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Localization of eigenfunctions via an effective potential

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ABSTRACT

We consider the localization of eigenfunctions for the operator $L = -\operatorname{div} A \operatorname{grad} + V$ on a Lipschitz domain Ω and, more generally, on manifolds with and without boundary. In earlier work, two authors of the present paper demonstrated the remarkable ability of the landscape, defined as the solution to $Lu = 1$, to predict the location of the localized eigenfunctions. Here, we explain and justify a new framework that reveals a richly detailed portrait of the eigenfunctions and eigenvalues. We show that the *reciprocal* of the landscape function, $1/u$, acts as an *effective potential*. Hence from the single measurement of u , we obtain, via $1/u$, explicit bounds on the exponential decay of the eigenfunctions of the system and estimates on the distribution of eigenvalues near the bottom of the spectrum.

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1. Introduction

The term localization refers to a wide range of phenomena in mathematics and condensed matter physics in which eigenfunctions of an elliptic system concentrate on a small portion of the original domain and nearly vanish in the remainder, hindering, or preventing wave propagation. For many decades, its different manifestations have been a source of wide interest, with an enormous array of applications. In addition to celebrated results concerning localization by disordered potentials [1–6], there is localization by randomness in the coefficients of $-\operatorname{div} A \nabla$ and of the Maxwell system [7, 8], localization by a quasiperiodic potential [9], and localization by fractal boundaries [10], to mention only a few examples. However, with the notable exception of the recent work [9] for a 1D almost Matthieu operator, these results do not address detailed, deterministic geometric features of the localized eigenfunctions.

The present paper changes the point of view through the introduction of a new *effective potential*, and applies it to establish the location, shape, and a detailed structure of the exponential decay of the eigenfunctions of the operator $L = -\operatorname{div} A \operatorname{grad} + V$ on a finite domain, as well as estimates on its spectrum.

In 2012, Filoche and Mayboroda introduced the concept of the *landscape*, namely the solution u to $Lu=1$ for an elliptic operator L , and showed that this single function has remarkable power to predict the shape and location of localized low energy eigenfunctions of L , whether the localization is triggered by the disorder of the potential, the geometry of the domain, or both (see [11]). These ideas led to beautiful new results in mathematics [12, 13], as well as theoretical and experimental physics [14].

In this paper and its companion papers [15] in physics and [16] in computational mathematics, we propose a new framework that greatly extends the predictive power of the landscape function u . We show that the reciprocal $1/u$ of the landscape function should be viewed as an *effective quantum potential* revealing detailed structure of the eigenfunctions. The eigenfunctions of L reside in the wells of $1/u$ and decay exponentially across the barriers of $1/u$. Under hypotheses on the behavior of u that can be confirmed easily and efficiently numerically, the original domain splits into independently vibrating regions, and the global eigenfunctions are exponentially close to eigenfunctions of subregions. As a corollary, we prove an approximate diagonalization of the operator and confirm that localization according to $1/u$ gives an accurate eigenvalue count up to exponential errors.

Predicting the eigenvalue count or “density of states” is an important goal linking this paper to the other two. The proposal in [15] to use $1/u$ to estimate the density of states, starting from the very bottom of the spectrum, has provoked a burst of applications beyond the scope of the single-particle Schrödinger equation. In particular, in the context of the Poisson-Schrödinger system, the paper [17] finds an iterative algorithm that speeds up the time it takes to compute the performance of the type of semiconductor used in LED devices from one year to one day. The key to this acceleration is that at each step of the iteration, a new potential is computed as a function of the density of states. This modifies in turn the operator L and therefore the effective potential $1/u$ from which the next density of states is derived, without ever solving the Schrödinger equation. In the companion article [16] in computational mathematics, we explore systematically efficient shortcuts leading from the effective potential to the density of states.

Although some of our applications are to random regimes, the effective potential $1/u$ is a deterministic tool. It is not designed to replace probabilistic methods, but to complement and enhance them by providing a new way to detect the quantum geometry of disordered materials. Statistical mechanics often treats the source of disorder as a black box, whereas this mechanism allows us to enter the box and identify detailed deterministic features of the disorder.

Put another way, this paper does not aim to prove localization, but rather to describe a new mechanism to measure it. The results are conditional on the separation of the potential wells of the effective potential, but this separation can be confirmed efficiently numerically. As a result, for many families of random potentials V , one can learn from the effective potential $1/u$ what the eigenfunctions look like. Finally, the paper does not address wave interference, another suggested cause of localization. It does, however, capture quantitatively effects of quantum tunneling.

The paper is organized as follows. In [Section 2](#), we state our results in a special case and illustrate their numerical significance. In [Section 3](#), we give our main definitions

and state some preliminary estimates on the landscape function and eigenfunctions. In [Section 4](#), we derive our exponential decay estimates, known as Agmon estimates, in the setting of bounded domains in \mathbb{R}^n . In [Section 5](#), we deduce the approximate diagonalization into localized eigenfunctions and estimates on the eigenvalue distribution. Finally, in [Section 6](#), we describe how to generalize our theorems to manifolds and prove the boundary regularity theorems stated in [Section 3](#). We also address the difficulty that Agmon metrics are only defined for continuous coefficient matrices A ; because our estimates are independent of the modulus of continuity, we are able to use a fairly straightforward procedure to approximate bounded measurable coefficient matrices by continuous ones.

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2. Outline of results and comparison with numerical examples

To describe our results we consider the very special case in which the operator is (minus) the ordinary Laplace operator plus a nonnegative, bounded potential,

$$L = -\Delta + V \quad (0 \leq V(x) \leq \bar{V}; \quad \bar{V} := \sup V)$$

acting on periodic functions, that is, on the manifold $M = \mathbb{R}^n/T\mathbb{Z}^n$. It is crucial to applications that the estimates be independent of the “size” T of the manifold M as $T \rightarrow \infty$. What makes them even more valuable is that they are essentially universal, as we shall discuss later in this section.¹

Assume that V is positive on a set of positive measure. Then the landscape function u , the solution to $Lu = 1$ on M , exists and is unique. Moreover, $u > 0$ by the maximum principle. Our starting point is the conjugation of the operator L by multiplication by u :

$$\tilde{L}g := \frac{1}{u}L(gu) = -\frac{1}{u^2} \operatorname{div}(u^2 \nabla g) + \frac{1}{u}g.$$

The operator \tilde{L} has a similar form to L but with the new potential $1/u$ replacing V . Writing the quadratic form associated with the operator L in terms of \tilde{L} , we find the identity ([Lemma 4.1](#))

$$\int_M [|\nabla f|^2 + Vf^2] dx = \int_M \left(u^2 |\nabla(f/u)|^2 + \frac{1}{u} f^2 \right) dx, \quad (2.1)$$

which holds for all $f \in W^{1,2}(M)$. In particular,

$$\int_M [|\nabla f|^2 + Vf^2] dx \geq \int_M (1/u)f^2 dx. \quad (2.2)$$

Inequality (2.2) suggests that we can replace V with a new *effective potential function* $1/u$. In fact, we will need the full identity (2.1) to demonstrate this. The identity reflects a trade in kinetic and potential energy, enabling $1/u$ to capture effects of both the kinetic term $|\nabla f|^2$ and the potential term Vf^2 rather than only the potential energy.

¹Furthermore, in the body of the paper, we will treat operators with bounded measurable coefficients on Lipschitz and more general domains and on compact C^1 manifolds with and without boundary; see [Sections 3](#) and [6](#).

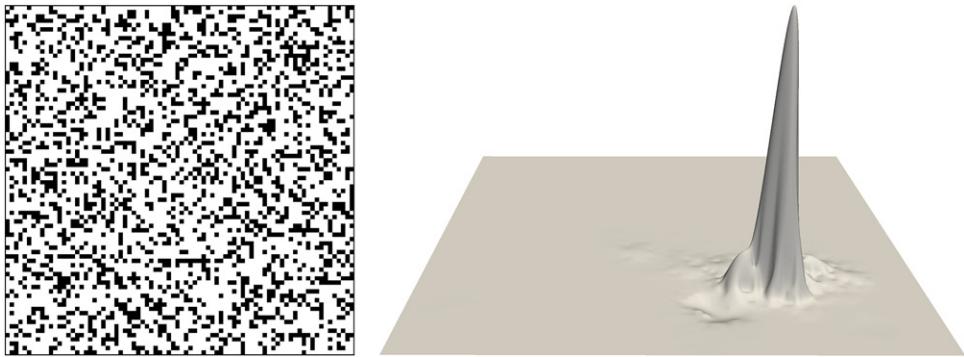


Figure 1. Bernoulli potential (left) and the fifth eigenfunction (right).

An example of the localization we are trying to predict and control is shown in Figure 1, which depicts a potential V on $\mathbb{R}^2/T\mathbb{Z}^2$ with $T=80$ and constant values on unit squares, $V=0$ on white squares and $V=4$ on black squares. The values were chosen independently, with probability 30% for $V=4$ and 70% for $V=0$. In other words, V is a Bernoulli random variable on unit squares. At the right is the graph of fifth eigenfunction. In spite of the fact that the zero set of V percolates everywhere, this eigenfunction and dozens of others are highly localized.

2.1. Exponential decay

The first main result of this paper is the rigorous proof that the steep decay in Figure 1 comes from the barriers of the effective potential. We do this by formulating and proving appropriate exponential decay estimates of Agmon type (see [18, 19]). Roughly speaking, these theorems say that if (2.2) holds, then eigenfunctions of eigenvalue λ have “most” of their mass in the region

$$E(\lambda + \delta) = \{x \in M : 1/u(x) \leq \lambda + \delta\}$$

for a suitable small $\delta > 0$, and exponential decay in the complementary region.

To formulate our estimate precisely, consider the weights

$$w_\lambda(x) := \max\left(\frac{1}{u(x)} - \lambda, 0\right).$$

Exponential decay is expressed in terms of the so-called Agmon distance, traditionally built from V , but for our purposes arising from $1/u$. We define our version of Agmon distance, which we will refer to loosely as the *effective distance*, as the degenerate metric on M given by

$$\rho_\lambda(x, y) = \inf_{\gamma} \int_0^1 w_\lambda(\gamma(t))^{1/2} |\dot{\gamma}(t)| dt,$$

with the infimum taken over absolutely continuous paths $\gamma : [0, 1] \rightarrow M$ from $\gamma(0) = x$ to $\gamma(1) = y$.

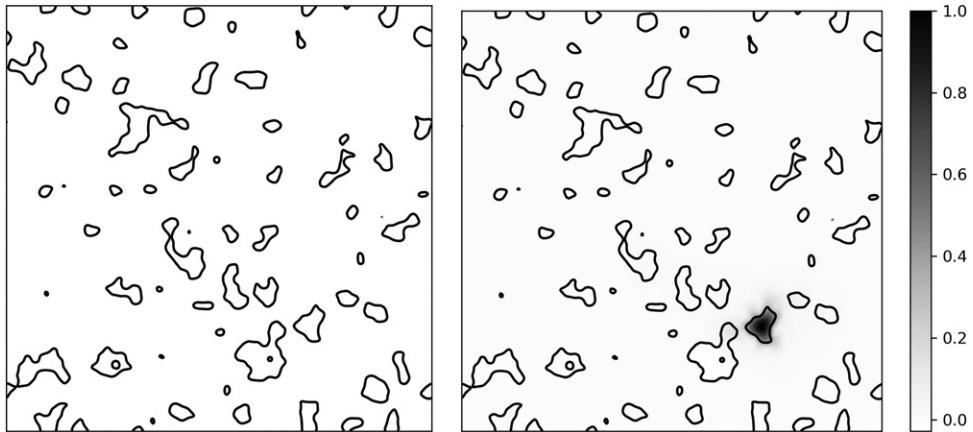


Figure 2. $E(\lambda_5 + \delta) = \{1/u(x) \leq \lambda_5 + \delta\}$ (left) with fifth eigenfunction superimposed in gray scale (right).

Theorem 2.1. (see [Corollary 4.5](#)) Let ψ be an eigenfunction: $L\psi = \lambda\psi$ on M . Let

$$h(x) := h_{\lambda, \delta}(x) = \inf \left\{ \rho_\lambda(x, y) : y \in E(\lambda + \delta) \right\}$$

be the effective distance from x to $E(\lambda + \delta)$. Then

$$\int_{\{h \geq 1\}} e^h (|\nabla \psi|^2 + \bar{V}\psi^2) \, dx \leq 50(\bar{V}/\delta) \int_M \bar{V}\psi^2 \, dx. \quad (2.3)$$

The theorem says that the square density and energy of the eigenfunction are at most of size e^{-h} , with h the effective distance from $E(\lambda + \delta)$. The theorem only guarantees decay insofar as the function h grows. But in numerical examples, the growth of h and the way it matches the decay of eigenfunctions is very evident, as we will illustrate shortly using the eigenfunction in [Figure 1](#). Later in this section, we will discuss the typical behavior as $T \rightarrow \infty$.

The main difficulty of the proof is to compensate for the price we paid for replacing V with $1/u$, namely that the gradient term $|\nabla f|^2$ has been replaced by $u^2|\nabla(f/u)|^2$ in [\(2.1\)](#). We can't afford this dependence on u , and a crucial feature of the estimate we obtain in [\(2.3\)](#) is that this part of the dependence on u disappears, leaving only the effects of $1/u$.

Remarkably, we get a uniform bound, independent of the dimension n and the size T of the manifold. It is universal in that it depends only on the effective distance and the scale-invariant ratio δ/\bar{V} . The parameter δ is at our disposal, but a natural choice is to take δ less than the distance between successive eigenvalues, in which case δ should be viewed as a spectral gap. Given that the dependence on parameters is so explicit and scale-invariant, the estimates can be interpreted easily both numerically and physically across a wide family of contexts.

To illustrate this exponential decay, we compute the effective potential $1/u(x)$ for the Bernoulli potential in [Figure 1](#). [Figure 2](#) shows the contour of $E(\lambda_5 + \delta)$ on the left with $\lambda_5 = 0.45508$, the fifth eigenvalue. (The value $\delta = 0.005$ was chosen as the average spacing between eigenvalues in the vicinity of the fifth.) Overlaid on the right in gray scale are the values of fifth eigenfunction ψ_5 . Note that most of ψ_5 occupies just one

component of the set $E(\lambda_5 + \delta)$. In fact, dozens of eigenfunctions coincide essentially with single components or clusters of components.

2.2. Approximate diagonalization

So far, estimate (2.3) only guarantees that ψ_5 is supported primarily in a union of wells, that is, it is mostly a linear combination of highly localized functions, whereas Figures 1 and 2 show that the eigenfunction is primarily a single spike. We want to show that eigenfunctions are single spikes or clusters of spikes and justify implicitly the numerical procedure for finding the eigenfunctions in order by examining the wells separately, starting from the deepest (see [16]).

To prove that eigenfunctions localize to a single well or a cluster, we establish an approximate diagonalization. This will require an extra assumption on spectral gaps. For the purposes of localization and diagonalization, near multiplicity, or resonance, is the enemy. Eigenfunctions with nearly the same eigenvalue can, in fact, share wells.

We introduce a space of localized eigenfunctions as follows. Consider a threshold $\bar{\mu}$ that will be used to handle eigenvalues $\lambda \leq \bar{\mu} - \delta$. Choose any subdivision² of $E = E(\bar{\mu} + \delta)$ into a finite collection of disjoint closed subsets

$$E = \bigcup_{\ell} E_{\ell}.$$

Let \bar{S} denote the smallest effective distance $\rho_{\bar{\mu}}$ between distinct pairs of sets E_{ℓ} and $E_{\ell'}$. Let Ω_{ℓ} be the $\bar{S}/2$ neighborhood³ of E_{ℓ} in the effective distance $\rho_{\bar{\mu}}$. Let $\varphi_{\ell,j}, j = 1, \dots$, be the orthonormal basis of $L^2(\Omega_{\ell})$ of eigenfunctions of L satisfying the Dirichlet condition $\varphi = 0$ on $M \setminus \Omega_{\ell}$. By results analogous to the exponential bounds for ψ , these functions $\varphi_{\ell,j}$ are concentrated near E_{ℓ} and decay exponentially in the larger region Ω_{ℓ} , provided the corresponding eigenvalue satisfies $\mu_{\ell,j} \leq \bar{\mu}$. In other words, such functions are *localized to a single well or cluster E_{ℓ} in M* .

Denote by $\Phi_{(a,b)}$ the orthogonal projection onto the subspace of $L^2(M)$ spanned by $\varphi_{\ell,j}$ with eigenvalues between a and b , and $\Psi_{(a,b)}$ the corresponding spectral projection for eigenfunctions of L . Denote by $\|\cdot\|$ the norm of $L^2(M)$. Our main result is the following.

Theorem 2.2. (see Theorem 5.1) *If ψ is an eigenfunction of L with eigenvalue λ on M and $\lambda \leq \bar{\mu} - \delta$, then*

$$\|\psi - \Phi_{(\lambda-\delta, \lambda+\delta)}\psi\|^2 \leq 300 \left(\frac{\bar{V}}{\delta} \right)^3 e^{-\bar{S}/2} \|\psi\|^2. \quad (2.4)$$

If $\varphi = \varphi_{\ell,j}$ is a localized eigenfunction with eigenvalue $\mu = \mu_{\ell,j} \leq \bar{\mu} - \delta$, then

$$\|\varphi - \Psi_{(\mu-\delta, \mu+\delta)}\varphi\|^2 \leq 300 \left(\frac{\bar{V}}{\delta} \right)^3 e^{-\bar{S}/2} \|\varphi\|^2.$$

²The E_{ℓ} are typically connected components of E , but since the theorem is stronger when the minimum separation \bar{S} is larger, it is sometimes useful to merge nearby wells into one set E_{ℓ} .

³The sets Ω_{ℓ} can also be chosen to be somewhat larger, provided each Ω_{ℓ} is separated by at least $\rho_{\bar{\mu}}$ distance $\bar{S}/2$ from $E_{\ell'}$ for every $\ell' \neq \ell$. They are roughly in the spirit of Voronoi cells.

The interpretation is that the eigenfunctions ψ are linear combinations of localized $\varphi_{\ell,j}$ associated with the spectral band $\lambda \pm \delta$. In particular, if the projection has rank one, then ψ lives primarily in one well or cluster E_ℓ . This is the kind of localization we see in numerical simulation.

Let us make the spectral gap condition required for the projection to have rank one more explicit. If we choose δ so that

$$\delta/\bar{V} \geq 30e^{-\bar{S}/6}, \quad (2.5)$$

then the constant on the R.H.S. of (2.4) is $\leq 1/90$. If there is only one eigenvalue $\mu_{\ell,j}$ in the range $(\lambda - \delta, \lambda + \delta)$, then the projection has rank one, and the eigenfunction ψ is localized. Up to the factor $1/6$ in the exponent, this is the best result of its kind that one can hope for. If the spectral gap δ between eigenvalues $\mu_{\ell,j}$ in adjacent E_ℓ is smaller than $\bar{V}e^{-c\bar{S}}$ for some sufficiently large c , then the eigenfunction may be a linear combination with significant contributions from more than one E_ℓ .

Finally, we describe the correspondence between actual eigenvalues and localized eigenvalues $\mu_{\ell,j}$ up to exponential errors. This, combined with [Theorem 2.2](#), gives the full picture of the correspondence between actual eigenfunctions and localized eigenfunctions $\varphi_{\ell,j}$ up to exponential errors for low eigenvalues. Denote by $N_0(\lambda)$ the cumulative eigenvalue counting function for the union of the $\varphi_{\ell,j}$ and by $N(\lambda)$ the counting function for the original operator L .

Corollary 2.3. (see [Corollary 5.2](#)) Suppose that $\delta, \bar{\mu}$ and \bar{N} are chosen so that

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{\bar{N}} \leq \bar{\mu} - \delta, \quad 300\bar{N} \left(\frac{\bar{V}}{\delta} \right)^3 e^{-\bar{S}/2} < 1. \quad (2.6)$$

Then

$$N_0(\lambda - \delta) \leq N(\lambda) \leq N_0(\lambda + \delta), \quad \text{for } \lambda \leq \bar{\mu} - \delta.$$

The corollary follows readily from [Theorem 2.2](#). It says that the two eigenvalue counts coincide up to δ with $\delta \approx \bar{V}^{-1/3} e^{-\bar{S}/6}$, parallel to (2.5).

The constants in our estimates depend only on the spectral ratio δ/\bar{V} , so we can easily see the exponential take control as T increases with the help of numerical experiments on $\mathbb{R}/T\mathbb{Z}$. For each of $T = 2^5, 2^6, \dots, 2^{19}$, we carried out 200 realizations of a potential V with constant values on unit intervals, chosen independently and uniformly distributed between 0 and $\bar{V} = 4$. We found that the gap $\lambda_2 - \lambda_1$ is typically⁴ greater than $1/T$. (This is nearly the same, by (2.6), as the spectral gap between the first two *localized* eigenvalues $\mu_{\ell,j}$.) The minimum separation S between consecutive connected components of $E(\lambda_1 + 1/T)$ conforms very well to the power law $\text{median}(S) \sim .69 T^{.59}$. For $T = 2^{15}$, the values $\delta_1 = 1/T, \bar{V} = 4$, and the median $S_1 = .69 T^{.59}$, we have

⁴Lower bounds on spectral gaps are called Wegner type estimates. In [6], Fröhlich and Spencer showed that for large disorder, the gap is bounded below by a multiple of $1/T^n$ with high probability in the discrete Anderson model on \mathbb{Z}^n with uniformly distributed V . A similar conclusion holds with a larger power of T in many cases in which V has a singular continuous distribution (see [20]).

$$300 \left(\frac{\bar{V}}{\delta_1} \right)^3 e^{-S_1/2} \ll 10^{-50}.$$

Thus, (2.4) typically shows that the ground state ψ_1 is extremely close to a single spike.

Theorem 2.2 is motivated by work of Helffer and Sjöstrand [21, 22] and Simon [23, 24] on resonance for smooth potentials V in the semi-classical regime, $-h^2\Delta + V$ as $h \rightarrow 0$, but our potentials are much more irregular and our eigenfunctions have a different shape. A new feature of our methods is that we use weak eigenfunction equations and derive error estimates in the dual space to the standard Sobolev space $W^{1,2}(M)$ (see (5.2)). By relying only on dual space estimates, we can eliminate all dependence on smoothness, and express our results explicitly in terms of the spectral gap ratio δ/\bar{V} . The dual estimates are just barely strong enough to yield estimates for the spectral projection and eigenvalue distribution.

Because our exponential decay result is relative to distance to all of $E(\lambda + \delta)$ rather than to a single well, it does not address directly the further decay we see numerically as we pass through the second and third effective barrier, etc. Our subsequent estimates show that resonance is the main issue. The natural conjecture is that the interaction of pairs of eigenfunctions depends primarily on the effective distance between the wells or cluster to which they belong, rather than the minimum distance \bar{S} between all pairs of wells. The proof can be expected to depend on multi-scale analysis and a more detailed spectral gap hypothesis like the condition (2.5) above, localized to pairs or groupings of wells. Showing that such a hypothesis is satisfied with high probability should employ tools associated with so-called Wegner estimates in the theory of Anderson localization.

3. Main assumptions and preliminary estimates

Let Ω be a bounded, connected, open subset of \mathbb{R}^n such that at each boundary point the domain is locally equivalent to a half space via a bi-Lipschitz mapping. (In Section 5, we will replace the ambient space \mathbb{R}^n with a compact C^1 manifold \hat{M} .)

Set $M = \bar{\Omega}$, and let $m \in L^\infty(\Omega)$ be a real-valued density satisfying uniform upper and lower bounds

$$\frac{1}{C} \leq m(x) \leq C,$$

for some positive constant C . Let $A = (a_{ij}(x))_{i,j=1}^n$ be a bounded measurable, real symmetric matrix-valued function, satisfying the uniform ellipticity condition

$$\frac{1}{C} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq C |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n. \quad (3.1)$$

for some $C < \infty$. We define the elliptic operator L acting formally on real-valued functions φ by

$$L\varphi = -\frac{1}{m} \operatorname{div} (mA\nabla\varphi) + V\varphi = -\frac{1}{m} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(m a_{ij} \frac{\partial \varphi}{\partial x_j} \right) + V\varphi.$$

The operator L will always be used in the weak sense, defined as follows.

Definition 3.1. A function $\varphi \in W^{1,2}(\Omega)$ satisfies $L\varphi = f$ weakly on Ω (respectively, on $M = \bar{\Omega}$) if

$$\int_{\Omega} [(A\nabla\varphi) \cdot \nabla\eta + V\varphi\eta] \ m \ dx = \int_{\Omega} f\eta \ m \ dx \quad (3.2)$$

for every $\eta \in W_0^{1,2}(\Omega)$ (respectively, for every $\eta \in W^{1,2}(\Omega)$).

Here the space $W^{1,2}(\Omega) = W^{1,2}(M)$ is the usual Sobolev space, namely the closure of $C^1(M)$ in the function space with square norm given by

$$\int_{\Omega} (|\nabla\varphi|^2 + \varphi^2) \ dx.$$

The space $W_0^{1,2}(\Omega)$ is the closure in the same norm of the subspace $C_0^1(\Omega)$ of continuously differentiable functions that are compactly supported in Ω .

The weak equation on $M = \bar{\Omega}$ imposes, in addition to the interior condition, a weak form of the Neumann boundary condition on φ . If there is sufficient smoothness to justify integration by parts, then the Neumann condition can be written

$$\nu(x) \cdot A(x)\nabla\varphi(x) = 0, \quad x \in \partial\Omega,$$

with ν the normal to $\partial\Omega$. In fact, in the case of Lipschitz boundaries, the Neumann condition is valid almost everywhere with respect to surface measure on $\partial\Omega$ for suitable right hand sides f . But, we will only need the weak form, not this strong version of the boundary condition. (For now we confine ourselves to Neumann boundary conditions; we will say a few words about Dirichlet and mixed boundary conditions later.)

We assume further that V is non-degenerate in the sense that it is strictly positive on a subset of positive measure of Ω . By ellipticity of A and the fact that Ω is a connected, bounded bi-Lipschitz domain, we have the coercivity inequality

$$\int_M [(A\nabla\varphi) \cdot \nabla\varphi + V\varphi^2] \ m \ dx \geq c \int_M (|\nabla\varphi|^2 + \varphi^2) \ dx,$$

for some $c > 0$. In other words, the formal $L^2(M, m \ dx)$ inner product $\langle L\varphi, \varphi \rangle$ is comparable to the square of the $W^{1,2}(\Omega) = W^{1,2}(M)$ norm of φ . By the Fréchet–Riesz theorem (identifying a Hilbert space with its dual), this implies that for every $f \in L^2(M, m \ dx)$, there is a unique solution $v \in W^{1,2}(M)$ to the weak equation $Lv = f$ on M . The *landscape function* u is defined as the solution to

$$Lu = 1 \quad \text{weakly on } M.$$

In other words, u is the unique weak solution to the inhomogeneous Neumann problem with R.H.S. the constant 1.

Proposition 3.2. Let V be nondegenerate and satisfy $0 \leq V \leq \bar{V}$ for some constant \bar{V} . Then the landscape function u satisfies $u \geq 1/\bar{V}$ on M . Moreover $u \in C^\alpha(M)$ for some $\alpha > 0$.

Proof. Consider the weak solution to $Lv = f$ on M for bounded measurable f . Hölder regularity of v at interior points of M follows from a version of the theorem of De Giorgi, Nash, and Moser (see Theorem 8.24, [25]). Near each boundary point, one can

define an “even” reflection of v that satisfies a uniformly elliptic equation in a full neighborhood; hence v is C^α up to the boundary for some $\alpha > 0$. This reflection argument is presented in the last section in the more general context of manifolds (see [Proposition 6.1](#)). In particular, $u \in C^\alpha(M)$.

Next, we prove a version of the maximum principle, namely that $v \geq 0$ provided $f \geq 0$. Since v is continuous, the set $\Omega^- = \{x \in \Omega : v(x) < 0\}$ is open. Since v minimizes

$$\int_{\Omega} ((A\nabla\varphi) \cdot \nabla\varphi + V\varphi^2 - 2f\varphi) \, m \, dx$$

among all $\varphi \in W^{1,2}(M)$, we have

$$\begin{aligned} & \int_{\Omega} ((A\nabla v) \cdot \nabla v + Vv^2 - 2fv) \, m \, dx \\ & \leq \int_{\Omega} ((A\nabla v_+) \cdot \nabla v_+ + Vv_+^2 - 2fv_+) \, m \, dx \end{aligned}$$

for $v_+(x) = \max(v(x), 0)$. Consequently,

$$\int_{\Omega^-} ((A\nabla v) \cdot \nabla v + Vv^2 - 2fv) \, m \, dx \leq 0.$$

Because $V \geq 0$ and $f \geq 0$, we have $Vv^2 - 2fv \geq 0$ on Ω^- . Therefore,

$$\int_{\Omega^-} (A\nabla v) \cdot \nabla v \, m \, dx \leq 0.$$

Since A is coercive, $\nabla v = 0$ a.e. on Ω^- , and v is a strictly negative constant on each connected component of Ω^- . If any such component is a proper subset of Ω , then the continuity of v contradicts the fact that $v \geq 0$ on $\Omega \setminus \Omega^-$. On the other hand, if $\Omega^- = \Omega$, then $v \equiv -a$, for some constant $a > 0$. But in that case, $Lv = -aV$, which cannot equal $f \geq 0$. Thus, the only possibility is that Ω^- is empty.

Finally, to conclude proof of the proposition, consider u , the weak solution to $Lu = 1$ on M . Then

$$v = u - \frac{1}{V} \quad \text{solves} \quad Lv = 1 - \frac{V}{V} \geq 0.$$

Therefore, by the maximum principle, $v \geq 0$, and $u \geq 1/V$. \square

By the bi-Lipschitz assumption on Ω and the Rellich-Kondrachov lemma, the inclusion mapping $W^{1,2}(M) \hookrightarrow L^2(M)$ is compact. Thus, by the spectral theorem for compact operators, there is a complete orthonormal system of eigenfunctions to the Neumann problem for L , that is, an orthonormal basis ψ_j of $L^2(M)$ such that $\psi_j \in W^{1,2}(M)$, and

$$L\psi_j = \lambda_j \psi_j \quad \text{weakly on } M.$$

The non-degeneracy of V implies that the eigenvalues λ_j are strictly positive,

We will compare these eigenfunctions to localized eigenfunctions of Dirichlet or mixed boundary value problems. Let K be a compact subset of M . Let U be a connected component of $M \setminus K$. We say that $L\varphi = f$ weakly on U if Eq. (3.2) holds for all test functions $\eta \in C^1(M)$ such that the support of η is contained in U . We will denote the closure of this set of test functions in the usual $W^{1,2}(\mathbb{R}^n)$ norm by $W_0^{1,2}(U)$. Formally, solutions to $L\varphi = f$ on U satisfy mixed boundary conditions

$$\varphi(x) = 0, \quad x \in K \cap \partial U; \quad \nu(x) \cdot A(x) \nabla \varphi(x) = 0, \quad x \in \partial \Omega \cap \partial U.$$

In the special case $K \supset \partial \Omega$, the problem is no longer mixed because we only have Dirichlet boundary conditions. We won't need the Neumann boundary equations in strong form, only the weak, integrated form. On the other hand, we will use continuity of the solutions up to the boundary. In fact, we will obtain C^α regularity.

To ensure the Hölder regularity of solutions we make an additional assumption on the compact set $K \subset M$. We will say that K satisfies the *bi-Lipschitz cone condition* if there are $r > 0$ and $\varepsilon > 0$ such that at every point $x_0 \in \partial K$ there is a mapping $F: B_r(x_0) \rightarrow \mathbb{R}^n$ with $F(x_0) = 0$, bi-Lipschitz bounds $\varepsilon|x-y| \leq |F(x)-F(y)| \leq (1/\varepsilon)|x-y|$, and such that

$$F(K) \supset \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'| < \varepsilon x_1 < \varepsilon^2\}.$$

The constants in our main theorems do not depend on r , ε or the bi-Lipschitz constants of Ω because continuity of the solutions is only used in a qualitative way.

Proposition 3.3. *Suppose that K is a non-empty compact subset of M satisfying the bi-Lipschitz cone condition. Let U be a connected component of $M \setminus K$. Then there is an orthonormal basis φ_j of $L^2(U, m \, dx)$ of eigenfunctions solving $L\varphi_j = \mu_j \varphi_j$ weakly on U , $\mu_j > 0$. After extending the functions φ_j from U to the rest of M by $\varphi_j = 0$ on $M \setminus U$, they satisfy $\varphi_j \in C^\alpha(M) \cap W^{1,2}(M)$ for some $\alpha > 0$.*

The proof of the existence of the complete orthonormal basis of eigenfunctions is the same as in the case of $K = \emptyset$, that is, the case of ψ_j above. See [Proposition 6.1](#) for the proof C^α regularity. (At interior points the proof is similar to the case of $Lv = f$ above. The boundary regularity is proved by reducing to a Dirichlet problem using an even reflection.)

4. Agmon estimates

We will frequently write

$$\nabla_A = A^{1/2} \nabla$$

in which $A^{1/2} = A^{1/2}(x)$ is the positive definite square root of the matrix $A(x)$ and ∇ is a column vector. Thus, we have

$$\nabla_A \varphi \cdot \nabla_A \eta = (A \nabla \varphi) \cdot (\nabla \eta); \quad |\nabla_A \varphi|^2 = (A \nabla \varphi) \cdot \nabla \varphi.$$

Lemma 4.1. *Assume that f and u belong to $W^{1,2}(M)$, that V , f , and $1/u$ belong to $L^\infty(M)$, and that u satisfies $Lu = 1$ weakly on M . Then*

$$\int_M (|\nabla_A f|^2 + Vf^2) \, m \, dx = \int_M \left(u^2 |\nabla_A(f/u)|^2 + \frac{1}{u} f^2 \right) \, m \, dx.$$

Proof. The function f^2/u belongs to $W^{1,2}(M)$, so we may use it as a test function in the weak form of $Lu = 1$ to obtain

$$\int_M [(\nabla_A u \cdot \nabla_A (f^2/u)) + Vu(f^2/u)] m dx = \int_M (f^2/u) m dx.$$

Substituting the identity $\nabla_A u \cdot \nabla_A (f^2/u) = |\nabla_A f|^2 - u^2 |\nabla_A (f/u)|^2$ (from the product rule), this becomes

$$\int_M (|\nabla_A f|^2 - u^2 |\nabla_A (f/u)|^2 + Vf^2) m dx = \int_M (f^2/u) m dx,$$

which, after moving a term from the left to the right, is the desired result. \square

Given the importance of [Lemma 4.1](#) to this paper, we wish to elaborate on it, recapitulating the introduction with more details. Recall that

$$Lf = -\frac{1}{m} \operatorname{div} (mA \nabla f) + Vf$$

in the weak sense. Define the operator \tilde{L} by

$$\tilde{L}g := \frac{1}{u} L(gu).$$

In other words, \tilde{L} is the conjugation of L by the operator multiplication by u . If the functions m and A are differentiable, then one can use equation $Lu = 1$ to compute that

$$\tilde{L}g = -\frac{1}{mu^2} \operatorname{div} (mu^2 A \nabla g) + \frac{1}{u} g.$$

Note that the operator \tilde{L} is of the same form as L but with a different density and potential. The key point is that the potential V in L has been replaced by the potential $1/u$ in \tilde{L} . Mechanisms of this type are familiar in the theory of second order differential equations. Conjugation of operators of the form $-\Delta + V$ using an auxiliary solution is a standard device leading to the generalized maximum principle (see Theorem 10, p. 73 [[26](#)]). A similar device appears even earlier in work of Jacobi on conjugate points and work of Sturm on oscillation of eigenfunctions. In all of these cases, the multipliers are eigenfunctions or closely related supersolutions rather than solutions to the equation $Lu = 1$.

Consider the space $L^2(M, m dx)$ with inner product $\langle \cdot, \cdot \rangle$. The operators L and $u^2 \tilde{L}$ are self adjoint in this inner product. Using the formula for \tilde{L} above, one could derive the lower bound $\langle Lf, f \rangle \geq \langle (1/u)f, f \rangle$ formally by substituting $f = gu$:

$$\langle Lf, f \rangle = \langle u^2 \tilde{L}g, g \rangle \geq \langle u^2 (1/u)g, g \rangle = \langle (1/u)f, f \rangle.$$

[Lemma 4.1](#) implies that the identity $\langle Lf, f \rangle = \langle u^2 \tilde{L}g, g \rangle$ is valid in weak form. Indeed, it says that

$$\langle Lf, f \rangle = \int_M (|\nabla_A f|^2 + Vf^2) m dx = \int_M \left[u^2 |\nabla_A (f/u)|^2 + \frac{1}{u} f^2 \right] m dx,$$

and so, since $g = f/u$,

$$\langle Lf, f \rangle = \int_M u^2 \left[|\nabla_A g|^2 + \frac{1}{u} g^2 \right] m dx = \langle u^2 \tilde{L}g, g \rangle.$$

Although conjugation and the calculation of \tilde{L} leads to our identity, the weak form has considerable advantages. It is easier to check the weak formula than the differential formula for \tilde{L} because it only involves first derivatives. Moreover, because we only

differentiated once and didn't integrate by parts, our proof of [Lemma 4.1](#) was not only shorter but also more general in that it applied to bounded measurable m and A .

We will now derive estimates of Agmon type from [Lemma 4.1](#).

Lemma 4.2. *Suppose φ belongs to $W^{1,2}(M) \cap C(M)$, $\varphi = 0$ on a compact subset K of M and $L\varphi = \mu\varphi$ weakly on $M \setminus K$. Let u be as in [Lemma 4.1](#) and let g be a Lipschitz function on M . Then*

$$\int_M \left[u^2 |\nabla_A(g\varphi/u)|^2 + \left(\frac{1}{u} - \mu \right) (g\varphi)^2 \right] m \, dx = \int_M |\nabla_A g|^2 \varphi^2 m \, dx. \quad (4.1)$$

Furthermore, setting $g = \chi e^h$ with h and χ Lipschitz functions on M , we have

$$\begin{aligned} & \int_M u^2 \left| \nabla_A \left(\frac{\chi e^h \varphi}{u} \right) \right|^2 m \, dx + \int_M \left(\frac{1}{u} - \mu - |\nabla_A h|^2 \right) (\chi e^h \varphi)^2 m \, dx \\ &= \int_M (|\chi \nabla_A h + \nabla_A \chi|^2 - |\chi \nabla_A h|^2) (e^h \varphi)^2 m \, dx. \end{aligned} \quad (4.2)$$

Proof. Since $g^2 \varphi \in W^{1,2}(M)$ and $g^2 \varphi = 0$ on K , it can be used as a test function for the equation $L\varphi = \mu\varphi$, yielding

$$\int_M (V - \mu) g^2 \varphi^2 m \, dx = - \int_M \nabla_A \varphi \cdot \nabla_A (g^2 \varphi) m \, dx. \quad (4.3)$$

Substituting $f = g\varphi$ in [Lemma 4.1](#), gives

$$\begin{aligned} & \int_M \left[|\nabla_A(g\varphi)|^2 + (V - \mu) g^2 \varphi^2 \right] m \, dx \\ &= \int_M \left[u^2 |\nabla_A(g\varphi/u)|^2 + \left(\frac{1}{u} - \mu \right) g^2 \varphi^2 \right] m \, dx. \end{aligned}$$

On the other hand, (4.3) implies that

$$\begin{aligned} & \int_M \left[|\nabla_A(g\varphi)|^2 + (V - \mu) g^2 \varphi^2 \right] m \, dx \\ &= \int_M \left[|\nabla_A(g\varphi)|^2 - \nabla_A \varphi \cdot \nabla_A (g^2 \varphi) \right] m \, dx = \int_M \varphi^2 |\nabla_A g|^2 m \, dx. \end{aligned}$$

This proves (4.1). The second formula, (4.2), follows from the first, by setting $g = \chi e^h$, and using the formula

$$|\nabla_A g|^2 = |\nabla_A(\chi e^h)|^2 = (\chi e^h)^2 |\nabla_A h|^2 + (|\chi \nabla_A h + \nabla_A \chi|^2 - |\chi \nabla_A h|^2) e^{2h}.$$

□

Let w be a nonnegative, continuous function on M . Assume the elliptic matrix A is continuous on M . Denote the entries of $B = A^{-1}$ by $b_{ij}(x)$. We define the distance $\rho(x, y)$ on M for the degenerate Riemannian metric $ds^2 = w(x) \sum b_{ij} dx_i dx_j$ by

$$\rho(x, y) = \inf_{\gamma} \int_0^1 \left(w(\gamma(t)) \sum_{i,j=1}^n b_{ij}(\gamma(t)) \dot{\gamma}_i(t) \dot{\gamma}_j(t) \right)^{1/2} dt,$$

where the infimum is taken over all absolutely continuous paths $\gamma : [0, 1] \rightarrow M$ such

that $\gamma(0) = x$ and $\gamma(1) = y$. (Note that the distance between points in a connected component of the set $\{w = 0\}$ is zero.)

With these notations, we have the following lemma.

Lemma 4.3 ([18], Theorem 4, p. 18]). *If h is real-valued and $|h(x) - h(y)| \leq \rho(x, y)$ for all $x, y \in M$, then h is a Lipschitz function, and*

$$|\nabla_A h(x)|^2 \leq w(x) \quad \text{for all } x \in M.$$

In particular, this holds when

$$h(x) = \inf_{y \in E} \rho(x, y),$$

for any nonempty set $E \subset M$.

The lemma is stated in [18] for w strictly positive. Considering the case $w(x) + \epsilon$ and taking the limit as $\epsilon \searrow 0$ gives the result for non-negative w .

Recall that V is a measurable function on M such that $0 \leq V(x) \leq \bar{V}$, and V is non-zero on a set of positive measure and u is the unique weak solution to $Lu = 1$ on M , the landscape function.

Fix $\mu \geq 0$, and set

$$w_\mu(x) = \left(\frac{1}{u(x)} - \mu \right)_+ = \max \left(\frac{1}{u(x)} - \mu, 0 \right).$$

With our additional assumption that the elliptic matrix A has continuous coefficients on M , we can define $\rho_\mu(x, y)$ as the Agmon distance associated to the weight $w_\mu(x)$. For any $E \subset M$, denote

$$\rho_\mu(x, E) = \inf_{y \in E} \rho_\mu(x, y).$$

Theorem 4.4. *Let $0 \leq \mu \leq \nu \leq \bar{V}$ be constants. With u the landscape function as above, denote*

$$E(\nu) = \left\{ x \in M : \frac{1}{u(x)} \leq \nu \right\}.$$

Let K be a compact subset of M . Denote

$$h(x) = \rho_\mu(x, E(\nu) \setminus K), \quad x \in M,$$

and

$$\chi(x) = \begin{cases} h(x), & h(x) < 1, \\ 1, & h(x) \geq 1. \end{cases}$$

Suppose φ belongs to $W^{1,2}(M) \cap C(M)$, $\varphi = 0$ on K , and $L\varphi = \mu\varphi$ weakly on $M \setminus K$. Then for $0 < \alpha < 1$,

$$\begin{aligned} & \int_M u^2 \left| \nabla_A \left(\frac{\chi e^{\alpha h} \varphi}{u} \right) \right|^2 m \, dx + (1 - \alpha^2) \int_M \left(\frac{1}{u} - \mu \right)_+ \left(\chi e^{\alpha h} \varphi \right)^2 m \, dx \\ & \leq (1 + 2\alpha) e^{2\alpha} (\bar{V} - \mu) \int_{\{0 < h < 1\}} \varphi^2 m \, dx. \end{aligned} \tag{4.4}$$

Furthermore, if $\nu = \mu + \delta, \delta > 0$, we have

$$\int_{h \geq 1} e^{2\alpha h} (|\nabla_A \varphi|^2 + \bar{V} \varphi^2) m dx \leq \left(450 + \frac{130 \bar{V}}{(1-\alpha)\delta} \right) \bar{V} \int_M \varphi^2 m dx. \quad (4.5)$$

Proof. Using (4.2) with αh in place of h , the first term on the L.H.S. is the same as in (4.4). Since $\chi = 0$ on $E_\mu \setminus K$ and $\varphi = 0$ on K , we have $\chi \varphi = 0$ on E_μ . Moreover, by Lemma 4.3 $|\nabla_A h|^2 \leq w_\mu(x)$. Thus,

$$\begin{aligned} & \int_M \left(\frac{1}{u} - \mu - \alpha^2 |\nabla_A h|^2 \right) (\chi e^{\alpha h} \varphi)^2 m dx \\ &= \int_{M \setminus E_\mu} \left(\frac{1}{u} - \mu - \alpha^2 |\nabla_A h|^2 \right) (\chi e^{\alpha h} \varphi)^2 m dx \\ &\geq (1-\alpha^2) \int_{M \setminus E_\mu} \left(\frac{1}{u} - \mu \right)_+ (\chi e^{\alpha h} \varphi)^2 m dx \\ &= (1-\alpha^2) \int_M \left(\frac{1}{u} - \mu \right)_+ (\chi e^{\alpha h} \varphi)^2 m dx. \end{aligned}$$

The R.H.S. integrand of (4.2) is zero almost everywhere on the set $\nabla_A \chi = 0$, so we may restrict the integral to the set $\{0 < h < 1\}$. There we have $\chi \equiv h$, so

$$|\chi \alpha \nabla_A h + \nabla_A \chi|^2 - |\chi \alpha \nabla_A h|^2 = [(2\alpha + 1)^2 - \chi^2 \alpha^2] |\nabla_A h|^2 \leq (2\alpha + 1) |\nabla_A h|^2.$$

Finally, $|\nabla_A h|^2 \leq w_\mu(x) \leq \bar{V} - \mu$, by Lemma 4.3 and Proposition 3.2. This concludes the proof of (4.5).

It remains to prove (4.5). For convenience, normalize φ so that its $L^2(M, m dx)$ norm is 1:

$$\|\varphi\|^2 := \int_M \varphi^2 m dx = 1.$$

Let $f = \chi e^{\alpha h} \varphi$. Since $f = 0$ on $E(\nu)$, $(1/u - \mu) \geq \delta$ on $M \setminus E(\nu)$, and $\mu \geq 0$, (4.4) implies

$$\int_M u^2 |\nabla_A (f/u)|^2 m dx + (1-\alpha^2) \delta \int_M f^2 m dx \leq (1+2\alpha) e^{2\alpha} \bar{V}. \quad (4.6)$$

Since ∇f and ∇u belong to $L^2(M)$, and $1/u$ and f belong to $L^\infty(M)$, f^2/u is a permissible test function. Thus, using $Lu = 1$, $1/u(x) \leq \bar{V}$, $V(x) \geq 0$, and (4.6), we have

$$\begin{aligned} & \int_M \nabla_A u \cdot \nabla_A (f^2/u) m dx = \int_M (1 - Vu) (f^2/u) m dx \\ & \leq \bar{V} \int_M f^2 m dx \leq \frac{(1+2\alpha) e^{2\alpha}}{(1-\alpha^2)\delta} \bar{V}^2 \leq \frac{3e^2}{2(1-\alpha)\delta} \bar{V}^2. \end{aligned} \quad (4.7)$$

Next,

$$\begin{aligned}
\int_M |\nabla_A u|^2 (f/u)^2 m dx &= - \int_M 2(f/u)(\nabla_A u) \cdot (u \nabla_A (f/u)) m dx \\
&\quad + \int_M \nabla_A u \cdot \nabla_A (f^2/u) m dx \\
&\leq \int_M \left[\frac{1}{2} (f/u)^2 |\nabla_A u|^2 + 2u^2 |\nabla_A (f/u)|^2 + \nabla_A u \cdot \nabla_A (f^2/u) \right] m dx.
\end{aligned}$$

Hence, after subtracting the term with factor 1/2 and multiplying by 2,

$$\begin{aligned}
\int_M |\nabla_A u|^2 (f/u)^2 m dx &\leq \int_M \left[4u^2 |\nabla_A (f/u)|^2 + 2\nabla_A u \cdot \nabla_A (f^2/u) \right] m dx \\
&\leq 4(1+2\alpha)e^{2\alpha}\bar{V} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} \\
&\leq 12e^2\bar{V} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_M |\nabla_A f|^2 m dx &= \int_M |u \nabla_A (f/u) + (f/u) \nabla_A u|^2 m dx \\
&\leq 2 \int_M u^2 |\nabla_A (f/u)|^2 m dx + 2 \int_M |\nabla_A u|^2 (f/u)^2 m dx \\
&\leq 2(1+2\alpha)e^{2\alpha}\bar{V} + 2 \left[12e^2\bar{V} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} \right] m dx \\
&\leq 30e^2\bar{V} + 6e^2 \frac{\bar{V}^2}{(1-\alpha)\delta}.
\end{aligned} \tag{4.8}$$

Finally, since $e^{\alpha h}\varphi = f$ on $\{h \geq 1\}$, and $|\nabla_A h|^2 \leq \bar{V}$, we have (by (4.7) and (4.8) in particular)

$$\begin{aligned}
\int_{\{h \geq 1\}} e^{2\alpha h} |\nabla_A \varphi|^2 m dx &= \int_{\{h \geq 1\}} |\nabla_A (e^{\alpha h}\varphi) - \alpha(\nabla_A h)e^{\alpha h}\varphi|^2 m dx \\
&\leq 2 \int_{\{h \geq 1\}} |\nabla_A (e^{\alpha h}\varphi)|^2 m dx + 2 \int_{\{h \geq 1\}} \alpha^2 |\nabla_A h|^2 (e^{\alpha h}\varphi)^2 m dx \\
&\leq 2 \int_{\{h \geq 1\}} |\nabla_A f|^2 m dx + 2\bar{V} \int_{\{h \geq 1\}} f^2 m dx \\
&\leq 60e^2\bar{V} + 12e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} + 3e^2 \frac{\bar{V}^2}{(1-\alpha)\delta}.
\end{aligned} \tag{4.9}$$

Thus, by (4.7) again,

$$\begin{aligned} \int_{\{h \geq 1\}} e^{2\alpha h} (|\nabla_A \varphi|^2 + \bar{V} \varphi^2) \ m \ dx &\leq 60e^2 \bar{V} + 15e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} + \frac{3}{2}e^2 \frac{\bar{V}^2}{(1-\alpha)\delta} \\ &\leq \left(450 + \frac{130\bar{V}}{(1-\alpha)\delta} \right) \bar{V}. \end{aligned}$$

□

Theorem 4.4 displays the dependence of the constant as $\alpha \rightarrow 1$. We state next a variant for $\alpha = 1/2$ in the form we will use below.

Corollary 4.5. *Let $0 < \mu \leq \bar{\mu}$ and $0 < \delta \leq \bar{V}/10$ be constants. Suppose that $\bar{\mu} + \delta \leq \bar{V}$. Let K be a compact subset of M , and set*

$$h_K(x) = \bar{\rho}(x, E(\bar{\mu} + \delta) \setminus K), \quad x \in M,$$

with $\bar{\rho} = \rho_{\bar{\mu}}$ the Agmon metric associated to the weight $\bar{w}(x) = (1/u(x) - \bar{\mu})_+$. Suppose φ belongs to $W^{1,2}(M) \cap C(M)$, $\varphi = 0$ on K , and $L\varphi = \mu\varphi$ weakly on $M \setminus K$. Then

$$\int_{h_K \geq 1} e^{h_K} (|\nabla_A \varphi|^2 + \bar{V} \varphi^2) \ m \ dx \leq 18e \left(\frac{\bar{V}}{\delta} \right) \bar{V} \int_M \varphi^2 \ m \ dx. \quad (4.10)$$

In particular, in the case $K = \emptyset$, the corollary says that for eigenfunctions ψ satisfying $L\psi = \lambda\psi$ weakly on all of M for which $\lambda \leq \bar{\mu}$, we have

$$\int_{h \geq 1} e^h (|\nabla_A \psi|^2 + \bar{V} \psi^2) \ m \ dx \leq 18e \left(\frac{\bar{V}}{\delta} \right) \bar{V} \int_M \psi^2 \ m \ dx. \quad (4.11)$$

with

$$h(x) = \bar{\rho}(x, E(\bar{\mu} + \delta)), \quad x \in M.$$

Proof. Corollary 4.5 is not, strictly speaking, a corollary of Theorem 4.4, but rather the specialization of the inequalities in the proof to the case $\alpha = 1/2$. Note also the theorem is proved for $\mu = \bar{\mu}$, but the corollary is also valid for any larger value of $\bar{\mu}$. This because increasing $\bar{\mu}$ gives rise to a weaker conclusion: it decreases h_K .

Rather than repeat the proof, we indicate briefly the arithmetic that ensues from setting $\alpha = 1/2$ in the proof of Theorem 4.4. With $f = \chi e^{h_K/2} \varphi$ and the normalization $\|\varphi\| = 1$, we have

$$\int_{\{h_K \geq 1\}} e^{h_K} \bar{V} \varphi^2 \ m \ dx \leq \bar{V} \int_M f^2 \ m \ dx \leq \frac{8e}{3} \frac{\bar{V}^2}{\delta},$$

as in the second line of (4.7),

$$\int_M |\nabla_A f|^2 \ m \ dx \leq \left(20 + \frac{3\bar{V}}{\delta} \right) e \bar{V},$$

by the proof of (4.8), and (as for (4.9))

$$\int_{\{h_K \geq 1\}} e^{h_K} |\nabla_A \varphi|^2 \, m \, dx \leq \left(40 + \frac{34\bar{V}}{3\delta} \right) e\bar{V}.$$

Therefore, again with the normalization $\|\varphi\| = 1$,

$$\int_{\{h_K \geq 1\}} e^{h_K} (|\nabla_A \varphi|^2 + \bar{V} \varphi^2) \, m \, dx \leq \left(40 + 14 \frac{\bar{V}}{\delta} \right) e\bar{V} \leq 18e \left(\frac{\bar{V}}{\delta} \right) \bar{V},$$

where we have used $\delta \leq \bar{V}/10$ to obtain the last inequality. \square

5. Localized approximate eigenfunctions

We have already proved a theorem about exponential decay of the eigenfunctions ψ . We will now show, roughly speaking, that if the landscape function predicts localization, then an eigenfunction with eigenvalue λ is localized in the components of $\{1/u \leq \lambda\}$ where an appropriate localized problem has an eigenvalue in the range $\lambda \pm \delta$.

Let $\bar{\mu}$ and δ be as in [Corollary 4.5](#). Consider any finite decomposition of the sublevel set $E(\bar{\mu} + \delta)$ into subsets:

$$E(\bar{\mu} + \delta) = \left\{ x \in M : \frac{1}{u(x)} \leq \bar{\mu} + \delta \right\} = \bigcup_{\ell=1}^R E_\ell.$$

We regard the sets E_ℓ as potential wells. It is easiest to visualize E_ℓ as the (closed) connected components of $E(\bar{\mu} + \delta)$. In practice, such connected wells often yield the optimal result. But there is no requirement that E_ℓ be connected. Rather each E_ℓ should be chosen to consist of a collection of “nearby” wells. It is occasionally useful to merge nearby wells because what is important is to choose the sets E_ℓ so as to have a large separation between them, where the separation \bar{S} is defined by

$$\bar{S} = \inf \left\{ \bar{\rho}(x, y) : x \in E_\ell, y \in E_{\ell'}, \ell \neq \ell' \right\},$$

that is, the smallest effective distance between wells. Here, as before, $\bar{\rho} = \rho_{\bar{\mu}}$ denotes the Agmon metric associated to the weight $\bar{w}(x) = (1/u(x) - \bar{\mu})_+$. Whether or not a decomposition into small, well-separated wells exists depends on the level set structure of $1/u(x)$ and the size of $\bar{\mu} + \delta$.

Let $S_1 < \bar{S}$ (as near to \bar{S} as we like). We claim that there is a compact set $K_\ell \subset M = \bar{\Omega}$ satisfying the hypothesis of [Proposition 3.3](#) and such that

$$\{x \in M : \bar{\rho}(x, E_\ell) \geq \bar{S}/2\} \subset K_\ell \subset \{x \in M : \bar{\rho}(x, E_\ell) > S_1/2\}. \quad (5.1)$$

In fact, as we will show in [Lemma 6.2](#), for any compact $K \subset M$ and any neighborhood $U \supset K$ (that is, U is relatively open in M) there is an intermediate set $K \subset K' \subset U$ such that K' satisfies the bi-Lipschitz cone condition.

Define Ω_ℓ as the connected component of $M \setminus K_\ell$ containing E_ℓ . Because the sets E_ℓ are at least distance \bar{S} apart, the sets Ω_ℓ are disjoint.

Denote by $W_0^{1,2}(\Omega_\ell)$ the closure in $W^{1,2}(M)$ norm of the space of smooth functions that are compactly supported on Ω_ℓ . Note that these functions can be extended by zero on $M \setminus \Omega_\ell$ and regarded as belonging to $W^{1,2}(M)$. But the notation is slightly misleading, because Ω_ℓ is not necessarily open, and may contain parts of ∂M that do not lie in

K_ℓ . On those parts, functions of $W_0^{1,2}(\Omega_\ell)$ do not need to vanish. In other words, our definition of $W_0^{1,2}(\Omega_\ell)$ includes a Dirichlet condition on $K_\ell \cap \partial\Omega_\ell$ only.

The operator L is self-adjoint with our mixture of Dirichlet and Neumann conditions, and for each ℓ there a complete system of orthonormal eigenfunctions $\varphi_{\ell,j} \in W_0^{1,2}(\Omega_\ell)$ satisfying

$$\int_M [\nabla_A \varphi_{\ell,j} \cdot \nabla \zeta + V \varphi_{\ell,j} \zeta] m \, dx = \mu_{\ell,j} \int_M \varphi_{\ell,j} \zeta m \, dx$$

for all test functions ζ in $W_0^{1,2}(\Omega_\ell)$. We have Dirichlet conditions on $K_\ell \cap \partial\Omega_\ell$. If $\partial\Omega_\ell \cap \partial M$ is non-empty, then on that portion of the boundary, the weak equation is interpreted as a Neumann condition. But we will never have to use normal derivatives, only the weak equation. The purpose of inserting the somewhat nicer domain Ω_ℓ is so that the eigenfunctions $\varphi_{\ell,j}$ are continuous (in fact Hölder continuous) on M . We do this so that the integrals in the lemmas above are well defined. None of our inequalities with exponential weights depend on the Lipschitz constant of Ω_ℓ , just as they don't depend on the ellipticity constant or modulus of continuity of A .

Let ψ_j denote the complete system of orthonormal eigenfunctions of L on M with eigenvalues λ_j . Let $\Psi_{(a,b)}$ denote the orthogonal projection in $L^2(M, m \, dx)$ onto the span of eigenvectors ψ_j with eigenvalue $\lambda_j \in (a, b)$. Let $\Phi_{(a,b)}$ be the orthogonal projection onto the span of the eigenvectors $\varphi_{\ell,j}$ with eigenvalue $\mu_{\ell,j} \in (a, b)$. Thus the range of $\Phi_{(0,\infty)}$ is the subspace of $L^2(M, m \, dx)$ of functions supported on $\cup_\ell \Omega_\ell$.

Theorem 5.1. *Let $0 < \delta \leq \bar{V}/10$. If φ is one of the $\varphi_{\ell,j}$ with eigenvalue $\mu = \mu_{\ell,j}$ and $\mu \leq \bar{\mu} - \delta$, and \bar{S} is the effective distance separating wells, defined above, then*

$$\|\varphi - \Psi_{(\mu-\delta, \mu+\delta)} \varphi\|^2 \leq 300 \left(\frac{\bar{V}}{\delta} \right)^3 e^{-\bar{S}/2} \|\varphi\|^2,$$

where here and below, $\|\cdot\|$ denotes the norm in $L^2(M, m \, dx)$. If ψ is one of the ψ_j with eigenvalue $\lambda = \lambda_j \leq \bar{\mu} - \delta$, then

$$\|\psi - \Phi_{(\lambda-\delta, \lambda+\delta)} \psi\|^2 \leq 300 \left(\frac{\bar{V}}{\delta} \right)^3 e^{-\bar{S}/2} \|\psi\|^2.$$

Note that this theorem only has content if \bar{S} is sufficiently large that

$$\frac{\bar{V}}{\delta} < 300^{-1/3} e^{\bar{S}/6}.$$

The separation \bar{S} increases as $\bar{\mu}$ decreases. Recall, also, that we have the flexibility to choose the sets E_ℓ so as to merge nearby wells that are not sufficiently separated. Numerical experiments show that the partition into well-separated wells does occur with high probability for many classes of random potentials V .

Proof. Here and in the remainder of the paper all eigenfunctions are normalized to have $L^2(m \, dx)$ norm 1. Consider φ such that $L\varphi = \mu\varphi$ in the weak sense on Ω_ℓ . Let

$$\eta(x) = f(\bar{p}(x, E_\ell))$$

be defined by

$$f(t) = \begin{cases} 1, & t \leq \frac{S_1}{2} - 1 \\ \frac{S_1}{2} - t, & \frac{S_1}{2} - 1 \leq t \leq \frac{S_1}{2} \\ 0, & \frac{S_1}{2} \leq t \end{cases}$$

Let r be the distribution satisfying the equation

$$L(\eta\varphi) = \mu\eta\varphi + r$$

in the weak sense on M . In other words, r is defined by

$$r(\zeta) := \int_M [\nabla_A(\eta\varphi) \cdot \nabla_A \zeta + (V - \mu)\eta\varphi\zeta] m \, dx$$

for all ζ smooth functions on M . Since $\eta\zeta$ is a suitable test function for $L\varphi = \mu\varphi$ in Ω_ℓ , we have

$$\int_M [\nabla_A(\varphi) \cdot \nabla_A(\eta\zeta) + (V - \mu)\eta\varphi\zeta] m \, dx = 0.$$

Subtracting this formula from the previous one for r , we find that

$$r(\zeta) = \int_M [\varphi \nabla_A \eta \cdot \nabla_A \zeta - \zeta \nabla_A \varphi \cdot \nabla_A \eta] m \, dx.$$

Observe that if $\nabla_A \eta(x) \neq 0$, then $\frac{S_1}{2} - 1 \leq \bar{\rho}(x, E_\ell) \leq \frac{S_1}{2}$. Furthermore, since the distance from E_ℓ to $E_{\ell'}$, $\ell' \neq \ell$, is greater than S_1 , $\bar{\rho}(x, E_{\ell'}) \geq S_1/2$. Thus, since $E(\bar{\mu} + \delta) = \bigcup_{\ell=1}^R E_\ell$, we have $\bar{\rho}(x, E(\bar{\mu} + \delta)) \geq S_1/2 - 1$. In particular, for any set K , $\nabla_A \eta(x) \neq 0$ implies

$$h_K(x) = \bar{\rho}(x, E_{\bar{\mu} + \delta} \setminus K) \geq \frac{S_1}{2} - 1.$$

We use this, (4.10) with $K = M \setminus \Omega_\ell$, and $|\nabla_A \eta|^2 \leq \bar{V}$ to obtain (recall the normalization $\|\varphi\| = 1$)

$$\begin{aligned} |r(\zeta)| &\leq (\sup |\nabla_A \eta|) \|\nabla_A \zeta\| \left(\int_{\{\nabla_A \eta \neq 0\}} \varphi^2 m \, dx \right)^{1/2} \\ &\quad + (\sup |\nabla_A \eta|) \|\zeta\| \left(\int_{\{\nabla_A \eta \neq 0\}} |\nabla_A \varphi|^2 m \, dx \right)^{1/2} \\ &\leq \|\nabla_A \zeta\| \left(\int_{\{\nabla_A \eta \neq 0\}} \bar{V} \varphi^2 m \, dx \right)^{1/2} + \bar{V}^{1/2} \|\zeta\| \left(\int_{\{\nabla_A \eta \neq 0\}} |\nabla_A \varphi|^2 m \, dx \right)^{1/2} \\ &\leq (\|\nabla_A \zeta\|^2 + \bar{V} \|\zeta\|^2)^{1/2} \left(\frac{18e^2 \bar{V}^2}{\delta e^{S_1/2}} \right)^{1/2}. \end{aligned}$$

We will abbreviate this inequality by

$$r(\zeta)^2 \leq \varepsilon \bar{V} [||\nabla_A \zeta||^2 + \bar{V} ||\zeta||^2], \quad \varepsilon := 18e^2 \frac{\bar{V}}{\delta} e^{-s_1/2}. \quad (5.2)$$

Since $V(x) \geq 0$,

$$||\nabla_A \psi_j||^2 + \bar{V} ||\psi_j||^2 \leq \int_M [|\nabla_A \psi_j|^2 + (V + \bar{V}) \psi_j^2] m \, dx = \lambda_j + \bar{V}.$$

Let J be any finite list of indices j such that $|\lambda_j - \mu| \geq \delta$ and let

$$\zeta = \sum_{j \in J} \gamma_j \psi_j$$

be any linear combination of the ψ_j . By density considerations, such a ζ is admissible. Then, since $V \geq 0$,

$$||\nabla_A \zeta||^2 + \bar{V} ||\zeta||^2 \leq \int_M [|\nabla_A \zeta|^2 + (V + \bar{V}) \zeta^2] m \, dx = \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2.$$

Consequently, it follows from (5.2) that

$$r(\zeta)^2 \leq \varepsilon \bar{V} [||\nabla_A \zeta||^2 + \bar{V} ||\zeta||^2] \leq \varepsilon \bar{V} \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2.$$

Denote by

$$\beta_j = \int_M \eta \varphi \psi_j m \, dx = \langle \eta \varphi, \psi_j \rangle$$

the coefficients of $\eta \varphi$ in the basis. Because $(L - \lambda_j) \psi_j = 0$ in the weak sense,

$$\begin{aligned} r(\zeta) &= \sum_{j \in J} \gamma_j \int_M [\nabla_A \psi_j \nabla_A (\eta \varphi) + (V - \lambda_j) \psi_j \eta \varphi] m \, dx \\ &\quad + \int_M \gamma_j (\lambda_j - \mu) \psi_j \eta \varphi m \, dx \\ &= \sum_{j \in J} \gamma_j (\lambda_j - \mu) \beta_j. \end{aligned}$$

Thus,

$$\left| \sum_{j \in J} \gamma_j (\lambda_j - \mu) \beta_j \right|^2 = r(\zeta)^2 \leq \varepsilon \bar{V} \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2.$$

Setting $\gamma_j = \beta_j (\lambda_j + \bar{V})^{-1/2} \operatorname{sgn}(\lambda_j - \mu)$, we find that

$$\left| \sum_{j \in J} \frac{|\lambda_j - \mu|}{\sqrt{\lambda_j + \bar{V}}} \beta_j^2 \right|^2 \leq \varepsilon \bar{V} \sum_{j \in J} \beta_j^2.$$

Since $\lambda_j \geq 0$ and $|\lambda_j - \mu| \geq \delta$,

$$\frac{|\lambda_j - \mu|}{\sqrt{\lambda_j + \bar{V}}} \geq \frac{\delta}{\sqrt{\bar{V}}}.$$

Therefore,

$$\sum_{j \in J} \beta_j^2 \leq \varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Since the set J is an arbitrary finite subset of j such that $|\lambda_j - \mu| \geq \delta$, we have

$$\|\eta\varphi - \Psi_{(\mu-\delta, \mu+\delta)}(\eta\varphi)\|^2 \leq \varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Next, it follows from (4.11) and $1 - \eta(x) = 0$ on $\{\bar{\rho}(x, E_\ell) \leq S_1/2 - 1\}$ that

$$\bar{V} \|(1 - \eta)\varphi\|^2 \leq \bar{V} \int_{h_K \geq \frac{S_1}{2} - 1} \varphi^2 m \, dx \leq \varepsilon \bar{V} \quad (K = M \setminus \Omega_\ell),$$

which, since the projection $I - \Psi_{(\mu-\delta, \mu+\delta)}$ has operator norm 1, implies that

$$\|(1 - \eta)\varphi - \Psi_{(\mu-\delta, \mu+\delta)}((1 - \eta)\varphi)\|^2 \leq \varepsilon.$$

Finally, adding the bounds for $\varphi = (1 - \eta)\varphi + \eta\varphi$ and using $\delta \leq \bar{V}/10$, we get

$$\|\varphi - \Psi_{(\mu-\delta, \mu+\delta)}\varphi\|^2 \leq 2\varepsilon \frac{\bar{V}^2}{\delta^2} + 2\varepsilon < 300 \frac{\bar{V}^3}{\delta^3} e^{-\bar{S}/2}.$$

This is the first claim of the theorem (recall the normalization $\|\varphi\| = 1$).

The second claim has a similar proof with the roles of φ and ψ reversed. We will sketch each step, but the reader will need to refer regularly to the previous proof. Let ψ be a normalized eigenfunction of L on M with eigenvalue $\lambda \leq \bar{\mu}$. We use the same cut-off functions

$$\eta_\ell(x) = f(\bar{\rho}(x, E_\ell)),$$

introducing the subscript ℓ since ℓ is no longer fixed. Then define

$$\tilde{\eta} = \sum_\ell \eta_\ell.$$

Note that $\tilde{\eta}\psi$ is compactly supported in the union of the Ω_ℓ , and the Ω_ℓ are disjoint. Define the distribution \tilde{r} by the equation

$$L(\tilde{\eta}\psi) = \lambda \tilde{\eta}\psi + \tilde{r}.$$

By similar reasoning to the proof of the first claim, using (4.11) we have the analog of (5.2), that for all $\zeta \in W^{1,2}(M)$,

$$\tilde{r}(\zeta)^2 \leq \varepsilon \bar{V} \left[\|\nabla_A \zeta\|^2 + \bar{V} \|\zeta\|^2 \right], \quad \varepsilon = 18e^2 \frac{\bar{V}}{\delta} e^{-S_1/2}.$$

Take any finite set \tilde{J} of indices (ℓ, j) and denote

$$\tilde{\zeta} = \sum_{(\ell, j) \in \tilde{J}} \gamma_{\ell j} \varphi_{\ell j}.$$

In the same way as before, we deduce

$$\tilde{r}(\tilde{\zeta})^2 \leq \varepsilon \bar{V} \sum_{\tilde{J}} (\mu_{\ell j} + \bar{V}) \gamma_{\ell j}^2.$$

Moreover, as before, if we define

$$\beta_{\ell j} = \int_M \tilde{\eta} \psi \varphi_{\ell j} m dx.$$

We claim that

$$\tilde{r}(\tilde{\zeta}) = \sum_{\tilde{J}} \gamma_{\ell j} (\mu_{\ell j} - \lambda) \beta_{\ell j}.$$

This last identity is the only place where the proof is slightly different. Observe that because $(L - \mu_{\ell j}) \varphi_{\ell j} = 0$ in the weak sense on Ω_ℓ and $\eta_{\ell'}$ has support disjoint from Ω_ℓ for all $\ell' \neq \ell$,

$$\begin{aligned} & \int_M \left[\nabla_A (\tilde{\eta} \psi) \nabla_A \varphi_{\ell j} + (V - \mu_{\ell j}) \tilde{\eta} \psi \varphi_{\ell j} \right] m dx \\ &= \int_{\Omega_\ell} \left[\nabla_A (\eta_\ell \psi) \nabla_A \varphi_{\ell j} + (V - \mu_{\ell j}) \eta_\ell \psi \varphi_{\ell j} \right] m dx = 0. \end{aligned}$$

This is the only aspect of the proof of the formula for $\tilde{r}(\tilde{\zeta})$ that differs from the one for $r(\zeta)$ above.

Now suppose that for every $(\ell, j) \in \tilde{J}$, $|\mu_{\ell j} - \lambda| \geq \delta$. Then, setting

$$\gamma_{\ell j} = \beta_{\ell j} \left(\mu_{\ell j} + \bar{V} \right)^{1/2} \operatorname{sgn}(\mu_{\ell j} - \lambda),$$

we obtain

$$\sum_{\tilde{J}} \beta_{\ell j}^2 \leq \varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Since $\tilde{\eta} \psi$ is supported in the union $\cup_\ell \Omega_\ell$ and the $\varphi_{\ell j}$ are an orthonormal basis for L^2 on that set, and L is an arbitrary finite subset of indices such that $|\mu_{\ell j} - \lambda| \geq \delta$, we have

$$\|\tilde{\eta} \psi - \Phi_{(\lambda - \delta, \lambda + \delta)}(\tilde{\eta} \psi)\|^2 \leq \varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Next, it follows from the fact that $(1 - \tilde{\eta}(x)) = 0$ on the set where $h(x) = \bar{\rho}(x, E(\bar{\mu} + \delta)) \leq S_1/2 - 1$ and (4.11) that

$$\bar{V} \|(1-\bar{\eta})\psi\|^2 \leq \bar{V} \int_{h \geq \frac{S_1}{2}-1} \psi^2 m \, dx \leq \varepsilon \bar{V}.$$

The rest of the proof is similar. \square

Theorem 5.1 shows that when the landscape potential $1/u(x)$ defines wells that are separated by a large number S , then the eigenfunctions are located in these wells (with a single eigenfunction possibly occupying several wells). An easy consequence is the following corollary saying that the graphs of the two counting functions enumerating eigenvalues of L and eigenvalues localized to wells agree (modulo a shift $\pm\delta$) up to a number \bar{N} defined below.

Corollary 5.2. *Consider the counting functions*

$$N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\}; \quad N_0(\mu) = \#\{\mu_{\ell,j} : \mu_{\ell,j} \leq \mu\}.$$

Recall that $\bar{\mu}$ and δ are used to specify \bar{S} . Suppose that $\mu \leq \bar{\mu}$ and choose \bar{N} such that

$$300\bar{N} \left(\frac{\bar{V}}{\delta} \right)^3 < e^{\bar{S}/2}.$$

Then

$$\min(\bar{N}, N_0(\mu-\delta)) \leq N(\mu) \quad \text{and} \quad \min(\bar{N}, N(\mu-\delta)) \leq N_0(\mu).$$

Proof. Let

$$p = \min(\bar{N}, N(\mu-\delta))$$

Consider the first p eigenvectors ψ_1, \dots, ψ_p of L on M . Then $p \leq N(\mu-\delta)$ implies $\lambda_j \leq \mu-\delta$, and therefore

$$\|\psi_j - \Phi_{(0,\mu)}\psi_j\|^2 \leq 300 \left(\frac{\bar{V}^3}{\delta^3} \right) e^{-\bar{S}/2}.$$

For any nonzero linear combination $\psi = \sum_{j=1}^p \alpha_j \psi_j$, we have

$$\begin{aligned} \|\psi - \Phi_{(0,\mu)}\psi\| &\leq \sum_j |\alpha_j| \|\psi_j - \Phi_{(0,\mu)}\psi_j\| \\ &\leq \left(300 \left(\frac{\bar{V}^3}{\delta^3} \right) e^{-\bar{S}/2} \right)^{1/2} \sum_j |\alpha_j| \\ &\leq \left(300 \left(\frac{\bar{V}^3}{\delta^3} \right) e^{-\bar{S}/2} \right)^{1/2} \|\psi\| p^{1/2} < \|\psi\|, \end{aligned}$$

by the Cauchy-Schwarz inequality and because $p \leq \bar{N}$. Denote by Q the span of the ψ_j , $j = 1, \dots, p$. The inequality implies the restriction of $\Phi_{(0,\mu)}$ to Q is injective and the dimension $N_0(\mu)$ of $\Phi_{(0,\mu)}(Q)$ is at least p . In other words, $N_0(\mu) \geq p$. The proof of the lower bound for $N(\mu)$ is similar. \square

6. Manifolds and approximation

In this section, we discuss two generalizations of the results of Section 3: the extension to manifolds and the removal of the continuity assumption on the coefficients of A . We also prove the boundary regularity for mixed data referred to in Section 3 and construct the intermediate sets K_ℓ of (5.1) with the bi-Lipschitz cone condition.

Let us first see how to replace \mathbb{R}^n with an ambient space \hat{M} defined as a compact, connected C^1 manifold. Let V be a bounded measurable function satisfying $0 \leq V(x) \leq \bar{V}$ on \hat{M} . Let A be a symmetric two-tensor and let m be a density on \hat{M} . In a coordinate chart, x , A is represented locally by a symmetric matrix-valued function (which we shall still denote by A) and m is represented by a scalar function. Given a test function $\eta = \eta(x)$ compactly supported in the coordinate chart, and a function $\varphi = \varphi(x)$, we write

$$\langle A \nabla \varphi, \nabla \eta \rangle := \int (A \nabla \varphi) \cdot \nabla \eta \ m \ dx, \quad \langle \varphi, \eta \rangle := \int \varphi \ \eta \ m \ dx.$$

We extend these definitions to test functions on all of \hat{M} by using a partition of unity. The covariance property that makes this definition independent of the choice of coordinate charts is that in a new coordinate system y with $x = x(y)$, the expression for the corresponding matrix $\tilde{A}(y)$ and density $\tilde{m}(y)$ is

$$\tilde{A}(y) = B(y)^{-1} A(x(y)) \left(B(y)^{-1} \right)^T, \quad \tilde{m}(y) = |\det B| m(x(y)),$$

where B is the Jacobian matrix

$$B_{ij}(y) = \frac{\partial x_i}{\partial y_j}, \quad B = (B_{ij}).$$

For η supported in the intersection (in the x variable) of the two coordinate charts, denoting $\tilde{\eta}(y) = \eta(x(y))$, $\tilde{\varphi}(y) = \varphi(x(y))$, and $\tilde{V}(y) = V(x(y))$, we have

$$\int [(A \nabla \varphi) \cdot \nabla \eta + V \varphi \eta] \ m \ dx = \int [(\tilde{A} \nabla \tilde{\varphi}) \cdot \nabla \tilde{\eta} + \tilde{V} \tilde{\varphi} \tilde{\eta}] \ \tilde{m} \ dy.$$

Thus we obtain globally defined quantities $\langle A \nabla \varphi, \nabla \eta \rangle$ and $\langle V \varphi, \eta \rangle$.

We will assume that in some family of coordinate charts covering all of \hat{M} , A is represented by bounded measurable, uniformly elliptic matrices and that m is bounded above and below by positive constants. The constant of ellipticity and the constants bounding m from above and below depend on the coordinate charts. But since our estimates won't depend on these constants, this does not matter to us.

Let Ω be an open, connected subset of \hat{M} such that near each point of $\partial\Omega$, Ω is locally bi-Lipschitz equivalent to a half space. This includes as a special case, bi-Lipschitz images of Lipschitz domains in \mathbb{R}^n (for instance, bounded chord-arc domains in \mathbb{R}^2). It also includes the case $\Omega = \hat{M}$ in which the boundary is empty. Set $M = \bar{\Omega}$. Denote the inner product associated to $L^2(M)$ with density m by $\langle \cdot, \cdot \rangle$. Let K be a compact subset of M and let $W_0^{1,2}(M \setminus K)$ denote the closure in $W^{1,2}$ norm of the set of functions in $C^1(M)$ that vanish on K . For $\varphi \in W_0^{1,2}(M \setminus K)$ and $f \in L^2(M \setminus K)$, the weak equation $L\varphi = f$ on $M \setminus K$ is defined by

$$\langle A\nabla\varphi, \nabla\eta \rangle + \langle V\varphi, \eta \rangle = \langle f, \eta \rangle$$

for every $\eta \in W_0^{1,2}(M \setminus K)$.

We will now prove Hölder regularity of solutions up to the boundary for suitable K and f .

Proposition 6.1. *Suppose that $\Omega \subset \hat{M}$ is locally bi-Lipschitz equivalent to a half space at each boundary point. Suppose that K satisfies the bi-Lipschitz cone condition as defined above Proposition 3.3. There is $\alpha > 0$ such that if $f \in L^\infty(M)$ and $\varphi \in W^{1,2}(M)$, with $\varphi = 0$ on K , solves $(L - \mu)\varphi = f$ in the weak sense on $M \setminus K$, then $\varphi \in C^\alpha(M)$.*

Proof. Without loss of generality, we can replace V by $V - \mu$ and assume the constant $\mu = 0$. As we have already observed, the interior Hölder regularity follows from the theorem of De Giorgi-Nash-Moser. We handle the Neumann boundary conditions by performing an even reflection at the boundary of Ω .

It will suffice to consider a single coordinate chart denoted here by y . Let

$$B_r = \{y \in \mathbb{R}^n : |y| < r\}, \quad Q = \{y \in \mathbb{R}^n : y_1 \geq 0\}.$$

and let K be a compact subset of $\bar{B}_1 \cap Q$ satisfying the bi-Lipschitz cone condition. Let $W^{1,2}((B_1 \cap Q) \setminus K)$ be the closure in $W^{1,2}$ norm of functions of $C^1(\bar{B}_1 \cap Q)$ with support disjoint from K . If $\varphi \in W^{1,2}((B_1 \cap Q) \setminus K)$, then the extension of φ by 0 on K belongs to $W^{1,2}(B_1 \cap Q)$. For $f \in L^\infty((B_1 \cap Q))$ we say φ solves $L\varphi = f$ weakly on $(B_1 \cap Q) \setminus K$ if

$$\int_{B_1 \cap Q} [(A\nabla\varphi) \cdot \nabla\eta + V\varphi\eta] m dy = \int_{B_1 \cap Q} f\eta m dy$$

for all $\eta \in C^1(B_1 \cap Q)$ with support disjoint from K . (The fact that η need not vanish on $y_1 = 0$ is what imposes the Neumann condition in the weak sense.) Here, as usual, A is a bounded measurable symmetric matrix, f , V and m bounded measurable functions defined in $B_1 \cap Q$. Moreover, A is elliptic (see (3.1)) and $1/C \leq m(y) \leq C$.

We extend m , V , φ and f to B_1 by reflection as follows. Let R be the reflection,

$$R(y_1, y_2, \dots, y_n) = (-y_1, y_2, \dots, y_n).$$

Set $\tilde{m}(y) = m(y)$, $\tilde{V}(y) = V(y)$, $\tilde{\varphi}(y) = \varphi(y)$, $\tilde{f}(y) = f(y)$, for $y \in B_1 \cap Q$, and

$$\tilde{m}(y) = \tilde{m}(Ry), \quad \tilde{V}(y) = \tilde{V}(Ry), \quad \tilde{\varphi}(y) = \tilde{\varphi}(Ry), \quad \tilde{f}(y) = \tilde{f}(Ry).$$

Define $\tilde{K} = K \cup RK$, then $\tilde{\varphi} = 0$ on \tilde{K} . We extend A to B_1 by

$$\tilde{A}(y) = R\tilde{A}(Ry)R.$$

Note that this is just the appropriate covariance for the changes of variable R since $R = R^T = R^{-1}$. In this way, we extend the definition of L to an operator \tilde{L} on B_1 .

We claim that $\tilde{L}\tilde{\varphi} = \tilde{f}$ weakly on $B_1 \setminus \tilde{K}$. To prove this, let $\eta \in C^1(\bar{B}_1)$ be such that the support of η is disjoint from $\tilde{K} \cup \partial B_1$. Denote

$$\eta_*(y) = \frac{1}{2}(\eta(y) + \eta(Ry)), \quad y \in B_1.$$

Observe that the $*$ operation symmetrizes η , whereas $\tilde{\varphi}$ and \tilde{f} are defined so that they have this symmetry already: $\tilde{\varphi}_* = \tilde{\varphi}$ and $\tilde{f}_* = \tilde{f}$.

Denote the inner products on $L^2(B_1, \tilde{m} dx)$ and $L^2(B_1 \cap Q, m dx)$ by $\langle \cdot, \cdot \rangle_{B_1}$ and $\langle \cdot, \cdot \rangle_Q$, respectively. Since $\tilde{\varphi} = \tilde{\varphi}_*$,

$$\langle \tilde{A} \nabla \tilde{\varphi}, \nabla \eta \rangle_{B_1} = \langle \tilde{A} \nabla \tilde{\varphi}_*, \nabla \eta \rangle_{B_1} = \langle \tilde{A} \nabla \tilde{\varphi}, \nabla \eta_* \rangle_{B_1} = 2 \langle A \nabla \varphi, \nabla \eta_* \rangle_Q$$

Furthermore, using the fact that $\eta_*(y) = 0$ on K and the weak equation for φ on $B_1 \cap Q$, we have

$$2 \langle A \nabla \varphi, \nabla \eta_* \rangle_Q = 2 \langle f, \eta_* \rangle_Q - 2 \langle V \varphi, \eta_* \rangle_Q = \langle \tilde{f}, \eta \rangle_{B_1} - \langle \tilde{V} \tilde{\varphi}, \eta \rangle_{B_1}.$$

Combining these two equations,

$$\langle \tilde{A} \nabla \tilde{\varphi}, \nabla \eta \rangle_{B_1} + \langle \tilde{V} \tilde{\varphi}, \eta \rangle_{B_1} = \langle \tilde{f}, \eta \rangle_{B_1}.$$

In other words, $\tilde{L} \tilde{\varphi} = \tilde{f}$ weakly on $B_1 \setminus \tilde{K}$, which was what we claimed.

We are now in a position to quote local boundary regularity theorems of Gilbarg and Trudinger. Theorems 8.25 and 8.26 of [25] imply that

$$\sup_{B_{1/2}} |\varphi| \leq C \left(\|\varphi\|_{L^1(B_1)} + \|f\|_{L^\infty(B_1)} \right)$$

with a constant C depending only on the ellipticity constants. (Note that the appropriate notion of supremum for $W^{1,2}$ functions, based on truncation, is defined just before Theorem 8.25.)

Next, the local Dirichlet boundary regularity theorem, Theorem 8.27 [25], implies that since K satisfies the bi-Lipschitz cone condition,⁵ there is $\alpha > 0$ such that for $r \leq 1/4$,

$$\operatorname{osc}_{B_r} \varphi \leq C r^\alpha \sup_{B_{1/2}} |\varphi|.$$

This proves Hölder continuity up to the boundary. □

Lemma 6.2. *Let K be a compact subset of M . Let U be a (relatively) open set in M such that $K \subset U$. Then there is a compact set K' , such that $K \subset K' \subset U$, K' satisfies the bi-Lipschitz cone condition.*

Proof. To find K' given K , cover M with a finite number of coordinate charts each of which is the bi-Lipschitz image of a closed cube, some of them interior to Ω and others with a boundary face on $\partial\Omega$. Fix $\varepsilon > 0$, and subdivide each closed cube of the covering dyadically to get a finite covering by cubes of diameter less than ε . Note that although this is not a disjoint covering because of the overlap of the coordinate charts, it is a finite covering. Define K' as the union of cubes in the subdivision that intersect K .

For ε sufficiently small $K' \subset U$. Each individual bi-Lipschitz cube satisfies the bi-Lipschitz cone condition, so this finite union also satisfies the condition □

The last difficulty that we wish to address is that the Agmon length of paths is not defined for discontinuous A . Suppose that A is bounded and measurable (and symmetric and uniformly elliptic as in (3.1)). Using convolution on coordinate charts and a

⁵To apply the theorem as stated one has to make a bi-Lipschitz change of variables to produce an actual cone. This changes the ellipticity constant by a fixed factor. There is an additional term in the estimate in Theorem 8.27, namely, the oscillation of φ over $K \cap B_{\sqrt{r}}$. But in our case, this is zero.

partition of unity, we find a sequence of A^ε of continuous uniformly elliptic two-tensors such that A^ε tends pointwise to A as $\varepsilon \rightarrow 0$. Denote by L and L_ε the operators on M corresponding formally in local coordinates to $-(1/m)\operatorname{div}(mA\nabla) + V$ and $-(1/m)\operatorname{div}(mA^\varepsilon\nabla) + V$.

Proposition 6.3. *Let λ_ε be a bounded sequence, and suppose that $L_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon$ in the weak sense on M , and normalize the eigenfunctions by $\|\psi_\varepsilon\| = 1$ in $L^2(M)$. Then there is a subsequence $\varepsilon_j \rightarrow 0$ such that*

- a. ψ_{ε_j} has a limit ψ in $W^{1,2}(M)$ norm and in $C^\alpha(M)$ norm for some $\alpha > 0$.
- b. λ_{ε_j} has a limit λ and $L\psi = \lambda\psi$ in the weak sense on M .

Proof. By the nondegeneracy of V , the sequence ψ_ε is uniformly bounded in $W^{1,2}(M)$ norm. Moreover by de Giorgi-Nash-Moser regularity the sequence is bounded in $C^\beta(M)$ norm for some $\beta > 0$. Note that β can be chosen independently of ε because ellipticity constants of A^ε are uniformly controlled. By the compactness of $C^\beta(M)$ in $C^\alpha(M)$ for $\alpha < \beta$ and the weak compactness of the unit ball of $W^{1,2}(M)$, there is a subsequence $\varepsilon_j \rightarrow 0$ such that ψ_{ε_j} converges in $C^\alpha(M)$ norm to a function $\psi \in C^\alpha(M) \cap W^{1,2}(M)$. Moreover, $\nabla\psi_{\varepsilon_j} \rightarrow \nabla\psi$ weakly in $L^2(M)$ and $\lambda_{\varepsilon_j} \rightarrow \lambda$ as $j \rightarrow \infty$. Hence, taking the weak limit in the equation $L_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon$, we obtain, $L\psi = \lambda\psi$.

It remains to show that $\nabla\psi_{\varepsilon_j}$ tends to $\nabla\psi$ in $L^2(M)$ norm. Indeed, by the dominated convergence theorem,

$$\|(A^\varepsilon - A)\nabla\psi\| \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (6.1)$$

From now on, we will omit the subscript j from ε with the understanding that we have passed to a subsequence of the A^ε and the ψ_ε . It follows that, along this subsequence,

$$\langle (A - A^\varepsilon)\nabla\psi, \nabla\psi \rangle \rightarrow 0 \quad \text{and} \quad \langle A^\varepsilon\nabla\psi, \nabla\psi \rangle \rightarrow \langle A\nabla\psi, \nabla\psi \rangle.$$

Furthermore, since $\|\nabla\psi_\varepsilon\|$ is uniformly bounded and by (6.1),

$$\langle (A - A^\varepsilon)\nabla\psi, \nabla\psi_\varepsilon \rangle \rightarrow 0.$$

This combined with the weak limit $\langle A\nabla\psi, \nabla\psi_\varepsilon \rangle \rightarrow \langle A\nabla\psi, \nabla\psi \rangle$ yields

$$\langle A^\varepsilon\nabla\psi, \nabla\psi_\varepsilon \rangle \rightarrow \langle A\nabla\psi, \nabla\psi \rangle.$$

Using the identity $L_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon$, we write

$$\langle A^\varepsilon\nabla\psi_\varepsilon, \nabla\psi_\varepsilon \rangle = \lambda_\varepsilon - \langle V\psi_\varepsilon, \psi_\varepsilon \rangle \rightarrow \lambda - \langle V\psi, \psi \rangle.$$

Finally,

$$\langle A^\varepsilon\nabla(\psi_\varepsilon - \psi), \nabla(\psi_\varepsilon - \psi) \rangle = \langle A^\varepsilon\nabla\psi_\varepsilon, \nabla\psi_\varepsilon \rangle - 2\langle A^\varepsilon\nabla\psi, \nabla\psi_\varepsilon \rangle + \langle A^\varepsilon\nabla\psi, \nabla\psi \rangle.$$

The first term of this last expression, $\langle A^\varepsilon\nabla\psi_\varepsilon, \nabla\psi_\varepsilon \rangle \rightarrow \lambda - \langle V\psi, \psi \rangle$. The second term tends to $-2\langle A\nabla\psi, \nabla\psi \rangle$ and the third term to $\langle A\nabla\psi, \nabla\psi \rangle$. But $L\psi = \lambda\psi$ implies $\langle A\nabla\psi, \nabla\psi \rangle = \lambda - \langle V\psi, \psi \rangle$. Thus

$$\langle A^\varepsilon\nabla(\psi_\varepsilon - \psi), \nabla(\psi_\varepsilon - \psi) \rangle \rightarrow 0$$

along the subsequence and $\nabla\psi_\varepsilon$ tends in $L^2(M)$ norm to $\nabla\psi$. \square

Let A have bounded measurable coefficients and let A^ε be a continuous approximation as above. Then the compactness argument in the proposition also shows that the landscape function u_ε tends uniformly to the landscape function u along a suitable subsequence. Because the Agmon distance functions are uniformly Lipschitz, at the expense of a further subsequence, one can ensure that this distance also converges uniformly. Notice that different sequences could, in principle, yield different limiting Agmon distances. For any of the limits we can now deduce estimates analogous to the ones in the previous sections.

We illustrate with (4.11) and discuss the subsequent theorems later. Fix μ and let W_μ be the subspace of $L^2(M)$ spanned by the eigenfunctions of L with eigenvalue $\leq \mu$, and let N be the dimension of W_μ . Denote

$$\mu_\varepsilon = \sup_{\psi \in W_\mu} \frac{\langle L_\varepsilon \psi, \psi \rangle}{\langle \psi, \psi \rangle}.$$

Let $\psi_j^\varepsilon, j = 1, \dots, N$ be the first N eigenfunctions of L_ε , and let λ_j^ε be the corresponding eigenvalues. It follows from the min/max principle and the fact that W_μ has dimension N that $\lambda_j^\varepsilon \leq \mu_\varepsilon, j \leq N$.

We claim that

$$\limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon \leq \mu. \quad (6.2)$$

In fact, if ψ_j satisfying $L\psi_j = \lambda_j\psi_j, j = 1, \dots, N$, is an orthonormal basis of W_μ , then by the dominated convergence theorem, for every $\delta > 0$ there is $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$|\langle L_\varepsilon \psi_j, \psi_k \rangle - \delta_{jk} \lambda_j| \leq \delta.$$

Representing ψ as a linear combination of the ψ_j , we deduce from $\lambda_j \leq \mu$ that $\mu_\varepsilon \leq \mu + N^2\delta$. Hence (6.2) holds.

By Proposition 6.3, for a suitable subsequence of values of ε the orthonormal basis $\psi_j^\varepsilon, j \leq N$, tends in $C^\alpha(M)$ and $W^{1,2}(M)$ norm to an orthonormal set of eigenfunctions of L with eigenvalues $\leq \mu$. Since W_μ has dimension N , this limiting set must be a basis for W_μ . Moreover, these eigenfunctions inherit the inequality (4.11).

There is a difference between this statement and the preceding one, applicable to continuous A . Here we only claim that there exists a basis of the eigenfunctions that satisfies (4.11). If an eigenvalue has multiplicity then the estimate may not apply to all linear combinations of the particular eigenbasis we obtain by taking limits. Thus, we have not ruled out the possibility that there has to be an extra factor of the multiplicity of the eigenspace in inequality (4.11). Similarly, in the comparisons with localized eigenfunctions in Theorem 5.1, we can only deduce that they are valid for *some* basis of eigenfunctions ψ_j and $\varphi_{\ell,j}$.

We leave open whether in the case of discontinuous A , it is possible to recover the full theorem for continuous coefficients for eigenfunctions with multiplicity. Another question that we are leaving open in the discontinuous case is whether the limiting Agmon distance is unique, that is, does not depend on the choice of the sequence A^ε . Even if the limit is not unique, there could be an optimal (largest) choice of h satisfying the Agmon bound $|\nabla_A h|^2 \leq w_\mu(x)$.



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