WHITTAKER–FOURIER COEFFICIENTS OF CUSP FORMS

ZHENGYU MAO

ABSTRACT. We report on the work with E. Lapid about a conjecture relating Whittaker-Fourier coefficients of cusp forms to special values of $L-$functions.

CONTENTS

1. INTRODUCTION

Let G be a quasi-split reductive group over a number field F and $\mathbb{A}$ the ring of adeles of F. Let B be a Borel subgroup of G defined over F, N the unipotent radical of B and fix a non-degenerate character ψ_N of $N(\mathbb{A})$, trivial on $N(F)$. For a cusp form φ of $G(F)\backslash G(\mathbb{A})$ we consider the Whittaker–Fourier coefficient

$$\mathcal{W}(\varphi) = \mathcal{W}^{\psi_N}(\varphi) := (\mathrm{vol}(N(F)\backslash N(\mathbb{A})))^{-1} \int_{N(F)\backslash N(\mathbb{A})} \varphi(n)\psi_N(n)^{-1}\, dn.$$

If π is an irreducible cuspidal representation then $\mathcal{W}$, if non-zero, gives a realization of π in the space of Whittaker functions on $G(\mathbb{A})$, which by local multiplicity one depends only on π as an abstract representation. It therefore provides a useful tool for understanding π, both computationally and conceptually.

It is natural to study the size of $\mathcal{W}(\varphi)$. In [LM15], inspired by the work of Ichino-Ikeda [II10], with E. Lapid we made a conjecture relating the size of $\mathcal{W}(\varphi)$ with a special $L-$value of π. In this note, we report on the work with Lapid on this conjecture.

In modular form language, the notion of Whittaker-Fourier coefficient is roughly that of Fourier coefficient of a modular form. We mention two well known examples of relation between Fourier coefficient and $L-$value in this setting.

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Example 1.1. *If f is a weight 0 Maass form of level 1,*

$$f(z) = \sum_{n \neq 0} a(n)|n|^{-\frac{1}{2}} W(nz)$$

where W is a Whittaker function. Assume f is a Hecke eigenform whose Petersson inner product $\langle f, f \rangle = 1$, then Hoffstein-Lockhart showed [HL94, (0.5)]

$$Res_{s=1} L(s, f \times f) = \frac{2\pi}{3} |a(1)|^{-2}.$$

Example 1.2. *A famous example is what generally referred to as Waldspurger's formula for Fourier coefficients of half integral weight forms. We state a simple case. If f is weight $2k$ newform of level 1, the Shimura correspondence associates to f a level 4 weight $k + \frac{1}{2}$ form g such that*

$$g(z) = \sum_{n>0} c(n) e^{2\pi i n z}$$

with $c(n) = 0$ when $(-1)^k n$ is congruent to $2, 3$ modulo 4, (i.e $g(z)$ lies in Kohnen plus space). The form $g(z)$ is unique up to scalar multiple and there is the following formula of Kohnen-Zagier [KZ81]

$$\frac{L(f,k)}{\langle f, f \rangle} \frac{(k-1)!}{\pi^k} = \frac{|c(1)|^2}{\langle g, g \rangle}.$$

In §2 we give a statement of the main conjecture. In §3 some known examples of the conjecture are given. In §4 we give a more detailed discussion for the case of classical group and metaplectic group, in particular the approach to the conjecture using automorphic descent method.

2. Statement of the conjecture

2.1. **Fourier coefficient of matrix coefficient.** Our conjecture will relate a product of Whittaker functions to the ψ_N-th Fourier coefficient of a matrix coefficient of π. The integral for Fourier coefficient of a matrix coefficient

$$\int_{N(\mathbb{A})} (\pi(n)\varphi, \varphi^\vee)_{G(F)\backslash G(\mathbb{A})^1} \psi_N(n)^{-1} \, dn$$

does not converge, (here $(\cdot, \cdot)_{G(F)\backslash G(\mathbb{A})^1}$ is Petersson inner product). To regularize this integral we first consider the local analogue of the integral.

Let G be a quasi-split reductive group over a local field F of characteristic 0. If (π, V) is a representation of finite length, we write $(\pi^\vee, V^\vee)$ for the contragredient representation. Let $(\cdot, \cdot) = (\cdot, \cdot)_\pi$ be the canonical pairing on $V \times V^\vee$. For any pair $v \in V$, $v^\vee \in V^\vee$ we define the matrix coefficient $\mathrm{MC}_{v,v^\vee}(g) = (\pi(g)v, v^\vee)_\pi$.

We say that an irreducible representation π of G is essentially square-integrable if some twist of π by a (not necessarily unitary) character of G is square-integrable. Then the integral

$$\int_N \mathrm{MC}_{v,v^\vee}(n) \psi_N(n)^{-1} \, dn \tag{2.1}$$

is absolutely convergent when π is essentially square-integrable. In more general cases, this integral can be regularized as follows.

Case (1), the field F is p-adic. Recall the definition of a stable integral over a unipotent group U over F.

Definition 2.1. *Let f be a smooth function on U. We say that f has a* stable integral *over U if there exists a compact subgroup $U_1 \subset U$ such that for any compact subgroup $U_2 \subset U$ containing U_1 we have*

$$\int_{U_2} f(u)\ du = \int_{U_1} f(u)\ du. \tag{2.2}$$

In this case we write $\int_U^{st} f(u)\ du$ for the common value (2.2) and say that $\int_U^{st} f(u)\ du$ stabilizes at U_1.

The regularization of (2.1) is given by

Proposition 2.2. [LM15, Proposition 2.3] *Let (π, V) be an irreducible representation of G. Then for any $v \in V$, $v^\vee \in V^\vee$ the function $\psi_N^{-1} \cdot \mathrm{MC}_{v,v^\vee}\big|_N$ has a stable integral over N. The bilinear form*

$$(v, v^\vee)_\pi^{\psi_N} := \int_{N(F)}^{st} (\pi(n)v, v^\vee)_\pi \psi_N(n)^{-1}\ dn$$

is (N, ψ_N)-equivariant in v and (N, ψ_N^{-1})-equivariant in $v^\vee$.

Case (2), F is an archimedean field. First consider the case when π is tempered. Let N° be the derived group of N. The integral $\int_{N^\circ} (\pi(n\cdot)v, v^\vee)_\pi\ dn$ converges and defines an L^2 function on $N^\circ\backslash N$. Its Fourier transform is regular on the open set of non-degenerate characters of N. Its value at ψ_N^{-1} is by definition the value of the regularization of (2.1), denoted again by $(v, v^\vee)_\pi^{\psi_N}$. In the general case from Langlands classification $\pi = \mathrm{Ind}_P^G \sigma$ where $P = M \ltimes U$ is a standard parabolic subgroup of G and σ is an essentially tempered irreducible representation of G. The regularized value $(v, v^\vee)_\pi^{\psi_N}$ of (2.1) can be defined using the bilinear form defined on σ; details are given in [LM15, §2.5]. The bilinear form $(v, v^\vee)_\pi^{\psi_N}$ again satisfies the equivariance property in Proposition 2.2.

By Casselman-Shalika formula [CS80], we have the following formula for $(v, v^\vee)_\pi^{\psi_N}$ in the unramified setting.

Proposition 2.3. [LM15, Proposition 2.14] *Suppose that π is an irreducible unramified representation of G and ψ_N is unramified (in the sense of* [CS80, §3]*). Let v_0 and $v_0^\vee$ be unramified vectors in π and $\pi^\vee$ respectively. Then (K is maximal compact subgroup of G)*

$$(v_0, v_0^\vee)_\pi^{\psi_N} = \mathrm{vol}(N \cap K) \frac{(v_0, v_0^\vee)_\pi \Delta_G(1)}{L(1, \pi, \mathrm{Ad})}. \tag{2.3}$$

Here $\Delta_G(s)$ is the L-factor of the dual $M^\vee$ to the motive M introduced by Gross in [Gro97].

2.2. **A global constant.** Now we go to a global setting. Let π be an irreducible cuspidal representation of G. Given that the local bilinear forms $(\cdot,\cdot)_\pi^{\psi_N}$ satisfy the equivariance property specified in Proposition 2.2, by local multiplicity one for Whittaker functional and the unramified calculation in Proposition 2.3, there exists a constant $c_\pi^{\psi_N}$ depending on the space of π such that

$$\mathcal{W}^{\psi_N}(\varphi)\mathcal{W}^{\psi_N^{-1}}(\varphi^\vee) = (c_\pi^{\psi_N}\operatorname{vol}(G(F)\backslash G(\mathbb{A})^1))^{-1}\frac{\Delta_G^S(s)}{L^S(s,\pi,\mathrm{Ad})}\Big|_{s=1}\prod_{v\in S}(\varphi_v,\varphi_v^\vee)_{\pi_v}^{\psi_N} \tag{2.4}$$

for all $\varphi=\otimes\varphi_v\in\pi$, $\varphi^\vee=\otimes_v\varphi_v^\vee\in\pi^\vee$ and all S sufficiently large. Here we assume the existence and non-vanishing of $\frac{\Delta_G^S(s)}{L^S(s,\pi,\mathrm{Ad})}\big|_{s=1}$, (or equivalently, of $\lim_{s\to1}(s-1)^l L^S(s,\pi,\mathrm{Ad})$ where l is the dimension of the split part of the center of G). This is known to be true when G is a general linear group or a classical group, and is expected to hold in general.

Remark 2.4. *By Proposition 2.3 $c_\pi^{\psi_N}$ does not depend on the choice of S. It also does not depend on the choice of Haar measure on $G(\mathbb{A})$. However, it depends on the automorphic realization of π, not just on π as an abstract representation, unless of course π has multiplicity one in the cuspidal spectrum.*

Remark 2.5. *A similar relation holds for $\widetilde{\mathrm{Sp}}_n$, the two-fold cover of Sp_n. The inner product on $\mathrm{Sp}_n(F)\backslash\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ of two genuine functions is defined to be the inner product on $\mathrm{Sp}_n(F)\backslash\mathrm{Sp}_n(\mathbb{A})$. Let $\tilde\pi$ be an irreducible genuine cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_n$. For simplicity assume that the ψ-theta lift of $\tilde\pi$ to $\mathrm{SO}(2n-1)$ vanishes. In this case it is a consequence of the descent method of Ginzburg–Rallis–Soudry* [GRS11] *that $L^S(1,\tilde\pi,\mathrm{Ad})$ is defined. We can define local pairing $(\cdot,\cdot)_{\tilde\pi_v}^{\psi_N}$ in a similar way. In unramified setting using the result of Bump–Friedberg–Hoffstein* [BFH91] *in place of Casselman-Shalika formula, we get:*

$$(v_0,v_0^\vee)_{\tilde\pi}^{\psi_N} = \operatorname{vol}(N\cap K)\frac{(v_0,v_0^\vee)L_\psi(\frac12,\tilde\pi)\Delta_{\mathrm{Sp}_n}(1)}{L(1,\tilde\pi,\mathrm{Ad})}. \tag{2.5}$$

Here ψ is a character of F depending on ψ_N and the factor $L_\psi(\frac12,\pi)$ in the numerator is the Shimura unramified local factor corresponding to $\tilde\pi$ and ψ. It is equal to $L(\frac12,\tau)$ when τ is the ψ-lift of $\tilde\pi$ to GL_{2n}. (Cf. [GRS99] *for precise definitions.) The factor $L(1,\tilde\pi,\mathrm{Ad})$ in the denominator is defined to be $L(1,\tau,\mathrm{sym}^2)$. Thus*

$$\mathcal{W}^{\psi_N}(\varphi)\mathcal{W}^{\psi_N^{-1}}(\varphi^\vee) = (c_{\tilde\pi}^{\psi_N}\operatorname{vol}(\mathrm{Sp}_n(F)\backslash\mathrm{Sp}_n(\mathbb{A}))^{-1}L_\psi^S(\frac12,\tilde\pi)\frac{\Delta_{\mathrm{Sp}_n}^S(s)}{L^S(1,\tilde\pi,\mathrm{Ad})}(\varphi,\varphi^\vee)_{\tilde\pi_S}^{\psi_N} \tag{2.6}$$

where $c_{\tilde\pi}^{\psi_N}$ is independent of S or the Haar measure on $\mathrm{Sp}_n(\mathbb{A})$.

2.3. **The conjecture.** We admit the existence of the Langlands group $\mathcal{L}_F$, a locally compact group whose irreducible n-dimensional representations classify cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$ [Lan79]. Let $\Psi(G)$ be the set of elliptic Arthur's parameters, which are equivalence classes of homomorphisms $\phi:\mathcal{L}_F\times\mathrm{SL}_2(\mathbb{C})\to{}^LG$ (satisfying certain properties). Here ${}^LG=\widehat{G}\rtimes W_F$ is the L-group of G, where $\widehat{G}$ is the complex dual group of

$\mathbf{G}$ and W_F is the Weil group of F. A stronger version of Arthur's conjecture states the existence of a canonical orthogonal decomposition[1]

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1) = \widehat{\bigoplus_{\phi\in\Psi(G)}} \overline{\mathcal{H}_\phi} \tag{2.7}$$

into $\mathrm{Ad}\, G^{\mathrm{ad}}(F)$-invariant subspaces where the unramified components of the irreducible constituents of $\mathcal{H}_\phi$ are determined by ϕ.

We say a parameter ϕ is of Ramanujan type if its restriction to $\mathrm{SL}_2(\mathbb{C})$ is trivial. If ϕ is not of Ramanujan type then $\mathcal{W}^{\psi_N}$ vanishes on $\mathcal{H}_\phi$ (for local reasons – see [Sha11]). Suppose that ϕ is of Ramanujan type. Then every constituent of $\mathcal{H}_\phi$ is tempered almost everywhere, hence cuspidal [Wch84]. We will *assume* that the orthogonal complement $\mathcal{H}_\phi^{\psi_N}$ of the space

$$\{\varphi \in \mathcal{H}_\phi : \mathcal{W}^{\psi_N}(\cdot, \varphi) \equiv 0\}$$

in $\mathcal{H}_\phi$ is irreducible (and in particular non-zero) and we will denote by $\pi^{\psi_N}(\phi)$ the irreducible automorphic cuspidal representation of $G(\mathbb{A})$ on $\mathcal{H}_\phi^{\psi_N}$.

One can attach to an elliptic parameter ϕ a finite group $\mathcal{S}_\phi$. If G is split over a cyclic extension E/F and $\Gamma := \mathrm{Gal}(E/F)$, then $\mathcal{S}_\phi$ is simply $C_{\widehat{G}}(\phi)/Z(\widehat{G})^\Gamma$ where $C_{\widehat{G}}(\phi)$ is the centralizer of the image of ϕ in $\widehat{G}$, and

$$Z(\widehat{G}) = \mathrm{Ker}[\widehat{G} \to \widehat{G_{\mathrm{SC}}}]$$

where G_{SC} is the simply connected cover of the derived group of G.

Motivated by Ichino-Ikeda conjecture on Gross-Prasad periods [II10], we make the following conjecture.

Conjecture 2.6. [LM15, Conjecture 1.1] *For any elliptic Arthur's parameter ϕ of Ramanujan type we have $c^{\psi_N}_{\pi^{\psi_N}(\phi)} = |\mathcal{S}_\phi|$.*

3. Some known cases

- When G is a general linear group:

  Theorem 3.1. [LM15, Theorem 4.1] *We have $c_\pi^{\psi_N} = 1$ for any cuspidal irreducible automorphic representation π of GL_m.*

  Since $|\mathcal{S}_\phi| = 1$ for a Arthur parameter ϕ when $G = \mathrm{GL}_m$, this verifies Conjecture 2.6. This theorem is a consequence of the theory of Rankin-Selberg integrals developed by Jacquet-Piatetski-Shapiro-Shalika. It generalizes the result of Hoffstein-Lockart in Introduction (which is also proved using Rankin-Selberg theory).
- Let G, $\widetilde{G}$ and $\underline{G}$ be three quasi-split groups related as follows. $\widetilde{G}$ is a connected reductive group defined and quasi-split over F; G is a connected algebraic subgroup of $\widetilde{G}$ defined over F containing the derived group of $\widetilde{G}$; $\underline{G} = G/T$ where where T is

[1] Here $\overline{\mathcal{H}_\phi}$ denotes the L^2-closure

a central torus in G which is *induced* i.e., it is the product of restriction of scalars over extensions of F of split tori. We have

Theorem 3.2. [LM15, Corollary 3.7, 3.10] *Conjecture 2.6 holds for G if and only if it holds for $\widetilde{G}$. It holds for G if and only if it holds for $\underline{G}$.*

We can infer that if Conjecture 2.6 holds for quasi-split semisimple simply connected groups, then it holds for all quasi-split reductive groups.

- Combining the above two results, we see Conjecture 2.6 holds for the groups SL_m and PGL_m.
- We remark also that verity of the conjecture is independent of the choice of character ψ_N, see [LM15, Remark 3.4].
- The case of the metaplectic two-fold cover of SL_2 (i.e., $\widetilde{\mathrm{Sp}}_1$) goes back to the classical result of Waldspurger on the Fourier coefficients of half-integral weight modular forms [Wal81] which was later generalized by many authors [Gro87, KS93, KZ81, Shi93, Koh85, KM96, Koj04, BM10, Qiu]. Waldspurger used the Shimura correspondence as his main tool. A different approach, which was taken by Jacquet [Jac87] and completed by Baruch–Mao [BM07] is via the relative trace formula. An important step in generalizing the relative trace formula approach to higher rank metaplectic groups was taken by Mao–Rallis [MR10].

4. Case of classical group and metaplectic group: automorphic descent

A natural approach to Conjecture 2.6 for a quasi-split group G is to use functorial lifting to relate it to a situation on a general linear group, then apply Theorem 3.1. In the case of $\widetilde{\mathrm{Sp}}_1$, the two approaches–Shimura correspondence (which is a special case of theta correspodence) and relative trace formula, are two methods in functorial lifting. In case when G is a classical group, there is another method for functorial lifting–the automorphic descent method developed by Ginzburg-Rallis-Soudry. We will use the descent method to approach Conjecture 2.6 in case G is a classical group or a metaplectic group.

4.1. Descent for $\widetilde{\mathrm{Sp}}_n$.

We give a description of the descent method in the case of $\widetilde{\mathrm{Sp}}_n$. Let $P = M \ltimes U$ be the Siegel parabolic subgroup of Sp_{2n} with M its Levi subgroup and U the Siegel unipotent subgroup. Let π_i, $i = 1, \dots, k$ be distinct irreducible cuspidal representations of GL_{2n_i} satisfying $L(1, \pi_i, \wedge^2) = \infty$ and $L(\frac{1}{2}, \pi_i) \neq 0$, (and $n = n_1 + \cdots + n_k$). Let π be the space of Eisenstein series on $M(\mathbb{A})$ induced from $\pi_1 \otimes \cdots \otimes \pi_k$.

Let $\mathcal{A}(\pi)$ be the space of functions $\varphi : M(F)U(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}$ such that $m \mapsto \delta_P(m)^{-\frac{1}{2}}\varphi(mg)$, $m \in M(\mathbb{A})$ belongs to the space of π for all $g \in G(\mathbb{A})$. (δ_P is the modulus function on P). We define Eisenstein series

$$\mathcal{E}(\varphi, s) = \sum_{\gamma \in P(F)\backslash G(F)} \varphi_s(\gamma g), \quad \varphi \in \mathcal{A}(\pi).$$

It converges absolutely for $\Re s \gg 0$ and extend meromorphically to $\mathbb{C}$. If follows from [GRS11, Theorem 2.1] that $\mathcal{E}(\varphi, s)$ has a pole of order k at $s = \frac{1}{2}$. Let

$$\mathcal{E}_{-k}\varphi = \lim_{s \to \frac{1}{2}} (s - \frac{1}{2})^k \mathcal{E}(\varphi, s).$$

Recall the definition of the Fourier-Jacobi coefficient. Let V (resp., V_0) be the unipotent radical in Sp_{2n} of the standard parabolic subgroup with Levi $\mathrm{GL}_1^n \times \mathrm{Sp}_n$ (resp., $\mathrm{GL}_1^{n-1} \times \mathrm{Sp}_{n+1}$). Then V/V_0 is isomorphic to a Heisenberg group. Fixing a nontrivial character ψ of $\mathbb{A}/F$. Attached to ψ there is a unique Weil representation ω_ψ of $\mathrm{Sp}_n \ltimes V/V_0$ defined on the space $\mathcal{S}(\mathbb{A}^n)$ of Schwartz functions on $\mathbb{A}^n$. Define the theta function on $\widetilde{\mathrm{Sp}}_n \ltimes V$

$$\Theta_\psi^\Phi(vg) = \sum_{\xi \in F^n} \omega_\psi(vg)\Phi(\xi)\psi_V(v), \quad v \in V(\mathbb{A}), g \in \widetilde{\mathrm{Sp}}_n(\mathbb{A}).$$

Here $\psi_V(v) = \psi(v_{1,2} + \cdots + v_{n-1,n})^{-1}$. For any automorphic form φ on $\mathrm{Sp}_{2n}(\mathbb{A})$ and $\Phi \in \mathcal{S}(\mathbb{A}^n)$, the Fourier–Jacobi coefficient $\mathrm{FJ}_\psi(\varphi, \Phi)$ is a genuine function on $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ given by

$$\mathrm{FJ}_\psi(\varphi, \Phi)(\widetilde{g}) = \int_{V(F)\backslash V(\mathbb{A})} \varphi(v\eta(g))\Theta_{\psi^{-1}}^\Phi(v\widetilde{g})\, dv, \quad g \in \mathrm{Sp}_n(\mathbb{A}). \tag{4.1}$$

Here $\eta(g) = \mathrm{diag}(1_n, g, 1_n) \in \mathrm{Sp}_{2n}$ and $\widetilde{g} = (g, 1) \in \widetilde{\mathrm{Sp}}_n$.

By definition, the *descent* of π (with respect to ψ) is the space $\tilde{\pi} = \mathcal{D}_\psi(\pi)$ generated by $\mathrm{FJ}_\psi(\mathcal{E}_{-k}\varphi, \Phi)$, $\varphi \in \mathcal{A}(\pi)$, $\Phi \in \mathcal{S}(\mathbb{A}^n)$. The following result summarizes the work of Ginzburg–Rallis–Soudry, Ginzburg–Jiang–Soudry, and Cogdell–Kim–Piatetski-Shapiro–Shahidi.

Theorem 4.1. [GRS11, Theorem 2.3, 11.2].

(1) $\tilde{\pi}$ is an irreducible cuspidal representation of $\widetilde{\mathrm{Sp}}_n$ with non-trivial ψ_N Whittaker-Fourier coefficient. Here for N the standard maximal unipotent subgroup of Sp_n, $\psi_N(u) = \psi(u_{1,2} + \ldots + u_{n-1,n} - \frac{1}{2}u_{n,n+1})$.
(2) The ψ-lift of $\tilde{\pi}$ is π.
(3) Any irreducible cuspidal representation of $\widetilde{\mathrm{Sp}}_n$ with trivial ψ-theta lift to $SO(2n-1)$ can be constructed as $\mathcal{D}_\psi(\pi)$ for some π.

4.2. **Descent for classical groups.** Let G be either $\mathrm{SO}(n)$ (the special orthogonal group of a split or a quasi-split quadratic space), Sp_n (the symplectic group of a symplectic space of dimension $2n$), or $\mathrm{U}(n)$ (the quasi-split unitary group of a hermitian space of dimension n).

In the even orthogonal case let $D \in F^*/(F^*)^2$ be the discriminant of the quadratic space (with the sign convention so that $\mathbf{G}$ is split if and only if $D \in (F^*)^2$). Let χ_D be the Hecke character $(D, \cdot)$ where $(\cdot, \cdot)$ denotes the quadratic Hilbert symbol.

In the unitary case we write E for the quadratic extension of F over which $\mathbf{G}$ splits (i.e., over which the hermitian space is defined). In all other cases we set $E = F$.

Consider the set $\Pi^G(\mathrm{Res}_{E/F}\mathrm{GL}_m)$ whose elements are sets $\{\pi_1,\dots,\pi_k\}$ (mutually inequivalent representations) where π_i is a cuspidal irreducible representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$, $i=1,\dots,k$ such that

(1) $n_1+\cdots+n_k=m=\begin{cases}2n & \mathbf{G}=\mathrm{SO}(2n+1),\ \text{or}\ \mathrm{SO}(2n),\\ 2n+1 & \mathbf{G}=\mathrm{Sp}_n,\\ n & \mathbf{G}=\mathrm{U}(n).\end{cases}$

(2) $L(1,\pi_i,r)=\infty$ for all i where $r=\begin{cases}\wedge^2 & \mathbf{G}=\mathrm{SO}(2n+1),\\ \mathrm{sym}^2 & \mathbf{G}=\mathrm{Sp}_n\ \text{or}\ \mathrm{SO}(2n),\\ \mathrm{As}^- & \mathbf{G}=\mathrm{U}(2n),\\ \mathrm{As}^+ & \mathbf{G}=\mathrm{U}(2n+1).\end{cases}$

Here $\mathrm{As}^\pm$ are the so-called Asai representations (see e.g., [KK04, KK05] for the precise definition).

(3) for ω_{π_i} central characters of π_i, $\prod_{i=1}^k\omega_{\pi_i}=1$ if $\mathbf{G}=\mathrm{Sp}_n$; $\prod_{i=1}^k\omega_{\pi_i}=\chi_D$ if $\mathbf{G}=\mathrm{SO}(2n)$. (In all other cases we automatically have $\omega_{\pi_i}\big|_{\mathbb{A}^*}\equiv 1$ for all i.)

The descent method provides for any $\{\pi_1,\dots,\pi_k\}\in\Pi^G(\mathrm{Res}_{E/F}\mathrm{GL}_m)$, a cuspidal subrepresentation $\sigma=\sigma(\{\pi_1,\dots,\pi_k\})$ of $L^2(G(F)\backslash G(\mathbb{A}))$ such that

(1) Any irreducible constituent σ' of σ has functorial lift $\pi_1\boxplus\cdots\boxplus\pi_k$.
(2) No ψ_N-generic cuspidal representation whose functorial lift is $\pi_1\boxplus\cdots\boxplus\pi_k$ is orthogonal to σ in $L^2(G(F)\backslash G(\mathbb{A}))$.

It is expected that when G is not an even orthogonal group σ is irreducible. In the $\mathrm{SO}(2n)$ case, let θ be the outer automorphism; then σ is expected to be either irreducible and θ-invariant or that $\sigma=\tau\oplus\theta(\tau)$ where τ is irreducible and $\theta(\tau)\not\simeq\tau$.

4.3. **Conjecture for classical groups.** The descent construction is closely related to the (hypothetical) representations $\pi^{\psi_N}(\phi)$ of §2.3. In all cases except $\mathrm{SO}(2n)$ the set $\Pi^G(\mathrm{Res}_{E/F}\mathrm{GL}_m)$ corresponds exactly to elliptic Arthur parameters of Ramanujan type. If ϕ is the corresponding parameter then $\mathcal{S}_\phi\simeq(\mathbb{Z}/2\mathbb{Z})^{k-1}$ (see [Art11] and [Mok15]). One expects multiplicity one for G and therefore if $\{\pi_1,\dots,\pi_k\}\in\Pi^G(\mathrm{Res}_{E/F}\mathrm{GL}_m)$ and ϕ is the corresponding parameter then $\pi^{\psi_N}(\phi)=\sigma(\{\pi_1,\dots,\pi_k\})$. The case of $\mathrm{SO}(2n)$ is subtler, we refer to [LM15, §5] for discussion on this case.

Conjecture 4.2. [LM15, Conjecture 5.1] *Suppose that π is an irreducible constituent of $\sigma(\{\pi_1,\dots,\pi_k\})$ and let s be the size of the stabilizer of π under $\{1,\theta\}$. (In particular, $s=1$ unless $G=\mathrm{SO}(2n)$.) Then in (2.4) $c_\pi^{\psi_N}=2^{k-1}/s$.*

Remark 4.3. *Note that $L^S(s,\pi,\mathrm{Ad})=L^S(s,\pi_1\boxplus\cdots\boxplus\pi_k,\tilde r)$ where* $\tilde r=\begin{cases}\mathrm{sym}^2 & \mathbf{G}=\mathrm{SO}(2n+1),\\ \wedge^2 & \mathbf{G}=\mathrm{Sp}_n\ \text{or}\ \mathrm{SO}(2n),\\ \mathrm{As}^+ & \mathbf{G}=\mathrm{U}(2n),\\ \mathrm{As}^- & \mathbf{G}=\mathrm{U}(2n+1).\end{cases}$

Remark 4.4. *Using accidental isomorphism for classical groups of lower rank, we can derive from Theorem 3.1 that Conjecture 4.2 holds for the groups* $\mathrm{U}(1)$*,* $\mathrm{U}(2)$*,* Sp_1*,* $\mathrm{SO}(2)$*,* $\mathrm{SO}(3)$*,* $\mathrm{SO}(4)$ *and the split* $\mathrm{SO}(6)$ *(see* [LM15, §6]*).*

4.4. **Case of $\widetilde{\mathrm{Sp}}_n$.** Let $\tilde{\pi} = \mathcal{D}_\psi(\pi)$ with π given as in §4.1, then we expect in (2.6) $c_\pi^{\psi_N} = 2^k$. Namely

$$\tilde{\mathcal{W}}^{\psi_N}(\tilde{\varphi})\tilde{\mathcal{W}}^{\psi_N^{-1}}(\tilde{\varphi}^\vee) = 2^{-k}\Delta^S_{\mathrm{Sp}_n}(1)\frac{L^S(\frac{1}{2},\pi)}{L^S(1,\pi,\mathrm{sym}^2)} \int^{st}_{N(F_S)} (\tilde{\pi}(\tilde{n})\tilde{\varphi},\tilde{\varphi}^\vee)_{\mathrm{Sp}_n(F)\backslash\mathrm{Sp}_n(\mathbb{A})}\psi_N(n)^{-1}\,dn. \tag{4.2}$$

Theorem 4.5. [LM17b] *The identity* (4.2) *holds when the archimedean components of* $\tilde{\pi}$ *are discrete series representations.*

Below we give a sketch of the proof. The main idea is that the descent construction allows us to reduce to a local problem through *period transition.*

We introduce a notation for period transition. Let $\pi_i = \otimes_v \pi_{i,v}$ $(i = 1,\dots,l)$ be automorphic representations on G_i. Let L_1, L_2 be two linear forms on $\otimes_i\pi_i$. We write $L_1 \sim L_2$ if for each local place v there is a locally defined function $\alpha_v(\phi_1,\dots,\phi_l)$ depending only on the one dimensional spaces spanned by ϕ_i, such that at almost all places $\alpha_v = 1$ on unramified vectors, and whenever $\varphi_i = \otimes\varphi_{i,v}$ in the spaces of π_i for large enough set S

$$L_1(\otimes\varphi_i) = L_2(\otimes\varphi_i)\prod_{v\in S}\alpha_v(\varphi_{1,v},\dots,\varphi_{l,v})$$

(whenever both sides are nonzero). Then (4.2) can be regarded as the following period transition formula:

$$\tilde{\mathcal{W}}^{\psi_N}(\tilde{\varphi})\tilde{\mathcal{W}}^{\psi_N^{-1}}(\tilde{\varphi}^\vee) \sim 2^{-k}\Delta_{\mathrm{Sp}_n}(1)\frac{L(\frac{1}{2},\pi)}{L(1,\pi,\mathrm{sym}^2)}(\tilde{\varphi},\tilde{\varphi}^\vee)_{\mathrm{Sp}_n(F)\backslash\mathrm{Sp}_n(\mathbb{A})}. \tag{4.3}$$

If $\tilde{\varphi} = \mathrm{FJ}_\psi(\mathcal{E}_{-k}\varphi,\Phi)$ is a cusp form in $\tilde{\pi}$ constructed through descent, its Whittaker-Fourier coefficient $\mathcal{W}(\tilde{\varphi})$ can be described using descent data φ and Φ. The description is given by [GRS11, Theorem 9.7, part (1)](reformatted in a more explicit form in [LM17a, Theorem 6.3]). For an automorphic form ϕ on Sp_{2n}, its restriction to M is an automorphic form on GL_{2n}, we use $\mathcal{W}^M(\phi)$ to denote the Whittaker-Fourier coefficient of this restriction to GL_{2n}. Then we have the following period transition formula

$$\tilde{\mathcal{W}}(\mathrm{FJ}_\psi(\mathcal{E}_{-k}\varphi,\Phi)) \sim 2^{-k}\frac{L(\frac{1}{2},\pi)}{L(\frac{3}{2},\pi)}\frac{\lim_{s\to 1}(s-1)^k L(s,\pi,\wedge^2)}{L(2,\pi,\wedge^2)}\mathcal{W}^M(\Phi * \varphi). \tag{4.4}$$

Here $\Phi * \phi$ is a convolution which gives an automorphic form on Sp_{2n}.

Meanwhile from Rankin-Selberg theory [GRS11, Theorem 10.4, (10.6), (10.63)] we get another period transition identity:

$$\langle\tilde{\varphi},\mathrm{FJ}_\psi(\mathcal{E}_{-k}\varphi,\Phi)\rangle \sim \Delta_{\mathrm{Sp}_n}(1)^{-1}\frac{\lim_{s\to 1}(s-1)^k L(s,\pi\otimes\pi)}{L(\frac{3}{2},\pi)L(2,\pi,\wedge^2)}\mathcal{W}(\tilde{\varphi})\mathcal{W}^M(\Phi * \varphi). \tag{4.5}$$

The two identities (4.4) and (4.5) immediately yield (4.3), thus reduces the proof of (4.2) to a purely local problem. In fact a more detailed analysis of the locally defined functions shows that

$$\tilde{\mathcal{W}}^{\psi_N}(\tilde{\varphi})\tilde{\mathcal{W}}^{\psi_N^{-1}}(\tilde{\varphi}^\vee) = 2^{-k}\Delta^S_{\mathrm{Sp}_n}(1)\frac{L^S(\frac{1}{2},\pi)}{L^S(1,\pi,\mathrm{sym}^2)})(\prod_{v\in S} c_{\pi_v})$$
$$\int^{st}_{N(F_S)}(\tilde{\pi}(\widetilde{n})\tilde{\varphi},\tilde{\varphi}^\vee)_{\mathrm{Sp}_n(F)\backslash\mathrm{Sp}_n(\mathbb{A})}\psi_N(n)^{-1}\,dn$$

where c_{π_v} is a local constant determined by π_v.

We can determine the local constant c_{π_v} using *model transition*–a local analogue of period transition. In [LM17b], based on the work of [LM15b], it is shown that when F_v is a p-adic field

$$c_{\pi_v} = \epsilon(\frac{1}{2},\pi_v,\psi_v). \tag{4.6}$$

Remark 4.6. *When $\tilde{\pi}_v$ the local descent of π_v is a discrete series representation, the argument in* [ILM] *shows that the local identity* (4.6) *is equivalent to Hiraga-Ichino-Ikeda's formal degree conjecture (*[HII08]*) for $\tilde{\pi}_v$. Thus the above work implies the formal degree conjecture of $\widetilde{\mathrm{Sp}}_n$ over a $p-$adic field. Meanwhile the formal degree conjecture is known to hold over an archimedean field, thus* (4.6) *holds over an archimedean field when $\tilde{\pi}_v$ is a discrete series representation.*

The product of $\epsilon(\frac{1}{2},\pi_v,\psi_v)$ is 1 thus we get Theorem 4.5.

Remark 4.7. *The descent approach can be used to establish Conjecture 4.2 for classical groups. For the case of unitary groups, in* [LM16] *we use the descent to reduce the conjecture to a local problem; K. Morimoto has established the necessary local identity in $p-$adic case. The work on orthogonal group and symplectic group is ongoing.*

References

[Art11] James Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, A.M.S. Colloquium Publ. **61** (2013).

[BM07] Ehud Moshe Baruch and Zhengyu Mao, *Central value of automorphic L-functions*, Geom. Funct. Anal. **17** (2007), no. 2, 333–384.

[BM10] ———, *A generalized Kohnen-Zagier formula for Maass forms*, J. Lond. Math. Soc. (2) **82** (2010), no. 1, 1–16.

[BFH91] Daniel Bump, Solomon Friedberg, and Jeffrey Hoffstein, *p-adic Whittaker functions on the metaplectic group*, Duke Math. J. **63** (1991), no. 2, 379–397.

[CS80] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function*, Compositio Math. **41** (1980), no. 2, 207–231.

[GRS99] David Ginzburg, Stephen Rallis, and David Soudry, *Lifting cusp forms on* GL_{2n} *to* $\widetilde{\mathrm{Sp}}_{2n}$*: the unramified correspondence*, Duke Math. J. **100** (1999), no. 2, 243–266.

[GRS11] ———, *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[Gro87] Benedict H. Gross, *Heights and the special values of L-series*, Number theory (Montreal, Que., 1985), CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, 1987, pp. 115–187.

[Gro97] ———, *On the motive of a reductive group*, Invent. Math. **130** (1997), no. 2, 287–313.

[HII08] Kaoru Hiraga, Atsushi Ichino, and Tamotsu Ikeda, *Formal degrees and adjoint γ-factors*, J. Amer. Math. Soc. **21** (2008), no. 1, 283–304.

[HL94] Jeffrey Hoffstein and Paul Lockhart, *Coefficients of Maass forms and the Siegel zero*, Ann. Math. **140** (1994), 161–181.

[II10] Atsushi Ichino and Tamutsu Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425.

[ILM] Atsushi Ichino, Erez Lapid, and Zhengyu Mao, *On the formal degrees of square-integrable representations of odd special orthogonal and metaplectic groups*, Duke Math. J., to appear (arXiv:1404.2909).

[Jac87] Hervé Jacquet, *On the nonvanishing of some L-functions*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), no. 1-3, 117–155.

[KS93] Svetlana Katok and Peter Sarnak, *Heegner points, cycles and Maass forms*, Israel J. Math. **84** (1993), no. 1-2, 193–227.

[KM96] Kamal Khuri-Makdisi, *On the Fourier coefficients of nonholomorphic Hilbert modular forms of half-integral weight*, Duke Math. J. **84** (1996), no. 2, 399–452.

[KK04] Henry H. Kim and Muthukrishnan Krishnamurthy, *Base change lift for odd unitary groups*, Functional analysis VIII, Various Publ. Ser. (Aarhus), vol. 47, Aarhus Univ., Aarhus, 2004, pp. 116–125.

[KK05] ———, *Stable base change lift from unitary groups to* GL_n, IMRP Int. Math. Res. Pap. (2005), no. 1, 1–52.

[Koh85] Winfried Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), no. 2, 237–268.

[KZ81] W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), no. 2, 175–198.

[Koj04] Hisashi Kojima, *On the Fourier coefficients of Hilbert-Maass wave forms of half integral weight over arbitrary algebraic number fields*, J. Number Theory **107** (2004), no. 1, 25–62.

[Lan79] R. P. Langlands, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 205–246.

[LM15] Erez Lapid and Zhengyu Mao, *A conjecture on Whittaker–Fourier coefficients of cusp forms*, J. Number Theory **146** (2015), 448–505.

[LM15b] ———, *Model Transition for Representations of Metaplectic Type*, Int. Math. Res. Not. IMRN (2015), no. 19, 9486–9568.

[LM16] ———, *On Whittaker-Fourier coefficients of automorphic forms on unitary groups: reduction to a local identity*, Contemporary Math **664**, (2016), 295-320.

[LM17a] ———, *Whittaker-Fourier coefficients of cusp forms on $\widetilde{Sp}_n$: reduction to a local statement*, Amer. J. Math. **139**, (2017), 1-55

[LM17b] ———, *On an analogue of the Ichino–Ikeda conjecture for Whittaker coefficients on the metaplectic group*, Algebra and Number Theory, **11**, (2017), 713-765

[MR10] Zhengyu Mao and Stephen Rallis, *A relative trace identity between GL_{2n} and $\widetilde{\mathrm{Sp}}_n$*, Duke Math. J. **152** (2010), no. 2, 207–255.

[Mok15] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Memoir AMS, **235** No. 1108 (2015).

[Qiu] Yannan Qiu, *The Whittaker period formula on metaplectic $SL(2)$*, To appear in Trans. AMS, arXiv:1308.2353.

[Sha11] Freydoon Shahidi, *Arthur packets and the Ramanujan conjecture*, Kyoto J. Math. **51** (2011), no. 1, 1–23.

[Shi93] Goro Shimura, *On the Fourier coefficients of Hilbert modular forms of half-integral weight*, Duke Math. J. **71** (1993), no. 2, 501–557.

[Wal81] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. (9) **60** (1981), no. 4, 375–484.

[Wch84] N. R. Wallach, *On the constant term of a square integrable automorphic form*, Operator algebras and group representations, Vol. II (Neptun, 1980), Monogr. Stud. Math., vol. 18, Pitman, Boston, MA, 1984, pp. 227–237.

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, USA

E-mail address: zmao@rutgers.edu