as follows. The space of bilinear forms  $\mathcal{B}(V_1, V_2)$  has dimension  $n^2$  and contains  $p^{n^2}$  elements. The number of subspaces of dimension  $k$  can be bounded by  $p^{kn^2}$ . Recall that there exists the same number of spaces of dimension  $k$  and  $n^2 - k$  so the total number of subspaces can be bounded above by  $\sum_{k=0}^{n^2} |\{H \subset \mathcal{B}(V_1, V_2) : \dim(H) = k\}| \leq 2 \frac{p^{n^4/2+n^2}-1}{p^{n^2}-1}$  (if  $n$  is even this is clear and if it is odd the number of subspaces of dimension  $(n^2 + 1)/2$  is only counted once and the bound obtained is smaller than the one given). Now we argue by contradiction. The absence of counterexamples would force  $(p, n \geq 2)$

$$\left( \frac{p^{n-1}}{e} \right)^{p^{n-1}} \leq p^{n-1}! \leq 2 \frac{p^{n^4/2+n^2}-1}{p^{n^2}-1} \leq \frac{32}{15} p^{n^4/2}$$

which provides the contradiction we were seeking for  $n \geq n_0(p)$ . Indeed, we can take  $n_0(p) = 11$  for all  $p$  but this estimate can be improved if we allow  $p$  to be large enough and for instance  $n_0(p) = 2$  is enough for  $p \geq 13$ .

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