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STRICT LOCAL MARTINGALES VIA FILTRATION ENLARGEMENT

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A strict local martingale is a local martingale that is not a martingale. We investigate how such a process might arise from a true martingale as a result of an enlargement of the filtration and a change of measure. We study and implement a particular type of enlargement, initial expansion of filtration, for stochastic volatility models with and without jumps and provide sufficient conditions in each of these cases such that initial expansion can create a strict local martingale. We provide examples of initial enlargement that effect this change.

Keywords: Strict local martingales; filtration expansion; bubbles.

1. Introduction

We are interested in mechanisms by which strict local martingales can arise from martingales. A strict local martingale is a local martingale that is not a martingale. We study how expanding the original filtration with respect to which a process is a martingale can lead to a strict local martingale, i.e. if we begin with a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F}$ denotes $(\mathcal{F}_t)_{t\geq 0}$, and with an $\mathbb{F}$ martingale $M = (M_t)_{t\geq 0}$, and consider an expanded filtration $\mathbb{G}$ such that, for all t we have the inclusion $\mathcal{F}_t \subset \mathcal{G}_t$, when can we obtain a filtration $\mathbb{G}$ such that M becomes a strict local martingale, possibly under a different but equivalent probability measure Q?

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At first sight, it might seem like a strange construction to enlarge a filtration and change the probability measure. We will argue that it is a natural thing to do from the standpoint of mathematical finance.

Strict local martingales have recently been a popular subject of study. Some relatively recent papers concerning strict local martingales include Biagini *et al.* (2014), Chybiryakov (2007), Cox & Hobson (2005), Delbaen & Schachermayer (1998), Föllmer & Protter (2011), Lions & Musiela (2007), Hulley (2010), Keller-Ressel (2015), Klebaner & Liptser (2014), Kreher & Nikeghbali (2013), Larsson (2014), Madan & Yor (2006), Mijatovic & Urusov (2012), Protter (2005, 2015), Protter & Shimbo (2008), and Sin (1998), and from this list, we can infer a certain interest. Our motivation comes from the analysis of financial bubbles, as explained in Protter (2005), for example. The theory tells us that on a compact time set, the (nonnegative) price process of a risky asset is in a bubble, i.e. undergoing speculative pricing, if and only if the discounted price process (with respect to a fixed baseline security) is a strict local martingale under the risk neutral measure governing the situation. Therefore, one can model the formation of bubbles by observing when the price process changes from being a martingale to being a strict local martingale. This is discussed in detail in Jarrow *et al.* (2010), Biagini *et al.* (2014), and Protter (2005), for example.

The models we study are stochastic volatility models. We work with the setting examined in Lions & Musiela (2007), which provides sufficient conditions such that the solutions of such stochastic differential equations are strict local martingales. We assume always that a component of the stochastic volatility process is an Itô diffusion, so that we can use Feller's test for explosions in our quest to characterize the stochastic processes in question. This is similar to the techniques used in Biagini *et al.* (2014) and Sin (1998), but with the difference that we introduce a cause for bubbles (new information available to the market), and then show how this mathematically evolves into a bubble.

The expansion of filtration using initial expansion involves adding the information encoded in a random variable to the original σ-algebra at time zero. It then propagates throughout the filtration. This augmentation doesn't have to happen at time zero, however; it can happen at any finite valued stopping time τ. This is due to the fact that at τ we know what is happening, and thus we can think of an enlargement beginning at τ exactly analogously to one beginning at time $t = 0$, with simply the time τ playing the role of the time $t = 0$. From now on, however, we will deal with enlargements at time $t = 0$, for notational simplicity.

This type of enlargement of filtration from $\mathbb{F}$ to $\mathbb{G}$ changes the semi-martingale decomposition of the underlying price process, and therefore, leads to a change of a risk-neutral measure from P to an equivalent probability measure Q. Our stochastic process, which we will call S, which is assumed to be a martingale under $(P, \mathbb{F})$, under certain conditions can become a strict local martingale on a stochastic interval that depends on the choice of Q and the random variable that we add to $\mathbb{F}$. This random variable is denoted as L.

The case of initial expansions is particularly tractable, since Jacod *et al.* (1985) has developed the theory that provides us with the dynamics of the process under the enlarged filtration i.e. he provides us with the semi-martingale decomposition of the process in the enlarged filtration, which, under some circumstances, permits us to choose a risk neutral measure Q for the enlarged filtration that removes the enlargement created drift. Under that Q we can sometimes detect the presence of the strict local martingale property of the process, or lack thereof.

An outline of our paper is as follows. In Sec. 2, we connect our results of this paper to problems arising in the theory of Mathematical Finance, and in particular how financial bubbles may be led to form. Section 3 is the heart of the paper. Here we present the model of P. L. Lions and M. Musiela on stochastic volatility (in the style of what are known as Heston-type models), and we show how the addition of more information via an "expansion of the filtration" can lead what was originally a martingale to become a strict local martingale, under a risk neutral measure chosen from the infinite selection available in an incomplete market. Our main results are Theorems 1 and 3. In Sec. 4, we drop the hypothesis of continuous paths and extend our results to the case of discontinuous martingales replacing Brownian motions. Our main result in this section is Theorem 4.

2. Motivation: Connections to Mathematical Finance

The motivation for this work is to relate possible economic causes of financial bubbles to mathematical models of how they might arise naturally within the martingale-oriented absence-of-arbitrage framework. We use the economic cause of speculative pricing that comes from overexcitement of the market due to the disclosure of new information. Examples might be the announcement of a new medicine with major financial consequences (such as a "cure" for the common cold, to exaggerate a bit), a technological breakthrough [this is the thesis of Galbraith (1994)], a resolution of some sort of political instability, a weather event (such as an early frost for the Florida orange crop), etc. The obvious and intuitive manner to model such an event is by the addition of new observable events to the underlying filtration, and an established way to do that is via the theory of the "expansion of filtrations." This theory was developed in the 1980s, and a recent presentation can be found, for example, in Chapter VI of Protter (2005).

Indeed, when the new information forces a change to a new risk neutral measure, Q, the probabilities given by Q reflect any distortions in the market due to overexcitement resulting in speculative pricing. A current example was that the rumor that Amazon would locate a new headquarters in Queens, New York led to a rash of speculative housing activity in Long Island City and environs. This rumor was subsequently first seen to be correct, but then when faced with local resistance, Amazon pulled out and decided to locate their regional headquarters elsewhere, bursting the short lived housing bubble.

The theory of the expansion of filtrations and the martingale theory of an absence of arbitrage date back at least to the seminal work of Grorud & Pontier (1998). That they do not mesh well is detailed in papers of Imkeller (2002), Fontana *et al.* (2014) and the book of Aksamit & Jeanblanc (2017), which give other plentiful references. In particular, it often happens that the expansion introduces arbitrage opportunities. Therefore, one has to be careful both as to how one expands the filtration as well as to what one means by an absence of arbitrage. Here, we use the approach of an "initial expansion," although we interpret it as occurring at a random (stopping) time. We work in an incomplete market setting where there are an infinite number of risk neutral measures; in particular, we take a stochastic volatility framework. We show how the expansion of filtrations creates a drift even in a drift-free model (this is well known) and then we need to change the risk neutral measure to remove the drift created by the addition of new information. The insight is that under this new risk neutral measure with the new enlarged filtration, the price process changes from a martingale to a strict local martingale. This has financial significance. It has been shown over the last decade that on compact time sets, a discounted price process models a financial bubble if and only if it is a strict local martingale under the risk neutral measure; thus we have shown how a nonbubble price process can become a bubble price process after the arrival of new information (via an expansion of the filtration). Our ideas were inspired by the previous works of Sin (1998) and Biagini *et al.* (2014) who were interested in bubble formation, but did not relate it to the expansion of filtrations.

Finally, we remark that this is different from the modeling of insider information, another popular use of the expansion of filtrations; see, for example, Aksamit & Jeanblanc (2017). In our context, we are not creating a new filtration of the insider, but rather adding new information to the market as a whole. This new information necessitates a change to a new risk neutral measure, which in turn changes the prices of some contingent claims. These changes in prices of contingent claims have been explained as an enthusiasm of speculative fever [see, e.g. Galbraith (1994)], or as "irrational exuberance" (Greenspan 1996).

3. The Framework of Lions and Musiela

3.1. *Our first model*

Let us begin with the framework established in Lions & Musiela (2007) that treats the case of stochastic volatility. We will begin working on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. We assume that the stochastic process $S = (S_t)_{0\leq t\leq T}$, which we can think of as a discounted stock price, and the stochastic volatility satisfy SDEs of the following system of two equations:

$$\begin{aligned} dS_t &= S_t v_t dB_t; \quad S_0 = 1, \\ dv_t &= \mu(v_t) dW_t + b(v_t) dt; \quad v_0 = 1. \end{aligned} \tag{3.1}$$

Here, B and W are correlated Brownian motions, with correlation coefficient ρ. We assume that the filtration $\mathbb{F}$ is generated by the Brownian motions (B, W) with the usual addition of the null sets. Our time interval is assumed to be $[0, T]$. We will assume that μ and b are C^∞ functions on $[0, \infty)$ and that μ is Lipschitz continuous on $[0, \infty)$ such that

$$\begin{aligned}
&\mu(0) = 0, \\
&b(0) \geq 0, \\
&\mu(x) > 0 \quad \text{if } x > 0 \quad \text{and} \quad \mu(x) = x\tilde{\mu}(x), \\
&b(x) \leq C(1 + x) \quad \text{and} \quad b(x) = x\tilde{b}(x),
\end{aligned} \tag{3.2}$$

where $\tilde{\mu}$ and $\tilde{b}$ are continuous functions on $[0, \infty)$. Note that the assumptions that μ and b factor as $\mu(x) = x\tilde{\mu}(x)$ and $b(x) = x\tilde{b}(x)$ ensure a positive solution of the equation for v in (3.1).

We recall the conditions of Lions and Musiela, which allow us to determine whether the solution to (3.1) is a strict local martingale or an integrable, nonnegative martingale: If

$$\limsup_{x \to +\infty} \frac{\rho\, x\mu(x) + b(x)}{x} < \infty, \tag{3.3}$$

holds, then S is a nonnegative martingale.

For the same model, if the condition

$$\liminf_{x \to +\infty} (\rho\, x\mu(x) + b(x))\phi(x)^{-1} > 0, \tag{3.4}$$

holds, then S is not a martingale but a supermartingale and a strict local martingale. Here, ϕ is an increasing, positive, smooth function that satisfies the following condition:

$$\int_0^a \frac{1}{\phi(x)} dx < \infty, \tag{3.5}$$

where a is some positive constant.

Remark 1. Lions & Musiela (2007) assume that μ and b are both C^∞, but if one reads their proof, they do not use the force of that assumption assuming as we have done is enough for their proofs to work.

We would like to determine whether or not an enlargement of the filtration can give rise to a strict local martingale in the bigger filtration, when one begins with a true martingale in the smaller one. More specifically, we would like to answer the following question: beginning with a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and a price process S that is an $\mathbb{F}$ martingale, if we perform an initial expansion of $\mathbb{F}$, resulting in an enlarged filtration $\mathbb{G}$, can we obtain a $\mathbb{G}$ strict local martingale under an equivalent measure?

3.2. *The case of initial expansions*

We will consider the case of initial expansions, i.e. the expansion of the filtration $\mathbb{F}$ by adding a random variable $L \in \mathcal{F}$ to $\mathcal{F}_0$. The underlying filtration $\mathbb{F}$ is assumed to be the minimal filtration generated by the two Brownian motions B and W of Eq. (3.1) with the usual completions via the null sets. We assume that this random variable L takes values in $\mathbb{R}$. The new, enlarged filtration, which we will call $\mathbb{G}$ can be denoted as

$$\mathcal{G}_t = \bigcap_{\epsilon>0} (\mathcal{F}_{t+\epsilon} \vee \sigma(L)).$$

Let η be the distribution of L, and let $R_t(\omega, dx)$ be the regular conditional distribution of L, given $\mathcal{F}_t$. If we assume that the random variable L satisfies the following condition, which we will call Jacod's condition:

$$R_t(\omega, dx) \ll \eta(dx), \tag{3.6}$$

we can use the results of Jacod *et al.* (1985) on the initial expansion of filtrations, which gives us the existence of a family of $(P, \mathbb{F})$ martingales $\{q^x : x \in \mathbb{R}\}$ such that for η almost all x, $\forall t$, P almost surely

$$R_t(\omega, dx) = q_t^x \eta(dx). \tag{3.7}$$

For a locally square integrable $\mathbb{F}$ martingale M, Jacod proves the existence of an $\mathbb{F}$ predictable process $(k^{x,M})$ such that we have the relation

$$\langle q^x, M\rangle = (k^{x,M} q_-^x) \cdot \langle M, M\rangle.$$

The superscript M appearing in the process k is to indicate that the process k depends on the martingale M. Jacod's theorem [Theorem 2.5 in Jacod *et al.* (1985)] tells us next that the following process is a $\mathbb{G}$ local martingale:

$$\widetilde{M_t} = M_t - \int_0^t k_s^{L,M} d\langle M, M\rangle_s. \tag{3.8}$$

Note that we are able to substitute L for the parameter x of k in (3.8) above since this is justified by the classic measurability results of Stricker & Yor (1978) combined with a monotone class argument for the Stieltjes integral $d\langle M, M\rangle_s$ [see alternatively Protter (2005), Theorem 63 of Chap. IV].

By saying it is a $\mathbb{G}$ local martingale, we are not precluding that it is a martingale; we need to have extra conditions to conclude it is a strict local martingale.

With the price process in (3.1), in mind, let us illustrate this concept with an example, the case where the random variable L takes on only a finite number of values.

Let $A_1, A_2, \ldots, A_n$ be a sequence of events such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{\infty} A_i = \Omega, P(A_i) > 0$. The enlarged filtration, $\mathbb{G}$, is the filtration generated by $\mathbb{F}$ and the random variable $L = \sum_{i=1}^{\infty} a_i 1_{A_i}$.

In this case, we have

$$k_t^{L,S} q_t^L = \sum_{i=1}^{\infty} \frac{\xi_t^i}{P(A_i)} 1_{\{L=a_i\}},$$

and

$$q_t^L = \sum_{i=1}^{\infty} \frac{P(L = a_i \,|\, \mathcal{F}_t)}{P(A_i)} 1_{\{L=a_i\}}.$$

Here, ξ^i are processes arising from the Kunita–Watanabe inequality, which ensures absolute continuity of the paths. Let N^i be the $\mathbb{F}$-martingale $P(L = a_i \,|\, \mathcal{F}_t)$; we have that $d[N^i, S]_t = \xi_t^i d[S, S]_t$, where S solves (3.1).

We will henceforth work with initial expansions wherein Jacod's condition (3.6) is satisfied but we don't necessarily have a countable partition of the sample space.

Returning to the Lions–Musiela framework, we have that, under $(P, \mathbb{G})$, the price process and stochastic volatility satisfy

$$S_t = 1 + \int_0^t S_s v_s dB_s - \int_0^t k_s^{L,S} S_s^2 v_s^2 ds + \int_0^t k_s^{L,S} S_s^2 v_s^2 ds,$$

where

$$\int_0^t S_s v_s dB_s - \int_0^t k_s^{L,S} S_s^2 v_s^2 ds,$$

is a $(P, \mathbb{G})$ local martingale, and

$$\int_0^t k_s^{L,S} S_s^2 v_s^2 ds,$$

is a finite variation process. Define $\tilde{v}$ to be the $\mathbb{F}$ local martingale part of v. i.e.,

$$\tilde{v}_t = \int_0^t \mu(v_s) dW_s. \tag{3.9}$$

The stochastic volatility in turn satisfies

$$v_t = 1 + \int_0^t \mu(v_s) dW_s - \int_0^t k_s^{L,\tilde{v}} \mu^2(v_s) ds + \int_0^t k_s^{L,\tilde{v}} \mu^2(v_s) ds + \int_0^t b(v_s) ds. \tag{3.10}$$

Here,

$$\int_0^t \mu(v_s) dW_s - \int_0^t k_s^{L,\tilde{v}} \mu^2(v_s) ds,$$

is a $(P, \mathbb{G})$ local martingale, and

$$\int_0^t k_s^{L,\tilde{v}} \mu^2(v_s) ds + \int_0^t b(v_s) ds,$$

is a finite variation process.

Define the $(P,\mathbb{G})$ Brownian motions $\tilde{B}$ and $\tilde{W}$ by

$$\tilde{B}_t = B_t - \int_0^t k_s^{L,B} ds, \tag{3.11}$$

$$\tilde{W}_t = W_t - \int_0^t k_s^{L,W} ds. \tag{3.12}$$

Define the measure Q such that, for P integrable and càdlàg $\mathbb{G}$ predictable processes H and J,[a]

$$Z_t = E\left[\frac{dQ}{dP}\bigg|\mathcal{G}_t\right] = 1 + ZH\cdot\tilde{B}_t + ZJ\cdot\tilde{W}_t. \tag{3.13}$$

In the above, $\cdot$ represents stochastic integration.

We perform a Girsanov transform, to switch to the probability measure Q, which will be such that S is a local martingale. We need

$$k_t^{L,S}(S_t v_t)^2 + H_t + \rho J_t = 0. \tag{3.14}$$

Then, under $(Q,\mathbb{G})$, where the measure Q is equivalent to P, S possesses the following decomposition:

$$S_t = 1 + \int_0^t S_s v_s dB_s^*.$$

The volatility, in turn, has the following decomposition:

$$\begin{aligned} v_t &= 1 + \int_0^t \mu(v_s)dW_s - \int_0^t k_s^{L,\tilde{v}}\mu^2(v_s)ds - \int_0^t \frac{1}{Z_s}d[Z,\mu\cdot W]_s \\ &\quad + \int_0^t k_s^{L,\tilde{v}}\mu^2(v_s)ds + \int_0^t b(v_s)ds + \int_0^t \frac{1}{Z_s}d[Z,\mu\cdot W]_s ds \\ &= 1 + \int_0^t \mu(v_s)dW_s^* + \int_0^t k_s^{L,\tilde{v}}\mu^2(v_s)ds + \int_0^t b(v_s)ds + \int_0^t \frac{1}{Z_s}d[Z,\mu\cdot W]_s ds, \end{aligned} \tag{3.15}$$

where in the above, the $(Q,\mathbb{G})$ Brownian motions B^* and W^* are given by

$$B_t^* = \tilde{B}_t - \int_0^t (H_s + \rho J_s)ds, \tag{3.16}$$

and

$$W_t^* = \tilde{W} - \int_0^t (\rho H_s + J_s)ds. \tag{3.17}$$

[a]We do not need this fact, but we recall under Jacod's condition, we have the predictable representation for any $(P,\mathbb{G})$ martingale [see Fontana (2015)].

We have

$$v_t = 1 + \int_0^t \mu(v_s)dW_s^* + \int_0^t b(v_s)ds + \int_0^t k_s^{L,\tilde{v}}\mu^2(v_s)ds + \int_0^t (\rho\mu(v_s)H_s + \mu(v_s)J_s)ds. \quad (3.18)$$

Recall that under $(Q, \mathbb{G})$, we would like S to be a local martingale. This entails the finite variation part of the decomposition of S under $(Q, \mathbb{G})$ being zero. Given that there are two Brownian motions, there are infinitely many combinations of H and J that will work.

Remark 2. We have several obvious choices for H and J given in (3.14).

(1) We could take $H_t = -(S_t v_t)^2 k_t^{L,S}$ and $J_t \equiv 0$.
(2) We could take $H_t = -(S_t v_t)^2$ and $J_t = \frac{(S_t v_t)^2(1-k_t^{L,S})}{\rho}$.
(3) We could take $H_t \equiv 0$ and $J_t = \frac{(S_t v_t)^2 k_t^{L,S}}{-\rho}$.

Of course, we have an infinite number of other possibilities.

We are not as free as it would seem however in our choices for H and J because we need to have the new measure Q they produce via (3.13) to be a true probability measure that is equivalent to P. It is not *a priori* obvious that it will be one, and in fact, it is not one in general.

For any probability measure Q on $(\Omega, \mathbb{F})$, we will use the notation Q_t to denote the restriction of Q to the σ-field $\mathcal{G}_t$. We also let S^t denote the process S stopped at $t > 0$ i.e. $S_s^t = S_{s\wedge t}$. This leads us to define a local absence of arbitrage property:

Definition 1. We say there is *locally an absence of arbitrage* if $Q_t \sim P$ for every $t, 0 \le t < T$ implies that S^t is a Q local martingale, for $0 \le t < T$.

Remark 3. A classic result of Parthasarathy (1967) shows that a local absence of arbitrage on $[0, T)$ implies that there exists a probability measure Q such that each Q_t of Definition 1 is the restriction of Q to $\mathcal{G}_t$. The rub is that while we have $Q_t \sim P$ for each t, $0 \le t < T$, we do not in general have that $Q \sim P$. Indeed, Q can even be, in general, singular with respect to P. For Parthasarathy's result to be applicable here, we need to deal with *standard Borel spaces*, which we do not have in general on the canonical spaces of $\mathcal{C}(\mathbb{R}_+, E)$ and $\mathcal{D}(\mathbb{R}_+, E)$, where E is a Polish space. But, if we add a "cemetery" Δ to the spaces, then we are in a standard Borel framework. To be clear, we need to replace E with $E \cup \Delta$ to get a standard space. This, in general, does not create any problems.

This idea of a local absence of arbitrage is developed in Bilina Falafala (2014) to a small extent. We remark that while a local absence of arbitrage is sufficient for our purposes, the question of an absence of arbitrage when one initially expands a filtration has recently been an object of intensive study, and several deep results have been obtained: See Aksamit *et al.* (2015), Acciaio *et al.* (2016) and especially Fontana (2018), and the recent book of Aksamit & Jeanblanc (2017).

In particular, in Sec. 4.2 of the book of Aksamit & Jeanblanc (2017), one has an example where if one is to work on a closed interval $[0, T]$ with an initial expansion, in general, there can arise arbitrage opportunities, almost trivially. The idea of local arbitrage as presented here allows us to work on the semi-open interval $[0, T)$ and thus to finesse the issue of arbitrage occurring.

Finally, we note that Example 1 in Sec. 4 provides a simple example of a probability measure Q where this entire theory plays out.

Remark 4. The concept of a local absence of arbitrage implies the already well-established concept of No Unbounded Profits with Bounded Risk, with acronym NUPBR. This is equivalent to yet another condition, known as NA1. In the article of Ruf & Runggaldier (2014), the authors give several examples of how NUPBR can arise in "natural" situations. Our treatment of new information entering the market via a filtration expansion provides yet another example. See also Acciaio *et al.* (2016), Aksamit (2014), Aksamit *et al.* (2015) and Jarrow *et al.* (2010).

Recall that $k_0^L \in \mathcal{G}_0 = \sigma(L)$. We will make the following assumptions on the processes k, H and J:

Hypothesis 1.

$$\begin{aligned}
&\text{There exists } \varepsilon^{(1)} > 0 \text{ such that } Q(k_0^{L,\tilde{v}} > \varepsilon^{(1)}) > 0\\
&\text{We assume that } k^{L,S}, H, J, \text{ and } k^{L,\tilde{v}} \text{ all have right continuous paths}\\
&\text{There exists } \varepsilon^{(2)} > 0 \text{ such that } Q(|H_0 + \rho J_0| < \varepsilon^{(2)}) > 0\\
&Q \text{ is a true probability measure that has a local absence of arbitrage.}
\end{aligned} \tag{3.19}$$

We note that in Sec. 4.1, we give examples and also a framework where the important process $k^{L,S}$ has right continuous paths, a.s., which show that Hypothesis (3.19) is not unreasonable. Under the enlarged filtration, the drift of v, which we will call $\hat{b}$, satisfies

$$\hat{b}_t = b(v_t) + k_t^{L,\tilde{v}} \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t). \tag{3.20}$$

Notice that we can no longer represent the drift in deterministic terms as simply functions of the real variable x, so we cannot immediately invoke the results of Lions & Musiela. To address this, let us take $\varepsilon^{(1)}, \varepsilon^{(2)} > 0$ such that $Q(\varepsilon^{(1)} < k_0^{L,\tilde{v}}) > 0$ and $Q(|\rho H_0 + J_0| < \varepsilon^{(2)}) > 0$, and define the following $\mathbb{F}$ stopping times:

$$\begin{aligned}
\tau^k &= \inf\{t : k_t^{L,\tilde{v}} < \varepsilon^{(1)}\},\\
\tau^{H,J} &= \inf\{t : |\rho H_t + J_t| > \varepsilon^{(2)}\}.
\end{aligned}$$

Now, we define the stopping time τ to be

$$\tau = (\tau^k \wedge \tau^{H,J}). \tag{3.21}$$

By Hypothesis 1, we have that $Q(\tau > 0) > 0$.

On the stochastic interval $[0, \tau)$, we have the following lower bound on our drift coefficients:

$$\hat{b}_t = b(v_t) + k_t^{L,\tilde{v}} \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t) \geq b(v_t) + \varepsilon^{(1)} \mu^2(v_t) - \varepsilon^{(2)} \mu(v_t). \quad (3.22)$$

Before we state the next result, we state and prove a technical lemma.

Lemma 1. *Let B be a standard Brownian motion, and let r be any continuous, adapted, finite valued process such that $\int_0^t r_s^2 ds < \infty$ a.s. for each $t > 0$. Suppose σ is continuous and is such that S exists and is the unique solution of*

$$S_t = 1 + \int_0^t S_s \sigma(S_s) r_s dB_s.$$

Then, S is strictly positive for all $t \geq 0$ a.s.

Proof. We let $V = \inf\{t > 0 : S_t = 0\}$. It suffices to show that $P(V < \infty) = 0$.

Define stopping times $R_n = \inf\{t > 0 : S_t = 1/n \text{ or } S_t = n\}$. Note that $P(R_n > 0) = 1$ for $n \geq 2$ because $S_0 = 1$ a.s. and S is continuous. Use Itô's formula up to time R_n to get

$$\ln(S_{R_n}) = \ln(S_0) + \int_0^{R_n} \sigma(S_s) r_s dB_s - \frac{1}{2} \int_0^{R_n} \sigma^2(S_s) r_s^2 ds. \quad (3.23)$$

The stopping times R_n increase to V as n tends to ∞, so the left side of (3.23) tends to ∞ a.s. on the event $\{V < \infty\}$. On the event $\{V < \infty\}$ for a given sample path ω, we have that S takes values in a compact set, whence $t \mapsto \sigma(S_t(\omega))$ is bounded, for each ω. Using that the quadratic variation of the Brownian integral term is therefore finite a.s., we have that the right side however remains finite ($\sigma(x)$ is assumed continuous, and is therefore bounded on compact sets) on $\{V < \infty\}$, and the only way this can happen is if $P(V < \infty) = 0$. □

Remark 5. The work of Engelbert & Schmidt (1989, 1991) gives necessary and sufficient conditions for a solution to exist that is unique in law, at least when v is not present. Lemma 1 remains true under more general hypotheses, with the obvious modifications of the proof. As such it is a slight extension of Theorem 71 of Chap. V of Protter (2005).

We need one preliminary result: That of the definition of a continuous local martingale on a half-open stochastic integral. This is a topic considered decades ago, but perhaps it is worthwhile to recall it. We use the definition of Maisonneuve (Maisonneuve 1977).

Definition 2. A stochastic process M defined on $[0, \tau)$ is called a continuous local martingale on $[0, \tau)$ if there exist stopping times T_n increasing to τ such that for each n there exists a continuous martingale $(M_t^n)_{t \geq 0}$ such that $M_t = M_t^n$ on $\{t < T_n\}$.

Maisonneuve's definition was extended by Sharpe (1992) without the assumption of path continuity, but we will not have need of such generality here.

We remark that our definition of a local martingale on an interval of the form $[0, \tau)$ is compatible with the theory as expounded by Maisonneuve, itself a small correction of earlier work of Getoor and Sharpe, as well as of Kunita.

The above discussion gives us the following result.

Theorem 1. *Assume that μ is Lipschitz and $\mathcal{C}^1$, and that b is $\mathcal{C}^1$. Assume also that the conditions in Hypothesis* 1 *hold, as well as the following conditions*:

$$\limsup_{x\to+\infty} \quad \frac{\rho\ x\mu(x) + b(x)}{x} < \infty,$$

$$\liminf_{x\to+\infty}(\rho\ x\mu(x) + b(x) + \varepsilon^{(1)}\mu^2(x) - \varepsilon^{(2)}\mu(x))\phi(x)^{-1} > 0.$$

Assume also that B and W are correlated Brownian motions with correlation ρ. Let the process S be the unique strong solution of the SDE

$$dS_t = S_t v_t dB_t, \tag{3.24}$$

$$dv_t = \mu(v_t)dW_t + b(v_t)dt, \tag{3.25}$$

on $(P, \mathbb{F})$. The solution S is also the solution of

$$dS_t = S_t v_t dB_t^*,$$

$$dv_t = \mu(v_t)dW_t^* + b(v_t)dt + k_t^{L,\tilde{v}}\mu^2(v_t)dt + (\rho H_t + J_t)\mu(v_t)dt,$$

on $(Q, \mathbb{G})$, where B^* and W^* are $(Q, \mathbb{G})$ Brownian motions. Then, S is a positive $(P, \mathbb{F})$ martingale and a positive $(Q, \mathbb{G})$ strict local martingale on the stochastic interval $[0, \tau)$, where τ is given in (3.21). *More specifically, we have $E_Q[S_t^\tau] < S_0$.*

In the above, $\phi(x)$ is an increasing, positive, smooth function that satisfies the condition in (3.5).

Before we continue, let us recall a result [proved for example in Protter (2005)] that allows us to compare the values of solutions of stochastic differential equations. It is well known, but we include it here for the reader's convenience. Let us denote by $\mathcal{D}^n$ the set of $\mathbb{R}^n$ — valued càdlàg processes. We write $\mathcal{D}$ for $\mathcal{D}^1$. An operator $\mathbf{F}$ from $\mathcal{D}^n$ to $\mathcal{D}$ is said to be *process Lipschitz* if for all X, $Y \in \mathcal{D}^n$ and for all stopping times T

(1) $X_-^T = Y_-^T \Rightarrow F(X)_-^T = F(Y)_-^T$.
(2) There exists an adapted process K such that $\|F(X)_t - F(Y)_t\| \leq K_t\|X_t - Y_t\|$, where $\|\cdot\|$ denotes the sup norm.

In the above, X_-^T denotes the left limit of the process X stopped at T.

Theorem 2 [Comparison Theorem, p. 324, Protter (2005)]. *Let Z be a continuous semi-martingale, let F be process Lipschitz, and let A be adapted, increasing, and continuous. Assume that G and H are process Lipschitz functionals such that*

$G(X)_{t_-} > H(X)_{t_-}$ for all $t > 0$ and all semi-martingales X. Let $x_0 \geq y_0$, and X and Y be the unique solutions of

$$X_t = x_0 + \int_0^t G(X)_{s_-} dA_s + \int_0^t F(X)_{s-} dZ_s,$$

$$Y_t = y_0 + \int_0^t H(Y)_{s_-} dA_s + \int_0^t F(Y)_{s-} dZ_s.$$

Then, $P\{\exists t \geq 0 : X_t \leq Y_t\} = 0.$

Now, we may begin the proof of Theorem 1.

Before we prove Theorem 1, we state a lemma that we will use several times. In particular, recall the system of two equations given in (3.23). Recall that these two equations are satisfied under $(Q, \mathbb{G})$ for the process ν given in (3.18).

Lemma 2. *Let τ be the stopping time defined in* (3.21)*. Then, $E_Q(S_t^\tau) < S_0$, where S is as given in* (3.24)*.*

Proof. Note that S is a nonnegative supermartingale, so its expectation is non-increasing with time. Define the sequence of stopping times T_n by $\inf\{t : v_t \geq n\}$. We have that the stopped process $S_{t\wedge\tau\wedge T_n (0\leq t\leq T)}$ is a $(Q, \mathbb{G})$ martingale. The stopping time T_∞ is the explosion time of v. Therefore, we may write

$$S_0 = E_Q[S_{t\wedge\tau\wedge T_n}] = E_Q[S_{t\wedge\tau} 1_{\{t\wedge\tau\leq T_n\}}] + E_Q[S_{T_n} 1_{\{T_n<t\wedge\tau\}}]. \tag{3.26}$$

Since $E_Q[S_{t\wedge\tau} 1_{\{t\wedge\tau\leq T_n\}}]$ converges to $E_Q[S_{t\wedge\tau}]$, we would have that $E_Q[S_{t\wedge\tau}] < S_0$ for all t if we can show that $\liminf_{n\to+\infty} E_Q[S_{T_n} 1_{\{T_n\leq t\wedge\tau\}}] > 0$.

We have: $E_Q[S_{T_n} 1_{\{T_n<t\wedge\tau\}}] = \hat{P}(T_n < t \wedge \tau)$ where under the measure $\hat{P}$, v solves

$$dv_t = \mu(v_t) d\bar{W}_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^{L,\tilde{v}} \mu^2(v_t)dt + (\rho H_t + J_t)\mu(v_t)dt.$$

In the above, $\bar{W}$ is a $\hat{P}$-Brownian motion.

Now, the condition

$$\liminf_{x\to+\infty} (\rho\, x\mu(x) + b(x) + \varepsilon^{(1)}\mu^2(x) - \varepsilon^{(2)}\mu(x))\phi(x)^{-1} > 0,$$

is sufficient to guarantee that the explosion time of the stochastic differential equation

$$dv_t = \mu(v_t) d\bar{W}_t + b(v_t)dt + \rho v_t\mu(v_t)dt + \varepsilon^{(1)}\mu^2(v_t)dt - \varepsilon^{(2)}\mu(v_t)dt,$$

can be made as small as we wish. By this, we mean that, for any stopping time R,

$$\hat{P}(T_\infty < R) > 0. \tag{3.27}$$

That this holds for a stopping time and not just a deterministic time follows from a simple modification of the argument on page 4 of Lions & Musiela (2007).

It is easy to see that the comparison theorem stated above implies that the solution to the SDE

$$dv_t = \mu(v_t)d\bar{W}_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^{L,\tilde{v}}\mu^2(v_t)dt + (\rho H_t + J_t)\mu(v_t)dt,$$

is $\hat{P}$ almost surely greater than that of the SDE

$$dv_t = \mu(v_t)d\bar{W}_t + b(v_t)dt + \varepsilon^{(1)}\mu^2(v_t)dt - \varepsilon^{(2)}\mu(v_t)dt,$$

for all $t \in [0, \tau]$. Thus, since the explosion time of v in the SDE

$$dv_t = \mu(v_t)d\bar{W}_t + b(v_t)dt + \varepsilon^{(1)}\mu^2(v_t)dt - \varepsilon^{(2)}\mu(v_t)dt,$$

can be made as small as possible, the explosion time T_∞ of v in the SDE

$$dv_t = \mu(v_t)d\bar{W}_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^{L,\tilde{v}}\mu^2(v_t)dt + (\rho H_t + J_t)\mu(v_t)dt,$$

can be made as small as possible as well.

This means that, for all t, we have $\hat{P}(T_\infty < t \wedge \tau) > 0$. This implies that, for all t, we have

$$E_Q[S_t^\tau] < S_0. \qquad \square$$

Proof of Theorem 1. That S is positive follows from Lemma 1. Notice that the condition

$$\limsup_{x \to +\infty} \frac{\rho\, x\mu(x) + b(x)}{x} < \infty, \tag{3.28}$$

is sufficient to show that the solution S to the SDE

$$\begin{aligned} dS_t &= S_t v_t dB_t, \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt, \end{aligned} \tag{3.29}$$

is a true martingale. This is the very SDE satisfied by (S, v) under $(P, \mathbb{F})$, and so the first claim in the theorem is proved.

It now follows from Lemma 2 that $E_Q[S_t^\tau] < S_0$, implying that S is a $(Q, \mathbb{G})$ local martingale that is not a martingale, and hence a strict local martingale. $\square$

Remark 6. If $\rho > 0$, it can be checked that the functions $\mu(x) = x$ and $b(x) = x - \rho x^2$ satisfy the criteria

$$\limsup_{x \to +\infty} \frac{\rho\, x\mu(x) + b(x)}{x} < \infty,$$

$$\liminf_{x \to +\infty} (\rho\, x\mu(x) + b(x) + \varepsilon^{(1)}\mu^2(x) - \varepsilon^{(2)}\mu(x)\phi(x)^{-1} > 0.$$

In fact, for $k \geq 1$, the functions $\mu(x) = x^k$ and $b(x) = x - \rho x^{k+1}$ work as well, *if ρ is positive*. The reason we need ρ to be positive here is that we need the following condition on the drift, in order for it to have a nonexploding, positive solution:

$$b(0) \geq 0, \tag{3.30}$$

$$b(x) \leq C(1 + x). \tag{3.31}$$

for some $C \geq 0$.

Thus, if we work with an SDE with such diffusion and drift coefficients, we begin with a true martingale and end up with a strict local martingale due to initial expansions.

Indeed, one can check using Feller's test for explosions [see, for example, Karatzas & Shreve (1991)] that the SDE

$$dv_t = v_t^k dW_t + (v_t - \rho v_t^{k+1})dt,$$

does not explode (in other words, that the time of explosion is infinite, almost surely): If we assume our state space for v to be $(0, +\infty)$, we need only to show that the scale function

$$p(x) = \int_c^x e^{\{-2\int_c^\psi \frac{b(y)}{\mu^2(y)}dy\}} d\psi,$$

satisfies the following:

$$p(0) = -\infty,$$

$$p(\infty) = \infty.$$

Taking $\mu(x) = x^k$, and $b(x) = x - \rho x^{k+1}$, a quick computation shows that indeed

$$p(0) = -\infty,$$

and

$$p(\infty) = \infty.$$

Before we continue, we must ensure that the subprobability measure Q defined above is a *true* probability measure. Let us begin by defining the sequence of probability measures Q_m by

$$dQ_m = Z_{T\wedge T_m} dP,$$

where $T_m = \inf\{t : \int_0^t (H_s^2 + J_s^2 + 2\rho J_s H_s)ds \geq m\}$. We then have

$$E[e^{\frac{1}{2}\int_0^{t\wedge T_m}(H_s^2+J_s^2+2\rho J_s H_s)ds}] \leq e^{\frac{1}{2}m} < \infty.$$

Recall that the relation $k_t^{L,S}(S_t v_t)^2 + H_t + \rho J_t = 0$ holds true for all $t \geq 0$.

So, we have $Q_m \ll P$ on $[0, T_m]$ for each m, as well as that the Q_m are true probability measures, since Z^{T_m} is a true $\mathbb{F}$ martingale.

3.3. *A slightly more general model*

We next perform a similar analysis for the following case:

$$dS_t = S_t^\beta v_t^\delta dB_t, \tag{3.32}$$

$$dv_t = \alpha v_t^\gamma dW_t + b(v_t)dt. \tag{3.33}$$

Here, we make the following assumptions and restrictions on the parameters and functions: α, γ, β, and δ are all positive, $b(0) \geq 0$, b is Lipschitz on $[0, \infty)$ and satisfies, for all x

$$b(x) \leq C(1+x).$$

It is shown in Lions & Musiela (2007) that if $\beta < 1$, the process S is a true martingale possessing moments of all orders. If $\beta > 1$ or $\gamma > 1$, Feller's test for explosion can be used to show that the stochastic differential equation (3.32) or (3.33) becomes explosive, respectively [see Karatzas & Shreve (1991) p. 348]. The interesting case is thus when $\beta = 1$, with $\gamma \in (0,1]$. As in the previous section, we work with a new probability measure and enlarged filtration $(Q, \mathbb{G})$. The measure Q is defined by $E[\frac{dQ}{dP}]\mathcal{G}_t] = Z_t$. We choose Q such that it is a local martingale measure for S. Let Z take the form

$$Z_t = 1 + ZH \cdot \tilde{B}_t + Z_- J \cdot \tilde{W}_t, \tag{3.34}$$

for $(P, \mathbb{G})$ Brownian motions $\tilde{B}$ and $\tilde{W}$. Recalling that $d[B,W]_t = \rho dt$, we arrive at the $(Q, \mathbb{G})$ decomposition for the price process and the volatility after doing a calculation very similar to that done for the previous model

$$\begin{aligned} S_t &= 1 + \int_0^t S_s^\beta v_s^\delta dB_t^*, \\ v_t &= 1 + \int_0^t \alpha v_s^\gamma dW_s^* + \int_0^t b(v_s)ds + \int_0^t \alpha^2 k_s^{L,\tilde{v}} v_s^{2\gamma} ds \\ &\quad + \int_0^t (\alpha v_s^\gamma H_s \rho + \alpha J_s v_s^\gamma) ds, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} B_t^* &= B_t - \int_0^t k_s^{L,S} S_s^\beta v_s^\delta ds - \int_0^t (H_s + \rho J_s) ds, \\ W_t^* &= W_t - \int_0^t \alpha k_s^{L,\tilde{v}} v_s^\delta ds - \int_0^t (\rho H_s + J_s) ds, \end{aligned}$$

are $(Q, \mathbb{G})$ Brownian motions.

Under the enlarged filtration, the drift of v, call it $\hat{b}$ satisfies

$$\hat{b}_t = b(v_t) + \alpha^2 k_t^{L,\tilde{v}} v_t^{2\gamma} + \alpha v_t^\gamma (H_t \rho + J_t).$$

We will make the following assumptions on the processes k, H and J:

Hypothesis 2.

We assume that $k^{L,S}, H$ and J have right continuous paths $a.s.$

There exists $\varepsilon^{(1)} > 0$ such that $Q(\varepsilon^{(1)} < \alpha^2 k_0^{L,\tilde{v}}) > 0$

There exists $\varepsilon^{(2)} > 0$ such that $Q(|\alpha\rho H_0 + \alpha J_0|) < \varepsilon^{(2)}) > 0$

Q is a true probability measure that has a local absence of arbitrage.

Again, we can no longer represent the drift in deterministic terms as simply functions of the real variable x, so we cannot immediately invoke the results of Lions & Musiela. Keeping in mind the assumptions mentioned above, in 2, take $\varepsilon^{(1)}, \varepsilon^{(2)} > 0$ such that $Q(\varepsilon^{(1)} < \alpha^2 k_0^{L,\tilde{v}}) > 0$ and $Q(|\alpha\rho H_0 + \alpha J_0|) < \varepsilon^{(2)}) > 0$, and define the following random times:

Define the random times

$$\begin{aligned} \tau^k &= \inf\{t : \alpha^2 k_t^{L,\tilde{v}} < \varepsilon^1\}, \\ \tau^{J,H} &= \inf\{t : |\alpha H_t \rho + \alpha J_t| > \varepsilon^2\}. \end{aligned}$$

Define the stopping time τ to be

$$\tau = (\tau^k \wedge \tau^{J,H}). \tag{3.36}$$

Proceeding, we have, on the stochastic interval $[0, \tau)$, the following lower bound on our drift:

$$\hat{b}(v_t) \geq b(v_t) + \varepsilon^{(1)} v_t^{2\gamma} - \varepsilon^{(2)} v_t^{\gamma}.$$

Let us recall the conditions of Lions and Musiela on the coefficients and parameters of this system of stochastic differential equations

$$\begin{aligned} dS_t &= S_t^{\beta} v_t^{\delta} dB_t, \\ dv_t &= \alpha v_t^{\gamma} dW_t + b(v_t) dt, \end{aligned} \tag{3.37}$$

such that S is a martingale: $\rho > 0$, $\gamma + \delta > 1$ and

$$\limsup_{x \to +\infty} \frac{\rho\alpha x^{\gamma+\delta} + b(x)}{x} < \infty. \tag{3.38}$$

Let us also recall the conditions on the coefficients and parameters of this system such that the process S is a *strict* local martingale $\rho > 0$, $\gamma + \delta > 1$ and there exists $\phi(x)$, an increasing, positive, smooth function that satisfies the conditions in (3.5), and

$$\liminf_{x \to +\infty} \frac{\rho\alpha x^{\gamma+\delta} + b(x)}{\phi(x)} > 0.$$

Our discussion has given rise to the following theorem:

Theorem 3. *Assume that $\beta = 1$, as well as the conditions in Hypothesis 2. Assume also that the following conditions are satisfied:*

$$\limsup_{x \to +\infty} \frac{\rho\alpha x^{\gamma+\delta} + b(x)}{x} < \infty,$$

and

$$\liminf_{x \to +\infty} \frac{\rho\alpha x^{\gamma+\delta} + b(x) + \varepsilon^{(1)} x^{2\gamma} - \varepsilon^{(2)} x^{\gamma}}{\phi(x)} > 0.$$

Let W and B be the correlated Brownian motions with correlation ρ. Assume that $\rho > 0$ and that $\gamma + \delta > 1$. Let the process S be the unique strong solution of the SDE

$$\begin{aligned} dS_t &= S_t^{\beta} v_t^{\delta} dB_t, \\ dv_t &= \alpha v_t^{\gamma} dW_t + b(v_t)dt, \end{aligned} \tag{3.39}$$

on $(P, \mathbb{F})$. The solution S is also the solution of

$$\begin{aligned} dS_t &= S_t^{\beta} v_t^{\delta} dB_t^*, \\ dv_t &= \alpha v_t^{\gamma} dW_t^* + b(v_t)dt + \alpha^2 k_t^{L,\tilde{v}} v_t^{2\gamma} dt + (\alpha v_t^{\gamma} H_t \rho + \alpha J_t v_t^{\gamma})dt, \end{aligned} \tag{3.40}$$

on $(Q, \mathbb{G})$.

Then, S is positive, is a $(P, \mathbb{F})$ martingale and a $(Q, \mathbb{G})$ strict local martingale on the stochastic interval $[0, \tau)$, where τ is given in (3.36). More specifically, we have $E_Q[S_t^{\tau}] < S_0$. In the above, ϕ is an increasing, positive, smooth function that satisfies the conditions in (3.5).

Proof. That S is positive follows from Lemma 1. The condition

$$\limsup_{x \to +\infty} \frac{\rho \alpha x^{\gamma+\delta} + b(x)}{x} < \infty,$$

is sufficient to ensure that the solution to Eq. (3.39) is a true martingale. Now, let us consider Eqs. (3.40)–(3.41), solved by (S, v) under $(Q, \mathbb{G})$. Defining the sequence of stopping times $T_n = \inf\{t : v_t \geq n\}$, we have that the stopped process $S_{t \wedge \tau \wedge T_n (0 \leq t \leq T)}$ is a $(Q, \mathbb{G})$ martingale. The stopping time T_∞ is the explosion time of v. Therefore, we may write

$$S_0 = E_Q[S_{t \wedge \tau \wedge T_n}] = E_Q[S_{t \wedge \tau} 1_{\{t \wedge \tau \leq T_n\}}] + E_Q[S_{T_n} 1_{\{T_n < t \wedge \tau\}}].$$

Since $E_Q[S_{t \wedge \tau} 1_{\{t \wedge \tau \leq T_n\}}]$ increases to $E_Q[S_{t \wedge \tau}]$ as $n \to \infty$, we would have that $E_Q[S_{t \wedge \tau}] < S_0$ for all t if we can show that $\liminf_{n \to +\infty} E_Q[S_{T_n} 1_{\{T_n < t \wedge \tau\}}] > 0$.

By a slight modification of Lemma 2, we get $E_Q(S_t^{\tau}) < S_0$, where τ is as defined in (3.36). □

Remark 7. If we assume that there exists an $\epsilon > 0$ such that $\gamma \geq \frac{1+\epsilon}{2}$, we can use $\phi(x) = x^{1+\epsilon}$, and one can easily check that the following forms of $b(x)$ satisfy:

$$\limsup_{x \to +\infty} \frac{\rho \alpha x^{\gamma+\delta} + b(x)}{x} < \infty,$$

and

$$\liminf_{x \to +\infty} \frac{\rho \alpha x^{\gamma+\delta} + b(x) + \varepsilon^{(1)} x^{2\gamma} - \varepsilon^{(2)} x^{\gamma}}{\phi(x)} > 0,$$

$$b(x) = K \ln(x) - \rho \alpha x^{\gamma+\delta},$$

$$b(x) = K\sin(x) - \rho\alpha x^{\gamma+\delta},$$
$$b(x) = Ke^{-ax} - \rho\alpha x^{\gamma+\delta},$$
$$b(x) = Kx^{m} - \rho\alpha x^{\gamma+\delta}.$$

In the above, K and a are positive constants, and m is a constant satisfying $m \leq 1$.

Before we continue, we must ensure that the subprobability measure Q defined above is a *true* probability measure. Let us begin by defining the sequence of probability measures Q_m by

$$dQ_m = Z_{T\wedge T_m} dP, \tag{3.41}$$

where $T_m = \inf\{t : \int_0^t (H_s^2 + J_s^2 + 2\rho J_s H_s) ds \geq m\}$. We then have

$$E[e^{\frac{1}{2}\int_0^{t\wedge T_m}(H_s^2+J_s^2+2\rho J_s H_s)ds}] \leq e^{\frac{1}{2}m} < \infty.$$

In this case, H and J must satisfy

$$k_t^{L,S} S_t^2 v_t^{2\delta} + H_t S_t v_t^{2\delta} + \rho J_t S_t v^{\delta} = 0,$$

for all $t \geq 0$ since we have assumed Q to be a local martingale measure for S.

So, we have $Q_m \ll P$ on $[0, T_m]$ for each m, as well as that the Q_m are true probability measures, since $Z_t^{T_m}$ is a true $\mathbb{G}$ martingale.

4. The Discontinuous Case

Let us now turn to the discontinuous case, i.e. we assume that S and v follow SDEs of the form:

$$\begin{aligned} dS_t &= S_{t-} v_t^{\alpha} dM_t, \\ dv_t &= \mu(v_t) dB_t + b(v_t) dt. \end{aligned} \tag{4.1}$$

We will assume that μ and b are C^∞ functions on $[0, \infty)$ and that μ is Lipschitz continuous on $[0, \infty)$ such that

$$\mu(0) = 0,$$
$$b(0) \geq 0,$$
$$\mu(x) > 0 \quad \text{if } x > 0 \text{ and } \mu(x) = x\tilde{\mu}(x),$$
$$b(x) \leq C(1+x) \text{ and } b(x) = x\tilde{b}(x),$$

where $\tilde{\mu}$ and $\tilde{b}$ are continuous functions on $[0, \infty)$. Note that the assumptions that μ and b factor as $\mu(x) = x\tilde{\mu}(x)$ and $b(x) = x\tilde{b}(x)$ ensure a positive solution of the equation for v in (4.1).

We assume α to be positive. In the above, B is a standard Brownian motion and $M = M^c + M^d$ is a discontinuous martingale such that $\langle M, M\rangle$ is locally in L^1

and such that $d\langle M,M\rangle_t = \lambda_t dt$. M^c and M^d are the continuous and discontinuous parts of M, respectively. Additionally, we will assume that $[M^c,B]_t = \rho t$.

Let us note that the conditions imposed on the coefficients b and μ of the volatility are sufficient to ensure the existence and uniqueness of a nonnegative solution v such that $E[\sup_{t\in[0,T]}|v_t^p|] < \infty$ for $1 \le p < \infty$. Last, we assume that the processes v and M satisfy

$$\Delta\left(\int_0^t v_s^\alpha dM_s\right) > -1, \tag{4.2}$$

i.e. for all t, ${v_t}^\alpha \Delta M_t > -1$. (We are using the standard notation that for a càdlàg process X that $\Delta X_t = X_t - X_{t-}$, the jump of X at time t.) The above condition (4.2) ensures that S remains positive for all $t \ge 0$.

Let us proceed to expand the filtration $\mathbb{F}$ to obtain $\mathbb{G}$ by an initial expansion, and compute the canonical expansion of S under $(P,\mathbb{G})$. We obtain the canonical decomposition of the process S under $\mathbb{G}$ via the theory of Jacod, in Jacod *et al.* (1985). [The reader can consult Chap. VI of Protter (2005) for a pedagogic treatment of the subject.] Jacod proves the existence of an $\mathbb{F}$ predictable process $k^{L,x}$ such that

$$\langle q^x, S\rangle = k^{x,S} q_-^x \cdot \langle S,S\rangle. \tag{4.3}$$

Jacod's theorem also tells us that the following process is a $(P,\mathbb{G})$ local martingale:

$$\widetilde{S}_t = S_t - \int_0^t k_t^{L,S} d\langle S,S\rangle_t. \tag{4.4}$$

We obtain, under $(P,\mathbb{G})$

$$\begin{aligned} S_t &= S_0 + \int_0^t S_{s-} v_s^\alpha dM_s - \int_0^t k_s^{L,S} S_s^2 v_s^{2\alpha} \lambda_s ds + \int_0^t k_s^{L,S} S_s^2 v_s^{2\alpha}\lambda_s ds,\\ v_t &= v_0 + \int_0^t \mu(v_s) dB_s - \int_0^t k_s^{L,\tilde{v}} \mu^2(v_s) ds + \int_0^t b(v_s) ds + \int_0^t k_s^{L,\tilde{v}}\mu^2(v_s) ds. \end{aligned} \tag{4.5}$$

Here, $\int_0^t S_{s-} v_s^\alpha dM_s - \int_0^t k_s^{L,S} S_s^2 v_s^{2\alpha}\lambda_s ds$ and $\int_0^t \mu(v_s) dB_s - \int_0^t k_s^{L,\tilde{v}}\mu^2(v_s)ds$ are $(P,\mathbb{G})$ local martingales, and

$$\int_0^t k_s^{L,S} S_s^2 v_s^{2\alpha}\lambda_s ds \quad \text{and} \quad \int_0^t b(v_s)ds + \int_0^t k_s^{L,\tilde{v}}\mu^2(v_s)ds, \tag{4.6}$$

are finite variation processes.

We perform a Girsanov transform, to switch to a probability measure Q which is equivalent to P, under which S is a local martingale. We can do this as long as we assume condition (4.12), given in Theorem 4. As in the previous cases, let $Z_t = E[\frac{dQ}{dP}\,|\,\mathcal{G}_t]$. Let Z take the form

$$Z_t = 1 + ZH\cdot \tilde{B}_t + Z_- J \cdot \tilde{M}_t,$$

for càdlàg $\mathbb{G}$ predictable processes J and H that maintain Z strictly positive and for $(P,\mathbb{G})$ local martingale $\tilde{M}_t = M_t - \int_0^t k_s^{L,M}\lambda_s ds$ and $(P,\mathbb{G})$ Brownian motion

$\tilde{B}_t = B_t - \int_0^t k_s^{L,B} ds$. We have the following decompositions for S and v under $(Q, \mathbb{G})$:

$$\begin{aligned} S_t = S_0 &+ \int_0^t S_{s-} v_s^\alpha dM_s - \int_0^t k_s^{L,S} S_s^2 v_s^{2\alpha} \lambda_s ds - \int_0^t (\rho H_s + \lambda_s J_s) S_s v_s^\alpha ds \\ &+ \int_0^t k_s^{L,S} S_s^2 v_s^{2\alpha} \lambda_s ds + \int_0^t (\rho H_s + \lambda_s J_s) S_s v_s^\alpha ds. \end{aligned} \tag{4.7}$$

$$\begin{aligned} v_t = v_0 &+ \int_0^t \mu(v_s) dB_s - \int_0^t k_s^{L,\tilde{v}} \mu^2(v_s) ds - \int_0^t (H_s + \rho J_s)\mu(v_s) ds \\ &+ \int_0^t b(v_s) ds + \int_0^t (H_s + \rho J_s)\mu(v_s) ds + \int_0^t k_s^{L,\tilde{v}} \mu^2(v_s) ds. \end{aligned} \tag{4.8}$$

Since we have assumed that under $(Q, \mathbb{G})$, S is a local martingale, we set the finite variation term in its decomposition to zero:

$$\lambda_t k_t^{L,S} S_t^2 v_t^{2\alpha} + (\rho H_t + \lambda_t J_t) S_t v_t^\alpha = 0. \tag{4.9}$$

Returning to the decomposition of the volatility we just arrived at, we again note that we can no longer represent the drift in deterministic terms as simply functions of the real variable x, so we cannot immediately invoke the results of Lions & Musiela. To address this, let us take $\varepsilon^{(1)}, \varepsilon^{(2)} > 0$ such that $Q(\varepsilon^{(1)} < k_0^{L,\tilde{v}}) > 0$ and $Q(|H_0 + \rho J_0|) < \varepsilon^{(2)}) > 0$, and define the following $\mathbb{F}$ stopping times:

$$\begin{aligned} \tau^k &= \inf\{t : k_t^{L,\tilde{v}} < \varepsilon^{(1)}\}, \\ \tau^{H,J} &= \inf\{t : |H_t + \rho J_t| > \varepsilon^{(2)}\}. \end{aligned}$$

Now, define the stopping time τ to be

$$\tau = (\tau^k \wedge \tau^{H,J}). \tag{4.10}$$

On the stochastic interval $[0, \tau)$, we have the following lower bound on our drift coefficient, $\hat{b}_t$:

$$\hat{b}_t = b(v_t) + k_t^{L,\tilde{v}} \mu^2(v_t) + (H_t + \rho J_t)\mu(v_t) \geq b(v_t) + \varepsilon^{(1)} \mu^2(v_t) - \varepsilon^{(2)} \mu(v_t).$$

From this discussion, we have arrived at the following theorem.

Theorem 4. *Let S be the strong solution under $(P, \mathbb{F})$ of*

$$\begin{aligned} dS_t &= S_{t-} v_t^\alpha dM_t, \\ dv_t &= \mu(v_t) dB_t + b(v_t) dt. \end{aligned} \tag{4.11}$$

S is also the solution, under $(Q, \mathbb{G})$ of

$$\begin{aligned} dS_t &= S_{t-} v_t^\alpha d\tilde{M}_t, \\ dv_t &= \mu(v_t) d\tilde{B}_t + b(v_t) dt + k_t^{L,\tilde{v}} \mu(v_t)^2 dt + (H_t + \rho J_t)\mu(v_t) dt. \end{aligned}$$

Assume

$$E_P[e^{\int_0^T v_s^{2\alpha} d\langle M^d, M^d\rangle_s + \frac{1}{2}\int_0^T v_s^{2\alpha} d\langle M^c, M^c\rangle_s}] < \infty \tag{4.12}$$

and that $\langle M^d, M^d\rangle$ *is locally bounded.*

Assume also that Hypothesis 1 *holds, and moreover that*

$$\liminf_{x\to+\infty}(b(x) + \varepsilon^{(1)}\mu^2(x) - \varepsilon^{(2)}\mu(x))\phi(x)^{-1} > 0.$$

Then, the process S *is a true* $(P, \mathbb{F})$ *martingale and a* $(Q, \mathbb{G})$ *strict local martingale. Specifically, we have* $E_Q[S_t^\tau] < S_0$ *where* τ *is given by* (4.10).

Proof of Theorem 4. First note that the strong assumption given in (4.2) ensures that S_- is positive. From Protter & Shimbo (2008), a sufficient condition for the solution S of $dS_t = S_t v_t^\alpha dM_t$ to be a $(P, \mathbb{F})$ martingale on $[0, T]$ is that

$$E_P[e^{\int_0^T v_s^{2\alpha} d\langle M^d, M^d\rangle_s + \frac{1}{2}\int_0^T v_s^{2\alpha} d\langle M^c, M^c\rangle_s}] < \infty. \tag{4.13}$$

Let us now display sufficient conditions for the solution S of (4.11) under $(Q, \mathbb{G})$ to be a strict local martingale.

Define a sequence of stopping times T_n by $\inf\{t : v_t \geq n\}$. We have that the stopped process $S_{t\wedge\tau\wedge T_n(0\leq t\leq T)}$ is a $(Q, \mathbb{G})$ martingale. The stopping time T_∞ is the explosion time of v. Therefore, we may write

$$S_0 = E_Q[S_{t\wedge\tau\wedge T_n}] = E_Q[S_t 1_{\{t\wedge\tau\leq T_n\}}] + E_Q[S_{T_n} 1_{\{T_n < t\wedge\tau\}}]. \tag{4.14}$$

As we saw in the continuous case, since $E_Q[S_{t\wedge\tau} 1_{\{t\wedge\tau\leq T_n\}}]$ increases to $E_Q[S_{t\wedge\tau}]$, we would have that $E[S_{t\wedge\tau}] < S_0$ for all t if we can show that $\liminf_{n\to+\infty} E_Q[S_{T_n} 1_{\{T_n < t\wedge\tau\}}] > 0$.

The rest of the argument is analogous to that used in Theorem 1 with a slight modification of Lemma 2, where τ is defined as in (4.10). □

Corollary. *Let* M *be a Lévy martingale. Then, by the Lévy–Itô decomposition,*

$$M_t = W_t + \int_{|x|<1} x(N([0,t], dx) - t\nu(dx)) + \sum_{0<s<t} \Delta M_s 1_{\{|\Delta M_s|\geq 1\}} - \alpha t. \tag{4.15}$$

In the above, $N([0,t], \lambda) = N_t(\Lambda)$ *is a Poisson random measure*

$$\alpha t = E\left[\sum_{0<s<t} \Delta M_s 1_{|\Delta M_s|\geq 1}\right],$$

and $\nu(dx)$ *is the Lévy measure of the process* M *i.e.* $\nu(\Lambda) = E[N_1(\Lambda)]$. *The martingale* M *satisfies*

$$d\langle M, M\rangle_t = \left(1 + \int_{\mathbb{R}} x^2\nu(dx)\right)dt = cdt.$$

Assume that $E_P[\mathrm{e}^{\int_0^T(\frac{1}{2}+\int_{\mathbb{R}}x^2\nu(dx))v_s^{2\alpha}ds}]<\infty$. *This is satisfied if* $\int v_s^\alpha dM_s$ *is locally square integrable. Assume also that* (4.2) *holds and that*

$$\liminf_{x\to+\infty}(\rho\, x\mu(x)+b(x)+\varepsilon^{(1)}\mu^2(x)-\varepsilon^{(2)}\mu(x))\phi(x)^{-1}>0. \tag{4.16}$$

Then, the process S *of* (4.1) *is a true* $(P,\mathbb{F})$ *martingale and a* $(Q,\mathbb{G})$ *strict local martingale.*

Remark 8. We now give an alternative way to ensure that when we change probabilities from P to Q after a filtration enlargement, that Q is indeed a true probability measure and not a subprobability measure. This is an alternative to assuming that the continuous paths equivalent of (4.12) holds, although it is related. Let us ensure that, in the discontinuous case we have just encountered, the subprobability measure Q we defined is a *true* probability measure. We will begin by defining the sequence of probability measures Q_m by

$$dQ_m=Z_{T\wedge T_m}dP, \tag{4.17}$$

where Z is the Doléans–Dade exponential of

$$\left(\int_0^t H_s dB_s+\int_0^t J_s dM_s\right)\quad\text{and}$$
$$T_m=\inf\left\{t:\int_0^t\left(H_s^2+J_s^2+2\rho J_sH_s+J_s^2\int_{\mathbb{R}}x^2\nu(dx)\right)ds\geq m\right\}. \tag{4.18}$$

We then have

$$E[e^{\frac{1}{2}\int_0^{t\wedge T_m}(H_s^2+J_s^2+2\rho J_sH_s+H_s^2(\int_{\mathbb{R}}x^2\nu(dx)))ds}]\leq e^{\frac{1}{2}m}<\infty. \tag{4.19}$$

Recall that the relation

$$\lambda_t k_t^{L,S}S_t^2v_t^{2\alpha}+\rho J_tS_tv_t^\alpha+\lambda_tH_tS_tv_t^\alpha=0, \tag{4.20}$$

holds true for all $t\geq 0$.

Continuing, we have $Q_m\ll P$ on $[0,T_m]$ for each m, as well as that the Q_m are true probability measures, since $Z_t^{T_m}$ is a true $\mathbb{G}$ martingale.

4.1. *Examples*

Let us now consider a few examples.

Example 1 [Mansuy & Yor (2006) p. 20]. S and v solve

$$dS_t=S_tv_tdB_t;\quad S_0=1$$
$$dv_t=\mu(v_t)dW_t+b(v_t)dt;\quad v_0=1,$$

and $[B,W]=\rho t$. and $L=B_T$. In this case, we have

$$k_t^{L,\tilde{v}}=\rho\mu(v_t)\frac{B_T-B_t}{T-t}. \tag{4.21}$$

We have $k_0^{L,\tilde{v}}=\rho\mu(v_0)\frac{B_T}{T}$, and indeed, $Q(k_0^L>0)>0$. Here, it is immediately apparent that the process k has right-continous paths.

Remark 9. Let us show that in the above example, Jacod's condition is satisfied for $0 \leq t < T$. i.e.

$$Q_t(\omega, dx) = \mathcal{L}(L|\mathcal{F}_t)(dx) \ll \eta(dx) = \mathcal{L}(L), \quad 0 \leq t < T.$$

To do a simple calculation, we have for any bounded Borel function f

$$E(f(B_T)|\mathcal{F}_t) = E(f(B_T)|\sigma(B_t, W_t)) \quad \text{by the Markov property.}$$

Using basic measure theory, there exists a Borel function g such that $E((f(B_T)|\sigma(B_t, W_t)) = g(B_t, W_t)$. In this simple case, we can calculate g, and we get

$$g(x, y) = E(f(B_T)|B_t = x, W_t = y) = E^{x,y}(f(B_{T-t}) = P_{T-t}f(x, y),$$

where in general,

$$P_s f(x) = E^{x,y}(f(B_s, W_s)).$$

We can describe $Q_t(\omega, dxdy)$ by describing its action on any bounded, Borel function f

$$\int f(x, y)Q_t(\omega, dxdy) = P_{T-t}f(B_t, W_t) = E^{B_t,W_t}(f(B_{T-t}, W_{T-t})).$$

Since $\mathcal{L}(L)$ equals $N(0, T)$, a Gaussian law, it has the same sets of Lebesgue measure zero as does λ, which represents Lebesgue measure. We need only to show that the conditional distribution of L given $\mathcal{F}_t$ is absolutely continuous with respect to λ. To that end, let A be a subset of $\mathbb{R}$ such that $\lambda(A) = 0$. We have

$$\begin{aligned} P_s 1_{A\times\mathbb{R}}(B_t, W_t) &= E^{B_t,W_t}(1_{A\times\mathbb{R}}(B_s, W_s)) \\ &= E^{B_t,W_t}(1_A(B_s)1_{\mathbb{R}}(W_s)) = E^{B_t,W_t}(1_A(B_s)) \\ &= E^{x,y}(1_A(B_s))|_{(x,y)=(B_t,W_t)}. \end{aligned}$$

But $E^{x,y}(1_A(B_s)1_{\mathbb{R}}(W_s)) = 0$ for every starting point $x \in \mathbb{R}$, since $E^0(1_A(B_s)) = 0$ (because it's an integral of the Gaussian density over a null set), and Lebesgue measure is translation invariant. Since $E^x(1_A(B_s)) = 0$ for all x, we have also $E^{xy}(1_A(B_s)1_{\mathbb{R}}(W_s))|_{(x,y)=(B_t,W_t)} = 0$, and this gives the desired absolute continuity. Therefore, Jacod's condition is verified.

Example 2 [Mansuy & Yor (2006)]. S and v solve

$$dS_t = S_t v_t dB_t; \quad S_0 = 1,$$

$$dv_t = \mu(v_t)dW_t + b(v_t)dt; \quad v_0 = 1.$$

We set $L = T_a$, the first hitting time of a of the Brownian motion B. In this case, we take the Brownian motions B and W to be independent. We have

$$k_t^{L,\tilde{v}} = \mu(v_t)\left(-\frac{1}{a - B_t} + \frac{a - B_t}{T_a - t}\right) \textbf{ on the event } \{T_a > t\}.$$

Again, it is immediately apparent that the process k has right-continous paths and that $Q(k_0^L > 0) > 0$. We refer the reader to pages 34 and 35 of Mansuy & Yor (2006), where this example explained in detail.

Example 3 [The Countable Partition Case]. Let S and v solve

$$dS_t = S_t v_t dB_t; \quad S_0 = 1,$$

$$dv_t = \mu(v_t)dW_t + b(v_t)dt; \quad v_0 = 1.$$

Let us assume that we have a countable partition of the sample space such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{\infty} A_k = \Omega$ and that the information encoded in L can be modeled as $L = \sum_{i=1}^{\infty} a_i 1_{A_i}$. Note that the vector process $\begin{bmatrix} S \\ v \end{bmatrix}$ is a strong Markov process. Let us define our partition in terms of this Markov process. Fix a time $T > 0$ and assume we have half open sets $(\alpha_i, \beta_i]$ such that $\bigcup_{i=1}^{\infty}(\alpha_i, \beta_i] = \mathbb{R}$ and $(\alpha_i, \beta_i] \cap (\alpha_j, \beta_j] = \phi$, $i \neq j$. Let $A_i = \{\omega : S_T \in (\alpha_i, \beta_i]\}$.

If, in this case, we have $a_i > 0$ and $P(A_i) > 0$ then we have that the process k satisfies $k_0^L > 0$. Consider the sequence of martingales $N_t^i = E[1_{A_i} \,|\, \mathcal{F}_t]$. By the Kunita–Watanabe inequality, there exists processes ξ^i such that $d[N^i, S]_t = \xi_t^i d[S, S]_t$. Now, the determination of whether or not k has right-continuous paths is tantamount to the determination of whether or not, for each i, the processes ξ^i possess right-continuous paths. Since we are in the Brownian framework, we can employ martingale representation to write

$$N_t^i = \int_0^t h_s^i dB_s + \int_0^t g_s^i dW_s.$$

Then, $[N^i, S]_t = \int_0^t (h_s^i + \rho g_s^i) S_s v_s ds$, ρ being the correlation of the Brownian motions B and W, and ξ_t^i is such that $(h_t^i + \rho g_t^i)dt = \xi_t^i S_t v_t dt$. If we can prove then, that for each i, the processes $h^i + \rho g^i$ possess right-continuous paths, then we are done.

We now apply the results of Jacod *et al.* (1985), specifically Corollary 2.5. For all i, we can write

$$f^i\left(\begin{bmatrix} S_T \\ v_T \end{bmatrix}\right) = \begin{bmatrix} 1_{\{(\alpha_i, \beta_i]\}}(S_T) \\ 0 \end{bmatrix}.$$

For each i, we need to find an approximating sequence of functions $f^{i,n}(x)$ such that

$$f^{i,n}\left(\begin{bmatrix} S_T \\ v_T \end{bmatrix}\right) \to f^i\left(\begin{bmatrix} S_T \\ v_T \end{bmatrix}\right) \quad \text{in } L^2(P).$$

For all i, $f^{i,n}(x)$ must be Borel functions, and $(t, y) \to P_t f^{i,n}(y)$ on $(0, \infty)$ must be once differentiable in t and twice differentiable x, all partial derivatives being continuous. P_t denotes the transition semigroup of the process $\begin{bmatrix} S \\ v \end{bmatrix}$. Note that this holds when the functions $f^{i,n}(x)$ are twice differentiable, with continuous second derivative, and with compact support. Note that this differentiability is just what we need to apply Theorem 3.2 of Ma *et al.* (2001), which gives us that the corresponding process $(h_s^{i,n} + \rho g_s^{i,n})_{0\leq s\leq T}$ has càdlàg paths, for each n.

We have that for each i an approximating sequence of functions $f^{i,n}(x)$ of $f^i(x)$ is given by $f^i(x) * \phi^n(x)$, where $\phi^n(x)$ is a sequence of mollifiers. For example, we can take

$$\phi^n(x) = n^2\phi(nx), \quad \text{where } \phi(x) = c\,\mathrm{e}^{-\frac{1}{1-\|x\|^2}}\chi_{[-1,1]}(x).$$

We have that $f^i(x) * \phi^n(x)$ is smooth and with compact support. It converges uniformly and thus in L^2 to $f^i(x)$. Moreover, we also have the uniform (in t) convergence of $P_t f^{i,n}(y) \to P_t f^i(y)$. Now, by Corollary 2.5, we have for each i the existence of an explicit representation of a version of the process $h^i + \rho g^i$ which indeed possesses càdlàg paths, since it is the uniform (in the time variable) limit of the càdlàg processes $h^{i,n} + \rho g^{i,n}$.

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