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Tatyana Shcherbina

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Tatyana Shcherbina[1]

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Abstract
The paper continues (Shcherbina and Shcherbina in Commun Math Phys 351:1009–1044, 2017); Shcherbina in Commun Math Phys 328:45–82, 2014) which study the behaviour of second correlation function of characteristic polynomials of the special case of $n \times n$ one-dimensional Gaussian Hermitian random band matrices, when the covariance of the elements is determined by the matrix $J = (-W^2\triangle + 1)^{-1}$. Applying the transfer matrix approach, we study the case when the bandwidth W is proportional to the threshold $\sqrt{n}$.

Keywords Band matrices · Characteristic polynomials · Transfer matrices

1 Introduction

As in [18,21], we consider Hermitian $n \times n$ matrices H whose entries H_{ij} are random complex Gaussian variables with mean zero such that

$$\mathbf{E}\{H_{ij} H_{lk}\} = \delta_{ik}\delta_{jl} J_{ij}, \tag{1.1}$$

where

$$J_{ij} = \left(-W^2\Delta + 1\right)^{-1}_{ij}, \tag{1.2}$$

and Δ is the discrete Laplacian on $\mathcal{L} = [1, n] \cap \mathbb{Z}$ with Neumann boundary conditions. It is easy to see that the variance of matrix elements J_{ij} is exponentially small when $|i - j| \gg W$, and so W can be considered as the width of the band.

The density of states ρ of the ensemble is given by the well-known Wigner semicircle law (see [3,16]):

$$\rho(E) = (2\pi)^{-1}\sqrt{4 - E^2}, \quad E \in [-2, 2]. \tag{1.3}$$

Random band matrices (RBM) provide a natural and important model to study eigenvalue statistic and quantum transport in disordered systems as they interpolate between classical

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✉ Tatyana Shcherbina
tshcherbyna@princeton.edu

1 Department of Mathematics, Princeton University, Princeton, USA

Wigner matrices, i.e. Hermitian random matrices with all independent identically distributed elements, and random Schrödinger operators, where only a random on-site potential is present in addition to the deterministic Laplacian on a regular box in d-dimension lattice. Such matrices have various application in physics: the eigenvalue statistics of RBM is in relevance in quantum chaos, the quantum dynamics associated with RBM can be used to model conductance in thick wires, etc.

One of the main long standing problem in the field is to prove a fundamental physical conjecture formulated in late 80th (see [8,10]). The conjecture states that the eigenvectors of $n \times n$ RBM are completely delocalized and the local spectral statistics governed by random matrix (Wigner-Dyson) statistics for large bandwidth W, and by Poisson statistics for a small W (with exponentially localized eigenvectors). The transition is conjectured to be sharp and for RBM in one spatial dimension occurs around the critical value $W = \sqrt{n}$. This is the analogue of the celebrated Anderson metal-insulator transition for random Schrödinger operators.

The conjecture on the crossover in RBM with $W \sim \sqrt{n}$ is supported by physical derivation due to Fyodorov and Mirlin (see [10]) based on supersymmetric formalism, and also by the so-called Thouless scaling. However, there are only partial results on the mathematical level of rigour (see reviews [5,17] and references therein for the details).

The only result that rigorously demonstrate the threshold around $W \sim \sqrt{n}$ for a certain eigenvalue statistics was obtain in [21] (regime $W \gg \sqrt{n}$), [18] (regime $W \ll \sqrt{n}$). Instead of eigenvalue correlation functions these papers deal with more simple object which is the second correlation functions of characteristic polynomials:

$$F_2(x_1, x_2) = \mathbf{E}\Big\{\det(x_1 - H)\det(x_2 - H)\Big\}. \tag{1.4}$$

Characteristic polynomials of random matrices were studied for many classical ensembles (see e.g. [1,2,4,6,7,11,15–20,22] and references therein). The interest to this spectral characteristic is stimulated by its connections to the number theory, quantum chaos, integrable systems, combinatorics, and representation theory.

The main results of [18,21] concern the asymptotic behaviour of F_2 with

$$x_{1,2} = E + \frac{\xi_{1,2}}{n\rho(E)}, \quad E \in (-2, 2), \quad \xi_1, \xi_2 \in [-C, C].$$

for RBM (1.1)–(1.2). Namely, let

$$D_2 = F_2(E, E), \quad \bar{F}_2(x_1, x_2) = D_2^{-1} \cdot F_2(x_1, x_2).$$

Then we have the following theorem

Theorem 1.1 [18,21] *For the 1d RBM of* (1.1)–(1.2) *we have*

$$\lim_{n\to\infty} \bar{F}_2\Big(E + \frac{\xi}{2n\rho(E)}, E - \frac{\xi}{2n\rho(E)}\Big) = \begin{cases} \dfrac{\sin \pi\xi}{\pi\xi}, & W \ge n^{1/2+\theta}; \\ 1, & 1 \ll W \le \sqrt{\dfrac{n}{C_* \log n}}, \end{cases}$$

where the limit is uniform in ξ varying in any compact set $C \subset \mathbb{R}$. Here $E \in (-2, 2)$, and $\rho(x)$ is defined in (1.3).

The purpose of the present paper is to complete Theorem 1.1 by the study of correlation functions of characteristic polynomials (1.4) near the threshold $W \sim \sqrt{n}$. The main result is

Theorem 1.2 *For the 1d RBM of* (1.1)–(1.2) *with* $n = C_* W^2$ *we have*

$$\lim_{n\to\infty} \bar{F}_2\Big(E + \frac{\xi}{2n\rho(E)}, E - \frac{\xi}{2n\rho(E)}\Big) = (e^{-C^*\Delta_U - i\xi\pi\hat{\nu}} \cdot 1, 1),$$

where $C^* = C_*/(2\pi\rho(E))^2$. *In this formula* $(\cdot,\cdot)$ *is an inner product on a 2-dimensional sphere* $\mathbb{S}^2$, Δ_U *is a Laplace operator on* $\mathbb{S}^2$

$$\Delta_U = -\frac{d}{dx}x(1-x)\frac{d}{dx}, \quad x = |U_{12}|^2, \tag{1.5}$$

U *is a* 2×2 *unitary matrix, and* $\hat{\nu}$ *is an operator of multiplication by*

$$\nu(U) = 1 - 2|U_{12}|^2 \tag{1.6}$$

on $\mathbb{S}^2$.

Remark 1.1 It is easy to see that if $W \gg \sqrt{n}$ (and so $C^* \to 0$), then we have

$$(e^{-C^*\Delta_U - \pi i\xi\hat{\nu}} \cdot 1, 1) \approx (e^{-\pi i\xi\hat{\nu}} \cdot 1, 1) = \frac{\sin \pi\xi}{\pi\xi}.$$

Similarly if $W \ll \sqrt{n}$ (and so $C^* \to \infty$), then we get

$$(e^{-C^*\Delta_U - \pi i\xi\hat{\nu}} \cdot 1, 1) \approx (e^{-C^*\Delta_U} \cdot 1, 1) = 1.$$

Thus the result of Theorem 1.2 "glues" together two parts of Theorem 1.1.

Remark 1.2 The study of eigenfunctions and spectral statistics in the critical regime (near the threshold) is of independent interest. Critical wave-functions at the point of the Anderson localization transition are expected to be multifractal. Moreover, multifractal structure occurs in a critical regime of power-law banded random matrices (see the review [9] and reference therein for the details). Although the correlation functions of characteristic polynomials (1.4) are not reach enough to feel this phenomena, the techniques developed in the paper can be useful in studying the usual correlation functions of 1d RBM near the threshold.

The proof of Theorem 1.2 is based on the techniques of [18]. Namely, we apply the version of transfer matrix approach introduced in [18] to the integral representation obtained in [21] by the supersymmetry techniques (note that the integral representation does not contain Grassmann integrals, see Proposition 2.1).

The paper is organized as follows. In Sect. 2 we rewrite F_2 as an action of the n-th degree of some transfer operator K_ξ (see (2.5) below) and outline the proof of Theorem 1.2. In Sect. 3 we collect all preliminaries results obtained in [18]. Section 4 deals with the proof of Theorem 1.2.

We denote by C, C_1, etc. various W and n-independent quantities below, which can be different in different formulas. To reduce the number of notations, we also use the same letters for the integral operators and their kernels.

2 Outline of the Proof of Theorem 1.2

First, we rewrite F_2 as an action of the $(n-1)$-th degree of some transfer operator, as it was done in [18].

For $X \in \mathrm{Herm}(2)$ define

$$f := \mathcal{F}(X) = \exp\Big\{ -\frac{1}{4}\,\mathrm{Tr}\,\Big(X + \frac{i\Lambda_0}{2}\Big)^2 + \frac{1}{2}\,\mathrm{Tr}\,\log\big(X - i\Lambda_0/2\big) - C_+\Big\}, \tag{2.1}$$

$$f_\xi := \mathcal{F}_\xi(X) = \mathcal{F}(X)\cdot\exp\Big\{ -\frac{i}{2n\rho(E)}\,\mathrm{Tr}\,X\hat{\xi}\Big\}$$

with $\hat{\xi} = \mathrm{diag}\,\{\xi, -\xi\}$, $\Lambda_0 = E \cdot I_2$,

$$a_\pm = \pm\sqrt{1 - E^2/4} \tag{2.2}$$

$$C_+ = \frac{1}{4}\,\mathrm{Tr}\,\Big(a_+ I + \frac{i\Lambda_0}{2}\Big)^2 - \frac{1}{2}\,\mathrm{Tr}\,\log\big(a_+ I - i\Lambda_0/2\big). \tag{2.3}$$

Set also $\mathcal{H} = L_2[\mathrm{Herm}(2)]$, and let $K, K_\xi : \mathcal{H} \to \mathcal{H}$ be operators with the kernels

$$K(X,Y) = \frac{W^4}{2\pi^2}\,\mathcal{F}(X)\,\exp\Big\{ -\frac{W^2}{2}\mathrm{Tr}\,(X-Y)^2\Big\}\,\mathcal{F}(Y); \tag{2.4}$$

$$K_\xi(X,Y) = \frac{W^4}{2\pi^2}\,\mathcal{F}_\xi(X)\,\exp\Big\{ -\frac{W^2}{2}\mathrm{Tr}\,(X-Y)^2\Big\}\,\mathcal{F}_\xi(Y). \tag{2.5}$$

As it was proved in [18], Sect. 2, we have

Proposition 2.1 ([18]) *The second correlation function of characteristic polynomials of (1.4) for 1D Hermitian Gaussian band matrices (1.1)–(1.2) can be represented as follows:*

$$F_2\Big(E + \frac{\xi}{n\rho(E)}, E - \frac{\xi}{n\rho(E)}\Big) = -C_n(\xi)\cdot W^{-4n}\det{}^{-2} J \cdot (K_\xi^{n-1} f_\xi, \bar{f}_\xi), \tag{2.6}$$

where $(\cdot,\cdot)$ *is a standard inner product in* $\mathcal{H}$

$$\big(f(X), g(X)\big) = \int f(X)\bar{g}(X)\,dX, \quad dX = dX_{11}\,dX_{22}\,d(\Re X_{12})d(\Im X_{12}),$$

ρ *is defined in (1.3), and*

$$C_n(\xi) = \exp\big\{2nC_+ + \xi^2/n\rho(E)^2\big\}$$

with C_+ *of (2.3).*

For an arbitrary compact operator M denote by $\lambda_j(M)$ the jth (by its modulo) eigenvalue of M, so that $|\lambda_0(M)| \ge |\lambda_1(M)| \ge \ldots$.

The idea of the transfer operator approach is very simple and natural. Let $\mathcal{K}(X,Y)$ be the matrix kernel of the compact integral operator in $L_2[X, d\mu(X)]$. Then

$$\int g(X_1)\mathcal{K}(X_1,X_2)\ldots\mathcal{K}(X_{n-1},X_n)f(X_n)\prod d\mu(X_i) = (\mathcal{K}^{n-1}f,\bar{g})$$
$$= \sum_{j=0}^{\infty}\lambda_j^{n-1}(\mathcal{K})c_j, \quad with \quad c_j = (f,\psi_j)(g,\tilde{\psi}_j),$$

where $\{\psi_j\}$ are eigenvectors corresponding to $\{\lambda_j(\mathcal{K})\}$, and $\{\tilde{\psi}_j\}$ are the eigenvectors of $\mathcal{K}^*$. Hence, to study the integral, it suffices to study the eigenvalues and eigenfunctions of the integral operator with the kernel $\mathcal{K}(X,Y)$.

The main difficulties in application of this approach to (2.6) are the complicated structure and non self-adjointness of the corresponding transfer operator K_ξ of (2.5).

In fact, since the analysis of eigenvectors of non self-adjoint operators is rather involved, it is simpler to work with the resolvent analog of (2.6)

$$(K_\xi^{n-1} f_\xi, \bar{f}_\xi) = -\frac{1}{2\pi i}\oint_{\mathcal{L}} z^{n-1}(G_\xi(z) f_\xi, \bar{f}_\xi)dz, \quad G_\xi(z) = (K_\xi - z)^{-1}, \tag{2.7}$$

where $\mathcal{L}$ is any closed contour which enclosed all eigenvalues of K_ξ.

To explain the idea of the proof, we start from the following definition

Definition 2.1 We shall say that the operator $\mathcal{A}_{n,W}$ is equivalent to $\mathcal{B}_{n,W}$ ($\mathcal{A}_{n,W} \sim \mathcal{B}_{n,W}$) on some contour $\mathcal{L}$ if

$$\int_{\mathcal{L}} z^{n-1}((\mathcal{A}_{n,W} - z)^{-1} f, \bar{g})dz = \int_{\mathcal{L}} z^{n-1}((\mathcal{B}_{n,W} - z)^{-1} f, \bar{g})dz\,(1 + o(1)), \quad n, W \to \infty,$$

with some particular functions f, g depending of the problem.

The aim is to find some operator equivalent to K_ξ whose spectral analysis we are ready to perform. Now we are going to discuss how this was done on the ideological level. The specific choice of the contour $\mathcal{L}$ and functions f, g for each step will be discussed in details in Sect. 4.

It is easy to see that the stationary points of the function $\mathcal{F}$ of (2.1) are

$$\begin{aligned} X_+ &= a_+ \cdot I_2, \quad X_- = a_- \cdot I_2;\\ X_\pm(U) &= a_+ ULU^*, \quad U \in \mathring{U}(2), \end{aligned} \tag{2.8}$$

where $a_\pm$ is defined in (2.2), $\mathring{U}(2) := U(2)/U(1)\times U(1)$, $L = \mathrm{diag}\,\{1, -1\}$. Notice also that the value of $|\mathcal{F}|$ at points (2.8) is 1.

The first step in the proof of Theorem 1.2 is to apply the saddle-point approximation. Roughly speaking, we show that if we introduce the projection P_s onto the $W^{-1/2}\log W$-neighbourhoods of the saddle points X_+, X_- and the saddle "surface" $X_\pm$, then in the sense of Definition 2.1

$$K_\xi \sim P_s K_\xi P_s =: K_{s,\xi}. \tag{2.9}$$

This step was done in [18]. The exact meaning of the projection operator P_s will be explained in Sect. 3 (see the definition of the operator P in (3.11)).

To study the operator $K_{s,\xi}$ near the saddle "surface" $X_\pm$ we use the "polar coordinates". Namely, introduce

$$t = (a_1 - b_1)(a_2 - b_2), \quad p(a,b) = \frac{\pi}{2}(a-b)^2, \tag{2.10}$$

and denote by dU the integration with respect to the Haar measure on the group $\mathring{U}(2)$: in the standard parametrization

$$U = \begin{pmatrix} \cos\varphi & \sin\varphi\cdot e^{i\theta}\\ -\sin\varphi\cdot e^{-i\theta} & \cos\varphi \end{pmatrix}, \tag{2.11}$$

we have

$$dU = \frac{1}{\pi} u\,du\,d\theta, \quad u = |\sin\varphi| \in [0,1], \quad \theta \in [0, 2\pi).$$

Consider the space $L_2[\mathbb{R}^2, p]\times L_2[\mathring{U}(2), dU]$. The inner product and the action of an integral operator in this space are

$$(f,g)_p = \int f(a,b)\bar{g}(a,b)p(a,b)\,da\,db;$$
$$(Mf)(a_1,b_1,U_1) = \int M(a_1,b_1,U_1;a_2,b_2,U_2)\,f(a_2,b_2,U_2)\,p(a_2,b_2)da_2\,db_2\,dU_2. \tag{2.12}$$

Changing the variables

$$X = U^*\Lambda U, \quad \Lambda = \mathrm{diag}\{a,b\}, \quad a > b, \quad U\in\mathring{U}(2),$$

we obtain that $K_\xi = K + \widetilde{K}_\xi$ can be represented as an integral operator in $L_2[\mathbb{R}^2, p]\times L_2[\mathring{U}(2), dU]$ defined by the kernel

$$K_\xi(X,Y) = K(a_1,b_1,U_1;a_2,b_2,U_2) + \widetilde{K}_\xi(a_1,b_1,U_1;a_2,b_2,U_2), \tag{2.13}$$

where

$$K(a_1,b_1,U_1;a_2,b_2,U_2) = t^{-1}A(a_1,a_2)A(b_1,b_2)K_*(t,U_1,U_2);$$
$$K_*(t,U_1,U_2) := W^2t\cdot e^{tW^2\mathrm{Tr}\,U_1U_2^*L(U_1U_2^*)^*L/4-tW^2/2};$$
$$\widetilde{K}_\xi(a_1,b_1,U_1;a_2,b_2,U_2) = K(a_1,b_1,U_1;a_2,b_2,U_2)\big(e^{-i\xi\pi\left(\nu(a_1-b_1,U_1)+\nu(a_2-b_2,U_2)\right)/n}-1\big);$$
$$\nu(x,U) = \frac{x}{4\pi\rho(E)}\mathrm{Tr}\,ULU^*L = \frac{x}{2\pi\rho(E)}(1-2|U_{12}|^2) \tag{2.14}$$

with t of (2.10). K_* here is a contribution of the unitary group $\mathring{U}(2)$ into operator K, and $\exp\{-i\xi\pi\nu(x,U)/n\}$ comes from the factor $\exp\{-\frac{i}{2n\rho(E)}\mathrm{Tr}\,X\hat{\xi}\}$ which is $1/n$ order perturbation of $\mathcal{F}$ appearing in $\mathcal{F}_\xi$ (see (2.1)). Operator A is a contribution of eigenvalues a, b and it has the form

$$A(x,y) = (2\pi)^{-1/2}We^{-g(x)/2}e^{-W^2(x-y)^2/2}e^{-g(y)/2};$$
$$g(x) = (x+iE/2)^2/2 - \log(x-iE/2) - C_+. \tag{2.15}$$

Note also that

$$\|\widetilde{K}_\xi\| \le C/n \tag{2.16}$$

with some absolute $C>0$.

Observe that the operator $\mathcal{K}_*(t,U_1,U_2)$ with some $t>0$ is self-adjoint and its kernel depends only on $|(U_1U_2^*)_{12}|^2$. Thus by the standard representation theory arguments (see e.g. [23]), its eigenfunctions are the classical spherical harmonics. More precisely:

Proposition 2.2 *Consider any self-adjoint integral operator M in $L_2[\mathring{U}(2), dU]$ (see (2.11)) with kernel $M(U_1,U_2)$ depending only on $|(U_1U_2^*)_{12}|^2$. Then its eigenvectors $\{\phi_{\bar{j}}(U)\}$ ($\bar{j} = (j,s)$, $j=0,1,\ldots$, $s=-j,\ldots,j$) are the standard spherical harmonics:*

$$\phi_{j,s}(U) = l_{j,s}\,P_j^s(\cos(2\varphi))\,e^{is\theta} = l_{j,s}\Big(\frac{d}{dx}\Big)^s P_j(x)\Big|_{x=1-2|U_{12}|^2}(2\bar{U}_{11}U_{12})^s, \tag{2.17}$$

where U has the form (2.11), and P_j^s is an associated Legendre polynomial

$$P_j^s(\cos x) = (\sin x)^s \Big(\frac{d}{d\cos x}\Big)^s P_j(\cos x), \quad P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}(x^2-1)^j,$$
$$l_{j,s} = \sqrt{\frac{(2j+1)(j-s)!}{(j+s)!}}.$$

Moreover, the subspace

$$L_2[u, dU] \subset L_2[\mathring{U}(2), dU] \tag{2.18}$$

of the functions depending on $|U_{12}|^2$ only is invariant under M, and the restriction of M to $L_2[u, dU]$ has eigenvectors

$$\phi_j(U) := \phi_{j,0}(U). \tag{2.19}$$

If $M(U_1, U_2) = K_*(t, U_1, U_2)$ of (2.14), then the corresponding eigenvalues $\{\lambda_j(t)\}_{j=0}^{\infty}$, if $t > d > 0$, where d is some absolute positive constant, have the form

$$\lambda_0(t) = 1 - e^{-W^2 t},$$
$$\lambda_j(t) = (1 - e^{-W^2 t})\Big(1 - \frac{j(j+1)}{W^2 t}(1 + O(j^2/W^2 t)\Big). \tag{2.20}$$

The proof of the proposition can be found in [18] (see the proof of Proposition 3.2 in Appendix).

Notice that since

$$\mathrm{Tr}\, U^* L U L = 2(1 - 2u^2),$$

functions $\mathcal{F}, \mathcal{F}_\xi$ do not depend on θ of (2.11), and hence according to Proposition 2.2 in what follows we can consider restrictions of K, K_* and $\widetilde{K}_\xi$ of (2.14) to $L_2[u, dU]$ (to simplify notations we will denote these restrictions by the same letters).

In addition, it follows from Proposition 2.2 that if we introduce the following basis in $L_2[\mathbb{R}^2, p] \times L_2[u, dU]$ (see (2.12))

$$\Psi_{\bar{k},j}(a, b, U) = \Psi_{\bar{k}}(a, b)\phi_j(U),$$
$$\Psi_{\bar{k}}(a, b) = \sqrt{\frac{2}{\pi}}\,(a-b)^{-1}\psi_{k_1}(a)\psi_{k_2}(b),$$

where $\bar{k} = (k_1, k_2)$, and $\{\psi_k(x)\}_{k=0}^{\infty}$ is a certain basis in $L_2[\mathbb{R}]$, then the matrix of K of (2.14) in this basis has a "block diagonal structure", which means that

$$(K\Psi_{\bar{k}',j}, \Psi_{\bar{k},j_1})_p = 0, \quad j \neq j_1$$
$$(K\Psi_{\bar{k}',j}, \Psi_{\bar{k},j})_p = (K_j \Psi_{\bar{k}'}, \Psi_{\bar{k}})_p$$
$$= \int \lambda_j(t) A(a_1, a_2) A(b_1, b_2)\psi_{k_1}(a_1)\psi_{k_2}(b_1)\psi_{k_1'}(a_2)\psi_{k_2'}(b_2) da_1 db_1 da_2 db_2. \tag{2.21}$$

The next step in the proof of Theorem 1.2 is to show that only the neighbourhood of the saddle "surface" $X_\pm$ gives the main contribution to the integral, and moreover we can restrict the number of ϕ_j to $l = [\log W]$. More precisely, we are going to show that in the sense of Definition 2.1

$$K_{s,\xi} \sim \mathcal{P}_l K_{s,\xi} \mathcal{P}_l =: K_{m,l,\xi}, \tag{2.22}$$

where $\mathcal{P}_l$ is the projection on the linear span of $\{\Psi_{\bar{k},j}(a, b, U)\}_{j \le l, |\bar{k}| \le m}$.

For the further resolvent analysis we want to change t in the definition of K_* and $a_1 - b_1$, $a_2 - b_2$ in the definition of $\widetilde{K}_\xi$ (see (2.10), (2.13)–(2.14)) by their saddle-point values $t^* = (a_+ - a_-)^2 = 4\pi^2\rho(E)^2$ and $a_+ - a_- = 2\pi\rho(E)$ correspondingly. More precisely we want to show that in the sense of Definition 2.1

$$K_{m,l,\xi} \sim \mathcal{A}_m \otimes K_{*\xi,l} \tag{2.23}$$

where

$$\begin{aligned} &K_{*\xi,l} = Q_l\, K_{*\xi}\, Q_l, \\ &K_{*\xi}(U_1, U_2) = W^2 t^* \cdot e^{t^* W^2 (\operatorname{Tr} U_1 U_2^* L (U_1 U_2^*)^* L - 2)/4} \cdot e^{-i\xi\pi(\nu(2\pi\rho(E), U_1) + \nu(2\pi\rho(E), U_2))/n} \end{aligned} \tag{2.24}$$

and Q_l is the projection on $\{\phi_j(U)\}_{j \le l}$. The operator $\mathcal{A}_m$ in (2.23) is defined as

$$\mathcal{A}_m = P_m A(a_1, a_2) A(b_1, b_2) P_m, \tag{2.25}$$

where P_m is the projection on $\{\Psi_{\bar{k}}(a, b)\}_{|\bar{k}| \le m}$.

Now (2.23), (2.7) and Definition 2.1 give

$$\begin{aligned} F_2\Big(E + \frac{\xi}{2n\rho(E)}, E - \frac{\xi}{2n\rho(E)}\Big) &= C_n\Big(\Big(\mathcal{K}_{*\xi,l}^{n-1} \otimes \mathcal{A}_m^{n-1}\Big) f_\xi, \bar{f}_\xi\Big)(1 + o(1)) \\ &= (\mathcal{A}_m^{n-1} f_1, \bar{f}_1)(\mathcal{K}_{*\xi,l}^{n-1} 1, 1)(1 + o(1)), \end{aligned}$$

where we used that f_ξ asymptotically can be replaced by $f_1 \otimes 1$, where f_1 does not depend on ξ and U_j. Similarly

$$D_2 = C_n(\mathcal{K}_{*0}^{n-1} \otimes \mathcal{A}_m^{n-1} f, \bar{f})(1 + o(1)) = (\mathcal{A}_m^{n-1} f_1, \bar{f}_1)(\mathcal{K}_{*0,l}^{n-1} 1, 1)(1 + o(1)).$$

According to Proposition 2.2, $\phi_0(U) = 1$ is an eigenvector of K_{*0} of (2.24) with $\xi = 0$ and the corresponding eigenvalue is 1, thus

$$(\mathcal{K}_{*0,l}^{n-1} 1, 1) = 1. \tag{2.26}$$

Hence

$$\bar{F}_2\Big(E + \frac{\xi}{2n\rho(E)}, E - \frac{\xi}{2n\rho(E)}\Big) = (\mathcal{K}_{*\xi,l}^{n-1} 1, 1)(1 + o(1)). \tag{2.27}$$

Recall that according to Proposition 2.2 the eigenvectors of $K_{*0,l}$ are (2.19) and the corresponding eigenvalues are (see (2.20))

$$\lambda_j := \lambda_j(t^*) = 1 - j(j+1)/t^* W^2 + O((j(j+1)/W^2)^2), \quad j = 0, 1 \ldots, l. \tag{2.28}$$

Moreover, it follows from (2.13)–(2.14) that

$$\mathcal{K}_{*\xi,l} = \mathcal{K}_{*0,l} - n^{-1}\pi i \xi \hat{\nu} + o(n^{-1}) \tag{2.29}$$

where $\hat{\nu}$ is the operator of multiplication by (1.6), and $o(1/n)$ means some operator whose norm is $o(1/n)$. Thus the eigenvalues of $\mathcal{K}_{*\xi,l}$ are in the n^{-1}-neighbourhood of λ_j. Therefore in the regime of localization $W^{-2} \gg n^{-1}$ considered in [18]

$$|\lambda_1(\mathcal{K}_{*\xi})| \le 1 - O(W^{-2}),$$

thus only $\lambda_0(\mathcal{K}_{*\xi})$ gives the contribution to (2.27). Since

$$(\hat{\nu}\, 1, 1) = 0,$$

we get

$$\lambda_0(\mathcal{K}_{*\xi}) = 1 + o(n^{-1}),$$

and so the limit of (2.27) is 1.

In the regime of delocalization considered in [21] (by another approach) all eigenvalues of $K_{*\xi,l}$ give contribution to (2.27), but $\mathcal{K}_{*0,l}^{n-1} \to I$ (roughly speaking, this means that the second term in the r.h.s. of (2.28) does not give a contribution). Hence we have

$$\mathcal{K}_{*\xi,l} \approx 1 - n^{-1} i\xi\pi\nu \Rightarrow (\mathcal{K}_{*\xi}^{n-1} 1, 1) \to (e^{-i\xi\pi\hat{\nu}} 1, 1) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

In the critical regime $W^{-2} = C_* n^{-1}$ considered in the current paper again all eigenvalues of $K_{*\xi,l}$ give contribution, but now both second term in the r.h.s. of (2.28) and $1/n$-order term in the r.h.s. of (2.29) make an impact.

It is well-known that the spherical harmonics (2.17) are the eigenfunctions of the spherical Laplace operator. In our case Δ_U of (1.5) depends on $|U_{12}|^2$ only, so its restriction to $L_2[u, dU]$ has eigenvectors (2.19) with corresponding eigenvalues

$$\lambda_j^* = j(j+1).$$

Thus $1 - n^{-1} C^* \Delta_U$ with $C^* = C_*/t^*$ has the same basis of eigenvectors with eigenvalues $1 - j(j+1)/t^* W^2$.

Recall that we are interested in $j \le l = [\log W]$ and Q_l is the projection on $\{\phi_j(U)\}_{j \le l}$. Hence, according to (2.28)–(2.29), in the regime $W^{-2} = C_* n^{-1}$ we can write

$$\mathcal{K}_{*\xi,l} = Q_l\big(1 - n^{-1}(C^*\Delta_U + i\xi\pi\nu)\big) Q_l + O(l^2 n^{-2}), \tag{2.30}$$

which implies

$$(\mathcal{K}_{*\xi,l}^{n-1} 1, 1) \to (e^{-C^*\Delta_U - i\xi\pi\hat{\nu}} 1, 1), \tag{2.31}$$

and finishes the proof of Theorem 1.2. The detailed proof of (2.31) is given in Sect. 4 (see Lemma 4.1).

3 Preliminary Results

Recall that stationary points X_+, X_-, and $X_\pm(U)$ of the function $\mathcal{F}$ of (2.1) are defined in (2.8). Now choose W, n-independent $\delta > 0$, which is small enough to provide that the domain

$$\Omega_\delta = \{X : |\mathcal{F}(X)| > 1 - \delta\}$$

contains three non-intersecting subdomains $\Omega_\delta^\pm, \Omega_\delta^+, \Omega_\delta^-$, such that each of $\Omega_\delta^+, \Omega_\delta^-$ contains one of the points X_+, X_-, and $\Omega_\delta^\pm$ contains the surface $X_\pm(U)$ of (2.8).

To study K, K_ξ near $\Omega_\delta^\pm$ we will use the representation described in (2.13)–(2.14).

Considering the operators $K(X, Y)$, $K_\xi(X, Y)$ near the points X_+ and X_-, we are going to extract the contribution from the diagonal elements of X, Y. To this end, put

$$X = \begin{pmatrix} a_1 & (x_1 + iy_1)/\sqrt{2} \\ (x_1 - iy_1)/\sqrt{2} & b_1 \end{pmatrix}, \quad Y = \begin{pmatrix} a_2 & (x_2 + iy_2)/\sqrt{2} \\ (x_2 - iy_2)/\sqrt{2} & b_2 \end{pmatrix}.$$

Now rewrite $K(X,Y)$, $K_\xi(X,Y)$ of (2.4)–(2.5) as

$$K_\xi(X,Y) = K(X,Y) + \widetilde{K}_\xi(X,Y),$$
$$K(X,Y) = A(a_1,a_2)\,A(b_1,b_2)\,A_1(X,Y), \tag{3.1}$$

where the kernels A (the contribution of the diagonal elements) is defined in (2.15), and A_1 (the contribution of the off-diagonal elements, which however depends on diagonal elements as well) has the form

$$A_1(X,Y) = (2\pi)^{-1}W^2F_1(X)\cdot\exp\{-W^2(x_1-x_2)^2/2 - W^2(y_1-y_2)^2/2\}\cdot F_1(Y);$$
$$F_1(X) = \exp\Big\{-\frac{1}{4}(x_1^2+y_1^2) + \frac{1}{2}\log\Big(1 - \frac{x_1^2+y_1^2}{2(a_1-iE/2)(b_1-iE/2)}\Big)\Big\}. \tag{3.2}$$

The perturbation kernel $\widetilde{K}_\xi$ in this coordinates is

$$\widetilde{K}_\xi(X,Y) = A(a_1,a_2)\,A(b_1,b_2)\,A_1(X,Y)\,\Big(e^{-\frac{i\xi}{2n\rho(E)}\big((a_1-b_1)+(a_2-b_2)\big)} - 1\Big). \tag{3.3}$$

It is easy to check that for g defined in (2.15)

$$g(a_\pm + x) - g(a_\pm) = c_\pm x^2 + c_{3\pm}x^3 + \dots$$

with

$$c_\pm = a_+(\sqrt{4-E^2} \pm iE)/2, \quad \Re c_+ = \Re c_- > 0, \tag{3.4}$$

and some constants $c_{3\pm}, c_{4\pm}, \dots$

As was mentioned above, the first step (2.9) in the analysis of operators K, K_ξ is based on the saddle-point approximation and was done in [18] (Sect. 4). The main idea was to prove that the main contribution was given by the "projections" onto the domains $\Omega_\delta^\pm$, Ω_δ^+, Ω_δ^-, where the operators A, A_1 are close to their quadratic approximations near the saddle-points $a_\pm$ of (2.2). Thus the information about the "projections" of K, K_ξ onto the domains $\Omega_\delta^\pm$, Ω_δ^+, Ω_δ^- can be obtained by the analysis of A, A_1 near the saddle points $a_\pm$ (see Sect. 3.1 of [18]) and information about K_* provided in Proposition 2.2. Following [18], define the useful basis in which the quadratic approximation of A near $a_\pm$ is an upper-triangular matrix (see Lemma 3.1 of [18]). Namely, consider the orthonormal in $L_2[\mathbb{R}]$ system of functions

$$\psi_0^\alpha(x) = e^{-\alpha Wx^2}\sqrt[4]{\alpha W/\pi};$$
$$\psi_k^\alpha(x) = h_k^{-1/2}e^{-\alpha Wx^2}e^{2\Re\alpha\cdot Wx^2}\Big(\frac{d}{dx}\Big)^k e^{-2\Re\alpha\cdot Wx^2} = e^{-\alpha Wx^2}p_k(x);$$
$$h_k^\alpha = k!(4\Re\alpha\cdot W)^{k-1/2}\sqrt{2\pi}, \quad k = 1,2,\dots \tag{3.5}$$

with some α such that $\Re\alpha > 0$. It is easy to see that for any $C > 0$

$$\psi_k(x) = O(e^{-cW}) \text{ for } |x| \ge C, \quad k \ll W. \tag{3.6}$$

Set

$$\psi_k^\pm(x) = \psi_k^{\alpha_\pm}(x - a_\pm) \tag{3.7}$$

with

$$\alpha_\pm = \sqrt{\frac{c_\pm}{2}}\Big(1 + \frac{c_\pm}{2W^2}\Big)^{1/2}.$$

To define the precise "projections" on the domains $\Omega_\delta^\pm$, Ω_δ^+, Ω_δ^-, set

$$m = [\log^2 W]. \tag{3.8}$$

The order of m is chosen in such a way that $m > C \log W$, and $m^p/W^\sigma \to 0$, as $W \to \infty$, for any $C, p, \sigma > 0$.

It follows from (3.6) that $\psi_k^\pm$ are exponentially small outside of the small neighbourhood of $a_\pm$ for any $k \ll W$, and so the projection on $\{\psi_k^\pm\}_{k=0}^m$ gives the "projection" onto the neighbourhoods of $a_\pm$. More accurately, consider the system of functions

$$\begin{aligned}&\{\Psi_{\bar{k},j,\delta}\}_{|\bar{k}|\le m,\, j\le (mW)^{1/2}},\\ &\bar{k} = (k_1,k_2),\ \ |\bar{k}| = \max\{k_1,k_2\},\end{aligned} \tag{3.9}$$

obtained by the Gram–Schmidt procedure from

$$\{1_{\Omega_\delta^\pm}\Psi_{\bar{k},j}\}_{|\bar{k}|\le m;\, j\le (mW)^{1/2}},$$

where

$$\begin{aligned}\Psi_{\bar{k},j}(a,b,U) &= \Psi_{\bar{k}}(a,b)\phi_j(U),\\ \Psi_{\bar{k}}(a,b) &= \sqrt{\frac{2}{\pi}}\,(a-b)^{-1}\psi_{k_1}^+(a)\psi_{k_2}^-(b).\end{aligned} \tag{3.10}$$

Notice that according to (2.28) the eigenvalues of $K_*(t,U_1,U_2)$ of (2.14) corresponding to $\phi_j(U)$ with $j > (mW)^{1/2}$ are smaller than $1 - Cm^2/W$ for any $t > d > 0$.

Similarly, consider the system of functions $\{\Psi_{\bar{k},\delta}^+\}_{|\bar{k}|\le m}$ (with $\bar{k} = (k_1,k_2,k_3,k_4)$, $|\bar{k}| = \max\{k_i\}$) obtained by the Gram-Schmidt procedure from

$$\{1_{\Omega_\delta^+}\,\psi_{k_1}^+(a)\,\psi_{k_2}^+(b)\,\psi_{k_3}^+(x+a_+)\,\psi_{k_4}^+(y+a_+)\}_{|\bar{k}|\le m},$$

and define $\{\Psi_{\bar{k},\delta}^-\}_{|\bar{k}|\le m}$ by the same way. Denote $P_\pm$, P_+, and P_- the projections on the subspaces spanned on these three systems. Evidently these three projection operators are orthogonal to each other. Set

$$P = P_\pm + P_+ + P_-,\quad \mathcal{H}_1 = P\mathcal{H},\quad \mathcal{H}_2 = (1-P)\mathcal{H},\quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{3.11}$$

where $\mathcal{H} = L_2[\text{Herm}(2)]$ (the projector P corresponds to projector P_s in (2.9)).

Besides, it is easy to see that for any φ supported in some domain Ω and any $C > 0$

$$(K\varphi)(X) = O(e^{-cW^2}) \text{ for } X : \text{dist}\{X,\Omega\} \ge C > 0 \tag{3.12}$$

with K of (2.4) (see (4.4) of [18]).

Consider the operator K as a block operator with respect to the decomposition (3.11). It has the form

$$\begin{aligned}K^{(11)} &= K_\pm + K_+ + K_- + O(e^{-cW}),\\ &\quad K_\pm := P_\pm K P_\pm,\quad K_+ = P_+ K P_+,\quad K_- := P_- K P_-,\\ K^{(12)} &= P_\pm K(I_\pm - P_\pm) + P_+ K(I_+ - P_+) + P_- K(I_- - P_-) + O(e^{-cW}),\\ K^{(21)} &= (I_\pm - P_\pm)KP_\pm + (I_+ - P_+)KP_+ + (I_- - P_-)KP_- + O(e^{-cW}),\\ K^{(22)} &= (1-P)K(1-P),\end{aligned} \tag{3.13}$$

where $I_\pm$, I_+, and I_- are operators of multiplication by $1_{\Omega_\delta^\pm}$, $1_{\Omega_\delta^+}$, and $1_{\Omega_\delta^-}$ respectively. Indeed, it is easy to see from (3.12) and (3.6) that, e.g. , $P_+KP_-f = O(e^{-cW})$, $P_\pm K(I_+ - P_+)f = O(e^{-cW})$, etc. Similar decomposition can be written for K_ξ.

Notice that by (2.21) $K_\pm$ also has a block diagonal structure:

$$K_\pm = \sum_{j=0}^{(mW)^{1/2}} K_\pm^{(j)}, \quad K_\pm^{(j)} = \mathcal{P}_j P_\pm K P_\pm \mathcal{P}_j. \tag{3.14}$$

Here and below we denote by $\mathcal{P}_j$ the projection on $\phi_j(U)$.

Remember that the final goal is to prove that $K_\xi \sim \mathcal{A}_m \otimes K_{*\xi,l}$ in a certain sense, where $\mathcal{A}_m$, $K_{*\xi,l}$ are defined in (2.25) and (2.24). As will be shown below (see Lemma 3.1), the top eigenvalue of $\mathcal{A}_m$ is close to $\lambda_0(K)$, and the spectral gap of $\mathcal{A}_m$ is of order $1/W$. Moreover, since $\|\tilde{K}_\xi\| \le C/n$ with some n, W-independent C, the eigenvalues of $K_{*\xi,l}$ lie in C/n-neighbourhoods (an so in C_1/W^2-neighbourhoods) of the eigenvalues (2.28) of $K_{*0,l}$, thus in C_1/W^2-neighbourhoods of points

$$\lambda_{j,*} = 1 - \frac{j(j+1)}{W^2(a_+ - a_-)^2}. \tag{3.15}$$

Hence the top eigenvalues of $\mathcal{A}_m \otimes K_{*\xi,l}$ lie in C/n-neighbourhoods (thus in C/W^2-neighbourhoods) of points $\lambda_{j,*} \cdot \lambda_0(K)$ with some n, W-independent C. For small j the distance between $\lambda_{j,*} \cdot \lambda_0(K)$ is also of order c/n, so such neighbourhoods cannot be distinguished, but for $j > D$ with sufficiently big D they do not have intersections. All other eigenvalues (corresponding to $j > l$) are smaller than $|\lambda_0(K)|\big(1 - \frac{\log^2 W}{(a_+ - a_-)^2 W^2}\big)$ and so do not important for us (see Lemma 4.2 below).

This motivates the following choice of the contour $\mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2, \tag{3.16}$$

where

$$\mathcal{L}_2 = \Big\{z : |z| = |\lambda_0(K)|\Big(1 - \frac{\log^2 W}{(a_+ - a_-)^2 W^2}\Big)\Big\}, \tag{3.17}$$

and

$$\begin{aligned} &\mathcal{L}_1 = L^0 \cup L^1, \\ &L^0 = \Big\{z : |z - \lambda_0(K)| = \frac{D^2}{(a_+ - a_-)^2 W^2}\Big\}; \\ &L^1 = \cup_{j=D}^{l-1} L_j, \quad L_j = \big\{z : \big|z - \lambda_{j,*} \cdot \lambda_0(K)\big| = \frac{\gamma}{W^2}\big\} \end{aligned} \tag{3.18}$$

with

$$l = \log W. \tag{3.19}$$

Here $\gamma > 0$ and $D > 0$ are sufficiently large (but $\gamma < D/2(a_+ - a_-)^2$). Notice that

$$\mathrm{dist}\{L^0, L^1\} \ge \frac{D}{3(a_+ - a_-)^2 W^2}, \tag{3.20}$$

$$\mathrm{dist}\{\mathcal{L}_1, \mathcal{L}_2\} \ge \frac{C \log W}{W^2}. \tag{3.21}$$

We have the following lemma

Lemma 3.1 *Given* (2.25),

$$\lambda_0(\mathcal{A}_m)=\lambda_0(K)+O(e^{-c\log^2 W}),\quad |\lambda_1(\mathcal{A}_m)|\le|\lambda_0(K)|-C/W. \tag{3.22}$$

Moreover, the contour $\mathcal{L}$ *encircles all eigenvalues of* $\mathcal{A}_m\otimes K_{*\xi,l}$ *and if we set*

$$G^0_\xi(z)=(\mathcal{A}_m\otimes K_{*\xi,l}-z)^{-1}, \tag{3.23}$$

then

$$|G^0_\xi(z)|\le C_1W^2 \tag{3.24}$$

for z *outside of* $\mathcal{L}$.

Proof of Lemma 3.1 According to (2.25) and the choice of $\Psi_{\bar{k}}$ in (3.10) we have

$$\mathcal{A}_m=A_m^{(+)}\otimes A_m^{(-)}+O(e^{-c\log^2 W}), \tag{3.25}$$

where

$$A_m^{(\pm)}=P_\pm AP_\pm,$$

where P_+ and P_- are the projections on the subspaces spanned on the systems $\{\psi^+_{k,\delta}\}_{k=0}^m$ and $\{\psi^-_{k,\delta}\}_{k=0}^m$ respectively (see (3.7)). The behaviour of $A_m^{(\pm)}$ was studied in [18], Sect. 3.1. In particular, it was proved in Lemma 3.3, [18] that

$$\lambda_0(A_m^{(\pm)})=1+O(1/W),\quad |\lambda_1(A_m^{(\pm)})|\le|\lambda_0(A_m^{(\pm)})|\cdot(1-c/W). \tag{3.26}$$

Since also (see [18], Eq. (4.22))

$$\lambda_0(K)=\lambda_0(A_m^{(+)})\cdot\lambda_0(A_m^{(-)})+O(e^{-c\log^2 W}), \tag{3.27}$$

we get (3.22).

Also according to (2.28)–(2.29) the eigenvalues of $K_{*\xi,l}$ lie in the C/n-neighbourhood of $\lambda_{j,*}$ of (3.15) with some uniformly (in W and n) bounded C. Now this and (3.22) imply (3.24) for the contour $\mathcal{L}$ with sufficiently big D,γ. □

It appears that eigenvalues of K,K_ξ also lie inside $\mathcal{L}$. Moreover, to implement the strategy described in Sect. 2, we need additional information about K,K_ξ described in the following theorem (based on consideration in [18]):

Theorem 3.1 *For the operators* K *defined in* (2.4) *we have*

(i) *the contour* $\mathcal{L}$ *of* (3.16) *encircles all eigenvalues of* K, *and for* z *outside of* $\mathcal{L}$ *we have* $||(K-z)^{-1}||\le CW^2$;

(ii) *Given* z *such that*

$$\big|z-\lambda_{j,*}\cdot|\lambda_0(K)|\big|\ge\frac{\gamma}{W^2},\quad |z|\ge|\lambda_0(K)|\Big(1-\frac{\log^2 W}{(a_+-a_-)^2W^2}\Big) \tag{3.28}$$

with sufficiently big $\gamma>0$, *consider* $G^{(j)}(z)=(K^{(j)}_\pm-z)^{-1}$. *Then*

$$||G^{(j)}||\le C_1W^2/\gamma \tag{3.29}$$

with some absolute constant C_1 *which does not depend on* γ.
In addition, for any z *such that*

$$|\lambda_0(K)|\Big(1-\frac{\log^2 W}{(a_+-a_-)^2W^2}\Big)\le|z|\le 1+C_2/n. \tag{3.30}$$

we have

$$\|(K_+ - z)^{-1}\| \le CW, \quad \|(K_- - z)^{-1}\| \le CW, \quad ||(K^{(22)} - z)^{-1}|| \le CW/m^{1/3}. \tag{3.31}$$

(iii) We have

$$||K^{(21)}|| \le Cm^{3/2}/W^{3/2}, \quad ||K^{(12)}|| \le Cm/W, \tag{3.32}$$

and for z outside of $\mathcal{L}$ we also have

$$||(K^{(11)} - z)^{-1}K^{(12)}|| \le Cm^p, \quad ||K^{(21)}(K^{(11)} - z)^{-1}|| \le Cm^p \tag{3.33}$$

with some $p > 0$.

Same statements are valid for K_ξ of (2.5).

Proof of Theorem 3.1 It is easy to see that the part $\mathcal{L}_1$ of the contour $\mathcal{L}$ of (3.16) satisfies (3.30), and $\mathcal{L}_2$ satisfies (3.28). Although (3.28) and the domain (3.30) is slightly different from eq. (4.33), [18] and

$$D_\gamma = \{z : |\lambda_0(K)| - \frac{\gamma}{W^2} \le |z| \le 1\}$$

considered in [18], it does not change anything in the proof of Proposition 4.1 and Lemmas 4.1 – 4.3 of [18], and thus the proof of Theorem 3.1 for K and (3.24) follows from those lemmas of [18].

To obtain the result for K_ξ set

$$\begin{aligned} G_{1,\xi} &= (K_\xi^{(11)} - z)^{-1} = (K^{(11)} + \widetilde{K}_\xi^{(11)} - z)^{-1}, \\ G_{2,\xi} &= (K_\xi^{(22)} - z)^{-1} = (K^{(22)} + \widetilde{K}_\xi^{(22)} - z)^{-1}, \end{aligned} \tag{3.34}$$

and denote by p and q some absolute exponents which could be different in different formulas.

Using the well-known Schur formula we get

$$(K_\xi - z)^{-1} = \begin{pmatrix} G_\xi^{(11)} & -G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi} \\ -G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)} & G_{2,\xi} + G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi} \end{pmatrix}, \tag{3.35}$$

where

$$G_\xi^{(11)} = (K_\xi^{(11)} - z - K_\xi^{(12)}G_{2,\xi}K_\xi^{(21)})^{-1} = (1 - G_{1,\xi}K_\xi^{(12)}G_{2,\xi}K_\xi^{(21)})^{-1}G_{1,\xi}.$$

Denoting

$$R = (1 - G_{1,\xi}K_\xi^{(12)}G_{2,\xi}K_\xi^{(21)})^{-1}, \tag{3.36}$$

we get

$$G_\xi^{(11)} = RG_{1,\xi}. \tag{3.37}$$

Notice that

$$\begin{aligned} \|G_{1,\xi}K_\xi^{(12)}\| &= \|(K^{(11)} - z + \widetilde{K}_\xi^{(11)})^{-1}(K^{(12)} + \widetilde{K}_\xi^{(12)})\| \\ &= \|(1 + (K^{(11)} - z)^{-1}\widetilde{K}_\xi^{(11)})^{-1}(K^{(11)} - z)^{-1}(K^{(12)} + \widetilde{K}_\xi^{(12)})\|. \end{aligned} \tag{3.38}$$

Moreover (3.13) and part (ii) of the Theorem for operator K yield

$$\|(K^{(11)} - z)^{-1}\| \le \frac{C_1 n}{\gamma}, \tag{3.39}$$

where γ is sufficiently big and C_1 does not depend on γ. Hence

$$\|(K^{(11)}-z)^{-1}\widetilde{K}_\xi^{(11)}\| \le C < 1.$$

Thus according to (3.38), (3.33) for K, and (2.16)

$$\begin{aligned}\|G_{1,\xi}K_\xi^{(12)}\| &\le C\|(K^{(11)}-z)^{-1}(K^{(12)}+\widetilde{K}_\xi^{(12)})\|\\ &\le C(\|(K^{(11)}-z)^{-1}K^{(12)}\|+\|(K^{(11)}-z)^{-1}\widetilde{K}_\xi^{(12)}\|)\\ &\le C(\log^p W+C_1)\le C\log^p W.\end{aligned}$$

Similarly

$$\|K_\xi^{(21)}G_{1,\xi}\| \le C\|(K^{(21)}+\widetilde{K}_\xi^{(21)})(K^{(11)}-z)^{-1}\| \le C\log^p W.$$

The bound (3.32) for K_ξ trivially follow from (3.32) for operator K and (2.16), which finishes the proof of (iii) for K_ξ.

In addition, due to the last bound of (3.31) for operator K and (2.16) we have

$$\begin{aligned}\|G_{2,\xi}\| &= \|(K^{(22)}+\widetilde{K}_\xi^{(22)}-z)^{-1}\|\\ &= \|(1+(K^{(22)}-z)^{-1}\widetilde{K}_\xi^{(22)})^{-1}(K^{(22)}-z)^{-1}\| \le CW/m^{1/3}\end{aligned}\tag{3.40}$$

which gives the last bound of (3.31) for operator K_ξ. This implies

$$\|G_{2,\xi}K_\xi^{(21)}\| \le \frac{\log^p W}{W^{1/2}}.\tag{3.41}$$

Thus

$$\|G_{1,\xi}K_\xi^{(12)}G_{2,\xi}K_\xi^{(21)}\| \le \|G_{1,\xi}K_\xi^{(12)}\|\cdot\|G_{2,\xi}K_\xi^{(21)}\| \le \frac{C\log^p W}{W^{1/2}},\tag{3.42}$$

and so

$$\|R\| \le C.$$

This, (3.36)–(3.37), and (3.39) yield

$$\|G_\xi^{(11)}\| \le Cn.\tag{3.43}$$

Similarly (3.37) gives

$$\|G_\xi^{(11)}K_\xi^{(12)}\| = \|RG_{1,\xi}K_\xi^{(12)}\| \le \|R\|\cdot\|G_{1,\xi}K_\xi^{(12)}\| \le C\log^p W,$$

which implies

$$\|G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}\| \le C\log^p W\cdot W.\tag{3.44}$$

It is easy to see that

$$D^{-1}C(A-BD^{-1}C)^{-1} = (D-CA^{-1}B)^{-1}CA^{-1},$$

thus

$$\begin{aligned}G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)} &= (K_\xi^{(22)}-z-K_\xi^{(21)}G_{1,\xi}K_\xi^{(12)})^{-1}K_\xi^{(21)}G_{1,\xi}\\ &= (1-G_{2,\xi}K_\xi^{(21)}G_{1,\xi}K_\xi^{(12)})^{-1}G_{2,\xi}K_\xi^{(21)}G_{1,\xi}.\end{aligned}$$

But

$$\|G_{2,\xi}K_\xi^{(21)}G_{1,\xi}K_\xi^{(12)}\| \le \|G_{2,\xi}K_\xi^{(21)}\| \cdot \|G_{1,\xi}K_\xi^{(12)}\| \le \frac{C\log^p W}{W^{1/2}},$$

hence using (3.33) for K_ξ we obtain

$$\|G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}\| \le C\|G_{2,\xi}\| \cdot \|K_\xi^{(21)}G_{1,\xi}\| \le C\log^p W \cdot W. \tag{3.45}$$

We also can write

$$\|G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}\| \le \|G_{2,\xi}\|^2 \cdot \|K_\xi^{(21)}\| \cdot \|G_\xi^{(11)}K_\xi^{(12)}\| \le C\log^p W \cdot W^{1/2} \tag{3.46}$$

which finishes the proof of (i) for K_ξ.

Bounds (3.29)–(3.31) for K_ξ can be obtained easily from those for K and from (2.16). □

4 Proof of Theorem 1.2

The key step in the proof of Theorem 1.2 is the following theorem

Theorem 4.1 *Given $G_\xi(z) = (K_\xi - z)^{-1}$ with K_ξ of (2.5), f_ξ of (2.1), and the contour $\mathcal{L}$ defined in (3.16)–(3.19), we can write for the integral in (2.7)*

$$\int_{\mathcal{L}} z^{n-1}(G_\xi(z)f_\xi, \bar{f}_\xi)dz$$
$$= \int_{\mathcal{L}} z^{n-1}(G_\xi^0(z)(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz + |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\Big(\frac{1}{\log W}\Big), \tag{4.1}$$

where

$$f_1 = Pf, \tag{4.2}$$

where P is the orthogonal projector to the the space $\mathcal{H}_1$ (see (3.11)), and G_ξ^0 is defined in (3.23). Here $f_{1,\pm}$ is a projection of f on the linear span of $\{\Psi_{\bar{k},0}(a,b), |k| \le m\}$ of (3.10).

In addition, given (2.25),

$$(\mathcal{A}_m^{n-1}f_{1,\pm}, f_{1,\pm}) = |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot (1 + o(1)). \tag{4.3}$$

Let us assume that Theorem 4.1 is proved and derive the assertion of Theorem 1.2.

Indeed, since $\mathcal{L}$ encircles all eigenvalues of $\mathcal{A}_m \otimes K_{*\xi,l}$ (see Lemma 3.1), according to the Cauchy theorem we get

$$-\frac{1}{2\pi i}\int_{\mathcal{L}} z^{n-1}(G_\xi^0(z)(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz = \big((\mathcal{A}_m \otimes K_{*\xi,l})^{n-1}(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1)\big)$$
$$= (\mathcal{A}_m^{n-1}f_{1,\pm}, \bar{f}_{1,\pm}) \cdot (K_{*\xi,l}^{n-1}1, 1).$$

Now let us prove

Lemma 4.1 *Given (2.24), if $n = C_*W^2$, $l = [\log W]$ we have*

$$(K_{*\xi,l}^{n-1}1, 1) \to (e^{-C^*\Delta_U - i\xi\pi\hat{\nu}}1, 1), \quad n, W \to \infty,$$

with $C^* = C_*/t^*$ and $\hat{\nu}$ as in Theorem 1.2.

Proof of Lemma 4.1 Recall that according to (2.30)

$$K_{*\xi,l} = Q_l\big(1 - n^{-1}(C^*\Delta_U + i\xi\pi\nu\big) + O(l^2n^{-2}))Q_l = Q_l e^{-n^{-1}Q_l(C^*\Delta_U + i\xi\pi\nu + O(l^2n^{-2}))Q_l}Q_l.$$

Here $O(l^2n^{-2})$ means an operator whose norm is bounded by Cl^2n^{-2}. Thus

$$K_{*\xi,l}^{n-1} = Q_l e^{-Q_l(C^*\Delta_U + i\xi\pi\nu + O(l^2n^{-1}))Q_l}Q_l = Q_l e^{-Q_l(C^*\Delta_U + i\xi\pi\nu)Q_l}(1 + O(l^2/n))Q_l,$$

ans so

$$(K_{*\xi,l}^{n-1}1, 1) = (e^{-Q_l(C^*\Delta_U + i\xi\pi\nu)Q_l}1, 1) + O(l^2/n).$$

Consider the basis $\{\phi_j\}$ of (2.19). In this basis Laplace operator Δ_U is diagonal, and operator $\hat{\nu}$ is three diagonal (since it corresponds to the multiplication by x in the space of Legendre polynomials). To simplify notations, let F be an operator of multiplication by $(i\pi\xi\nu)$ and $\Delta = C^*\Delta_U$. Set

$$\begin{aligned} D &= \Delta + F,\\ D^{(l)} &= \Delta + F^{(l)}, \end{aligned} \tag{4.4}$$

where $F^{(l)}$ be the matrix F where we put $F_{l,l+1} = F_{l+1,l} = 0$. It is evident that (recall $\phi_0 = 1$)

$$(e^{-D^{(l)}}\phi_0, \phi_0) = \Big(e^{-Q_lDQ_l}\phi_0, \phi_0\Big) = (e^{-Q_l(C^*\Delta_U + i\xi\pi\nu)Q_l}1, 1).$$

Thus we are left to prove that

$$\Big((e^{-D} - e^{-D^{(l)}})\phi_0, \phi_0\Big) \to 0. \tag{4.5}$$

Notice that both e^{-D}, $e^{-D^{(l)}}$ are bounded operators, and $|F| \le C$, $|F^{(l)}| \le C$. We will use the well-known Duhamel formula

$$e^{-tA_1} - e^{-tA_2} = \int\limits_0^t e^{-(t-s)A_2}(A_1 - A_2)e^{-sA_1}ds. \tag{4.6}$$

For $A_1 = D$, $A_2 = D^{(l)}$ and $t = 1$ it gives

$$\begin{aligned} \Big|(e^{-D} - e^{-D^{(l)}})\phi_0\Big| &= \Big|\int_0^1 e^{-(1-s)D^{(l)}}(F - F^{(l)})e^{-sD}\phi_0\,ds\Big|\\ &= \Big|\int_0^1 e^{-(1-s)D^{(l)}}(F_l \cdot E_{l,l+1} + F_{l+1}\cdot E_{l+1,l})e^{-sD}\phi_0\,ds\Big|\\ &= \Big|\int_0^1 e^{-(1-s)D^{(l)}}\Big(F_{l+1}\phi_{l+1}\big(e^{-sD}\phi_0, \phi_l\big) + F_l\phi_l\big(e^{-sD}\phi_0, \phi_{l+1}\big)\Big)\,ds\Big|\\ &\le C\Big(\big|\big(e^{-sD}\phi_0, \phi_l\big)\big| + \big|\big(e^{-sD}\phi_0, \phi_{l+1}\big)\big|\Big). \end{aligned}$$

Here $E_{l,l+1}$ is an operator whose matrix in the basis $\{\phi_j\}$ has 1 at $(l, l+1)$ place and zeros everywhere else, and $E_{l+1,l}$ is defined in a similar way. F_l, F_{l+1} are $(l, l+1)$ and $(l+1, l)$ elements of the matrix F in the same basis.

Now let us bound $\left|\left(e^{-sD}\phi_0, \phi_l\right)\right|$. To this end apply Duhamel's formula (4.6) $p = [l/2]$ times with $A_1 = D$ and $A_2 = \Delta$. We obtain

$$\left(e^{-sD}\phi_0, \phi_l\right) = \sum_{j=1}^{p} \int_{s_1+..+s_j\le s} (e^{-s_1\Delta} F e^{-s_2\Delta} F \dots e^{-s_j\Delta}\phi_0, \phi_l)\, ds_1..ds_j$$
$$+ \int_{s_1+\dots+s_p\le s} (e^{-s_1 D} F e^{-s_2\Delta} F \dots e^{-s_p\Delta}\phi_0, \phi_l)\, ds_1..ds_p.$$

Since $e^{-s\Delta}$ is diagonal in the basis $\{\phi_j\}$, and F is only three diagonal, the expression $e^{-s_1\Delta} F e^{-s_2\Delta} F \dots e^{-s_j\Delta}\phi_0$ is in the linear span of $\{\phi_k\}_{k=0}^{j}$, and thus the sum above is 0. Hence

$$\left|\left(e^{-sD}\phi_0, \phi_l\right)\right| \le \left|\int_{s_1+\dots+s_p\le s} (e^{-s_1 D} F e^{-s_2\Delta} F \dots e^{-s_p\Delta}\phi_0, \phi_l)\, ds_1..ds_p\right|$$
$$\le C^l \left|\int_{s_1+\dots+s_p\le s} ds_1..ds_p\right| = \frac{C^l s^l}{l!} \le C_1 e^{-l\log l} \to 0,$$

which finishes the proof of (4.5). □

Lemmas 4.1 and (4.3) imply that

$$-\frac{1}{2\pi i}\int_{\mathcal{L}} z^{n-1}(G_\xi^0(z)(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz$$

is of order

$$|\lambda_0(K)|^{n-1}\cdot \|f_{1,\pm}\|^2,$$

and so (4.1) can be rewritten as

$$-\frac{1}{2\pi i}\int_{\mathcal{L}} z^{n-1}(G_\xi(z) f_\xi, \bar{f}_\xi)dz = (\mathcal{A}_m^{n-1} f_{1,\pm}, \bar{f}_{1,\pm})\cdot (K_{*\xi,l}^{n-1}1, 1)(1+o(1)), \quad n\to\infty.$$

This, a similar relation with $\xi = 0$, (2.6), and (2.7), yield

$$D_2^{-1} F_2\Big(E + \frac{\xi}{2n\rho(E)}, E - \frac{\xi}{2n\rho(E)}\Big)$$
$$= \frac{(\mathcal{A}_m^{n-1} f_{1,\pm}, \bar{f}_{1,\pm})\cdot (K_{*\xi,l}^{n-1}1, 1)}{(\mathcal{A}_m^{n-1} f_{1,\pm}, \bar{f}_{1,\pm})\cdot (K_{*0,l}^{n-1}1, 1)}(1+o(1)) = (K_{*\xi,l}^{n-1}1, 1)(1+o(1)).$$

Here we used (2.26). This relation and Lemma 4.1 complete the proof of Theorem 1.2.

4.1 Proof of Theorem 4.1

We are left to prove Theorem 4.1.

First we decompose $f = (f_1, f_2)$ with respect to decomposition (3.11). Observe that since

$$|\mathcal{F}(X)| \le 1,$$

and $\mathcal{F}(X)$ exponentially decreases at ∞ (in eigenvalues a, b), we have $\|f\| = const \le 1$. Moreover it is easy to see that

$$\|f_1\|^2 \ge \|f_{1,\pm}\|^2 \ge \|\mathcal{F}(X)\Psi_{\bar{0},0}\|^2 = \Big|(2W\Re\alpha_\pm)^{1/2}\Big(\int e^{-f(a)/2}e^{-\alpha_\pm W(a-a_+)^2}da\Big)^2\Big|^2 \ge \frac{C}{W},$$

with $\Psi_{\bar{0},0}$ of (3.10). Therefore

$$\|f_1\| \ge \|f_{1,\pm}\| \ge C/W^{1/2}. \tag{4.7}$$

We start with the following simple lemma

Lemma 4.2 *The main contribution to the integral in (2.7) is given by the integral over the contour $\mathcal{L}_1$ of (3.18), i.e.*

$$\int_{\mathcal{L}} z^{n-1}(G_\xi(z)f_\xi, \bar{f}_\xi)dz = \int_{\mathcal{L}_1} z^{n-1}(G_\xi(z)f, \bar{f})dz + |\lambda_0(K)|^{n-1}\cdot\|f_1\|^2\cdot O\Big(\frac{\log W}{W}\Big),$$

where f is defined in (2.1). In addition,

$$\int_{\mathcal{L}_2} z^{n-1}(G^0_\xi(z)(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz = |\lambda_0(K)|^{n-1}\cdot\|f_1\|^2\cdot o\Big(e^{-C\log^2 W}\Big), \tag{4.8}$$

where $\mathcal{L}_2$ is defined in (3.17), and $G^0_\xi(z)$ is defined in (3.23).

Proof of Lemma 4.2 Since for $z \in \mathcal{L}_2$ we have

$$|z|^{n-1} \le |\lambda_0(K)|^{n-1}\cdot e^{-C\log^2 W},$$

we get using $\|G_\xi(z)\| \le CW^2$ (see part (i) of Theorem 3.1 for K_ξ) that

$$\begin{aligned}\Big|\int_{\mathcal{L}_2} z^{n-1}(G_\xi(z)f_\xi, \bar{f}_\xi)dz\Big| &\le C_1|\lambda_0(K)|^{n-1}\cdot e^{-C_2\log^2 W}\cdot W^2\\ &= |\lambda_0(K)|^{n-1}\cdot\|f_1\|^2\cdot o\Big(e^{-C\log^2 W}\Big).\end{aligned}$$

Here we used (4.7). Similarly one can obtain (4.8) from (3.24).

Besides,

$$|\mathcal{L}_1| \le C\log W/W^2, \tag{4.9}$$

and for $z \in \mathcal{L}_1$

$$|z|^{n-1} \le C|\lambda_0(K)|^{n-1}. \tag{4.10}$$

Thus, since $\|f - f_\xi\| \le C/n$, we get according to (4.7)

$$\begin{aligned}\Big|\int_{\mathcal{L}_1} z^{n-1}(G_\xi(z)(f_\xi - f), \bar{f}_\xi)dz\Big| &\le C|\lambda_0(K)|^{n-1}\cdot W^2\cdot\|f - f_\xi\|\cdot|\mathcal{L}_1|\\ &\le |\lambda_0(K)|^{n-1}\cdot\frac{\log W}{W^2} \le |\lambda_0(K)|^{n-1}\cdot\|f_1\|^2\cdot O\Big(\frac{\log W}{W}\Big),\end{aligned}$$

which gives the lemma. $\square$

Lemma 4.2 yields that we can prove (4.1) for $\mathcal{L}_1$ instead of $\mathcal{L}$.

The next step is to prove that we can consider only the upper-left block $K_\xi^{(11)}$ of K_ξ (see (3.13)). More precisely, we are going to prove

Lemma 4.3 *Given* (3.34) *and* (4.2), *we have*

$$\int_{\mathcal{L}_1} z^{n-1}(G_\xi(z)f,\bar{f})dz = \int_{\mathcal{L}_1} z^{n-1}(G_{1,\xi}(z)\,f_1,\bar{f}_1)dz + |\lambda_0(K)|^{n-1}\cdot\|f_1\|^2\cdot O\Big(\frac{\log^p W}{W^{1/2}}\Big).$$

Proof of Lemma 4.3 According to (3.35) we have

$$\begin{aligned}\int_{\mathcal{L}_1} z^{n-1}((K_\xi - z)^{-1}f,\bar{f})dz &= \int_{\mathcal{L}_1} z^{n-1}(G_\xi^{(11)}f_1,\bar{f}_1)dz\\ &-\int_{\mathcal{L}_1} z^{n-1}(G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}f_2,\bar{f}_1)dz\\ &-\int_{\mathcal{L}_1} z^{n-1}(G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}f_1,\bar{f}_2)dz\\ &+\int_{\mathcal{L}_1} z^{n-1}((G_{2,\xi}+G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi})f_2,\bar{f}_2)dz.\end{aligned}$$

Thus, we get using (3.44)–(3.45), (4.9)–(4.10), $\|f_2\|\le C$, and (4.7)

$$\begin{aligned}&\Big|\int_{\mathcal{L}_1} z^{n-1}(G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}f_2,\bar{f}_1)dz\Big| \le \|G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}\|\cdot\|f_1\|\cdot\|f_2\|\cdot\int_{\mathcal{L}_1}|z|^{n-1}|dz|\\ &\quad\le \frac{C\log^p W\cdot W}{W^2}\cdot|\lambda_0(K)|^{n-1}\cdot\|f_1\|\le O\Big(\frac{C\log^p W}{W^{1/2}}\Big)\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1},\\ &\Big|\int_{\mathcal{L}_1} z^{n-1}(G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}f_1,\bar{f}_2)dz\Big| \le \|G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}\|\cdot\|f_1\|\cdot\|f_2\|\cdot\int_{\mathcal{L}_1}|z|^{n-1}|dz|\\ &\quad\le \frac{C\log^p W\cdot W}{W^2}\cdot|\lambda_0(K)|^{n-1}\cdot\|f_1\|\le O\Big(\frac{C\log^p W}{W^{1/2}}\Big)\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1}.\end{aligned}$$

Notice that $G_{2,\xi}$ of (3.34) is analytic outside of $\mathcal{L}_2$ (see (3.31)), and so

$$\int_{\mathcal{L}_1} z^{n-1}(G_{2,\xi}f_2,\bar{f}_2)dz = 0.$$

Hence

$$\begin{aligned}&\int_{\mathcal{L}_1} z^{n-1}((G_{2,\xi}+G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi})f_2,\bar{f}_2)dz\\ &\quad=\int_{\mathcal{L}_1} z^{n-1}(G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}f_2,\bar{f}_2)dz.\end{aligned}$$

Thus (3.46) and (4.7) yield

$$\begin{aligned}&\Big|\int_{\mathcal{L}_1} z^{n-1}(G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}f_2,\bar{f}_2)dz\Big|\\ &\quad\le\|G_{2,\xi}K_\xi^{(21)}G_\xi^{(11)}K_\xi^{(12)}G_{2,\xi}\|\cdot\|f_2\|^2\cdot\int_{\mathcal{L}_1}|z|^{n-1}|dz|\\ &\quad\le\frac{C\log^p W\cdot W^{1/2}}{W^2}\cdot|\lambda_0(K)|^{n-1}\le O\Big(\frac{C\log^p W}{W^{1/2}}\Big)\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1}.\end{aligned}$$

Besides, according to (3.37) and (3.42)

$$\Big|\int_{\mathcal{L}_1} z^{n-1}((G_\xi^{(11)}-G_{1,\xi})f_1,\bar f_1)dz\Big| \le \|1-R\|\cdot\|G_{1,\xi}\|\cdot\|f_1\|^2\cdot\int_{\mathcal{L}_1}|z|^{n-1}|dz|$$
$$\le \frac{C\log^p W\cdot W^2}{W^{1/2}\cdot W^2}\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1}=O\Big(\frac{C\log^p W}{W^{1/2}}\Big)\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1}.$$

These bounds imply Lemma 4.3. □

Now write $K_\xi^{(11)}-z$, $K^{(11)}-z$ in the block form

$$K^{(11)}-z=\begin{pmatrix} M_1 & M_{12}\\ M_{21} & M_2\end{pmatrix},\quad K_\xi^{(11)}-z=\begin{pmatrix} M_{1,\xi} & M_{12,\xi}\\ M_{21,\xi} & M_{2,\xi}\end{pmatrix} \tag{4.11}$$

according to decomposition

$$\mathcal{H}_1=\mathcal{M}_1\oplus\mathcal{M}_2,$$

where $\mathcal{M}_1$ is a linear span of $\{\Psi_{j,k,\delta},\ j\le\log W, |k|\le m\}$ (see (3.9)). Then (see (3.13), (3.14))

$$M_1=\sum_{j=0}^{\log W}K_\pm^{(j)},\quad K_\pm^{(j)}=\mathcal{P}_jP_\pm KP_\pm\mathcal{P}_j,$$
$$M_2=K_++K_-+\sum_{j=\log W+1}^{(mW)^{1/2}}K_\pm^{(j)},$$
$$M_{12}=O(e^{-cW}),\quad M_{21}=O(e^{-cW}), \tag{4.12}$$

where $\mathcal{P}_j$ is the projection on $\{\Psi_{\bar k}(a,b)\phi_j(U)\}$.

Set

$$G_{1,l,\xi}(z)=(K_{m,l,\xi}-z)^{-1}=(M_{1,\xi})^{-1}, \tag{4.13}$$

where $K_{m,l,\xi}$ is defined in (2.22). Notice also that, since f_1 does not depend on $\{U_j\}$, the part of f_1 corresponding to $\mathcal{M}_1$ is $f_{1,\pm}\otimes 1$.

The next step is to show

Lemma 4.4 *The operator $K_\xi^{(11)}$ of (3.13) can be replaced by $K_{m,l,\xi}$ of (2.22), i.e. we can write*

$$\int_{\mathcal{L}_1} z^{n-1}(G_{1,\xi}(z)f_1,\bar f_1)dz$$
$$=\int_{\mathcal{L}_1} z^{n-1}(G_{1,l,\xi}(z)(f_{1,\pm}\otimes 1),(\bar f_{1,\pm}\otimes 1))dz+|\lambda_0(K)|^{n-1}\cdot\|f_1\|^2\cdot O\Big(\frac{1}{\log W}\Big).$$

Proof of Lemma 4.5 Denote

$$D_\xi=M_{1,\xi}-M_{12,\xi}M_{2,\xi}^{-1}M_{21,\xi},\quad D_{0,\xi}=1-M_{12,\xi}M_{2,\xi}^{-1}M_{21,\xi}M_{1,\xi}^{-1}$$

and write $f_1=(f_\pm\otimes 1, f_{12})$ according to the decomposition (4.11).

Using Schur's formula we get

$$G_{1,\xi}=\begin{pmatrix} D_\xi^{-1} & -D_\xi^{-1}M_{12,\xi}M_{2,\xi}^{-1}\\ -M_{2,\xi}^{-1}M_{21,\xi}D_\xi^{-1} & M_{2,\xi}^{-1}+M_{2,\xi}^{-1}M_{21,\xi}D_\xi^{-1}M_{12,\xi}M_{2,\xi}^{-1}\end{pmatrix} \tag{4.14}$$

Notice that according to (ii) of Theorem 3.1 $M_{2,\xi}^{-1}$ is analytic inside of $\mathcal{L}_1$, and so

$$\int_{\mathcal{L}_1} z^{n-1}(M_{2,\xi}^{-1} f_{12}, \bar{f}_{12})dz = 0,$$

thus

$$\begin{aligned}\int_{\mathcal{L}_1} z^{n-1}(G_{1,\xi}(z) f_1, \bar{f}_1)dz &= \int_{\mathcal{L}_1} z^{n-1}(D_\xi^{-1}(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz \\ &- \int_{\mathcal{L}_1} z^{n-1}(D_\xi^{-1} M_{12,\xi} M_{2,\xi}^{-1} f_{12}, (\bar{f}_{1,\pm}\otimes 1))dz \\ &- \int_{\mathcal{L}_1} z^{n-1}(M_{2,\xi}^{-1} M_{21,\xi} D_\xi^{-1}(f_{1,\pm}\otimes 1), \bar{f}_{12})dz \\ &+ \int_{\mathcal{L}_1} z^{n-1}(M_{2,\xi}^{-1} M_{21,\xi} D_\xi^{-1} M_{12,\xi} M_{2,\xi}^{-1} f_{12}, \bar{f}_{12})dz \end{aligned} \tag{4.15}$$

Let $z \in \mathcal{L}_1$. Then using (3.14) and (3.21) we can write (recall that $\log W \sim \log n$)

$$\|M_2^{-1}\| \le Cn/\log n.$$

In addition,

$$\begin{aligned} &\|K_\xi^{(11)} - K^{(11)}\| \le C/n, \\ &\|M_{2,\xi}^{-1}\| = \|M_2^{-1}(1 + (M_{2,\xi} - M_2)M_2^{-1})^{-1}\| \le \frac{C_1 n}{\log n}\cdot\Big(1 - \frac{C_2}{\log n}\Big)^{-1} \le Cn/\log n, \\ &\|M_{12,\xi}\| \le C/n, \quad \|M_{21,\xi}\| \le C/n. \end{aligned} \tag{4.16}$$

Here we used (2.16). Part (ii) of Theorem 3.1 also gives (recall $n = C_* W^2$)

$$\|M_{1,\xi}^{-1}\| \le Cn. \tag{4.17}$$

In addition, using the resolvent identity we obtain

$$D_\xi^{-1} - M_{1,\xi}^{-1} = M_{1,\xi}^{-1} M_{12,\xi} M_{2,\xi}^{-1} M_{21,\xi} M_{1,\xi}^{-1} D_{0,\xi}^{-1}. \tag{4.18}$$

According to (4.16)–(4.17) we get

$$\|M_{12,\xi} M_{2,\xi}^{-1} M_{21,\xi} M_{1,\xi}^{-1}\| \le C/\log n,$$

thus

$$\|D_{0,\xi}^{-1}\| \le C. \tag{4.19}$$

In view of (4.18)

$$\|D_\xi^{-1} - M_{1,\xi}^{-1}\| \le \frac{Cn}{\log n}.$$

Therefore, since according to (3.15), we have for $z \in L_j$ of (3.18)

$$|z|^{n-1} \le C_1|\lambda_0(K)|^{n-1}\cdot e^{-C_2 j(j+1)},$$

and $|L_j| = 2\pi\gamma/W^2$, we get

$$\Big|\int\limits_{\mathcal{L}_1} z^{n-1}\big((D_\xi^{-1} - M_{1,\xi}^{-1})(f_{1,\pm}\otimes 1), (f_{1,\pm}\otimes 1)dz\Big|$$
$$\le \frac{Cn}{\log n}\|f_{1,\pm}\|^2\cdot|\lambda_0(K)|^{n-1}\cdot\sum_{j=D}^{l}|L_j|\cdot e^{-C_2 j(j+1)}$$
$$\le \frac{C}{\log n}\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1}\cdot\sum_{j=D}^{l}e^{-C_2 j(j+1)} \le \frac{C}{\log n}\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1}.$$

Now consider another integrals in (4.15). Using $D_\xi = D_{0,\xi}^{-1}M_{1,\xi}^{-1}$, we obtain similarly

$$\Big|\int\limits_{\mathcal{L}_1} z^{n-1}(D_\xi^{-1}M_{12,\xi}M_{2,\xi}^{-1}f_{12}, (\bar{f}_{1,\pm}\otimes 1))dz\Big|$$
$$\le \frac{Cn}{\log n}\cdot\|f_{1,\pm}\|\cdot\|f_{12}\|\cdot|\lambda_0(K)|^{n-1}\cdot\sum_{j=D}^{l}|L_j|\cdot e^{-C_2 j(j+1)}$$
$$\le \frac{C}{\log n}\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1},$$

and by the same argument

$$\Big|\int\limits_{\mathcal{L}_1} z^{n-1}(M_{2,\xi}^{-1}M_{21,\xi}D_\xi^{-1}(f_{1,\pm}\otimes 1), \bar{f}_{12})dz\Big| \le \frac{C}{\log n}\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1},$$
$$\Big|\int\limits_{\mathcal{L}_1} z^{n-1}(M_{2,\xi}^{-1}M_{21,\xi}D_\xi^{-1}M_{12,\xi}M_{2,\xi}^{-1}f_{12}, \bar{f}_{12})dz\Big| \le \frac{C}{\log^2 n}\cdot\|f_1\|^2\cdot|\lambda_0(K)|^{n-1}.$$

This implies the lemma. □

Now we have the integral

$$\int\limits_{\mathcal{L}_1} z^{n-1}(G_{1,l}(z)(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz.$$

The last step is to show

Lemma 4.5 *The operator $K_{m,l,\xi}$ of (2.22) can be replaced by $\mathcal{A}_m\otimes K_{*\xi,l}$ (see (2.24)–(2.25)), i.e. we have*

$$\int\limits_{\mathcal{L}_1} z^{n-1}(G_{1,l}(z)(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz$$
$$= \int\limits_{\mathcal{L}_1} z^{n-1}(G_\xi^0(z)(f_{1,\pm}\otimes 1), (\bar{f}_{1,\pm}\otimes 1))dz + |\lambda_0(K)|^{n-1}\cdot\|f_1\|^2\cdot O\Big(\frac{\log^p W}{W^{1/2}}\Big),$$

where G_ξ^0 is defined in (3.23).

Proof of Lemma 4.5 Using the resolvent identity we can write

$$G_{1,l}(z) - G^0_\xi(z) = -G^0_\xi(z)(M_{1,\xi} - \mathcal{A}_m \otimes K_{*\xi,l})G_{1,l}(z)$$

Since for (3.5)

$$\psi^\alpha_k(x) = O(e^{-c\log^2 W}), \quad |x| \geq 2W^{-1/2}\log W, \quad k \leq m,$$

we get that both $K_{m,l,\xi}$, $\mathcal{A}_m \otimes K_{*\xi,l}$ are concentrated in the $\log W/W^{1/2}$-neighbourhoods of $a_\pm$ (see [18], for details). In this neighbourhood

$$a_1 - b_1 = a_+ - a_- + O\Big(\frac{\log W}{W^{1/2}}\Big), \quad a_2 - b_2 = a_+ - a_- + O\Big(\frac{\log W}{W^{1/2}}\Big),$$
$$t = (a_+ - a_-)^2 + O\Big(\frac{\log W}{W^{1/2}}\Big) = t_* + O\Big(\frac{\log W}{W^{1/2}}\Big).$$

Thus according to (2.20)

$$\|K_{m,l,0} - \mathcal{A}_m \otimes K_{*0,l}\| \leq \frac{C\log W}{W^{5/2}},$$

where $K_{m,l,0}$, $\mathcal{A}_m \otimes K_{*0,l}$ are $K_{m,l,\xi}$, $\mathcal{A}_m \otimes K_{*\xi,l}$ with $\xi = 0$. In addition, in this neighbourhood

$$\|\widetilde{K}_\xi(X,Y) - \widetilde{K}_\xi(X,Y)\big|_{X=Y=X_\pm}\| \leq \frac{C\log W}{n\sqrt{W}}.$$

Hence, since $n \sim W^2$, we get

$$\|K_{m,l,\xi} - \mathcal{A}_m \otimes K_{*\xi,l}\| \leq \frac{C\log W}{W^{5/2}},$$

and so

$$\Big|\int\limits_{\mathcal{L}_1} z^{n-1}\Big((G_{1,l}(z)(f_{1,\pm}\otimes 1), (\bar f_{1,\pm}\otimes 1)) - (G^0_\xi(z)(f_{1,\pm}\otimes 1), (\bar f_{1,\pm}\otimes 1))\Big)dz\Big|$$
$$\leq C|\mathcal{L}_1| \cdot \frac{CW^4 \cdot \log^p W}{W^{5/2}} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1} \leq \frac{C\log^p W}{W^{1/2}} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1}$$

□

We are left to prove (4.3).

It follows from (3.25)–(3.27) that

$$(\mathcal{A}^{n-1}_m f_{1,\pm}, \bar f_{1,\pm}) = \lambda_0(K)^{n-1} \cdot |(f_1, \Psi_{\bar 0,0})|^2(1 + o(1)),$$

where we used that $(f_{1,\pm}, \Psi_{\bar 0,0}) = (f_1, \Psi_{\bar 0,0})$.

According to the definition of $\{\Psi_{\bar k}\}_{|\bar k|\leq m}$ it is also easy to see that

$$\|f_1\|^2 = |(f_1, \Psi_{\bar 0,0})|^2(1 + O(1/W)).$$

Thus

$$(\mathcal{A}^{n-1}_m f_{1,\pm}, f_{1,\pm}) = \lambda_0(K)^{n-1} \cdot \|f_1\|^2(1 + o(1)),$$

which completes the proof of Theorem 4.1.

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