A Splitting Theorem for Scalar Curvature

OTIS CHODOSH
Princeton University and Institute for Advanced Study

MICHAEL EICHMAIR
University of Vienna

AND

VLAD MORARU
University of L'Aquila

We dedicate this paper to Gregory Galloway on the occasion of his 70th birthday.

Abstract

We show that a Riemannian 3-manifold with nonnegative scalar curvature is flat if it contains an area-minimizing cylinder. This scalar-curvature analogue of the classical splitting theorem of J. Cheeger and D. Gromoll (1971) was conjectured by D. Fischer-Colbrie and R. Schoen (1980) and by M. Cai and G. Galloway (2000). © 2018 the Authors. *Communications on Pure and Applied Mathematics* is published by the Courant Institute of Mathematical Sciences and Wiley Periodicals, Inc.

1 Introduction

Let (M, g) be a connected, orientable, complete Riemannian 3-manifold with nonnegative scalar curvature. D. Fischer-Colbrie and R. Schoen show in [10] that a connected, orientable, complete stable minimal immersion into (M, g) is conformal to a plane, a sphere, a torus, or a cylinder. They conjecture that (M, g) is flat if the immersion is conformal to the cylinder; cf. remark 4 in [10]. M. Cai and G. Galloway point out a counterexample obtained from flattening standard $\mathbb{R} \times \mathbb{S}^2$ near $\mathbb{R} \times \{\text{great circle}\}$ in their concluding remark in [7]. They ask if the conjecture holds under the additional assumption that the immersion be "suitably" area-minimizing. In this paper, we prove the following result:

THEOREM 1.1. *Let (M, g) be a connected, orientable, complete Riemannian 3-manifold with nonnegative scalar curvature. Assume that (M, g) contains a properly embedded surface $S \subset M$ that is both homeomorphic to the cylinder and absolutely area-minimizing. Then (M, g) is flat. In fact, a cover of (M, g) is isometric to standard $\mathbb{S}^1 \times \mathbb{R}^2$ upon scaling.*

Note that this result is in satisfying analogy with the classical splitting theorem of J. Cheeger and D. Gromoll [9] in dimension 3, where scalar curvature takes the

place of Ricci curvature and where area-minimizing cylinders stand in for length-minimizing geodesic lines.[1]

Theorem 1.1 follows from the work of M. Anderson and L. Rodríguez [2] when we impose the much stronger assumption of bounded, nonnegative Ricci curvature. The strategy of M. Anderson and L. Rodríguez [2] was refined by G. Liu [13] to classify complete, noncompact Riemannian 3-manifolds with nonnegative Ricci curvature. These ideas have been developed by the first- and second-named authors to establish the following *scalar-curvature* rigidity result for asymptotically flat 3-manifolds, which had been conjectured by R. Schoen.

THEOREM 1.2 ([8]). *The only asymptotically flat Riemannian 3-manifold with nonnegative scalar curvature that admits a noncompact, area-minimizing boundary is flat* $\mathbb{R}^3$.

Our proof of Theorem 1.1 in this paper is a further development of these ideas. We now discuss challenges in the proof that are not present in the previously discussed works.

The goal is to construct a *foliation* of (M, g) by area-minimizing cylinders. The leaves of this foliation arise as limits of solutions of certain Plateau problems for least area. A major challenge we face here that has no substantial analogue in the proof of Theorem 1.2 is the a priori possible appearance of stable minimal planes or spheres (rather than cylinders or tori) in these limits. This scenario is addressed in Figure 3.2 below. The ambient scalar curvature may well be positive along such surfaces, as the examples in remark 3 of [10] show.

We need an approach that is sensitive to the topology of solutions of the Plateau problems in the construction. At the same time, we have to make sure that these solutions pass to limits in a reasonable way. Recall from, e.g., [21] that stable minimal surfaces in Riemannian 3-manifolds satisfy local curvature estimates that are independent of area bounds. In particular, sequences of such surfaces admit subsequential limits as pointed immersions. If each surface in the sequence is an area-minimizing boundary, then so is the limit. (If a small ambient ball intersects such a surface in two components, then these sheets have opposite orientation and are almost parallel. This scenario can be ruled out by a cut-and-paste argument.) However, limits of general area-minimizing surfaces can exhibit much greater complexity—think of condensing closed geodesics on the torus and compare with Remark A.1 below.

As such, the use of solutions of Plateau problems in the class of *all* (oriented) competitors risks the loss of local area bounds in the limit. On the other hand, the use of solutions in the class of boundaries risks the appearance of planes or spheres in the limit. These threats taken together force us to select the various classes of surfaces considered in the proof of Theorem 1.1 with great care.

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[1] As the example of the doubled Schwarzschild manifold shows, the classical splitting theorem fails when we relax the assumption of nonnegative Ricci curvature to nonnegative scalar curvature.

We review the notions of area-minimizing surfaces that are used in this paper in Appendix A.

Subsequent to the paper of M. Cai and G. Galloway [7], there have been many further works establishing scalar curvature rigidity results in the presence of *compact* area-minimizing surfaces, including [1, 4–6, 11, 14–16, 18, 19]. We anticipate that the techniques developed here will lead to alternative proofs of these results. We plan to explore this possibility in forthcoming work.

Finally, we mention that parts of the strategy of M. Anderson and L. Rodríguez [2] depend on the assumption of nonnegative Ricci curvature in a subtle but essential way. In particular, we do not see how to carry over the crucial area estimate [2, eq. (1.5)] to the nonnegative scalar curvature setting.

2 Tools

The following result is due to R. Schoen and S.-T. Yau [22, sec. 5], except for the conclusion that the induced metric $\varphi^* g$ on S is flat when S is a torus, which was proven by D. Fischer-Colbrie and R. Schoen [10, theorem 3].

LEMMA 2.1. *Let (M, g) be a Riemannian 3-manifold. Let $\varphi\colon S \to M$ be an orientable, complete, two-sided stable minimal immersion of a closed surface S such that $R \circ \varphi \geq 0$, where R is the scalar curvature of (M, g). Then S is topologically either a sphere or a torus. In the case where S is a torus, the immersion is totally geodesic and $R \circ \varphi = 0$; moreover, the induced metric $\varphi^* g$ on S is flat.*

The following result due to D. Fischer-Colbrie and R. Schoen is part of theorem 3 in [10].

LEMMA 2.2. *Let (M, g) be a Riemannian 3-manifold. Let $\varphi\colon S \to M$ be an orientable, complete, noncompact, two-sided stable minimal immersion such that $R \circ \varphi \geq 0$ where R is the scalar curvature of (M, g). Then S with the induced metric $\varphi^* g$ is conformal to either the plane or the cylinder.*

We refer to [3, 17, 20, 23] as well as appendix C of [8] for discussions and proofs of the following refinement of Lemma 2.2.

LEMMA 2.3. *Let (M, g) be a Riemannian 3-manifold. Let $\varphi\colon S \to M$ be an orientable, complete, noncompact, two-sided stable minimal immersion such that $R \circ \varphi \geq 0$, where R is the scalar curvature of (M, g). If S is a cylinder, then the immersion is totally geodesic, the induced metric $\varphi^* g$ is flat, and $R \circ \varphi = 0$.*

The following rigidity result is due to M. Cai and G. Galloway [7, theorem 2].

LEMMA 2.4. *Let (M, g) be a connected, complete Riemannian 3-manifold with possibly empty weakly mean-convex boundary. We also assume that (M, g) has nonnegative scalar curvature. If (M, g) contains a two-sided torus that has least area in its isotopy class, then (M, g) is flat.*

3 Proofs

We need the following lifting property for absolutely area-minimizing surfaces.

LEMMA 3.1. *Let (M, g) be an orientable, complete Riemannian 3-manifold and let $S \subset M$ be a properly embedded, orientable, absolutely area-minimizing surface without boundary. There is a covering $p : \widetilde{M} \to M$ with $p_*(\pi_1(\widetilde{M})) = i_*(\pi_1(S))$ where $i : S \to M$ is the inclusion map. The inclusion map lifts to a proper embedding $\widetilde{i} : S \to \widetilde{M}$ and $\widetilde{i}_* : \pi_1(S) \to \pi_1(\widetilde{M})$ is surjective. Moreover, $\widetilde{S} = \widetilde{i}(S) \subset \widetilde{M}$ is absolutely area-minimizing in $(\widetilde{M}, \widetilde{g})$ where $\widetilde{g} = p^* g$.*

PROOF. The asserted existence of covering and lift are standard, see Propositions 1.36 and 1.33 in [12]. To see that $\widetilde{S}$ is area-minimizing, note that p is injective along $\widetilde{S}$ and that every (properly embedded) competing surface for $\widetilde{S}$ projects to a properly immersed, competing surface for S. Using cut-and-paste arguments with small changes in area, we obtain a properly embedded competitor downstairs. □
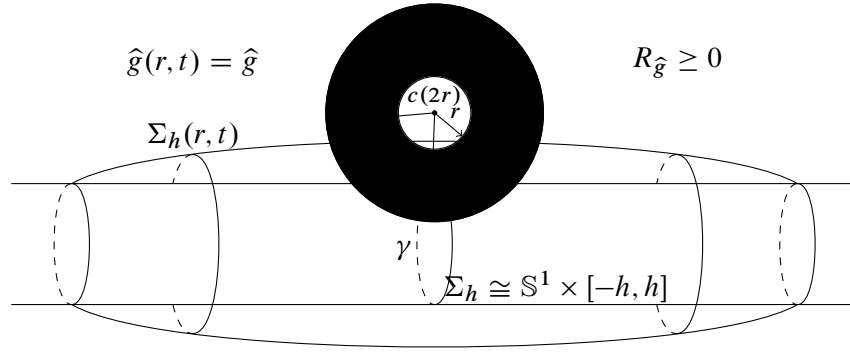
PROOF OF THEOREM 1.1. Let $S \subset M$ be a properly embedded cylinder that is absolutely area-minimizing in (M, g).

In view of Lemma 3.1, we may assume that the inclusion map $i \colon S \to M$ induces a surjection $i_* \colon \pi_1(S) \to \pi_1(M)$. It follows from Lemma 2.3 that $S \subset M$ is intrinsically flat. By scaling (M, g) if necessary, we may assume that $S \subset M$ is isometric to standard $\mathbb{S}^1 \times \mathbb{R}$.

If $S \subset M$ is separating, then $M \setminus S$ has two components. We cut M along S and make a choice of component to obtain a connected, complete Riemannian 3-manifold whose boundary is connected. If $S \subset M$ doesn't separate, then $M \setminus S$ is connected. We cut M along S to obtain a connected, complete Riemannian 3-manifold whose boundary has exactly two components, of which we choose one. Either way, we denote the new Riemannian 3-manifold by $(\widehat{M}, \widehat{g})$ and the chosen component of its boundary by Σ. Note that $\Sigma \subset \widehat{M}$ is isometric to S and absolutely area-minimizing in $(\widehat{M}, \widehat{g})$. We denote the closed curve on Σ that corresponds to $\mathbb{S}^1 \times \{0\}$ by γ. We write Σ_h for the portion of Σ corresponding to $\mathbb{S}^1 \times [-h, h]$.

Fix a unit speed geodesic $c \colon [0, \varepsilon) \to \widehat{M}$ with $c(0) \in \gamma$ and $\dot{c}(0) \perp \mathrm{T}_{c(0)} \Sigma$. As in the proof of Theorem 1.2 in appendix J of [8], we find a family of smooth Riemannian metrics $\{\widehat{g}(r, t)\}_{r, t \in (0, \varepsilon)}$ on $\widehat{M}$ with the following properties (illustrated in Figure 3.1):

(i) $\widehat{g}(r, t) \to \widehat{g}$ in C^3 as $t, r \searrow 0$;

(ii) $\widehat{g}(r, t) \to \widehat{g}$ smoothly as $t \searrow 0$ for $r \in (0, \varepsilon)$ fixed;

(iii) $\widehat{g}(r, t) = \widehat{g}$ on $\{x \in \widehat{M} \colon \mathrm{dist}_{\widehat{g}}(x, c(2r)) \geq 3r\}$;

(iv) $\widehat{g}(r, t) < \widehat{g}$ as quadratic forms in $\{x \in \widehat{M} \colon \mathrm{dist}_{\widehat{g}}(x, c(2r)) < 3r\}$;

(v) $\widehat{g}(r, t)$ has positive scalar curvature in $\{x \in \widehat{M} \colon r < \mathrm{dist}_{\widehat{g}}(x, c(2r)) < 3r\}$;

(vi) $\widehat{M}$ is weakly mean-convex with respect to $\widehat{g}(r, t)$.

FIGURE 3.1. The scalar curvature of $\widehat{g}(r,t)$ is positive in the shaded region.

Fix $h > 1$. Let B_h denote a precompact open set with smooth boundary in $\widehat{M}$ and such that $\{x \in \widehat{M}: \mathrm{dist}_{\widehat{g}}(x, \Sigma_h) < 2h\} \subset B_h$. We modify the metric $\widehat{g}(r,t)$ near the boundary of B_h to $\widehat{g}(r,t,h)$ so B_h is weakly mean-convex with respect to $\widehat{g}(r,t,h)$ and

$$(1-\delta)\widehat{g}(r,t) \leq \widehat{g}(r,t,h) \leq (1+\delta)\widehat{g}(r,t)$$

where $\delta \in (0,1)$ is chosen to satisfy (3.1) below. Among all compact, oriented surfaces in B_h with boundary $\partial\Sigma_h$ that bound an open subset of $\widehat{M}$, there is one whose area with respect to $\widehat{g}(r,t,h)$ is least. Choose one such area-minimizing surface and denote it by $\Sigma_h(r,t)$. We claim $\Sigma_h(r,t)$ intersects $\{x \in \widehat{M}: \mathrm{dist}_{\widehat{g}}(x, c(2r)) < 3r\}$. For if not, we have that $\widehat{g} = \widehat{g}(r,t)$ along $\Sigma_h(r,t)$ and we may compute, as in [8, eq. (12)], that

$$0 < \mathrm{area}_{\widehat{g}}(\Sigma_h) - \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h) \leq \mathrm{area}_{\widehat{g}}(\Sigma_h(r,t)) - \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h)$$

$$= \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h(r,t)) - \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h)$$

$$\leq \frac{1}{1-\delta}\,\mathrm{area}_{\widehat{g}(r,t,h)}(\Sigma_h(r,t)) - \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h)$$

$$\leq \frac{1}{1-\delta}\,\mathrm{area}_{\widehat{g}(r,t,h)}(\Sigma_h) - \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h)$$

$$\leq \frac{1+\delta}{1-\delta}\,\mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h) - \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h)$$

$$= \frac{2\delta}{1-\delta}\,\mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h).$$

This is a contradiction if we choose $\delta = \delta(r,t,h) > 0$ with

$$(3.1) \qquad \frac{2\delta}{1-\delta} < \frac{\mathrm{area}_{\widehat{g}}(\Sigma_h) - \mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h)}{\mathrm{area}_{\widehat{g}(r,t)}(\Sigma_h)}.$$

A comparison with small geodesic spheres gives local area bounds for the surfaces $\Sigma_h(r,t)$ that are independent of all parameters; cf. Remark A.1. Using standard results in geometric measure theory, we may pass these surfaces to a subsequential limit as $h \to \infty$ to obtain a properly embedded surface $\Sigma(r,t)$. Note that $\Sigma(r,t)$ is a boundary in $\widehat{M}$. It follows from the construction that this boundary is homologically* area-minimizing[2] with respect to $g(r,t)$.

When $r,t > 0$ are sufficiently small, the surface $\Sigma(r,t)$ contains a closed curve $\gamma(r,t)$ that intersects $\{x \in \widehat{M}: \mathrm{dist}_{\hat{g}}(x,c(2r)) \leq 3r\}$ and which is close to γ. Indeed, there is a small ball with center at $c(0) \in \gamma$ where a piece of $\Sigma(r,t)$ appears as a graph above $T_{c(0)}\Sigma$. The geometric Harnack principle allows us to continue this piece of $\Sigma(r,t)$ into a ribbon as we traverse γ. The ribbon must close up as we travel around γ, as otherwise there would be two nearby sheets—a scenario that contradicts the area-minimizing property of $\Sigma(r,t)$. The component $\widehat{\Sigma}(r,t)$ of $\Sigma(r,t)$ that contains this curve $\gamma(r,t)$ converges to Σ as $r,t \searrow 0$. Observe that $\widehat{\Sigma}(r,t)$ is a boundary in $\widehat{M}$ that is homologically* area-minimizing with respect to $\hat{g}(r,t)$; cf. lemma 33.4 in [24].

We claim that $\widehat{\Sigma}(r,t)$ is *neither* a plane *nor* a sphere if $r,t > 0$ are small enough. Suppose otherwise. In this case, $\gamma(r,t)$ bounds an embedded disk in $\widehat{M}$. Consequently, so does γ. In particular, inclusion induces the trivial map $\pi_1(\Sigma) \to \pi_1(\widehat{M})$. By Lemma 3.2 below, every connected, closed surface in $\widehat{M}$ is separating. If $\widehat{\Sigma}(r,t)$ is a sphere, it bounds a possibly unbounded region in $\widehat{M}$. This contradicts its homologically* area-minimizing property. Assume that $\widehat{\Sigma}(r,t)$ is a plane (see Figure 3.2). Let $\Delta(r,t) \subset \widehat{\Sigma}(r,t)$ be the disk bounded by $\gamma(r,t)$. Since $\widehat{\Sigma}(r,t)$ converges to Σ as $r,t \searrow 0$, we have that $\mathrm{area}_g(\Delta(r,t)) \to \infty$ as $r,t \searrow 0$. Choose $0 < r_2 \ll r_1$ and $0 < t_2 \ll t_1$ such that $\widehat{\Sigma}(r_1,t_1)$ and $\widehat{\Sigma}(r_2,t_2)$ are planes. Note that $\Delta(r_2,t_2)$, $\Delta(r_1,t_1)$, and a small neck connecting $\gamma(r_1,t_1)$ and $\gamma(r_2,t_2)$ bound an open set. This contradicts the homologically* area-minimizing property of $\widehat{\Sigma}(r_1,t_1)$.
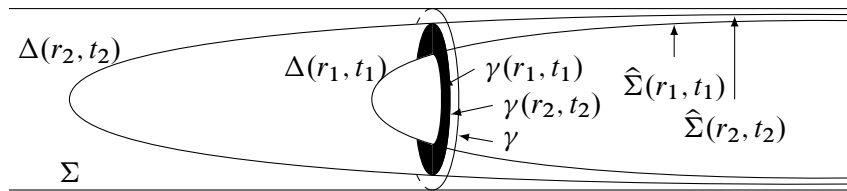

FIGURE 3.2. The possibility that $\widehat{\Sigma}(r,t)$ is a plane for $r,t > 0$ small enough can be ruled out by comparing the areas of the disks $\Delta(r,t)$ or by incompressibility of Σ.

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[2] This nonstandard area-minimizing property is discussed in Appendix A. Its full strength is needed in the cut-and-paste argument used to rule out planes and spheres below.

We see from (v), Lemma 2.1, Lemma 2.2, and Lemma 2.3 that, for $r, t > 0$ small enough, $\widehat{\Sigma}(r, t)$ either intersects $\{x \in \widehat{M} \colon \operatorname{dist}_{\hat{g}}(x, c(2r)) \leq r\}$, or $\widehat{\Sigma}(r, t)$ intersects $\{x \in \widehat{M} \colon \operatorname{dist}_{\hat{g}}(x, c(2r)) = 3r\}$ but not $\{x \in \widehat{M} \colon \operatorname{dist}_{\hat{g}}(x, c(2r)) < 3r\}$. Fix $r > 0$ small and pass to a geometric subsequential limit as $t \searrow 0$. We obtain a properly embedded boundary $\Sigma(r) \subset \widehat{M}$ that has a connected component $\widehat{\Sigma}(r)$ that intersects $\{x \in \widehat{M} \colon \operatorname{dist}_{\hat{g}}(x, c(2r)) \leq 3r\}$ and that is disjoint from Σ. As before, we see that $\widehat{\Sigma}(r)$ is a properly embedded boundary in $\widehat{M}$ that is homologically* area-minimizing with respect to $\hat{g}$. Clearly, $\widehat{\Sigma}(r)$ is disjoint from Σ and contains a closed embedded curve $\gamma(r)$ close to γ. The argument of the preceding paragraph shows that when $r > 0$ is small enough, the surface $\widehat{\Sigma}(r)$ is diffeomorphic to either a torus or a cylinder.

If $\widehat{\Sigma}(r)$ is a torus, then $(\widehat{M}, \hat{g})$ is isometric to either standard $\mathbb{S}^1 \times \mathbb{R} \times [0, \infty)$ or standard $\mathbb{S}^1 \times \mathbb{R} \times [0, a]$ for some $a > 0$ by Lemma 2.4. We may thus assume that $\widehat{\Sigma}(r)$ is cylindrical for all $r > 0$ small. By Lemma 2.3, $\widehat{\Sigma}(r) \subset \widehat{M}$ is intrinsically flat, totally geodesic, and the ambient Ricci tensor evaluated in the normal direction vanishes along $\widehat{\Sigma}(r)$. Note that $\widehat{\Sigma}(r)$ converges to Σ as $r \searrow 0$. We now argue exactly as in the proof of [8, theorem 1.6] to show that the ambient Riemann tensor vanishes along Σ.

We may repeat the above argument starting with any of the surfaces $\widehat{\Sigma}(r)$ for $r > 0$ sufficiently small, using that they are homologically* area-minimizing in $(\widehat{M}, \hat{g})$.[3] A continuity argument then gives that $(\widehat{M}, \hat{g})$ is either standard $\mathbb{S}^1 \times \mathbb{R} \times [0, \infty)$ or standard $\mathbb{S}^1 \times \mathbb{R} \times [0, a]$ for some $a > 0$. $\square$

LEMMA 3.2. *The assumptions and notation used here are as in the proof of Theorem* 1.1 *above. If the inclusion* $\Sigma \subset \widehat{M}$ *induces a trivial map* $\pi_1(\Sigma) \to \pi_1(\widehat{M})$, *then every connected, closed surface in* $\widehat{M}$ *is separating.*

PROOF. It follows from the hypothesis and the construction of $\widehat{M}$ from $M \setminus S$ that M is simply connected. (Here we are using that the map $\pi_1(S) \to \pi_1(M)$ induced by the inclusion $S \subset M$ is surjective.) Standard intersection theory gives that every connected, closed embedded surface $N \subset M$ separates M. Indeed, if we assume that $N \subset M$ is not separating, there is a closed embedded curve in M that intersects N transversely and exactly once. Such a curve cannot be homotopically trivial. Assume now that $\widehat{N} \subset \widehat{M}$ is a connected and closed surface disjoint from the boundary of $\widehat{M}$. It corresponds to a connected and closed surface $N \subset M$ that is disjoint from S. Using that N separates M and the construction of $\widehat{M}$ from M, we conclude that $\widehat{N}$ separates. $\square$

A variation of this proof of Theorem 1.1 gives the following result.

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[3] The advantage of the *absolutely* area-minimizing property in the above proof is that it lifts to covers. We only used this at the very beginning, when we applied Lemma 3.1.

THEOREM 3.3. *Let (M, g) be a connected, orientable, complete Riemannian 3-manifold with nonnegative scalar curvature. Assume that (M, g) contains a properly embedded, incompressible[4] surface $S \subset M$ that is homeomorphic to the cylinder and homologically area-minimizing. Then (M, g) is flat. In fact, a cover of (M, g) is isometric to standard $\mathbb{S}^1 \times \mathbb{R}^2$ upon scaling.*

PROOF. We follow the proof of Theorem 1.1 above, except for the following changes:

(a) We do not pass to a covering.
(b) We work with surfaces $\Sigma_h(r, t)$ with $\partial \Sigma_h(r, t) = \partial \Sigma_h$ that have least area with respect to $\widehat{g}(r, t, h)$ and which together with Σ_h bound an open set in B_h.
(c) To argue that $\widehat{\Sigma}(r, t)$ is neither a plane nor a sphere, we use that Σ is incompressible.[5]   □

Remark 3.4. We do not know if Theorem 3.3 holds if we drop the assumption that S is incompressible.

Appendix A   Notions of Area-Minimizing Surfaces

Let (M, g) be an orientable Riemannian manifold, possibly with boundary.

Let $\Sigma \subset M$ be an oriented, properly embedded hypersurface.

Recall that $\Sigma \subset M$ is *absolutely area-minimizing* in (M, g) if for every $U \subset M$ open with compact closure, we have that

$$\text{area}(U \cap \Sigma) \leq \text{area}(U \cap \widetilde{\Sigma})$$

whenever $\widetilde{\Sigma} \subset M$ is an oriented, properly embedded hypersurface with $\partial \widetilde{\Sigma} = \partial \Sigma$ (matching orientations) and

$$\widetilde{\Sigma} \setminus U = \Sigma \setminus U.$$

Recall that $\Sigma \subset M$ is *homologically area-minimizing* in (M, g) if for every $U \subset M$ open with compact closure, we have that

$$\text{area}(U \cap \Sigma) \leq \text{area}(U \cap \widetilde{\Sigma})$$

whenever $\widetilde{\Sigma} \subset M$ is an oriented, properly embedded hypersurface such that

$$\widetilde{\Sigma} = \Sigma + \partial \Omega_1 + \cdots + \partial \Omega_N$$

in the sense of Stokes' theorem for $\Omega_1, \ldots, \Omega_N \subset M$ compact top-dimensional submanifolds with $\Omega_i \subset U$.

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[4] More precisely, we require that every loop $\gamma \subset S$ that bounds an embedded disk $\Delta \subset M$ is contractible in S.

[5] It is not clear that the comparison surfaces we use in the cut-and-paste argument in the proof of Theorem 1.1 above are homologous to the original surface. A priori, they may differ by an *unbounded* open set.

Remark A.1. There is a standard a priori area bound for homologically area-minimizing boundaries; cf. [24, §37.2]. The properties of such boundaries are preserved under convergence. There is no such a priori bound for general area-minimizing surfaces. Indeed, every stack of finitely many parallel planes $\mathbb{R}^n \times \{z_i\}$ with standard orientation and where $z_1, \ldots, z_m \in \mathbb{R}$ is absolutely area-minimizing in $\mathbb{R}^{n+1}$.

Finally, we discuss a nonstandard notion that plays a pivotal role in the proof of Theorem 1.1.

We say that $\Sigma \subset M$ is *homologically* area-minimizing* in (M, g) if for every $U \subset M$ open with compact closure, we have that

$$\text{area}(U \cap \Sigma) \leq \text{area}(U \cap \widetilde{\Sigma})$$

whenever $\widetilde{\Sigma} \subset M$ is an oriented, properly embedded hypersurface such that

$$\widetilde{\Sigma} = \Sigma + \partial\Omega_1 + \cdots + \partial\Omega_N$$

in the sense of Stokes' theorem, where $\Omega_1, \ldots, \Omega_N \subset M$ are properly embedded top-dimensional submanifolds with $\partial\Omega_i \subset U$. *The point is that we do* not *require that $\Omega_i \subset U$ or even that Ω_i be bounded here.*

Example A.2. Consider the embedded curves

$$\gamma_1 = \{(e^{i\theta}, 0) \colon 0 \leq \theta \leq \pi/2\}, \quad \gamma_2 = \{(e^{i\theta}, 0) \colon \pi/2 \leq \theta \leq 2\pi\},$$

in the standard cylinder $\mathbb{S}^1 \times \mathbb{R}$. γ_1 is absolutely length-minimizing; γ_2 is not. Both γ_1 and γ_2 are homologically length-minimizing. γ_1 is homologically* length-minimizing; γ_2 is not.

We conclude by describing how this nonstandard area-minimizing property is tailored to a delicate aspect in the proof of Theorem 1.1. As we have just discussed, we cannot expect local area bounds for sequences of absolutely area-minimizing surfaces. For this reason, we want to fix the homology class of the surfaces we work
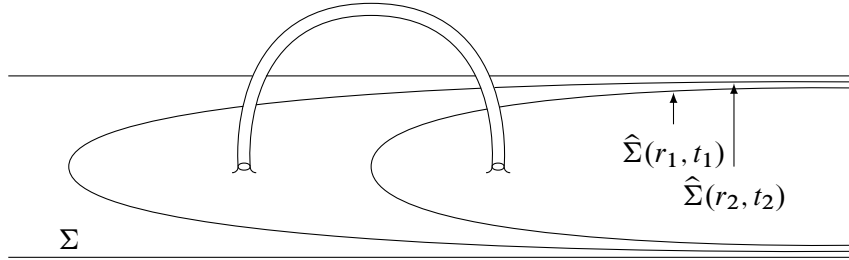

FIGURE A.1. This figure shows a hypothetical scenario if we had not passed to the cover (using Lemma 3.1). We want to compare the areas of $\widehat{\Sigma}(r_1, t_1)$ and $\widehat{\Sigma}(r_2, t_2)$ in a bounded set, using their respective minimizing properties. However, due to the presence of a *neck*, the surfaces are neither homologically nor homologically* related.
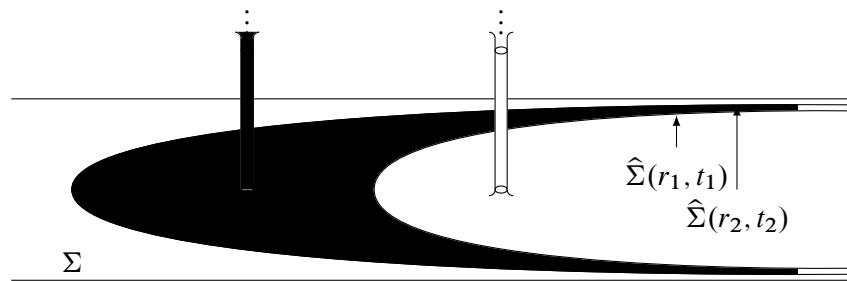
FIGURE A.2. A depiction of the situation in Figure A.1 after passing
to the appropriate cover (using Lemma 3.1). Now, compact pieces of
$\widehat{\Sigma}(r_1, t_1)$ and $\widehat{\Sigma}(r_2, t_2)$ are homologically* but not homologically com-
parable. This is because they differ by an open set (shaded grey) which
is necessarily unbounded.

with, or at least keep it in check. If we work with homological area-minimizers, it
is not clear how to rule out minimizing planes or spheres in the proof of Theorem
1.1 by cut-and-paste arguments. Figure A.1 shows a scenario where the planes we
worry about are *not* homologically related. To deal with this scenario, we pass
to the cover using Lemma 3.1 at the beginning of the proof. The situation after
passing to the cover is shown in Figure A.2. Now, the surfaces whose areas we
want to compare bound a *noncompact* region. They are homologically* related,
but not homologically related.

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OTIS CHODOSH
Princeton University
Department of Mathematics
Fine Hall
Washington Rd.
Princeton, NJ 08544
USA
    and
Institute for Advanced Study
School of Mathematics
Princeton, NJ 08540
USA
E-mail: `ochodosh@`
      `math.princeton.edu`

MICHAEL EICHMAIR
University of Vienna
Faculty of Mathematics
Oskar-Morgenstern-Platz 1
1090 Vienna
AUSTRIA
E-mail: `michael.eichmair@univie.ac.at`

VLAD MORARU
Department of Information Engineering
Computer Science and Mathematics
University of L'Aquila
Via Vetoio - Coppito
67100 l'Aquila
ITALY
E-mail: `vlad.moraru@univaq.it`