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An exponential lower bound for the degrees of invariants of cubic forms and tensor actions ☆

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Using the Grosshans Principle, we develop a method for proving lower bounds for the maximal degree of a system of generators of an invariant ring. This method also gives lower bounds for the maximal degree of a set of invariants that define Hilbert's null cone. We consider two actions: The first is the action of $\mathrm{SL}(V)$ on $S^3(V)^{\oplus 4}$, the space of 4-tuples of cubic forms, and the second is the action of $\mathrm{SL}(V) \times \mathrm{SL}(W) \times \mathrm{SL}(Z)$ on the tensor space $(V \otimes W \otimes Z)^{\oplus 9}$. In both these cases, we prove an exponential lower degree bound for a system of invariants that generate the invariant ring or that define the null cone.

1. Introduction

For simplicity, we choose the set $\mathbb{C}$ of complex numbers as our ground field, although most results are valid for arbitrary fields of characteristic 0. Let V be a rational repre-

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sentation of a reductive group G and denote the ring of polynomial functions on V by $\mathbb{C}[V]$. The group G also acts on $\mathbb{C}[V]$ and the ring of invariants is

$$\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] \mid g \cdot f = f \text{ for all } g \in G\}.$$

It is well known that the ring of invariants $\mathbb{C}[V]^G = \bigoplus_{d=0}^{\infty} \mathbb{C}[V]_d^G$ is a finitely generated graded subring of the polynomial ring $\mathbb{C}[V]$ (see [18–20,32]). All representations in this paper will be rational representations by default. A fundamental question in invariant theory is to describe the generators of an invariant ring and their relations.

Invariant rings play a central role in the Geometric Complexity Theory (GCT) approach to the P vs NP problem. This connection to computational complexity results in new problems in invariant theory, albeit with a different flavor. As one might expect, these problems are more quantitative in nature, asking for how easy or hard the invariant ring is from a computational perspective. There are well understood notions of hardness of computation in computational complexity. We refer to [30] for precise details, as well as numerous conjectures and open problems in invariant theory that are inspired by computational complexity. From the perspective of GCT, a central problem of interest is the problem of degree bounds for generators.

The problem of finding strong upper bounds for the degrees of generators has been studied. An approach via understanding the null cone was proposed by Popov (see [33, 34]), and improved by the first author, see [7]. The zero set of a set of polynomials $S \subseteq \mathbb{C}[V]$ is

$$\mathbb{V}(S) = \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}.$$

Hilbert's null cone $\mathcal{N} \subseteq V$ is defined by $\mathcal{N} = \mathbb{V}(\bigoplus_{d=1}^{\infty} \mathbb{C}[V]_d^G)$.

Definition 1.1. We define $\sigma_G(V)$ to be the smallest integer D such that the non-constant homogeneous invariants of degree $\leq D$ define the null cone, so

$$\sigma_G(V) = \min \left\{ D \,\middle|\, \mathcal{N} = \mathbb{V}\big(\bigoplus_{d=1}^{D} \mathbb{C}[V]_d^G \big) \right\}.$$

General upper bounds for $\sigma_G(V)$ were first given by Popov (see [33,34]), and improved by the first author in [7].

Remark 1.2. The number $\sigma_G(V)$ can also be defined as the smallest integer D such that $\mathbb{C}[V]^G$ is a module-finite extension over the subalgebra generated by $\oplus_{d=0}^{D} \mathbb{C}[V]_d^G$.

We define $\beta_G(V)$ to be the smallest integer D such that invariants of degree $\leq D$ generate $\mathbb{C}[V]^G$, i.e.,

$$\beta_G(V) = \min \left\{ D \,\middle|\, \bigoplus_{d=0}^{D} \mathbb{C}[V]_d^G \text{ is a generating set for } \mathbb{C}[V]^G \right\}.$$

The number $\beta_G(V)$ can also be seen as the largest degree of a minimal set of (homogeneous) generators for $\mathbb{C}[V]^G$. It is easy to see that $\beta_G(V) \geq \sigma_G(V)$. The first author showed in [7] that $\beta_G(V) \leq \max\{2, \frac{3}{8}r\sigma_G(V)^2\}$, where r is the Krull dimension of $\mathbb{C}[V]^G$, which is bounded above by $\dim V$.

In this paper, we focus instead on lower bounds. The key idea is to compare two invariant rings via a surjective map between them.

Lemma 1.3. *Suppose U_1, U_2 are representations of G and H respectively, such that we have a degree non-increasing surjective homomorphism $\phi : \mathbb{C}[U_1]^G \twoheadrightarrow \mathbb{C}[U_2]^H$. Then we have*

$$\beta_G(U_1) \geq \beta_H(U_2) \text{ and } \sigma_G(U_1) \geq \sigma_H(U_2).$$

Proof. It is clear that $\beta_G(U_1) \geq \beta_H(U_2)$ since surjections preserve generating sets. For the null cone, the argument is slightly more involved, but follows from Remark 1.2 since surjections preserve finite extensions. $\square$

The source of such surjective maps for us will be Grosshans principle (see [17]).[1]

Theorem 1.4 *(Grosshans principle). Let W be a representation of G, and let H be a closed subgroup of G. Then we have an isomorphism*

$$\psi : (\mathbb{C}[G]^H \otimes \mathbb{C}[W])^G \longrightarrow \mathbb{C}[W]^H.$$

We will derive the following result from Grosshans principle.

Theorem 1.5. *Let V, W be representations of G. Suppose $v \in V$ is such that its orbit $G \cdot v$ is closed. Let $H = \operatorname{Stab}_G(v) = \{g \in G \mid g \cdot v = v\}$ be a closed reductive subgroup of G. Then we have a degree non-increasing surjection*

$$\phi : \mathbb{C}[V \oplus W]^G \twoheadrightarrow \mathbb{C}[W]^H.$$

In particular, we have

$$\beta_G(V \oplus W) \geq \beta_H(W) \text{ and } \sigma_G(V \oplus W) \geq \sigma_H(W).$$

In order to use this method for finding invariant rings for G with large degree lower bounds, there are mainly three steps, each of which is relatively challenging. First, we have to show that the orbit of a certain point v is closed. Next, we must compute its stabilizer H. Finally, we need to find a G-representation W for which $\beta_H(W)$ is large.

[1] Grosshans principle has been used in the context of degree bounds before, see for example [4].

We develop the techniques in this paper in a general setup as we believe they are likely useful in many situations. To show that orbits are closed, we will use a criterion involving the moment map (see Theorem 6.5). We will pick our point carefully, so as to ensure that its stabilizer is a torus. For torus actions, it is relatively easier to construct examples with exponential lower bounds.

At this juncture, we make a remark to clarify the significance of exponential lower bounds. An important algorithmic problem in GCT is the null cone membership problem – decide if a given point is in the null cone. For torus actions, there are polynomial time algorithms for the null cone membership problem and this is connected to linear programming. However, for the actions of non-commutative groups (such as SL_n), the null cone membership is significantly harder. Mulmuley suggests a very general approach using the notion of a succinct encoding. A key conjecture in this approach predicts that generators for invariant rings can be packed into a polynomial sized succinct encoding.[2] While not strictly necessary, polynomial degree bounds can be very helpful in constructing such encodings (for example in the case of matrix semi-invariants [9,30]). Further, in specific cases, polynomial degree bounds have played a crucial role in obtaining polynomial time algorithms for null cone membership (and the more general orbit closure intersection), see [1,11,12,14,22,23]. For the above reasons, it is important to try and understand which representations have polynomial bounds and which do not.

In this paper, we prove exponential bounds in two cases, i.e., cubic forms and tensor actions. We will now proceed to state our results and explain the relevance and significance of these two particular cases in the context of complexity theory and in particular GCT.

1.1. Cubic forms

For a vector space V, we denote by $S^3(V)$ the third symmetric power (which has a natural action of $\mathrm{SL}(V)$) and loosely refer to it as cubic forms. We prove the following exponential lower bound for cubic forms.

Theorem 1.6. *Let V be a vector space of dimension $3n$. Then*

$$\beta_{\mathrm{SL}(V)}(S^3(V)^{\oplus 4}) \geq \sigma_{\mathrm{SL}(V)}(S^3(V)^{\oplus 4}) \geq \tfrac{2}{3}(4^n - 1).$$

We note that $\dim(S^3(V)^{\oplus 4}) = O(n^3)$, and $\dim(\mathrm{SL}(V)) = O(n^2)$. So, the group and the representation are polynomially sized in n, while the lower bound for the degree of generators is exponential in n.

The lower bound for cubic forms in Theorem 1.6 is meant to convince the GCT community that one should not expect polynomial bounds in any reasonable generality.

[2] While this strong version of Mulmuley's conjecture has been disproved recently (see [16]), a weaker formulation for separating invariants is potentially true, and degree bounds for generating invariants and degree bounds for separating invariants are polynomially related as was proved by the first author in [7].

First, let us note that it is not so difficult to produce representations with exponential degree bounds for SL_n actions. Indeed, take any representation W of SL_n for which one can prove exponential degree lower bounds for invariants with respect to a maximal torus (for e.g., $W = S^3(\mathbb{C}^n)$). Then using Theorem 1.5, one can show exponential lower degree bounds for the ring of SL_n-invariants for $\mathrm{Ad} \oplus W$, where Ad denotes the adjoint representation.[3] However, while the adjoint representation is a very simple one from an algebraic perspective, it is not the case from a complexity-theoretic perspective. The adjoint representation is a degree n representation and the next question would inevitably be to understand whether we have polynomial bounds for constant degree representations. Our lower bound for cubic forms shows that we do not since cubic forms are representations of degree 3, i.e., constant degree representations.

It is easy to see that degree 1 representations of SL_n have polynomial degree bounds and it is an interesting question to understand whether polynomial bounds hold for degree 2 representations or not. Using a theorem of Weyl [38], we can restrict our attention to precisely one representation.

Conjecture 1.7. Consider the action of $G = \mathrm{SL}_n$ on $V = (\mathbb{C}^n \otimes \mathbb{C}^n)^{\oplus n^2}$. Then $\beta_G(V)$ is bounded above by a polynomial in n.

1.2. Tensor actions

We now turn our attention to tensor actions. By a tensor action, we mean the action of $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \cdots \times \mathrm{SL}(V_d)$ on $(V_1 \otimes V_2 \otimes \cdots \otimes V_d)^{\oplus m}$ defined on each copy of $V_1 \otimes V_2 \otimes \cdots \otimes V_d$ by

$$(g_1, g_2, \ldots, g_d) \cdot v_1 \otimes \cdots \otimes v_d = g_1 v_1 \otimes \cdots \otimes g_d v_d.$$

The invariant ring in the case of $d = 2$ is often referred to as matrix semi-invariants. The polynomial degree bounds proved in [9,10] for matrix semi-invariants were instrumental in giving an algebraic polynomial time algorithm for the null cone membership and orbit closure algorithms in this case, see [9,12,22,23]. As a consequence, a polynomial time algorithm for non-commutative rational identity testing was obtained. These advances have resulted in numerous applications, e.g., to Brascamp-Lieb inequalities [15], Paulsen problem [26], entanglement [27], approximate polynomial identity testing [3] and nilpotency index of nil-algebras [13] to name a few.

The case when $d \geq 3$ are of interest because of connections with the algorithmic problem of tensor scaling and the quantum marginal problem (see, e.g., [5]). Polynomial degree bounds for tensor actions would have potentially helped in getting a polynomial time algorithm for tensor scaling and this could have generalized many of the aforementioned applications of the $d = 2$ case. Hence, our lower bound for tensor actions can be

[3] This follows because the orbit of a generic point in the adjoint representation is closed and its stabilizer is a maximal torus. We thank David Wehlau for pointing this out to us.

informally thought of as a barrier for current algorithmic (and optimization) techniques. Further, tensor actions (already in the case of $d = 3$) play an important role in problems of a computational nature in subjects ranging from tensor rank lower bounds to matrix multiplication to understanding equivalence classes for entangled states in quantum information theory. Let us state our result.

Theorem 1.8. *Suppose V, W, Z are vector spaces of dimension $3n$. Then, for the tensor action of $G = \mathrm{SL}(V) \times \mathrm{SL}(W) \times \mathrm{SL}(Z)$ on $(V \otimes W \otimes Z)^{\oplus 9}$, we have*

$$\beta_G(V) \geq \sigma_G(V) \geq 4^n - 1$$

Again, let us point out that the dimension of the group and representation are polynomial in n, but the lower bounds on the degree of generation is exponential in n.

1.3. Organization

In Section 2, we collect some preliminary linear algebraic calculations that will be used in later sections. In Section 3, we recall the invariant theory for torus actions and prove degree lower bounds for certain specific torus actions that we will need in the proofs of Theorem 1.6 and Theorem 1.8. The proof of the main technical result, i.e., Theorem 1.5 is given in Section 4. We quickly recall some notions from root systems and Lie algebras which are needed for computations in Section 5 and we discuss a criterion for closed orbits using the moment map (a generalized form of Dadok–Kac) in Section 6. The proofs of Theorem 1.6 and Theorem 1.8 are in Section 7 and Section 8 respectively. Finally, in Section 9, we discuss the challenges that need to be addressed to extend the technique to positive characteristic.

2. Preliminaries from linear algebra

We will first setup some preliminaries and computations from linear algebra. These computations will be used in proving degree lower bounds for torus actions in cases that we are interested in. An $n \times m$ matrix A should be interpreted as a linear map $A : \mathbb{Q}^m \to \mathbb{Q}^n$. The null space of A is defined as

$$\mathcal{Z}(A) = \{v \in \mathbb{Q}^m \mid Av = 0\}.$$

We will be interested in non-negative integral points in the null space. So, we define

$$\mathcal{I}(A) = \mathcal{Z}(A) \cap \mathbb{Z}^m_{\geq 0}.$$

Observe that $\mathcal{I}(A)$ is a monoid under addition. Further, we will be interested in the minimal generators of the monoid $\mathcal{I}(A)$. So, we define

$$\mathcal{GI}(A) = \{v \in \mathcal{I}(A) \mid v \neq w_1 + w_2 \ \forall \ w_1, w_2 \in \mathcal{I}(A) \setminus \{0\}\}.$$

It is easy to see that $\mathcal{GI}(A)$ is a minimal generating set for the monoid $\mathcal{I}(A)$.

We will be interested in computing this in two specific cases. The first is the $n \times (n+1)$ matrix

$$M = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & -4 & 3 \\ -4 & 1 & \ddots & & 0 & 0 & 0 \\ 0 & -4 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & -4 & 1 & 0 \end{pmatrix} \tag{1}$$

Lemma 2.1. *We have* $\mathcal{Z}(M) = \mathbb{Q} \cdot \left(1, 4, 16, \dots, 4^{n-1}, \frac{4^n - 1}{3}\right)^t$.

Proof. It is clear that the matrix M has full rank, i.e., $\text{rk}(M) = n$. By the rank-nullity theorem, we know that $\mathcal{Z}(M)$ is 1-dimensional. The lemma follows by checking that M kills $\left(1, 4, 16, \dots, 4^{n-1}, \frac{4^n-1}{3}\right)^t$. $\square$

Corollary 2.2. *The set $\mathcal{GI}(M)$ consists of only one vector. Further, we have*

$$\mathcal{GI}(M) = \left\{ \left(1, 4, 16, \dots, 4^{n-1}, \frac{4^n - 1}{3}\right)^t \right\}$$

Proof. Since $\mathcal{Z}(M)$ is 1-dimensional, the set $\mathcal{GI}(M)$ consists of at most one element. This will be smallest non-negative integral element in $\mathcal{Z}(M)$, and this is the one given in the statement of the corollary. $\square$

The second case we will be interested in is the $3n \times (3n - 1)$ matrix

$$N = \begin{pmatrix} B & I_3 & & & \\ & P & I_3 & & \\ & & P & \ddots & \\ & & & \ddots & I_3 \\ & & & & P \quad A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, P = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}.$$

Lemma 2.3. *We have* $\mathcal{Z}(N) = \mathbb{Q} \cdot \left(1, 2, 2, 2, 8, 8, 8, \dots, 2^{2n-3}, 2^{2n-3}, 2^{2n-3}, 2^{2n-1}\right)^t$.

Proof. Suppose $v = (v_1, \dots, v_{3n-1})$ is such that $Nv = 0$. Let us look at this as a system of $3n$ equations in $3n - 1$ variables. As is well understood, each row gives one equation. Let us assume $v_1 = \alpha$. Now, we will go through the equations corresponding to the rows from top to bottom to deduce what v_i have to be for $i \geq 1$.

The first three rows imply that $v_2 = v_3 = v_4 = 2\alpha$. The fourth row implies that $v_5 = 2v_2 + v_3 + v_4 = 4(2\alpha)$. Similarly the fifth and sixth rows imply $v_6 = v_7 = 8\alpha$. The process repeats until we get $v_{3n-4} = v_{3n-3} = v_{3n-2} = 2^{2n-3}\alpha$. The last three equations all imply that $v_{3n-1} = 2^{2n-1}\alpha$. In other words, we have $v = \alpha \cdot \left(1, 2, 2, 2, 8, 8, 8, \dots, 2^{2n-3}, 2^{2n-3}, 2^{2n-3}, 2^{2n-1}\right)^t$. $\square$

Using a similar argument to the case of M, we get:

Corollary 2.4. *The set $\mathcal{GI}(N)$ consists of only one vector. Further, we have*

$$\mathcal{GI}(N) = \left\{ \left(1, 2, 2, 2, 8, 8, 8, \dots, 2^{2n-3}, 2^{2n-3}, 2^{2n-3}, 2^{2n-1}\right)^t \right\}$$

3. Invariants for torus actions

We will briefly recall invariant theory for torus actions. Let $T = (\mathbb{C}^*)^n$ be an n-dimensional (complex) torus. A group homomorphism $T \to \mathbb{C}^*$ is called a character of T. Given two characters $\lambda, \mu : T \to \mathbb{C}^*$, we define a character $\lambda + \mu : T \to \mathbb{C}^*$ defined by $(\lambda + \mu)(t) = \lambda(t)\mu(t)$. With this operation, the set of characters of T form a group called the character group, which we denote by $\mathcal{X}(T)$.

To each $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, we can associate a character also denoted λ by abuse of notation. The character $\lambda : T \to \mathbb{C}^*$ is defined by $\lambda(t) = \prod_{i=1}^n t_i^{\lambda_i}$. This gives an isomorphism of groups $\mathbb{Z}^n \xrightarrow{\sim} \mathcal{X}(T)$. Characters of the torus are often called weights, and we will use this terminology as well.

Let V be a rational representation of T. We make the identification $\mathcal{X}(T) = \mathbb{Z}^n$. For a weight $\lambda \in \mathbb{Z}^n$, the weight space $V_\lambda = \{v \in V \mid t \cdot v = \lambda(t)v \ \forall t \in T\}$. A vector $v \in V_\lambda$ is called a weight vector of weight λ. Any representation V is a direct sum of its weight spaces, i.e., $V = \oplus_{\lambda \in \mathbb{Z}^n} V_\lambda$. In other words, we have a basis consisting of weight vectors.

Let $\mathcal{E} = (e_1, \dots, e_m)$ be an ordered basis of V consisting of weight vectors. Further, suppose each e_i is a weight vector of weight λ_i. Let $x_1, \dots, x_m$ denote the coordinate functions with respect to the basis $e_1, \dots, e_m$. The following are well known:

(1) A monomial $x^v = x_1^{v_1} x_2^{v_2} \dots x_m^{v_m}$ is an invariant monomial if and only if $\sum_i v_i \lambda_i = 0$.
(2) The ring of invariants $\mathbb{C}[V]^T$ is linearly spanned by such invariant monomials.

We will rewrite the above results in a slightly different language. We will first need a definition.

Definition 3.1. Let V be a representation of T with an (ordered) weight basis $\mathcal{E} = (e_1, \dots, e_m)$. Further, suppose each e_i is a weight vector of weight λ_i. Define $M_{\mathcal{E}}(V)$ to be the $n \times m$ matrix whose ith column is λ_i, i.e.,

$$M_{\mathcal{E}}(V) := \begin{pmatrix} | & | & \cdots & | \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ | & | & \cdots & | \end{pmatrix}$$

Remark 3.2. For a different choice of ordered weight basis $\mathcal{E}'$, the matrix $M_{\mathcal{E}'}(V)$ is obtained by a permutation of the columns of $M_{\mathcal{E}}(V)$. This is because the formal sum of the columns (i.e., $\sum_i e^{\lambda_i}$) is called the character of the representation V and independent of the choice of weight basis.

Proposition 3.3. *Let V be a representation of T. Let $\mathcal{E} = (e_1, \dots, e_m)$ be a weight basis, and let $x_1, \dots, x_m$ be the corresponding coordinate functions. Then*

(1) *For $v = (v_1, \dots, v_m) \in \mathcal{I}(M_{\mathcal{E}}(V))$, $x^v = x_1^{v_1} \dots x_m^{v_m}$ is an invariant monomial;*
(2) *The set $\{x^v \mid v \in \mathcal{I}(M_{\mathcal{E}}(V))\}$ is a $\mathbb{C}$-linear spanning set of invariants;*
(3) *The set $\{x^v \mid v \in \mathcal{GI}(M_{\mathcal{E}}(V))\}$ is a minimal set of generators for $\mathbb{C}[V]^T$.*

Proof. The first two statements is simply a rephrasing of the discussion before Definition 3.1. The last follows from the fact that for any matrix A, the set $\mathcal{GI}(A)$ is a minimal generating set for the monoid $\mathcal{I}(A)$. $\square$

The above results are quite standard. For the interested reader, we refer to [8,37] for more details on invariant theory for torus actions.

Now, we consider two specific torus actions and show exponential degree bounds for them. These bounds will be needed in the proofs of Theorem 1.6 and Theorem 1.8.

Proposition 3.4. *Let T act on $V = \mathbb{C}^{n+1}$ such that for some weight basis $\mathcal{E}$, we have $M_{\mathcal{E}}(V) = M$, the matrix in Section 2. Then, we have*

$$\beta_T(V) = \sigma_T(V) = \tfrac{2}{3}(4^n - 1).$$

Proof. Let $\mathcal{E} = e_1, \dots, e_{n+1}$. Let $x_1, \dots, x_{n+1}$ be the coordinates with respect to this basis. From the above proposition, we know that $\{x^v \mid v \in \mathcal{GI}(M)\}$ is a minimal set of generators for the invariant ring. Corollary 2.2 tells us that $\mathcal{GI}(M)$ consists of precisely one element. The corresponding monomial is $f := x_1 x_2^4 x_3^{16} \dots x_n^{4^{n-1}} x_{n+1}^{(4^n-1)/3}$. To summarize, we have $\mathbb{C}[V]^T = \mathbb{C}[f]$.

It is clear that f has degree $(1 + 4 + \dots 4^{n-1} + \frac{4^n-1}{3}) = \frac{2}{3}(4^n - 1)$. It is easy to see that $\beta_T(V) = \sigma_T(V) = \deg(f) = \frac{2}{3}(4^n - 1)$. $\square$

A similar argument gives the following:

Proposition 3.5. *Let $T = (\mathbb{C}^*)^{3n}$ act on an $V = \mathbb{C}^{3n-1}$ such that for some weight basis $\mathcal{E}$, we have $M_{\mathcal{E}}(V) = N$, the matrix in Section 2. Then, we have*

$$\beta_T(V) = \sigma_T(V) = 4^n - 1$$

Finally, we end with a simple statement on degree bounds for subrepresentations for torus actions.

Proposition 3.6. *Suppose $V \subseteq W$ are two representations of T, then*

$$\beta_T(V) \leq \beta_T(W) \text{ and } \sigma_T(V) \leq \sigma_T(W).$$

Proof. Representations of tori are completely reducible, so we have $W = V \oplus V'$, where V' is also a subrepresentation of T. The inclusion $V \hookrightarrow W$ gives a surjection $\pi : \mathbb{C}[W] \to \mathbb{C}[V]$ that is clearly degree non-increasing. It is easy to check that π descends to a map of invariant rings $\mathbb{C}[W]^T \to \mathbb{C}[V]^T$. We claim that this is a surjection. Indeed, for $f \in \mathbb{C}[V]^T$, define $\widetilde{f}$ by $\widetilde{f}(v, v') = f(v)$ for all $(v, v') \in V \oplus V' = W$. Clearly $\widetilde{f} \in \mathbb{C}[W]^T$ and $\pi(\widetilde{f}) = f$. The fact that the surjection $\pi : \mathbb{C}[W]^T \twoheadrightarrow \mathbb{C}[V]^T$ is degree non-increasing implies both statements by Lemma 1.3. □

4. Main technical result

In this section, we will give a proof of Theorem 1.5. Before doing so, we will first discuss some gradings. For any vector space U, the coordinate ring $\mathbb{C}[U] = S(U^*)$ is a polynomial ring, and hence we have a grading $\mathbb{C}[U] = \oplus_{d=0}^{\infty} \mathbb{C}[U]_d$. We will call this the polynomial grading. For any vector space W, and any ring R, we can define a grading on $R \otimes \mathbb{C}[W]$ by setting $(R \otimes \mathbb{C}[W])_d = R \otimes \mathbb{C}[W]_d$. We will call this the W-grading.

We will also need Matsushima's criterion, see [28,2].

Theorem 4.1 *(Matsushima's criterion). Let G be a reductive group and H a closed subgroup. Then G/H is affine if and only if H is reductive.*

We now prove Theorem 1.5.

Proof of Theorem 1.5. We deduce that G/H is an affine variety from the aforementioned Matsushima's criterion. It follows immediately that $\mathbb{C}[G/H] = \mathbb{C}[G]^H$ since G/H is clearly a categorical quotient in this case. Thus, Grosshans principle in this case reads as:

$$\mathbb{C}[G/H \times W]^G \xrightarrow{\sim} (\mathbb{C}[G]^H \otimes \mathbb{C}[W])^G \xrightarrow{\sim} \mathbb{C}[W]^H.$$

Observe that $G/H \cong G \cdot v$ as affine varieties,[4] and thus we have

$$G/H \times W \xrightarrow{\sim} G \cdot v \times W \hookrightarrow V \oplus W.$$

This gives a surjection of invariant rings $\mathbb{C}[V \oplus W]^G \twoheadrightarrow \mathbb{C}[G/H \times W]^G$ (see [8, Corollary 2.3.4]). Combining with the above discussion, we have:

$$\phi : \mathbb{C}[V \oplus W]^G \twoheadrightarrow \mathbb{C}[G/H \times W]^G \xrightarrow{\sim} \mathbb{C}[W]^H$$

Recall the W-grading on $\mathbb{C}[G/H \times W] = \mathbb{C}[G/H] \otimes \mathbb{C}[W]$ and on $\mathbb{C}[V \oplus W] = \mathbb{C}[V] \otimes \mathbb{C}[W]$. The surjection $\mathbb{C}[V \oplus W] \twoheadrightarrow \mathbb{C}[G/H \times W]$ is degree non-increasing in the W-grading. The isomorphism $\mathbb{C}[G/H \times W]^G \xrightarrow{\sim} \mathbb{C}[W]^H$ given by Grosshans principle is also degree non-increasing in the W-grading. Hence, ϕ is also degree non-increasing in the W-grading.

The polynomial grading and W-grading are different on $\mathbb{C}[V \oplus W]$. If $f \in \mathbb{C}[V \oplus W]$ is homogeneous in degree d in the polynomial grading, then f need not be homogeneous in the W-grading. However, the homogeneous components of f in the W-grading will all be in degrees $\leq d$. On the other hand, the W-grading and the polynomial grading on $\mathbb{C}[W]^H$ agree. In particular this means that the surjection $\phi : \mathbb{C}[V \oplus W]^G \twoheadrightarrow \mathbb{C}[W]^H$ is degree non-increasing even when we consider the polynomial grading on $\mathbb{C}[V \oplus W]^G$. Applying Lemma 1.3 concludes the proof. □

5. Root systems

In this section, we will briefly recall some standard notions surrounding root systems as well as formulate some convenient definitions and notation. This language will be used heavily in the later sections. For more details on this subject, we refer the reader to standard texts, e.g., [35,21].

Let G be a complex reductive group, and K a maximal compact subgroup (also called a compact real form). Let $T_{\mathbb{R}}$ be a (real) maximal torus of K. The complexification of $T_{\mathbb{R}}$, denoted T, is a complex maximal torus for G. Let $\mathfrak{g}$ and $\mathfrak{t}$ denote the Lie algebras of G and T respectively.

For any representation V of G, we can view it as a representation of T, and hence we get a weight space decomposition

$$V = \bigoplus_{\lambda \in \mathcal{X}(T)} V_\lambda.$$

We make a convenient definition.

[4] This follows essentially from Zariski's main theorem, see for e.g. [36, Theorem 25.1.2(iv)].

Definition 5.1. For any $v \in V$, the decomposition $v = \sum_\lambda v_\lambda$ with $v_\lambda \in V_\lambda$ is called the weight decomposition of v. The weight decomposition is unique. Further the set $\{\lambda \mid v_\lambda \neq 0\}$ is called the support of v, and we denote it by $\mathrm{Supp}(v)$.

For the adjoint action of G on $\mathfrak{g}$, the weight space decomposition is called the root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

The set of non-zero weights β for which the weight space $\mathfrak{g}_\beta$ is non-zero form a finite collection of vectors in $\mathcal{X}(T)$ called the root system, which we denote by Φ.

We use the following example and the subsequent remark to develop notation that will be helpful at various stages of paper.

Example 5.2. Suppose $G = \mathrm{SL}_n(\mathbb{C})$. Then $K = \mathrm{SU}_n(\mathbb{C})$ is a compact real form. Let $\mathrm{diag}(a_1, \dots, a_n)$ denote a diagonal $n \times n$ matrix whose diagonal entries are $a_1, \dots, a_n$. Then $T_\mathbb{R} = \{\mathrm{diag}(a_1, \dots, a_n) \mid a_i \in \mathbb{C}, |a_i| = 1, \prod_i a_i = 1\}$ is a (real) maximal torus, its complexification $T = \{\mathrm{diag}(t_1, \dots, t_n) \mid t_i \in \mathbb{C}, \prod_i t_i = 1\}$ is a (complex) maximal torus. Let $\widetilde{e}_i \in \mathcal{X}(T)$ be defined by $\widetilde{e}_i \cdot \mathrm{diag}(t_1, \dots, t_n) = t_i$. Then, for the action of SL_n on $\mathbb{C}^n$ by left multiplication, the standard basis vector e_i is a weight vector with weight $\widetilde{e}_i$. The weights $\widetilde{e}_i$ do not form a basis for $\mathcal{X}(T)$. They satisfy one relation, i.e., $\sum_i \widetilde{e}_i = 0$. The root system $\Phi = \{\widetilde{e}_i - \widetilde{e}_j \mid 1 \leq i, j \leq n\}$.

Let us reformulate the above example with respect to a basis.

Remark 5.3. Suppose $G = \mathrm{SL}(V)$ with $\mathcal{B}$ a basis for V. Then, using the basis, we can identify $\mathrm{SL}(V)$ with SL_n. With this identification, we can define $K_\mathcal{B}, T_{\mathbb{R},\mathcal{B}}, T_\mathcal{B}, \mathfrak{t}_\mathcal{B}$ as in the above example. Under these choices, $\mathcal{B}$ consists of weight vectors. Let us denote the weight of $b \in \mathcal{B}$ by $\widetilde{b}$. These weights satisfy precisely one relation. i.e., $\sum_{b \in \mathcal{B}} \widetilde{b} = 0$. The root system $\Phi = \{\widetilde{b} - \widetilde{b'} \mid b, b' \in \mathcal{B}, b \neq b'\}$.

We make some useful definitions to aid in formulating later statements.

Definition 5.4 *(Root adjacent).* We say two weights $\lambda, \mu \in \mathcal{X}(T)$ are root adjacent if $\lambda - \mu \in \Phi$.

Definition 5.5 *(Uncramped sets of weights).* A subset of weights $I \subseteq \mathcal{X}(T)$ is called uncramped if no pair of weights in I is root adjacent.

We also make a simple observation regarding root adjacent weights for tensor actions. Let V_1, V_2, V_3 be vector spaces with basis $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$. Then $\mathcal{B} = \{b_1 \otimes b_2 \otimes b_3 \mid b_i \in \mathcal{B}_i\}$ forms a basis for $V_1 \otimes V_2 \otimes V_3$. Let us write $K = K_\mathcal{B}$, $K_i = K_{\mathcal{B}_i}$ (and similarly for

$T, \mathfrak{g}$, etc.). Observe that $K = K_1 \times K_2 \times K_3$, $\mathcal{X}(T) = \mathcal{X}(T_1) \times \mathcal{X}(T_2) \times \mathcal{X}(T_3)$, etc. Further, let $\Phi_i \subseteq \mathcal{X}(T_i)$ denote the root system for $\mathrm{SL}(V_i)$, then we can view Φ_i as a subset of $\mathcal{X}(T)$. With this identification, it is easy to see that the root system Φ for $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$ is given by $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$, from which the following lemma is an easy consequence.

Lemma 5.6. *Consider the tensor action of* $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$ *on* $V_1 \otimes V_2 \otimes V_3$. *Suppose* $\mathcal{B}_i$ *is a basis for* V_i *and we make all the standard choices for compact real form, tori etc. with respect to the basis* $\mathcal{B}_i$ *as in Remark 5.3. Let* $v = b_1 \otimes b_2 \otimes b_3$ *and* $w = b'_1 \otimes b'_2 \otimes b'_3$ *with* $b_i, b'_i \in \mathcal{B}_i$ *for all* i. *Suppose for at least two choices of* $i \in \{1, 2, 3\}$, *we have* $b_i \neq b'_i$. *Then* v *and* w *are weight vectors whose weights are not root adjacent.*

6. Moment map and a criterion for closed orbits

For this section, let G be a connected complex reductive group and let K be a compact real form. Let $T_{\mathbb{R}}$ a maximal (real) torus of K and let T denote its complexification.

In order to be able to use Theorem 1.5 effectively, we would need to prove that an orbit is closed. A criterion for detecting whether an orbit is closed is interesting by itself, and a good criterion could have a range of applications in both pure and applied mathematics. We approach the problem via the moment map, which suffices for our purposes. It is an interesting problem to understand whether the criterion we propose (see Theorem 6.5) has a suitable analogue in positive characteristic. We first define the moment map.

Definition 6.1. Let V be a representation of G, and let $\langle -, - \rangle$ be a K-invariant positive definite Hermitian form on V. The moment map $\mu_G : V \to \mathfrak{g}^*$ is defined by $\mu_G(v)(X) = \langle Xv, v \rangle$ for $v \in V$ and $X \in \mathfrak{g}$.

Proposition 6.2 *(Kempf-Ness). Suppose* $\mu_G(v) = 0$, *then the orbit* $G \cdot v$ *is closed.*

An even stronger statement holds, namely that every closed orbit contains a unique K-orbit at which the moment map vanishes. This is precisely why the GIT quotient $X /\!\!/ G$ agrees with the symplectic reduction $\mu^{-1}(0)/K$, which is known as the Kempf-Ness theorem. We refrain from getting into this beautiful subject, and refer to [24,31] for details.

Now, we turn towards discussing a criterion for the vanishing of the moment map in the language of root systems that is due to Dadok and Kac.

Proposition 6.3 *(Dadok and Kac [6]). Let* V *be a representation of* G. *Let* $\langle -, - \rangle$ *be a* K*-invariant positive definite Hermitian form on* V. *Let* $v \in V$. *Let* $v = \sum_{\lambda \in \mathrm{Supp}(v)} v_\lambda$ *be its weight decomposition. Suppose*

(1) $\mathrm{Supp}(v)$ *is uncramped (see Definition 5.5).*

(2) $\sum_{\lambda \in \mathrm{Supp}(v)} ||v_\lambda||^2 \lambda = 0$.

Then, $\mu_G(v) = 0$ and hence the orbit of v is closed.

The proof is essentially to compute $\mu_G(v)(X)$ separately for $X \in \mathfrak{g}_\alpha$ and $X \in \mathfrak{t}$. The only thing one needs to observe is that weight spaces are orthogonal, which follows from K-invariance of the form. We will need a slight generalization of the above result. We first make a definition.

Definition 6.4 *(Direct sum form).* Suppose W_i is a vector space with a bilinear form $\langle -.- \rangle_j$ for $j \in J$ for some set J. Then we define the direct sum form $\langle -, - \rangle$ on $\oplus_{j \in J} W_j$ by

$$\langle (a_j)_{j \in J}, (b_j)_{j \in J} \rangle = \prod_{j \in J} \langle a_j, b_j \rangle_j$$

Theorem 6.5. *Let W be a representation of G and $w \in W$. Let $W = \bigoplus_{j \in J} W_j$ be a decomposition into subrepresentations. Take a K-invariant positive definite Hermitian form on each W_j, and let $\langle -, - \rangle$ denote their direct sum form on W. Let $w = \sum_{j \in J} w_j$ with $w_j \in W_j$. Further, write $w_j = \sum_{\lambda \in \mathrm{Supp}(w_j)} w_{j,\lambda}$ be the weight decomposition for each w_j. Suppose*

(1) $\mathrm{Supp}(w_j)$ *is uncramped for all j;*
(2) $\sum_j \sum_{\lambda \in \mathrm{Supp}(w_j)} ||w_{j,\lambda}||^2 \lambda = 0$.

Then, $\mu_G(w) = 0$ and hence the orbit of w is closed.

Proof. We want to show that $\mu_G(w)(X) = 0$ for all $X \in \mathfrak{g}$. Again, it suffices to show it separately for $X \in \mathfrak{t}$ and $X \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$. A straightforward computation shows that for $X \in \mathfrak{t}$, we have $\mu_G(v)(X) = (\sum_j \sum_{\lambda \in \mathrm{Supp}(w_j)} ||w_{j,\lambda}||^2 \lambda)(X)$ which is 0 by the second condition.

Now suppose $X \in \mathfrak{g}_\alpha$. For all j, we have $\langle X w_j, w_j \rangle = 0$ since $\mathrm{Supp}(w_j)$ is uncramped and weight spaces are orthogonal (by K-invariance). Since the irreducibles W_j are orthogonal by the construction of the form, this shows that $\mu_G(w)(X) = \langle Xw, w \rangle = 0$ as required. $\square$

7. Cubic forms

Let us set up the situation for this section. Let V be a vector space of dimension $3n$, and let a basis for V be $\mathcal{B} = \{x_i, y_i, z_i\}_{1 \leq i \leq n}$. Consider $W = S^3(V)^{\oplus 4}$, and let

$$w = \Big(\sum_i x_i^2 z_i, \sum_i y_i^2 z_i, \sum_i \alpha_i x_i y_i z_i \Big),$$

where α_i are distinct complex numbers with $|\alpha_i| = 1$ and for all $i \neq j$, $\alpha_i \neq \pm\alpha_j$. There is a natural action of $\mathrm{SL}(V)$ on $S^3(V)$, and hence on W. We will write $w = (w_1, w_2, w_3)$ where $w_1 = \sum_i x_i^2 z_i$, $w_2 = \sum_i y_i^2 z_i$ and $w_3 = \sum_i \alpha_i x_i y_i z_i$.

Proposition 7.1. *The orbit* $\mathrm{SL}(V) \cdot w$ *is closed.*

Let us define a map $\phi : (\mathbb{C}^*)^n \to \mathrm{SL}(V)$. To define the map, it suffices to understand how $\phi(t = (t_1, \dots, t_n))$ acts on the basis $\{x_i, y_i, z_i\}_{1 \leq i \leq n}$. Define ϕ by $\phi(t) \cdot x_i = t_i x_i$, $\phi(t) \cdot y_i = t_i y_i$ and $\phi(t) \cdot z_i = t_i^{-2} z_i$. Let $H := \phi((\mathbb{C}^*)^n)$.

Proposition 7.2. *We have* $\mathrm{Stab}_{\mathrm{SL}(V)}(w) = H$*.*

It is also easy to see that H is a closed subgroup of G. It is also reductive because it is a torus. It is indeed necessary that the stabilizer is closed and reductive to be able to apply Theorem 1.5, as we will do in the proof of Theorem 1.6.

We postpone the proofs Proposition 7.1 and Proposition 7.2 and complete the proof of Theorem 1.6.

Consider the $n + 1$-dimensional subspace $U \subset S^3(V)$ spanned by $\{x_1 z_2^2, x_2 z_3^2, \dots, x_n z_1^2, x_1^3\}$. This is an invariant subspace under the action of $H \subset \mathrm{SL}(V)$ described in the previous section.

Lemma 7.3. *We have* $\beta_H(U) \geq \sigma_H(U) \geq \frac{2}{3}(4^n - 1)$*.*

Proof. The basis $\mathcal{E} = (x_1 z_2^2, x_2 z_3^2, \dots, x_n z_1^2, x_1^3)$ is a weight basis, and $M_{\mathcal{E}(W)} = M$, the matrix in Section 2. The lemma now follows from Proposition 3.4. □

Corollary 7.4. *We have* $\beta_H(S^3(V)) \geq \sigma_H(S^3(V)) \geq \frac{2}{3}(4^n - 1)$*.*

Proof. This follows from Proposition 3.6 since U is a subrepresentation of $S^3(V)$ for the action of H. □

Proof of Theorem 1.6. Let $G = \mathrm{SL}(V)$. Recall $w \in S^3(V)^{\oplus 3}$ from the previous section such that $\mathrm{Stab}_G(w) = H$. Thus, by Theorem 1.5 and the above corollary, we have

$$\beta_G(S^3(V)^{\oplus 3} \oplus S^3(V)) \geq \sigma_G(S^3(V)^{\oplus 3} \oplus S^3(V)) \geq \sigma_H(S^3(V)) \geq \frac{2}{3}(4^n - 1). \quad \square$$

Remark 7.5. If instead of w, one takes $(\sum_i x_i^2 z_i, \sum_i y_i^2 z_i) \in S^3(V)^{\oplus 2}$, then this also has a closed orbit. However, its stabilizer is not the torus H (defined above), but rather a finite extension of it. With some additional work, this can be used to show exponential lower bounds for $S^3(V)^{\oplus 3}$ (instead of $S^3(V)^{\oplus 4}$ as stated in Theorem 1.6). However, we feel that this modest improvement does not warrant the additional discussion on how to deal with finite extensions of tori, so we omit it.

7.1. Closedness of orbit

The strategy is to apply Theorem 6.5. But before proceeding to check the hypothesis, we need a little groundwork.

Definition 7.6 *(Type of a monomial).* Every monomial in the basis $\mathcal{B}$ can be written as $b_1^{a_1} b_2^{a_2} \dots b_k^{a_k}$, where the b_i represent distinct elements in the basis $\mathcal{B}$, and $a_1 \geq a_2 \geq \dots \geq a_k > 0$. We define its type to be $(a_1, \dots, a_k)$.

Example 7.7. The types of $x_i^2 z_i$ and $y_j^2 z_j$ are $(2,1)$, whereas the type of $x_i y_i z_i$ is $(1,1,1)$.

There is a positive definite Hermitian form $\langle -,- \rangle$ on $S^d(V)$ called the Bombieri form. Under this form, monomials are orthogonal. Further, for a monomial m of type $(a_1, \dots, a_k)$, we have $\langle m, m \rangle = \frac{\prod_i a_i!}{(\sum_i a_i)!}$. These two properties define the Bombieri form. Another way to think of the Bombieri form is to take the standard Hermitian form (with respect to $\mathcal{B}$) on V, which gives a natural Hermitian form on $V^{\otimes d}$. This Hermitian form on $V^{\otimes d}$ when restricted to the subspace of symmetric tensors (which can be identified with $S^d(V)$) is the Bombieri form. The Bombieri form is a $K_{\mathcal{B}}$-invariant positive definite Hermitian form.

We can now prove Proposition 7.1.

Proof of Proposition 7.1. We want to show that w satisfies the hypothesis of Theorem 6.5. Recall that $w = (w_1, w_2, w_3)$ where $w_1 = \sum_i x_i^2 z_i$, $w_2 = \sum_i y_i^2 z_i$ and $w_3 = \sum_i \alpha_i x_i y_i z_i$. Note that these are the weight space decompositions $w_j = \sum_{\lambda \in \mathrm{Supp}(w_j)} w_{j,\lambda}$.

We want to check that the hypothesis of Theorem 6.5 is satisfied. To check condition (1) of Theorem 6.5, we need to check that each $\mathrm{Supp}(w_j)$ is uncramped. But observe from the weight decompositions that $\mathrm{Supp}(w_1) = \{2\widetilde{x}_i + \widetilde{z}_i\}_{1 \leq i \leq n}$, $\mathrm{Supp}(w_2) = \{2\widetilde{y}_i + \widetilde{z}_i\}_{1 \leq i \leq n}$ and $\mathrm{Supp}(w_3) = \{\widetilde{x}_i + \widetilde{y}_i + \widetilde{z}_i\}_{1 \leq i \leq n}$. It is clear that these are uncramped from the description of the root system Φ in Remark 5.3.

Consider the Bombieri form on each copy of $S^3(V)$ and consider their direct sum form on $S^3(V)^{\oplus 3}$. All monomials of a certain type have the same norm as discussed above. Let M denote the norm of the monomials of type $(2,1)$ (e.g., $x_i^2 z_i$ and $y_j^2 z_j$) and let N denote the norm of the monomials of type $(1,1,1)$ (e.g., $x_i y_i z_i$).

We compute $\sum_{\lambda \in \mathrm{Supp}(w_1)} ||w_{1,\lambda}||^2 \lambda$.

$$\sum_{\lambda \in \mathrm{Supp}(w_1)} ||w_{1,\lambda}||^2 \lambda = \sum_{i=1}^n ||x_i^2 z_i||^2 (2\widetilde{x}_i + \widetilde{z}_i)$$

$$= \sum_i M^2 (2\widetilde{x}_i + \widetilde{z}_i)$$

Similarly, we have

$$\sum_{\lambda \in \mathrm{Supp}(w_2)} ||w_{2,\lambda}||^2 \lambda = \sum_i M^2 (2\widetilde{y_i} + \widetilde{z_i}),$$

and

$$\begin{aligned}\sum_{\lambda \in \mathrm{Supp}(w_3)} ||w_{3,\lambda}||^2 \lambda &= \sum_i ||x_i y_i z_i||^2 (\widetilde{x_i} + \widetilde{y_i} + \widetilde{z_i}) \\ &= N^2 (\sum_i \widetilde{x_i} + \widetilde{y_i} + \widetilde{z_i}).\end{aligned}$$

Hence, we have

$$\sum_{j=1}^{3} \sum_{\lambda \in \mathrm{Supp}(w_j)} ||w_{j,\lambda}||^2 \lambda = (2M^2 + N^2)(\sum_i \widetilde{x_i} + \widetilde{y_i} + \widetilde{z_i}) = (2M^2 + N^2) \sum_{b \in \mathcal{B}} \widetilde{b} = 0.$$

The last equality follows from Remark 5.3, as we are working with $\mathrm{SL}(V)$. Hence, w satisfies the hypothesis of Theorem 6.5, so the orbit of w is closed. □

7.2. Computation of stabilizer

Now, we turn towards computing the stabilizer. We will proceed in steps.

Lemma 7.8. *Suppose $g \in SL(V)$ such that $g \cdot w_1 = w_1$. Then $g \cdot x_i = c_i x_{\sigma(i)}$ for some permutation σ of $\{1, 2, \dots, n\}$ and non-zero scalars c_i.*

Proof. The space of partial derivatives of w_1 is $\langle x_1^2, \dots, x_n^2, x_1 z_1, \dots, x_n z_n \rangle$. This must be preserved by g. The squares in the space of partial derivatives are of the form $d_i x_i^2$ for some nonzero scalars d_i. Thus the image of x_i under the action of g must be a scalar multiple of x_j for some j. Since g is invertible, the lemma follows. □

Corollary 7.9. *Suppose $g \in \mathrm{Stab}_{\mathrm{SL}(V)}(w_1)$. Then for some permutation σ, we must have $g \cdot x_i = c_i x_{\sigma(i)}$ and $g \cdot z_i = c_i^{-2} z_{\sigma(i)}$ for some scalars c_i.*

Proof. From the above lemma, we already know that $g \cdot x_i = c_i x_{\sigma(i)}$ for some permutation σ and scalars c_i. Hence, we have

$$\sum_i (c_i x_{\sigma(i)})^2 (g \cdot z_i) = g \cdot w_1 = w_1 = \sum_i x_i^2 z_i = \sum_i x_{\sigma(i)}^2 z_{\sigma(i)}.$$

Thus, we have

$$\sum_i x_{\sigma(i)}^2 (c_i^2 g \cdot z_i - z_{\sigma(i)}) = 0.$$

Observe that monomials of degree 3 in $\{x_i, y_i, z_i\}_{1\leq i\leq n}$ are a basis for $S^3(V)$. Now, for any $p, q \in V$, $x_i^2 p$ and $x_j^2 q$ do not have any monomials in common. Hence, we must have $x_{\sigma(i)}^2(c_i^2 g\cdot z_i - z_{\sigma(i)}) = 0$ for all i. Hence, for all i, we must have $c_i^2 g\cdot z_i - z_{\sigma(i)} = 0$ or equivalently $g\cdot z_i = c_i^{-2} z_{\sigma(i)}$ as required. □

We can do a similar analysis for w_2, and we get:

Lemma 7.10. *Suppose $g \in \mathrm{Stab}_{\mathrm{SL}(V)}(w_2)$. Then for some permutation π and scalars d_i, we have $g\cdot y_i = d_i y_{\pi(i)}$ and $g\cdot z_i = d_i^{-2} z_{\pi(i)}$.*

Corollary 7.11. *Suppose $g \in \mathrm{Stab}_{\mathrm{SL}(V)}(w_1, w_2)$. Then for some permutation σ and scalars c_i, we have $g(x_i) = c_i x_{\sigma(i)}$, $g(y_i) = \pm c_i y_{\sigma(i)}$ and $g(z_i) = c_i^{-2} z_{\sigma(i)}$.*

Proof. Suppose $g \in \mathrm{Stab}_{\mathrm{SL}(V)}(w_1, w_2)$. Then from Corollary 7.9, we know that there is a permutation σ and scalars c_i such that $g(x_i) = c_i x_{\sigma(i)}$ and $g(z_i) = c_i^{-2} z_{\sigma(i)}$. By Lemma 7.10, there is a permutation π and scalars d_i such that $g(y_i) = d_i y_{\pi(i)}$ and $g(z_i) = d_i^{-2} z_{\pi(i)}$.

Thus, we have $g\cdot z_i = c_i^{-2} z_{\sigma(i)} = d_i^{-2} z_{\pi(i)}$ for all i. Hence, we must have $\sigma = \pi$ and $d_i = \pm c_i$. □

Proof of Proposition 7.2. Suppose $g \in \mathit{Stab}(w_1, w_2, w_3)$. Then since $g \in \mathit{Stab}(w_1, w_2)$, we know that there is a permutation σ and scalars c_i such that $g(x_i) = c_i x_{\sigma(i)}$, $g(y_i) = \pm c_i y_{\sigma(i)}$ and $g(z_i) = c_i^{-2} z_{\sigma(i)}$

In particular, this means that $g\cdot x_i y_i z_i = \pm x_{\sigma(i)} y_{\sigma(i)} z_{\sigma(i)}$. But now g also fixes $w_3 = \sum_i \alpha_i x_i y_i z_i$. However, we have

$$\sum_i \pm\alpha_i x_{\sigma(i)} y_{\sigma(i)} z_{\sigma(i)} = g\cdot w_3 = w_3 = \sum_i \alpha_i x_i y_i z_i$$

This means that $\pm\alpha_i = \alpha_{\sigma(i)}$. But recall that the choice of α_i's was such that $\alpha_i \neq \pm\alpha_j$ for all $i \neq j$. This means that σ is the identity permutation, and further that we must have $g\cdot x_i y_i z_i = x_i y_i z_i$. Hence, this implies $g\cdot y_i = c_i y_i$.

Thus we must have $g\cdot x_i = c_i x_i$, $g\cdot y_i = c_i y_i$ and $g\cdot z_i = c_i^{-2} z_i$. In other words, $g \in H$. Conversely, it is easy to observe that $H \subseteq \mathit{Stab}(w)$. □

8. Tensor actions

Let U, V, W be $3n$-dimensional vector spaces with basis $\mathcal{B}_u = \{u_1^k, u_2^k, u_3^k\}_{1\leq k\leq n}$, $\mathcal{B}_v = \{v_1^k, v_2^k, v_3^k\}_{1\leq k\leq n}$ and $\mathcal{B}_w = \{w_1^k, w_2^k, w_3^k\}_{1\leq k\leq n}$ respectively.

Let

$$F_1 = \sum_{k=1}^n u_1^k v_2^k w_3^k + u_2^k v_3^k w_1^k + u_3^k v_1^k w_2^k$$

$$G_1 = \sum_{k=1}^{n} \alpha_k u_1^k v_2^k w_3^k + \beta_k u_2^k v_3^k w_1^k + \gamma_k u_3^k v_1^k w_2^k$$

$$F_2 = \sum_{k=1}^{n} u_2^k v_1^k w_3^k + u_1^k v_3^k w_2^k + u_3^k v_2^k w_1^k$$

$$G_2 = \sum_{k=1}^{n} \alpha_k u_2^k v_1^k w_3^k + \beta_k u_1^k v_3^k w_2^k + \gamma_k u_3^k v_2^k w_1^k$$

$$F_3 = \sum_{k=1}^{n} u_1^k v_1^k w_3^k + u_2^k v_3^k w_2^k + u_3^k v_1^k w_1^k$$

$$G_3 = \sum_{k=1}^{n} \alpha_k u_1^k v_1^k w_3^k + \beta_k u_2^k v_3^k w_2^k + \gamma_k u_3^k v_1^k w_1^k$$

$$F_4 = \sum_{k=1}^{n} u_2^k v_2^k w_3^k + u_1^k v_3^k w_1^k + u_3^k v_2^k w_2^k$$

$$G_4 = \sum_{k=1}^{n} \alpha_k u_2^k v_2^k w_3^k + \beta_k u_1^k v_3^k w_1^k + \gamma_k u_3^k v_2^k w_2^k,$$

where $\alpha_k, \beta_k, \gamma_k$ are a collection of distinct scalars in $\mathbb{C}$ with unit norm. Consider

$$\underline{F} = (F_1, G_1, F_2, G_2, F_3, G_3, F_4, G_4) \in (U \otimes V \otimes W)^8.$$

The approach will be the same as cubic forms. First, we show:

Proposition 8.1. *The orbit of $\underline{F}$ for the action of $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$ is closed.*

Next, we compute the stabilizer. Let us define a map $\phi_U : ((\mathbb{C}^*)^3)^n \to \mathrm{GL}(U)$. To define such a map it suffices to understand the action of $t = (p_1, q_1, r_1, p_2, q_2, r_2, \ldots, p_n, q_n, r_n)$ on each basis vector $b \in \mathcal{B}_u$. The map ϕ_U is defined by

$$\phi_U(t)u_1^k = p_k u_1^k, \phi_U(t)u_2^k = p_k u_2^k \text{ and } \phi_U(t)u_3^k = (q_k r_k)^{-1} u_3^k.$$

Similarly define $\phi_V : ((\mathbb{C}^*)^3)^n \to \mathrm{GL}(V)$ by

$$\phi_V(t)v_1^k = q_k v_1^k, \phi_V(t)v_2^k = q_k v_2^k \text{ and } \phi_V(t)v_3^k = (p_k r_k)^{-1} v_3^k.$$

Finally, define $\phi_W : ((\mathbb{C}^*)^3)^n \to \mathrm{GL}(W)$ by

$$\phi_W(t)w_1^k = r_k w_1^k, \phi_W(t)w_2^k = r_k w_2^k \text{ and } \phi_W(t)w_3^k = (p_k q_k)^{-1} w_3^k.$$

Let $\phi = (\phi_U, \phi_V, \phi_W) : ((\mathbb{C}^*)^3)^n \to \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$. Let H denote the image of ϕ. Then, we have:

Proposition 8.2. *We have* $\mathrm{Stab}_{\mathrm{GL}(U)\times\mathrm{GL}(V)\times\mathrm{GL}(W)}(\underline{F}) = H$.

Again, it is easy to check that H is a closed subgroup of $\mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$. It is also reductive because it is a torus. The reader perhaps has noticed that we have computed the stabilizer in $\mathrm{GL}(U)\times\mathrm{GL}(V)\times\mathrm{GL}(W)$ rather than the stabilizer in $\mathrm{SL}(U)\times \mathrm{SL}(V) \times \mathrm{SL}(W)$. There are several ways to fix this, and we indicate one of them.

Consider the group

$$J := \{(g_1, g_2, g_3) \in \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W) \mid \det(g_1)\det(g_2)\det(g_3) = 1\}.$$

Indeed, the first thing to observe is that $H \subset J$. Now, we claim that the orbits of J and the orbits of $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$ in $U \otimes V \otimes W$ are the same. Let $h = (g_1, g_2, g_3) \in J$. Since $\det(g_1)\det(g_2)\det(g_3) = 1$, we can choose $c_1, c_2, c_3 \in \mathbb{C}$ with $c_1c_2c_3 = 1$ such that $\det(c_ig_i) = 1$. Thus, we have $h \cdot v = (c_1g_1, c_2g_2, c_3g_3) \cdot v$ for any $v \in U \otimes V \otimes W$. But $(c_1g_1, c_2g_2, c_3g_3) \in \mathrm{SL}(U)\times\mathrm{SL}(V)\times\mathrm{SL}(W)$, so this means that the J-orbit of v is contained in the $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-orbit of v. On the other hand, $J \supseteq \mathrm{SL}(U)\times\mathrm{SL}(V)\times\mathrm{SL}(W)$, so the orbits must be the same. The same argument works for $(U\otimes V\otimes W)^{\oplus m}$. Further observe that the quotient $GL(U)\times GL(V)\times \mathrm{GL}(W)/J = \mathbb{C}^*$, which is affine. Since J is clearly a closed subgroup of $\mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$, by Matsushima's criterion (see Theorem 4.1) we conclude that J is reductive. We summarize the above discussion as follows:

Proposition 8.3. *The J-orbit of $\underline{F}$ is closed. Further, the stabilizer of $\underline{F}$ in J is H. Moreover J is a reductive group.*

Further, since orbits of J are the same as the orbits of $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$, we also have that the invariant rings are equal, i.e.,

Corollary 8.4. *We have* $\mathbb{C}[U \otimes V \otimes W]^{\mathrm{SL}(U)\times\mathrm{SL}(V)\times\mathrm{SL}(W)} = \mathbb{C}[U \otimes V \otimes W]^J$.

Consider the action of H on $U \otimes V \otimes W$. Let L denote the subspace spanned by $\mathcal{E} = \{u_1^1v_1^1w_1^1\} \cup \{u_1^{k+1}v_3^kw_3^k, u_3^kv_1^{k+1}w_3^k, u_1^kv_1^kw_3^{k+1}\}_{1\leq k\leq n-1} \cup \{u_3^nv_3^nw_3^n\}$. Now, it is clear that for the action of H on L, the set $\mathcal{E}$ is a weight basis, and further one can check that $M_{\mathcal{E}}(L) = N$, the matrix in Section 2. Hence, from Proposition 3.5, we obtain:

Corollary 8.5. *We have*

$$\beta_H(U \otimes V \otimes W) \geq \sigma_H(U \otimes V \otimes W) \geq \sigma_H(L) \geq 4^n - 1.$$

Proof of Theorem 1.8. Proceed in exactly the same fashion as the proof of Theorem 1.6 to obtain the required lower bounds on $\sigma_J((U \otimes V \otimes W)^{\oplus 9})$ and $\beta_J((U \otimes V \otimes W)^{\oplus 9})$. Then using Corollary 8.4, we conclude that the same lower bounds hold for $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$. $\square$

8.1. Closedness of orbit

This section is devoted to the proof of Proposition 8.1. The strategy will again be to use Theorem 6.5. We have the basis $\mathcal{B}_u, \mathcal{B}_v$ and $\mathcal{B}_w$ for U, V, W respectively. For $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$, we choose $K := K_{\mathcal{B}_u} \times K_{\mathcal{B}_v} \times K_{\mathcal{B}_w}$ for a compact real form and $T = T_{\mathcal{B}_u} \times T_{\mathcal{B}_v} \times T_{\mathcal{B}_w}$ for a maximal torus.

Observe that $\mathcal{B} = \{b_u \otimes b_v \otimes b_w \mid b_u \in \mathcal{B}_u, b_v \in \mathcal{B}_v, b_w \in \mathcal{B}_w\}$ is a basis for $U \otimes V \otimes W$. Consider the hermitian form on $U \otimes V \otimes W$ given by asking for $\mathcal{B}$ to be an orthonormal basis. It is easy to check that this form is K-invariant.

Proof of Proposition 8.1. We use the form described above for each copy of $U \otimes V \otimes W$ and take the direct sum form. In order to use Theorem 6.5, the first step is to check that the supports $\mathrm{Supp}(F_d)$ and $\mathrm{Supp}(G_d)$ are uncramped. Let us only indicate the proof for F_1, as the other cases are similar. The defining decomposition of F_1 is its weight decomposition. It has three types of terms $u_1^k v_2^k w_3^k$, $u_2^k v_3^k w_1^k$, and $u_3^k v_1^k w_2^k$. We want to show that the support is uncramped. So, for any two such terms, we need to show that their weights are not root adjacent. But this follows easily from Lemma 5.6.

Let us now check the second condition in Theorem 6.5 i.e., we want:

$$\sum_{d=1}^{4} \left(\sum_{\lambda \in \mathrm{Supp}(F_d)} ||(F_d)_\lambda||^2 \lambda + \sum_{\mu \in \mathrm{Supp}(G_d)} ||(G_d)_\lambda||^2 \mu \right) = 0.$$

The defining decompositions of F_d and G_d are weight decompositions. All the coefficients appearing in F_d and G_d have absolute value 1. Further, observe that $\mathrm{Supp}(F_d) = \mathrm{Supp}(G_d)$. Thus we have

$$\begin{aligned}\sum_d \left(\sum_{\lambda \in \mathrm{Supp}(F_d)} ||(F_d)_\lambda||^2 \lambda + \sum_{\mu \in \mathrm{Supp}(G_d)} ||(G_d)_\lambda||^2 \mu \right) &= \sum_d \left(\sum_{\lambda \in \mathrm{Supp}(F_d)} \lambda + \sum_{\mu \in \mathrm{Supp}(G_d)} \mu \right) \\ &= 2 \sum_d \left(\sum_{\lambda \in \mathrm{Supp}(F_d)} \lambda \right).\end{aligned}$$

Recall that $\widetilde{u}_i^k$ denotes the weight for u_i^k for $\mathrm{SL}(U)$. Recall that $\sum_{i,k} \widetilde{u}_i^k = 0$ from Remark 5.3. Observe that each u_i^k appears a total of 4 times in all the terms of T_1, T_2, T_3, T_4. Similarly for v_i^k and w_i^k. This means that

$$\begin{aligned}\sum_d \left(\sum_{\lambda \in \mathrm{Supp}(F_d)} \lambda \right) &= 4\Big(\sum_{i,k} \widetilde{u}_i^k, \sum_{i,k} \widetilde{v}_i^k, \sum_{i,k} \widetilde{w}_i^k\Big) \\ &= 0.\end{aligned}$$

Hence, the second condition of Theorem 6.5 is satisfied for $\underline{F}$. This concludes the proof. $\square$

8.2. Computation of stabilizer

In spirit, the computation is very similar to the computation for cubic forms in the previous section. However, we will need slightly different arguments for this.

Tensors of the form $a \otimes b \otimes c \in U \otimes V \otimes W$ are called rank 1 tensors.

Lemma 8.6. *Suppose $T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i \in U \otimes V \otimes W$, where $\{a_i\}, \{b_i\}, \{c_i\}$ are linearly independent collections of vectors in U, V and W respectively. Then this is the unique decomposition of T into a sum of r rank 1 tensors.*

Proof. For $r = 1$, this is clear. For $r \geq 2$, this follows from Kruskal's theorem, see [25]. $\square$

The above lemma can also be proved by using just elementary linear algebra arguments without resorting to Kruskal's theorem.

Lemma 8.7. *Suppose $g \in \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ fixes T as in the previous lemma. Then g must permute the terms $a_i \otimes b_i \otimes c_i$.*

Proof. Applying g to the decomposition into a sum of r rank 1 tensors also yields a decomposition into a sum of r rank 1 tensors. Hence, by the above lemma, g must permute the terms. $\square$

Corollary 8.8. *Suppose $g \in \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ fixes F_1, then g must permute the terms in F_1.*

Corollary 8.9. *Suppose $g \in \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ fixes F_1 and G_1, then g must fix all the terms in F_1.*

Proof. Any non-trivial permutation of the terms in F_1 does not fix G_1. Hence g must fix all the terms. $\square$

Similar arguments hold for F_2, F_3 and F_4 as well. In summary, we obtain:

Corollary 8.10. *Suppose $g \in \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ fixes $\underline{F}$, then g must fix all the terms in F_1, F_2, F_3 and F_4.*

Let $I_k = \{u_i^k v_j^k w_3^k, u_i^k v_3^k w_j^k, u_3^k v_i^k w_j^k\}_{1 \leq i,j \leq 2}$. Then $\cup_k I_k$ are precisely the terms occurring in F_1, F_2, F_3 and F_4.

Lemma 8.11. *Suppose $g = (g_u, g_v, g_w) \in \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ fixes I_k. Then for some $p_k, q_k, r_k \in \mathbb{C}^*$, we have*

$$g_u(u_i^k) = p_k u_i^k \text{ for } i = 1,2 \text{ and } g_u(u_3^k) = (q_k r_k)^{-1} u_3^k,$$
$$g_v(v_i^k) = q_k v_i^k \text{ for } i = 1,2 \text{ and } g_v(v_3^k) = (p_k r_k)^{-1} v_3^k,$$
$$g_w(w_i^k) = r_k w_i^k \text{ for } i = 1,2 \text{ and } g_w(w_3^k) = (p_k q_k)^{-1} w_3^k.$$

Proof. It is clear that if g fixes $b_u \otimes b_v \otimes b_w$, then each g_x must scale b_x for each $x \in \{u, v, w\}$. So, we must have $g_u(u_1^k) = p_k u_1^k$, $g_v(v_1^k) = q_k v_1^k$ and $g_w(w_1^k) = r_k w_1^k$ for some $p_k, q_k, r_k \in \mathbb{C}^*$. Then, since $u_1^k v_1^k w_3^k \in I_k$ is fixed by g, we must have $g_w(w_3^k) = (p_k q_k)^{-1} w_3^k$. Since $u_1^k v_2^k w_3^k \in I_k$ is fixed by g, we must have $g_v(v_2^k) = q_k v_k$. Symmetric arguments complete the proof. $\square$

Proof of Proposition 8.2. From Corollary 8.10, we conclude that if g fixes $\underline{F}$, then it must fix all the terms in $\cup_k I_k$. From the previous lemma, one concludes that $g \in H$. Conversely, it is easy to check that H fixes $\underline{F}$. $\square$

9. Concluding remarks: positive characteristic

It is difficult to imagine that degree bounds in positive characteristic are better than those in characteristic zero. Nevertheless, we do not know yet how to prove exponential lower bounds for cubic forms or tensor actions in positive characteristic. We require characteristic zero at two instances in our proof techniques. First and foremost is that our criterion for closed orbits has no analog in positive characteristic. Second, in the proof of Theorem 1.5, we use characteristic zero in two places. The first is when we use Zariski's main theorem to deduce that $G/H \xrightarrow{\sim} G \cdot v$. This statement remains true in positive characteristic (see [29, Corollary 7.13]), but one has to take the stabilizer in the scheme theoretic sense, i.e., H will now be a group scheme that in general may not be reduced or smooth (see [29, Remark 7.14]). The second instance is that the closed embedding $G \cdot v \times W \hookrightarrow V \times W$ gives a surjection $\mathbb{C}[V \times W]^G \twoheadrightarrow \mathbb{C}[G \cdot v \times W]^G$. Unfortunately, this is not necessarily true in positive characteristic. Nevertheless, it is clear that if we take a collection of invariants that separate orbit closures in $V \times W$, then they also separate orbit closures in the closed subset $G \cdot v \times W$. This means that we get an inequality for bounds on degrees of separating invariants as opposed to generating invariants. This is however not a serious problem, because bounds for separating invariants are sandwiched between bounds for invariants defining the null cone and bounds for generating invariants. In our applications of Theorem 1.5, we mainly used exponential lower bounds for the null cone, i.e., $\sigma_H(W)$. This can be sufficient because the above discussion will give us $\beta_G(V \oplus W) \geq \sigma_H(W)$.

In the two examples of cubic forms and tensor actions discussed in this paper, we will not be able to apply a modified version of Theorem 1.5 in positive characteristic simply because we do not know how to prove orbits are closed. However, we can still prove exponential lower bounds on certain actions – take $\mathrm{SL}(V)$ acting on $\mathrm{Ad} \oplus S^3(V)$. Then, we take v to be a regular semisimple element in Ad, whose orbit is well known to be closed. Further, its stabilizer is precisely a maximal torus. For the action of a maximal torus on $S^3(V)$, we can show exponential lower bounds (for invariants defining the null cone) by following the ideas in this paper. In fact for torus actions, degree bounds (whether for generators or null cone) are independent of characteristic.

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