R. Han and S. Zhang (2020) “Large Deviation Estimates and Hölder Regularity of the Lyapunov Exponents for Quasi-periodic Schrödinger Cocycles,”
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–48
doi:10.1093/imrn/rnz319

Large Deviation Estimates and Hölder Regularity of the Lyapunov Exponents for Quasi-periodic Schrödinger Cocycles

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We consider one-dimensional quasi-periodic Schrödinger operators with analytic potentials. In the positive Lyapunov exponent regime, we prove large deviation estimates, which lead to refined Hölder continuity of the Lyapunov exponents and the integrated density of states, in both small Lyapunov exponent and large coupling regimes. Our results cover all the Diophantine frequencies and some Liouville frequencies.

1 Introduction and the Main Results

In this paper, we study the following one-dimensional discrete quasi-periodic operators on $\ell^2(\mathbb{Z})$:

$$(H(x)\varphi)(n) = \varphi(n-1) + \varphi(n+1) + v(x+n\omega)\varphi(n),\ n \in \mathbb{Z}, \tag{1.1}$$

where $x \in \mathbb{T} := [0,1]$ is called phase, $\omega \in \mathbb{T} \setminus \mathbb{Q}$ is called frequency, and the real-valued analytic function $v : \mathbb{T} \to \mathbb{R}$ is called potential.

For an energy $E \in \mathbb{R}$, the Schrödinger equation

$$\varphi(n-1) + \varphi(n+1) + v(x+n\omega)\varphi(n) = E\varphi(n) \tag{1.2}$$

Received March 9, 2018; Revised October 16, 2018; Accepted October 19, 2018
Communicated by Prof. Svetlana Jitomirskaya

can be rewritten in the form of the skew-product:

$$\begin{pmatrix} \varphi(n+1) \\ \varphi(n) \end{pmatrix} = A(\omega, E; x + n\omega) \begin{pmatrix} \varphi(n) \\ \varphi(n-1) \end{pmatrix}, \tag{1.3}$$

where

$$A(\omega, E; x) := \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

The dynamical system $(\omega, A(\omega, E; \cdot)) : \mathbb{T} \times \mathbb{C}^2 \to \mathbb{T} \times \mathbb{C}^2$, defined by

$$(\omega, A(\omega, E; \cdot))(x, v) = (x + \omega, A(\omega, E; x)v), \tag{1.5}$$

is called Schrödinger cocycle.

Let A be defined as in (1.4) and let

$$M_n(\omega, E; x) := A(\omega, E; x + n\omega)A(\omega, E; x + (n-1)\omega) \cdots A(\omega, E; x + \omega) \tag{1.6}$$

be the n-step transfer matrix, coming from n iterates of the Schrödinger cocycle (ω, A). Then, in view of (1.3), one clearly has

$$\begin{pmatrix} \varphi(n+1) \\ \varphi(n) \end{pmatrix} = M_n(\omega, E; x) \begin{pmatrix} \varphi(1) \\ \varphi(0) \end{pmatrix}. \tag{1.7}$$

Let

$$u_n(\omega, E; x) := \frac{1}{n} \log \|M_n(\omega, E; x)\|, \text{ and } L_n(\omega, E) := \int_{\mathbb{T}} u_n(\omega, E; x) \mathrm{d}x. \tag{1.8}$$

For any irrational $\omega \in [0, 1]$, the translation $x \mapsto x + \omega$ is ergodic. The Furstenberg–Kesten theorem implies that the following limit exists for a.e. x:

$$\lim_{n \to \infty} u_n(\omega, E; x) = \lim_{n \to \infty} L_n(\omega, E) =: L(\omega, E). \tag{1.9}$$

The limit $L(\omega, E)$ is called the *Lyapunov exponent*. Let us point out that in the definition of quasi-perodic cocycle, one could in general replace the one-dimensional rotation number $\omega \in \mathbb{T}$ by a higher dimensional vector $\omega \in \mathbb{T}^d$, and could also replace A by any $m \times m$ matrix-valued function, where $m \in \mathbb{N}$. The definition (1.9) then yields the maximal Lyapunov exponent $L(\omega, A)$.

Note that for any fixed $\kappa > 0, E, \omega$, the a.e. convergence in (1.9) implies

$$\operatorname{mes}\left\{x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \kappa\right\} \to 0 \text{ as } n \to \infty. \tag{1.10}$$

Thus, the question lies in the convergence rate w.r.t. n and the dependence on ω, E, and v. Such estimate is in general known as the *Large Deviation Theorem/Principle (LDT/LDP)* in probability theory. In this paper, we shall focus on the LDT for the monodromy matricies as introduced in (1.8). For the general LDT theory in probability theory, we refer readers to [15, 30].

Another important quantity in the spectral theory of Schrödinger operators is the *integrated density of states* (I.D.S.), denoted by N. It is also a function of the energy E. The I.D.S. gives the asymptotic distribution of eigenvalues of H restricted to large boxes. It is linked to $L(E) = L(\omega, E)$ via the Thouless formula, see for example, [12],

$$L(E) = \int \log|E - E'| \mathrm{d}N(E').$$

The I.D.S. is in general continuous in E, but this does not directly imply the continuity of the Lyapunov exponent. However, by virtue of the Hilbert transform, Hölder regularities of $N(E)$ and $L(E)$ pass from one to the other. For a proof of this fact, see for example, [19]. Therefore, we shall focus on the Lyapunov exponent in the rest of this paper.

Large deviation type estimates were introduced to study quasi-periodic Schrödinger operators in the late 1990s in a series of papers by Bourgain, Goldstein, and Schlag, [7, 19]. Their method has been well developed ever since and has shown to be sufficiently robust in the super-critical regime to deal with the following questions (not only restricted to the one-dimensional quasi-periodic Schrödinger case):

1. Regularity of the $L(E)$ and $N(E)$ in energy E (e.g., [4, 9, 19, 21, 27]),
2. Localization of the eigenfunctions (e.g., [7, 8, 23, 27]),
3. Eigenvalue separation and topological structure of the spectrum (e.g., [14, 22, 24]).

In this paper, we will focus on Problem 1. For more details about Problems 2 and 3, we refer readers to [5, 15, 20, 26] and references therein.

Proving regularity of $L(E)$ and $N(E)$ (in E) is considered difficult for any type of sequence of potentials, see [13]. Some weak regularity for general ergodic families was first proved in [12]. For quasi-periodic Schrödinger operators, the 1st breakthrough was made by Goldstein and Schlag in [19]. They developed a robust scheme, by combining

LDT with Avalanche Principle (AP), see Theorem C.1, to study the regularity problem. They proved Hölder regularity of $L(E)$ and $N(E)$ for typical frequencies in $\mathbb{T}$, assuming analyticity of the potential and positive Lyapunov exponents. Some weaker Hölder regularity was also obtained in the same paper for $\mathbb{T}^d$ with $d > 1$. Bourgain and Jitomirskaya proved in [9] that $L(\omega, E)$ is jointly continuous in (ω, E) at any irrational $\omega \in \mathbb{T}$ for analytic potentials; this result was obtained by Bourgain for $\mathbb{T}^d$ with $d > 1$ in [6]. More delicate estimates on sharp Hölder regularity for $\mathbb{T}$ were obtained by Goldstein and Schlag in [21]. In a recent monograph by Duarte–Klein [15], this scheme was extended systematically in depth and breadth, making it applicable to general cocycles, provided appropriate LDT estimates are available in the given setting.

For a general quasi-periodic analytic cocyle (ω, A), where A is an analytic $m \times m$ matrix-valued function, regularity of $L(\omega, A)$ is formulated in terms of the analytic norm of A. Joint continuity of $L(\omega, A)$ in (ω, A), without a modulus of continuity, was obtained in [3, 25] at any irrational $\omega \in \mathbb{T}$. The approaches of [3, 25] do not rely on LDT. Hölder regularity of $L(\omega, A)$ was obtained by LDT in [15, 16] for Diophantine $\omega \in \mathbb{T}^d$, under the gapped Lyapunov exponent assumption (equivalent to positive (maximal) Lyapunov exponent when $m = 2$).

In the subcritical regime with analytic potential, regularity results were proved often by reducibility method, cf. [1, 2]. In the low-regularity potential regime, fewer results were obtained with more restrictions on the potential and the frequency, see for example, [1, 11, 27, 28, 29, 32].

In this paper we follow the scheme developed by Goldstein and Schlag [19], namely by combining LDT and AP to obtain the Hölder continuity of $L(E)$ and $N(E)$:

$$|L(E) - L(E')| + |N(E) - N(E')| \leq |E - E'|^{\tau}, \quad |E - E'| \ll 1. \tag{1.11}$$

One of their key estimates for the one-dimensional case is

$$\operatorname{mes}\left\{x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \kappa L(\omega, E)\right\} \leq e^{-c(\omega, v, \kappa) L^2(\omega, E) n}, \tag{1.12}$$

under the positive Lyapunov exponent condition $L(\omega, E) > \gamma > 0$, for ω satisfying the strong Diophantine condition (S.D.C.), see (1.14). However, due to the $L^2(\omega, E)$ term in the exponential estimate on the right-hand side of (1.12), the Hölder exponent τ in (1.11) will tend to 0 as the lower bound γ approaches 0.

In [5], the LDT estimate (1.12) was improved to be

$$\text{mes}\{x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \kappa L(\omega, E)\} \le e^{-c(\omega, v, \kappa) L(\omega, E) n}, \tag{1.13}$$

in the small Lyapunov exponent regime, under the same assumption on ω. (Note that we use the same symbol $c(\omega, v, \kappa)$ in both (1.12) and (1.13), but they are not the same constants.) The improvement implies that the local Hölder exponent is independent of the lower bound γ.

As we mentioned above, both (1.12) and (1.13) were established for ω satisfying a S.D.C., which we define later. Going beyond S.D.C. is considered difficult for establishing LDT and Hölder continuity of the Lyapunov exponent in general. Our 1st result of this paper extends the LDT estimates to more frequencies in the best possible regime, see (1.17). Indeed, Hölder continuity fails for generic ω, see [2]. (See the paragraph below Theorem 1.2 of [3], for $v = \lambda \cos$ with $\lambda \neq 0$, Lyapunov exponent is discontinuous at rational ω's; thus, it is not Hölder for ω's that are well approximated by rationals.) Thus, the exponential decay (1.12) or (1.13) cannot hold for all frequencies.

In both (1.12) and (1.13), the dependence of $c(\omega, v, \kappa)$ on v are not written down explicitly. In our paper, we incorporate a refined Riesz representation of subharmonic functions of [21] into the proof of the LDT estimates. This leads to an explicit dependence of c on v. It turns out that the constant depends on the potential v in a "$\sup - \sup$" form, see (2.5). If $v = \lambda f$, the "$\sup - \sup$" yields a magical cancellation of λ. This leads to the 2nd result of our paper, see Corollary 1.3. Combining with AP, we obtain, for the 1st time, a λ-*independent* Hölder exponent in the large coupling regime for general non-trivial analytic potentials, see Theorem 1.10. Such kind of result was previously only known for trigonometric polynomials.

In order to formulate our results, we introduce the following notations: for any $x \in \mathbb{R}$, let $\|x\|_{\mathbb{T}} := \inf_{n \in \mathbb{Z}} |x - n|$. For any $\omega \in [0, 1] \setminus \mathbb{Q}$, let $[a_1, a_2, a_3, \ldots]$ be its continued fraction expansion. Let $\{p_s/q_s\}_{s=1}^{\infty}$ be its continued fraction approximants, defined by $p_s/q_s = [a_1, a_2, \ldots, a_s]$. It is well known that $\|q_s \omega\|_{\mathbb{T}} \le q_{s+1}^{-1}$. We say that ω satisfies an *S.D.C.* (or ω is strongly Diophantine), if for some constants $a > 1$, $c > 0$, the following holds for any $n \ge 1$,

$$\|n\omega\|_{\mathbb{T}} \ge \frac{c}{n(1 + \log n)^a}. \tag{1.14}$$

(We say that ω satisfies a Diophantine condition (D.C.) if $\|n\omega\|_{\mathbb{T}} \ge \frac{c}{n^{-a}}$ for all $n > 1$ and some $a > 1, c > 0$. Note that for any $a > 1$, a.e. ω satisfies a D.C. with some $c = c(\omega) > 0$.) Note that for any $a > 1$, a.e. ω satisfies S.D.C. for some $c = c(\omega) > 0$. It is also clear from the definition of S.D.C. that for strong Diophantine ω,

$$q_{s+1} \le c^{-1} q_s (\log q_s)^a. \tag{1.15}$$

Next we introduce an exponential growth exponent β defined as follows:

$$\beta(\omega) := \limsup_{s \to +\infty} \frac{\log q_{s+1}}{q_s} \in [0, \infty]. \tag{1.16}$$

It is then clear from (1.15) that S.D.C.$\subsetneq \{\omega : \beta(\omega) = 0\}$. Those ω with $\beta(\omega) > 0$ are usually called *Liouville numbers*.

Since our potential $v(x)$ is a real analytic function, it has a bounded extension to a strip $|\mathrm{Im} z| < \rho$ with width denoted by $\rho > 0$. Let $\mathcal{N}_v = [-2 - \|v\|_\infty, 2 + \|v\|_\infty]$ be the numerical range of the Schrödinger operator H. It is well known that $\sigma(H) \subset \mathcal{N}_v$ and $L(E)$ is a C^∞ function outside of the spectrum. Hence, we will only consider $E \in \mathcal{N}_v$ throughout the paper.

Theorem 1.1. Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$. There exist constants $c(v, \rho), \tilde{c}(v, \rho) \in (0, 1)$ such that, if

$$0 \le \beta(\omega) < c(v, \rho) \inf_{E \in [a,b]} L(\omega, E), \tag{1.17}$$

then there is $N = N(\omega, \inf_{E \in [a,b]} L(\omega, E), v, \rho) \in \mathbb{N}$ such that for any $n \ge N$ the following large deviation estimates hold uniformly in $E \in [a, b]$,

(a) If $0 < L(\omega, E) < 1$, then

$$\mathrm{mes} \left\{ x \in \mathbb{T} : | u_n(\omega, E; x) - L_n(\omega, E) | > \frac{1}{20} L(\omega, E) \right\} \le e^{-\tilde{c}(v,\rho) L(\omega,E)\, n}. \tag{1.18}$$

(b) If $L(\omega, E) \ge 1$, then

$$\mathrm{mes} \left\{ x \in \mathbb{T} : | u_n(\omega, E; x) - L_n(\omega, E) | > \frac{1}{20} L(\omega, E) \right\} \le e^{-\tilde{c}(v,\rho) L^2(\omega,E)\, n}. \tag{1.19}$$

Remark 1.2. The parameter $1/20$ in Theorem 1.1 can be replaced by any $0 < \kappa < 1$. The new constants $c_\kappa(v, \rho), \tilde{c}_\kappa(v, \rho)$ only differ from $c(v, \rho), \tilde{c}(v, \rho)$ by a constant multiple of κ^2. However, in order to apply AP to obtain Hölder continuity, κ can be taken to be at most $1/9$ due to technical reasons (see (C10)). We do not intend to improve the Hölder exponents in the paper by getting the best possible κ; thus, we take $\kappa = 1/20$ for simplicity. See more discussions about the sharp Hölder exponents after Theorem 1.5.

Corollary 1.3. Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $v(x)$ in (1.1) is given by $v(x) = \lambda f(x)$, where λ is a positive constant. There exist constants $0 < b = b(f, \rho) < 1, B = B(f, \rho) > 1$, and $\tilde{\lambda} = \tilde{\lambda}(f, \rho) > 0$ with the following properties: for any irrational ω with $0 \le \beta(\omega) < \infty$, suppose

$$\lambda > \max(\tilde{\lambda}, e^{B\beta(\omega)}),$$

then there is $N(\omega, \lambda, f, \rho) \in \mathbb{N}$ such that for any $n \ge N(\omega, \lambda, f, \rho)$, the following holds

$$\text{mes}\left\{x \in \mathbb{T} : | u_n(\omega, E; x) - L_n(\omega, E) | > \frac{1}{19} \log \lambda \right\} \le e^{-n\, b \log \lambda}. \tag{1.20}$$

Remark 1.4. The above exponential decay of the measure estimate w.r.t. $\log \lambda$ for large coupling λ is known for the 1st time even for $\beta(\omega) = 0$ or S.D.C. ω to the authors' knowledge.

As mentioned previously in (1.11), a direct consequence of the above large deviation estimates is the Hölder regularity of the Lyapunov exponents. With the refined parameters in the LDT estimates (1.18)–(1.20), we have the following Hölder continuity of the Lyapunov exponents.

Theorem 1.5. Let $c = c(v, \rho), \tilde{c} = \tilde{c}(v, \rho)$ be the constants in Theorem 1.1. There exists a constant $\tau > 0$ depending explicitly (and only) on $\tilde{c}(v, \rho)$ that satisfies the following property: if $(\omega_0, E_0) \in (\mathbb{R} \setminus \mathbb{Q}) \times \mathcal{N}_v$ is a point with $L(\omega_0, E_0) = \gamma > 0$, and $U \times I$ is a neighborhood of (ω_0, E_0) such that $L(\omega, E) \in [\frac{18}{19}\gamma, \frac{20}{19}\gamma]$, then for any $\omega \in U$ with

$$0 \le \beta(\omega) < \frac{1}{2} c\, \gamma,$$

there is $\eta = \eta(\omega, I, \gamma, v)$ such that the following holds for any $E, E' \in I$ and $|E - E'| < \eta$,

$$|L(\omega, E) - L(\omega, E')| \le |E - E'|^{\tau}. \tag{1.21}$$

Remark 1.6. By [9], $L(\omega, E)$ is jointly continuous in (ω, E) at (ω_0, E_0). Hence, the neighborhood $U \times I$ always exists.

Remark 1.7. Theorem 1.5 shows that the exponent τ is independent of the lower bound of the Lyapunov exponent γ. This generalizes the result in [5] for general analytic potentials from ω satisfying S.D.C. to $0 \le \beta(\omega) \lesssim \gamma$.

Remark 1.8. For trigonometric polynomial potentials, there are results on sharp Hölder exponents that only depend on the degree of the polynomial: $\frac{1}{2}$-Hölder if $v = \lambda\cos$, $\lambda \neq 0, 1$, [2, 4]; and $(\frac{1}{2k}-\epsilon)$-Hölder if v is a small C^∞ perturbation of a trigonometric polynomial of degree k [21]. Our current approach does not lead to such kind of sharp exponent for general analytic potentials, even for S.D.C. ω.

Remark 1.9. If v is of the form λf, with a general analytic f, in the small coupling regime $\lambda < \lambda_0(f)$, $\frac{1}{2}$-Hölder exponents were obtained in [2] using a reducibility method. However, there is no such kind of result for the large coupling regime. (For general analytic potential $v = \lambda f$, if one applies the LDT (1.12) in [19] and check all the constants explicitly, the Hölder exponent behaves like $O((\log\lambda)^{-1})$ for large λ even for S.D.C. ω, see more explanation in [33].) Our Theorem 1.10 is the 1st one in this regime, by giving a λ-independent Hölder exponent for general analytic f.

If $v = \lambda f$, we have the following:

Theorem 1.10. Under the same condition of Corollary 1.3, let $\tilde{\lambda}(f,\rho), b(f,\rho), B(f,\rho)$ be the constants given there. There exists a constant $\tilde{\tau} > 0$ depending explicitly (and only) on b (hence independent of λ) such that for any irrational ω with $0 \leq \beta(\omega) < \infty$, if $\lambda > \max(\tilde{\lambda}, e^{B\beta(\omega)})$, then there exists $\tilde{\eta} = \tilde{\eta}(\omega,\lambda,f,\rho) > 0$, such that for any $E, E' \in \mathcal{N}_{\lambda f}$ and $|E - E'| \leq \tilde{\eta}$, we have

$$|L(E) - L(E')| \leq |E - E'|^{\tilde{\tau}}. \tag{1.22}$$

The rest of the paper is organized as follows: in Section 2, we state all the important technical lemmas. In Section 3, we prove the three large deviation estimates using the lemmas in Section 2. Our Hölder continuity follows directly from LDT and a standard argument combined with the AP. For the sake of completeness, we sketch the proof in Section 5. Many details of this part are included in the Appendix for the reader's convenience.

2 Useful Lemmas

Let $\mathcal{N}_v = [-2 - \|v\|_\infty, 2 + \|v\|_\infty]$; as we mentioned before, we will only consider $E \in \mathcal{N}_v$ throughout the paper. Recall that $u_n(\omega, E; x)$ is defined as in (1.8).

This section contains lemmas that will be used in the proofs of Theorems 1.1 and 1.3. The proofs of these lemmas will be included in Section 4.

Let

$$\Lambda_v := \log(3 + 2\|v\|_{L^\infty(\mathbb{T})}). \tag{2.1}$$

Simple computations yield that

$$\sup_{E\in\mathcal{N}_v} \left\|u_n(\omega, E; \cdot)\right\|_{L^\infty(\mathbb{T})} \le \Lambda_v \tag{2.2}$$

holds uniformly in $\omega \in \mathbb{T}$ and $1 \le n \in \mathbb{N}$.

Since in our model, v is assumed to have bounded analytic extension to $\mathbb{T}_\rho := \{z : |\mathrm{Im} z| < \rho\}$, u_n has subharmonic extension on $\mathbb{T}_\rho$ with a uniform upper bound

$$\sup_{E\in\mathcal{N}_v} \sup_{n\in\mathbb{N}} \left\|u_n(\omega, E; \cdot)\right\|_{L^\infty(\mathbb{T}_\rho)} \le \log(3 + 2\|v\|_{L^\infty(\mathbb{T}_\rho)}) < \infty.$$

2.1 Estimates of the Fourier coefficients $\hat{u}_n(\omega, E; k)$

The function $u_n(\omega, E; x)$ is 1-periodic on $\mathbb{R}$ and we denote its Fourier coefficients by

$$\hat{u}_n(\omega, E; k) = \int_{\mathbb{T}} u_n(\omega, E; x) e^{-2\pi i k x} \mathrm{d}x. \tag{2.3}$$

The following estimate of the Fourier coefficient is well known and crucial to establishing our LDT, see for example, Bourgain's monograph [5, Corollary 4.7]. For a version of this estimate written precisely in the "sup − sup" form below, see [17, Lemma 2.8]. To obtain this "sup − sup" estimate, one needs to invoke a refined Riesz representation theorem [21, Lemma 2.2]. See details in Section 4.1.

Lemma 2.1. There is a constant $\alpha(\rho) > 0$ depending on ρ only, such that for any $k \ne 0$,

$$|\hat{u}_n(\omega, E; k)| \le \frac{\alpha(\rho)}{|k|} \left(\sup_{|\mathrm{Im} z|<\rho} u_n(\omega, E; z) - \sup_{|\mathrm{Im} z|<\rho/2} u_n(\omega, E; z) \right). \tag{2.4}$$

Corollary 2.2. Let

$$C(v, \rho) := \alpha(\rho) \sup_{E\in\mathcal{N}_v} \left(\sup_{|\mathrm{Im} z|<\rho} u_n(\omega, E; z) - \sup_{|\mathrm{Im} z|<\rho/2} u_n(\omega, E; z) \right) < \infty. \tag{2.5}$$

We then have that for any $k \ne 0$ and $E \in \mathcal{N}_v$,

$$|\hat{u}_n(\omega, E; k)| \le \frac{C(v, \rho)}{|k|}. \tag{2.6}$$

When v is given as λf, we can bound the above constant $C(\lambda f, \rho)$ by a constant independent of λ. This turns out to be crucial to our proof of Corollary 1.3.

Lemma 2.3. Let $C(v, \rho)$ be the constant defined in (2.5). Suppose that $v = \lambda f$. Then there is $C_0(f, \rho) > 0$, independent of λ, such that for any $\lambda > 0$,

$$C(\lambda f, \rho) \leq C_0(f, \rho). \tag{2.7}$$

Besides the Fourier decay estimate in Lemma 2.1, we also prove a new estimate as follows. This estimate improves that of Lemma 2.1 for small $|k|$ when n is large. It will play a crucial role in our proof of part (a) of Theorem 1.1.

Lemma 2.4. Let Λ_v be the constant defined in (2.1). We have the following bounds of the Fourier coefficients, for any $k \neq 0$,

$$|\hat{u}_n(\omega, E; k)| \leq \frac{\Lambda_v}{2n\|k\omega\|_{\mathbb{T}}}.$$

2.2 $\|u_n(\omega, E; \cdot)\|_{L^\infty(\mathbb{T})}$ under small Lyapunov exponent condition

We present an upper bound of $\|u_n(\omega, E; x)\|$, see Lemma 2.6 below. This can be viewed as a generalization of [5, Lemma 8.18], where a similar bound was proved for Diophantine ω. Compared to a trivial bound $\|u_n(\omega, E; x\| \leq \Lambda_v$, the new bound is much more effective when the Lyapunov exponent is small.

Compared to [5, Lemma 8.18], our improvement lies in the fact that we can relax the D.C. on ω. Indeed we give explicit dependence of the upper bound on the continued fraction approximants of ω, through the $\log q_{s+1}/q_s$ term. This improvement enables us to cover Liouville frequencies.

For $R \in \mathbb{N}$, let $u_n^{(R)}$ be the average of u_n along a trajectory with length $\sim R$, defined as

$$u_n^{(R)}(\omega, E; x) := \sum_{|j|<R} \frac{R - |j|}{R^2} u_n(\omega, E; x + j\omega). \tag{2.8}$$

Lemma 2.5. Let $C(v, \rho)$, C_3 be the constants in (2.5) and (4.19). Assuming that $0 < L(\omega, E) < 1$, we have the following upper bound of $u_n^{(R)}(\omega, E; x)$,

$$\|u_n^{(R)}(\omega, E; \cdot)\|_{L^\infty(\mathbb{T})} \leq L_n(\omega, E) + (2 + 8C(v, \rho) + 4\pi C_3 C(v, \rho))L(\omega, E) + 120C(v, \rho)\frac{\log q_{s+1}}{q_s}, \tag{2.9}$$

which holds for

$$n \geq 2\Lambda_v L(\omega,E)^{-2} \sup_{1\leq|k|\leq L(\omega,E)^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}}, \quad \text{and} \quad R \geq 144L(\omega,E)^{-5}.$$

Lemma 2.5 leads to the following

Lemma 2.6. Let $C(v,\rho)$, C_3 be the constants in (2.5) and (4.19). Assuming that $0 < L(\omega,E) < 1$, we have the following upper bound of $u_n(\omega,E;x)$,

$$\|u_n(\omega,E;\cdot)\|_{L^\infty(\mathbb{T})} \leq L_n(\omega,E) + C_1 L(\omega,E) + 120C(v,\rho)\frac{\log q_{s+1}}{q_s}, \tag{2.10}$$

which holds for $n \geq N_0(\omega, L(\omega,E), v, \rho)$, where C_1 explicitly depends on $C(v,\rho)$, Λ_v as

$$C_1 := 2 + \Lambda_v + 8C(v,\rho) + 4\pi C_3 C(v,\rho) \tag{2.11}$$

and

$$N_0(\omega, L(\omega,E), v, \rho) := L(\omega,E)^{-2} \max\left(2\Lambda_v \sup_{1\leq|k|\leq L(\omega,E)^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}},\ 49L(\omega,E)^{-4}\right). \tag{2.12}$$

2.3 Two estimates of $\|u_n(\omega,E;\cdot) - u_n^{(R)}(\omega,E;\cdot)\|_{L^\infty(\mathbb{T})}$

The following lemmas give upper bounds of $\|u_n - u_n^{(R)}\|_{L^\infty(\mathbb{T})}$ under different conditions.

Lemma 2.7. Let Λ_v be the constant defined in (2.1). For any n, R, ω, we have

$$\left\|u_n(\omega,E;\cdot) - u_n^{(R)}(\omega,E;\cdot)\right\|_{L^\infty(\mathbb{T})} \leq 2\Lambda_v \frac{R}{n}.$$

Recall the following uniform convergence in [9].

Lemma 2.8. [9, Corollary 3] Suppose v is analytic. Then

$$\limsup_{n\to\infty} u_n(\omega,E;x) \leq L(\omega,E) \tag{2.13}$$

uniformly in x and E in a compact set.

A direct consequence is the following:

Lemma 2.9. Suppose $L(\omega, E) > 0$ for all $E \in [a, b]$. There exists $\widetilde{N}_0(\omega, [a, b], v)$ such that for any $n > \widetilde{N}_0(\omega, [a, b], v)$, any $x \in \mathbb{T}$ and $E \in [a, b]$, we have

$$u_n(\omega, E; x) \leq \left(1 + \frac{1}{20}\right) L(\omega, E) \tag{2.14}$$

and

$$L_n(\omega, E) \leq \left(1 + \frac{1}{20}\right) L(\omega, E). \tag{2.15}$$

A more delicate upper bound of the difference $u_n - u_n^{(R)}$, when $L(\omega, E)$ is small, is given as follows. This upper bound will be the key to Theorem 1.1, part (a). Let N_0 be as in (2.12) and $\widetilde{N}_0$ be as in Lemma 2.9. Define

$$N_1(\omega, [a, b], L(\omega, E), v, \rho) := \max(N_0(\omega, L(\omega, E), v, \rho),\ \widetilde{N}_0(\omega, [a, b], v) + 1). \tag{2.16}$$

Using Lemmas 2.6 and 2.9, we obtain the following:

Lemma 2.10. Let C_1, N_1 be the constants in (2.11) and (2.16), and $C(v, \rho)$, Λ_v be the constants in (2.5) and (2.1), respectively. Suppose $0 < L(\omega, E) < 1$. For $R = \lfloor (400\,(C_1 + 2))^{-1}\, n \rfloor + 1$, we have that

$$\left\| u_n(\omega, E; \cdot) - u_n^{(R)}(\omega, E; \cdot) \right\|_{L^\infty(\mathbb{T})} \leq \frac{1}{100} L(\omega, E) + \frac{1}{5} C(v, \rho) \frac{\log q_{s+1}}{q_s}$$

holds for $n \geq N_2(\omega, [a, b], L(\omega, E), v, \rho)$, where

$$N_2(\omega, [a, b], L(\omega, E), v, \rho) := \max\left(150 \Lambda_v N_1 L(\omega, E)^{-1},\ 400(C_1 + 2) N_1 + 1\right). \tag{2.17}$$

Remark 2.11. We point out that $N_1(\omega, [a, b], L(\omega, E), v, \rho)$ is a decreasing function in the 3rd parameter $L(\omega, E)$, and so is $N_2(\omega, [a, b], L(\omega, E), v, \rho)$. This is clear from the definitions (2.12), (2.16), and (2.17).

3 Large Deviation Estimates.

For simplicity, from this point on, when there is no ambiguity, we will sometimes write $u_n(x) = u_n(\omega, E; x)$, $L_n = L_n(\omega, E)$, and $L = L(\omega, E)$.

3.1 Preparation

Let $\hat{u}_n(k)$ and $u_n^{(R)}(x)$ be defined as in (2.3) and (2.8). Let

$$F_R(k) := \sum_{|j|<R} \frac{R - |j|}{R^2} e^{2\pi i k j \omega}. \tag{3.1}$$

Let us recall the following estimates of $F_R(k)$ in [5, 9, 33], whose proofs are included in the Appendix E.

$$0 \le F_R(k) \le \min\left(1, \frac{2}{1 + R^2 \|k\omega\|_{\mathbb{T}}^2}\right), \tag{3.2}$$

$$\sum_{1 \le |k| < q/4} \frac{1}{1 + R^2 \|k\omega\|_{\mathbb{T}}^2} \le 2\pi \frac{q}{R}, \tag{3.3}$$

$$\sum_{|k| \in [\ell q/4,\ (\ell+1)q/4)} \frac{1}{1 + R^2 \|k\omega\|_{\mathbb{T}}^2} \le 2 + 4\pi \frac{q}{R}, \quad \forall \ell \in \mathbb{N}, \tag{3.4}$$

in which p/q is any continued fraction approximant of ω.

Direct computation shows that

$$u_n^{(R)}(x) = L_n + \sum_{k \in \mathbb{Z}, k \ne 0} \hat{u}_n(k) F_R(k) e^{2\pi i k x}. \tag{3.5}$$

Let p_s/q_s, p_{s+1}/q_{s+1} be any two consecutive continued fraction approximants of ω. For $0<\delta\le 1$, let us consider

$$\begin{aligned} u_n(x)-L_n &= u_n(x)-u_n^{(R)}(x)+u_n^{(R)}(x)-L_n \\ &= u_n(x)-u_n^{(R)}(x) \quad (=:\mathcal{U}_1(x)) \\ &\quad + \sum_{1\le|k|<\delta^{-1}} \hat{u}_n(k)F_R(k)e^{2\pi ikx} \quad (=:\mathcal{U}_2(x)) \\ &\quad + \sum_{\delta^{-1}\le|k|<q_s/4} \hat{u}_n(k)F_R(k)e^{2\pi ikx} \quad (=:\mathcal{U}_3(x)) \\ &\quad + \sum_{q_s/4\le|k|<q_{s+1}/4} \hat{u}_n(k)F_R(k)e^{2\pi ikx} \quad (=:\mathcal{U}_4(x)) \\ &\quad + \sum_{q_{s+1}/4\le|k|<K} \hat{u}_n(k)F_R(k)e^{2\pi ikx} \quad (=:\mathcal{U}_5(x)) \\ &\quad + \sum_{|k|\ge K} \hat{u}_n(k)F_R(k)e^{2\pi ikx} \quad (=:\mathcal{U}_6(x)). \end{aligned} \tag{3.6}$$

By Lemma 2.4, we have some refined estimates of $\mathcal{U}_2(x)$ and $\mathcal{U}_3(x)$:

Proposition 3.1. Let $\Lambda_v, C(v,\rho)$ be given as in (2.1) and (2.5). For any $n\ge 1$ and $R\in[q_s,q_{s+1})$, we have

$$\|\mathcal{U}_2(\cdot)\|_{L^\infty(\mathbb{T})} \le \frac{\Lambda_v}{\delta n}\cdot \sup_{1\le k\le\delta^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}} \tag{3.7}$$

and

$$\|\mathcal{U}_3(\cdot)\|_{L^\infty(\mathbb{T})} \le 4\pi\delta C(v,\rho) \tag{3.8}$$

Proof. By Lemma 2.4 and (3.2), we have

$$\|\mathcal{U}_2(\cdot)\|_{L^\infty(\mathbb{T})} \le \sum_{1\le|k|<\delta^{-1}} |\hat{u}_n(k)| \le \frac{\Lambda_v}{2n}\sum_{1\le|k|<\delta^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}} \le \frac{\Lambda_v}{\delta n}\sup_{1\le k\le\delta^{-1}}\frac{1}{\|k\omega\|_{\mathbb{T}}}. \tag{3.9}$$

By Lemma 2.1, (3.2), (3.3), and $q_s\le R$, we obtain

$$\begin{aligned} \|\mathcal{U}_3(\cdot)\|_{L^\infty(\mathbb{T})} &\le 2\sum_{\delta^{-1}\le|k|<q_s/4} |\hat{u}_n(k)|\frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} \le 2\sum_{\delta^{-1}\le|k|<q_s/4} \frac{C(v,\rho)}{\delta^{-1}}\frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} \\ &\le 2C(v,\rho)\cdot\delta\cdot\sum_{1\le|k|<q_s/4}\frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} \le 4\pi C(v,\rho)\cdot\delta\cdot\frac{q_s}{R} \le 4\pi\delta C(v,\rho), \end{aligned} \tag{3.10}$$

as desired. ∎

We have some general estimates for $\mathcal{U}_4(x) + \mathcal{U}_5(x)$ and $\mathcal{U}_6(x)$.

Proposition 3.2. Let $C(v,\rho)$ be given as in (2.5). For any $n \geq 1$, and $q_s \leq R < q_{s+1} \leq K$, we have

$$\|\mathcal{U}_4(\cdot) + \mathcal{U}_5(\cdot)\|_{L^\infty(\mathbb{T})} \leq 120C(v,\rho)\left(\frac{\log q_{s+1}}{q_s} + \frac{\log K}{R}\right), \tag{3.11}$$

and

$$\|\mathcal{U}_6(\cdot)\|^2_{L^2(\mathbb{T})} \leq C^2(v,\rho)\frac{2}{K}. \tag{3.12}$$

This part has been proved in [33], but we sketch the proof below for the reader's convenience.

Proof. By Lemma 2.1, (3.2), (3.4), and the choice of $R \in [q_s, q_{s+1})$, we have

$$\begin{aligned}
\|\mathcal{U}_4(\cdot)\|_{L^\infty(\mathbb{T})} &\leq 2 \sum_{q_s/4 \leq |k| < q_{s+1}/4} |\hat{u}_n(k)| \frac{1}{1 + R^2\|k\omega\|^2_{\mathbb{T}}} \\
&\leq 2 \sum_{\ell=1}^{\lfloor q_{s+1}/q_s \rfloor + 1} \sum_{|k| \in [\ell q_s/4, (\ell+1)q_s/4)} |\hat{u}_n(k)| \frac{1}{1 + R^2\|k\omega\|^2_{\mathbb{T}}} \\
&\leq 8C(v,\rho) \sum_{\ell=1}^{\lfloor q_{s+1}/q_s \rfloor + 1} \sum_{|k| \in [\ell q_s/4, (\ell+1)q_s/4)} \frac{1}{\ell q_s} \cdot \frac{1}{1 + R^2\|k\omega\|^2_{\mathbb{T}}} \\
&\leq 8C(v,\rho) \sum_{\ell=1}^{\lfloor q_{s+1}/q_s \rfloor + 1} \frac{1}{\ell q_s}\left(2 + 4\pi\frac{q_s}{R}\right) \\
&\leq 16C(v,\rho)\,(1 + 2\pi)\,\frac{\log q_{s+1}}{q_s}.
\end{aligned} \tag{3.13}$$

In view of $\mathcal{U}_5$, we have by Lemma 2.1 and (3.4) that

$$\begin{aligned}
\|\mathcal{U}_5(\cdot)\|_{L^\infty(\mathbb{T})} &\leq 2 \sum_{q_{s+1}/4 \leq |k| \leq K} |\hat{u}_n(k)| \frac{1}{1 + R^2\|k\omega\|^2_{\mathbb{T}}} \\
&\leq 8C(v,\rho) \sum_{\ell=1}^{\lfloor 4K/q_{s+1} \rfloor + 1} \sum_{|k| \in [\ell q_{s+1}/4, (\ell+1)q_{s+1}/4)} \frac{1}{\ell q_{s+1}} \cdot \frac{1}{1 + R^2\|k\omega\|^2_{\mathbb{T}}} \\
&\leq 16C(v,\rho)\,(1 + 2\pi)\,\frac{\log K}{R}.
\end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), and using that $16(1 + 2\pi) < 120$, we prove (3.11).

For $\mathcal{U}_6$, we have that by Lemma 2.1,

$$\|\mathcal{U}_6(\cdot)\|_{L^2(\mathbb{T})}^2 \le \sum_{|k|>K} |\hat{u}_n(k)|^2 \le C^2(v,\rho) \sum_{|k|>K} \frac{1}{k^2} \le C^2(v,\rho)\frac{2}{K}, \tag{3.15}$$

as claimed. ■

3.2 Proof of Theorem 1.1

Let

$$\underline{L}(\omega,[a,b]) = \inf_{E\in[a,b]} L(\omega,E), \quad \text{and} \quad \underline{\tilde{L}}(\omega,[a,b]) = \min(\underline{L}(\omega,[a,b]),1). \tag{3.16}$$

For simplicity, we will sometimes omit the dependence on ω and $[a,b]$ and write $\underline{L}$ and $\underline{\tilde{L}}$ instead.

Recall our notations: N_2 as in (2.17), and $\Lambda_v, C(v,\rho), C_1$ as in (2.1), (2.5), and (2.11). We choose c and $\tilde{c}$ in the statement of the theorem as follows:

$$c(v,\rho) = \big(36000C(v,\rho)\big)^{-1}, \quad \tilde{c}(v,\rho) = \big(2\times 10^7(C_1+2)C(v,\rho)\big)^{-1}. \tag{3.17}$$

By our condition,

$$\beta(\omega) = \limsup_{k\to\infty} \frac{\log q_{k+1}}{q_k} \le c(v,\rho)\underline{L}(\omega).$$

Hence, there exists $s_0 = s_0(\omega,[a,b],v,\rho)$ such that for any $k \ge s_0$,

$$\frac{\log q_{k+1}}{q_k} \le 2c(v,\rho)\underline{L}(\omega,[a,b]). \tag{3.18}$$

Let $n \ge N$, with N defined as follows:

$$N(\omega,\underline{L},v,\rho) := \max \begin{cases} (i).\ 400(C_1+2)q_{s_0}, \\ (ii).\ N_2(\omega,[a,b],\underline{L},v,\rho), \\ (iii).\ 1.6\times 10^5\pi\Lambda_v C(v,\rho)\underline{\tilde{L}}^{-2} \sup_{1\le k\le 800\pi C(v,\rho)\underline{\tilde{L}}^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}}, \\ (iv).\ 2\times 10^7(C_1+2)C(v,\rho)\underline{\tilde{L}}^{-1}\log\Big(2\times 10^4 C^2(v,\rho)\underline{\tilde{L}}^{-2}+e\Big). \end{cases} \tag{3.19}$$

This gives four lower bounds of n.

Remark 3.3. By Remark 2.11, N_2 is decreasing in $\underline{L}$. It is also clear that both (iii) and (iv) are decreasing in $\underline{L}$. Hence, N is non-increasing in $\underline{L}$.

3.2.1 *Parameters for part (a)*

In this case, $L < 1$; hence,

$$\tilde{\underline{L}} = \underline{L}. \tag{3.20}$$

In our decomposition of $u_n(x) - L_n$ in (3.6), we choose the following parameters:

$$\begin{aligned} \delta &= \frac{\underline{L}}{800\pi C(v,\rho)}, \qquad R = \left[\frac{n}{400(C_1+2)}\right] + 1, \\ K &= \left[\exp\left(\frac{RL}{1.2\times 10^4 C(v,\rho)}\right)\right], \qquad s = \max\left\{s \in \mathbb{N} : \ q_s \le R\right\}. \end{aligned} \tag{3.21}$$

It is clear from the choice of s that $q_s \le R < q_{s+1}$. Let us also note that with δ defined above, the lower bound (iii) in (3.19) becomes

$$\frac{200\Lambda_v}{\delta \underline{L}} \sup_{1\le k\le \delta^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}}. \tag{3.22}$$

Indeed, by (i) of (3.19), we have

$$R > \left(400(C_1+2)\right)^{-1} n \ge q_{s_0}.$$

By our definition of s, see (3.21), we clearly have $s \ge s_0$. This, by (3.18), implies

$$\frac{\log q_{s+1}}{q_s} \le 2c(v,\rho)\underline{L}. \tag{3.23}$$

An upper bound of q_{s+1} could be derived from (3.23). Indeed,

$$q_{s+1} \le \exp\left(2c(v,\rho)\underline{L}q_s\right) \le \exp\left(2c(v,\rho)\underline{L}R\right) \le \exp\left(\frac{LR}{1.8\times 10^4 C(v,\rho)}\right). \tag{3.24}$$

By (iv) of (3.19),

$$n \ge 2\times 10^7 (C_1+2) C(v,\rho) \underline{L}^{-1};$$

hence, we have

$$\exp\left(\frac{LR}{1.8\times 10^4 C(v,\rho)}\right) \ge \exp\left(\frac{\underline{L}n}{7.2\times 10^6 (C_1+2)C(v,\rho)}\right) \ge \exp\left(\frac{2\times 10^7}{7.2\times 10^6}\right) > 16.$$

Using the fact that $x < x^{\frac{3}{2}} - 1$, for $x > 3$, we have

$$\exp\left(\frac{LR}{1.8\times 10^4 C(v,\rho)}\right) < \exp\left(\frac{LR}{1.2\times 10^4 C(v,\rho)}\right) - 1 \le K. \tag{3.25}$$

Combining (3.24) with (3.25), we arrive at

$$q_{s+1} \le K. \tag{3.26}$$

3.2.2 *Proof of part (a)*

By (ii) of (3.19) and Remark 2.11, we have

$$n \ge N \ge N_2(\omega, [a,b], \underline{L}(\omega), v, \rho) \ge N_2(\omega, [a,b], L(\omega, E), v, \rho).$$

Hence, by Lemma 2.10, and (3.23), we have,

$$\|\mathcal{U}_1(\cdot)\|_{L^\infty(\mathbb{T})} \le \frac{1}{100}L + \frac{1}{5}C(v,\rho)\frac{\log q_{s+1}}{q_s} \le \frac{1}{100}L + \frac{2}{5}C(v,\rho)c(v,\rho)L = \left(\frac{1}{100} + \frac{1}{9\times 10^4}\right)L. \tag{3.27}$$

By Proposition 3.1 and our choice of δ, we have

$$\|\mathcal{U}_2(\cdot) + \mathcal{U}_3(\cdot)\|_{L^\infty(\mathbb{T})} \le \frac{\Lambda_v}{\delta n}\sup_{1\le k\le \delta^{-1}}\frac{1}{\|k\omega\|_{\mathbb{T}}} + 4\pi\delta C(v,\rho) \le \frac{1}{100}L, \tag{3.28}$$

in which we used (iii) of (3.19), see also (3.22),

$$n \ge N \ge \frac{200\Lambda_v}{\delta \underline{L}}\sup_{1\le k\le \delta^{-1}}\frac{1}{\|k\omega\|_{\mathbb{T}}} \ge \frac{200\Lambda_v}{\delta L}\sup_{1\le k\le \delta^{-1}}\frac{1}{\|k\omega\|_{\mathbb{T}}}.$$

Note that (3.26) verifies the condition $q_{s+1} \le K$ of Proposition 3.2. Hence, Proposition 3.2 implies that,

$$\|\mathcal{U}_4(\cdot) + \mathcal{U}_5(\cdot)\|_{L^\infty(\mathbb{T})} \le 120C(v,\rho)\left(\frac{\log q_{s+1}}{q_s} + \frac{\log K}{R}\right) \le 120C(v,\rho)\frac{\log q_{s+1}}{q_s} + \frac{1}{100}L.$$

Taking (3.23) into account, we have

$$\|\mathcal{U}_4(\cdot) + \mathcal{U}_5(\cdot)\|_{L^\infty(\mathbb{T})} \le 240C(v,\rho)c(v,\rho)L + \frac{1}{100}L = \frac{1}{60}L. \tag{3.29}$$

Combining (3.27), (3.28), and (3.29) with our choice of $c(v, \rho)$, see (3.17), we have

$$\left\| \sum_{j=1}^{5} \mathcal{U}_j(\cdot) \right\|_{L^\infty(\mathbb{T})} \leq \frac{1}{25} L. \tag{3.30}$$

By (3.12) and (3.25),

$$\begin{aligned} \|\mathcal{U}_6(\cdot)\|^2_{L^2(\mathbb{T})} \leq C^2(v,\rho)\frac{2}{K} &\leq 2C^2(v,\rho)\exp\left(-\frac{RL}{1.8\times 10^4 C(v,\rho)}\right) \\ &< 2C^2(v,\rho)\exp\left(-\frac{nL}{10^7(C_1+2)C(v,\rho)}\right). \end{aligned} \tag{3.31}$$

Combining (3.6) and (3.30) with (3.31), and using Markov's inequality, we obtain

$$\begin{aligned} \text{mes}\left\{x\in\mathbb{T} : \left|u_n(x) - L_n\right| > \frac{1}{20}L\right\} &\leq \text{mes}\left\{x\in\mathbb{T} : \left|\mathcal{U}_6(x)\right| > \frac{1}{100}L\right\} \\ &\leq 2\times 10^4 C^2(v,\rho)L^{-2}\exp\left(-\frac{nL}{10^7(C_1+2)C(v,\rho)}\right) \\ &\leq \exp\left(-\frac{nL}{2\times 10^7(C_1+2)C(v,\rho)}\right) \\ &= \exp\left(-\tilde{c}(v,\rho)nL\right), \end{aligned}$$

in which we used (iv) of (3.19),

$$n \geq 2\times 10^7(C_1+2)C(v,\rho)L^{-1}\log(2\times 10^4 C^2(v,\rho)L^{-2}).$$

This proves part (a) of Theorem 1.1.

3.2.3 *Parameters for part (b)*

In our decomposition of $u_n(x) - L_n$ in (3.6), we choose parameters as follows:

$$\begin{aligned} &\delta = \frac{1}{800\pi C(v,\rho)}, \qquad R = \left[\frac{nL}{400\Lambda_v}\right] + 1, \\ &K = \left[\exp\left(\frac{RL}{1.2\times 10^4 C(v,\rho)}\right)\right], \qquad s = \max\{s\in\mathbb{N} : \ q_s \leq R\}. \end{aligned} \tag{3.32}$$

It is clear that $q_s \leq R < q_{s+1}$.

Use the fact that $C_1 > \Lambda_v$, see (2.11), and $\tilde{\underline{L}} \leq 1$, (3.19) implies

$$n \geq \begin{cases} (i'). \ 400(\Lambda_v + 1)q_{s_0}. \\ \\ (iii'). \ \frac{200\Lambda_v}{\delta} \sup_{1 \leq k \leq \delta^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}}, \\ \\ (iv'). \ 2 \times 10^7 \Lambda_v C(v,\rho) \log\left(2 \times 10^4 C^2(v,\rho) + e\right). \end{cases} \tag{3.33}$$

Note that (i′) implies that

$$R > (400\Lambda_v)^{-1} nL \geq (400\Lambda_v)^{-1} n \geq q_{s_0}. \tag{3.34}$$

By our definition of s, we have $s \geq s_0$. This, by (3.18), implies

$$\begin{aligned} q_{s+1} \leq & \ \exp\left(2c(v,\rho)\underline{L}q_s\right) \leq \exp\left(2c(v,\rho)Lq_s\right) \\ \leq & \ \exp\left(2c(v,\rho)LR\right) \leq \exp\left(\frac{LR}{1.8 \times 10^4 C(v,\rho)}\right). \end{aligned} \tag{3.35}$$

By (iv′) of (3.33),

$$n \geq 2 \times 10^7 \Lambda_v C(v,\rho) \log\left(2 \times 10^4 C^2(v,\rho) + e\right) \geq 2 \times 10^7 \Lambda_v C(v,\rho);$$

hence,

$$\exp\left(\frac{RL}{1.8 \times 10^4 C(v,\rho)}\right) \geq \exp\left(\frac{nL^2}{7.2 \times 10^6 \Lambda_v C(v,\rho)}\right) \geq \exp\left(\frac{n}{7.2 \times 10^6 \Lambda_v C(v,\rho)}\right) > 16.$$

Thus, similar to (3.25), using the fact $x < x^{\frac{3}{2}} - 1$, for $x > 3$, we have

$$\exp\left(\frac{RL}{1.8 \times 10^4 C(v,\rho)}\right) \leq \exp\left(\frac{RL}{1.2 \times 10^4 C(v,\rho)}\right) - 1 \leq K. \tag{3.36}$$

Combining (3.35) with (3.36), we obtain, similar to (3.26), that

$$q_{s+1} \leq K. \tag{3.37}$$

3.2.4 *Proof of part (b)*

We use the trivial upper bound in Lemma 2.7 for $\mathcal{U}_1$,

$$\|\mathcal{U}_1(\cdot)\|_{L^\infty(\mathbb{T})} \leq 2\Lambda_v \frac{R}{n} \leq \frac{1}{200}L + \frac{2\Lambda_v}{n} \leq \frac{1}{100}L, \tag{3.38}$$

in which we used, see (i') of (3.33), that

$$n \geq 400(\Lambda_v + 1)q_{s_0} \geq 400\Lambda_v L^{-1}.$$

Proposition 3.1 yields that

$$\begin{aligned} \|\mathcal{U}_2(\cdot) + \mathcal{U}_3(\cdot)\|_{L^\infty(\mathbb{T})} &\leq \frac{\Lambda_v}{\delta n} \sup_{1\leq k\leq \delta^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}} + 4\pi C(v,\rho)\delta \\ &\leq \frac{1}{200}L + \frac{1}{200}L = \frac{1}{100}L, \end{aligned} \tag{3.39}$$

in which we used (iii') of (3.33),

$$n \geq \frac{200\Lambda_v}{\delta} \sup_{1\leq k\leq \delta^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}} \geq \frac{200\Lambda_v}{\delta L} \sup_{1\leq k\leq \delta^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}}. \tag{3.40}$$

Note that we have verified the condition $q_{s+1} \leq K$ in (3.37); Proposition 3.11 implies that

$$\begin{aligned} \|\mathcal{U}_4(\cdot) + \mathcal{U}_5(\cdot)\|_{L^\infty(\mathbb{T})} &\leq 120C(v,\rho)\left(\frac{\log q_{s+1}}{q_s} + \frac{\log K}{R}\right) \\ &\leq 120C(v,\rho)\frac{\log q_{s+1}}{q_s} + \frac{1}{100}L. \end{aligned}$$

By (3.23), we then have

$$\|\mathcal{U}_4(\cdot) + \mathcal{U}_5(\cdot)\|_{L^\infty(\mathbb{T})} \leq 240C(v,\rho)c(v,\rho)L + \frac{1}{100}L = \frac{1}{60}L. \tag{3.41}$$

In view of $\mathcal{U}_6$, (3.12) and (3.36) yield that

$$\begin{aligned} \|\mathcal{U}_6(\cdot)\|^2_{L^2(\mathbb{T})} &\leq C^2(v,\rho)\frac{2}{K} \leq 2C^2(v,\rho)\exp\left(-\frac{RL}{1.8\times 10^4 C(v,\rho)}\right) \\ &\leq 2C^2(v,\rho)\exp\left(-\frac{nL^2}{10^7\Lambda_v C(v,\rho)}\right). \end{aligned} \tag{3.42}$$

Combining (3.38),(3.39), and (3.41) with (3.42), we get that by Markov's inequality,

$$\begin{aligned}\operatorname{mes}\left\{x\in\mathbb{T}:\left|u_n(x)-L_n\right|>\frac{1}{20}L\right\}&\le \operatorname{mes}\left\{x\in\mathbb{T}:\ \left|\mathcal{U}_6(x)\right|>\frac{1}{100}L\right\}\\&\le 2\times 10^4C^2(v,\rho)L^{-2}\exp\left(-\frac{nL^2}{10^7\Lambda_v C(v,\rho)}\right)\\&\le \exp\left(-\frac{nL^2}{2\times 10^7\Lambda_v C(v,\rho)}\right),\end{aligned}$$

in which we used (iv′) of (3.33). Using that $C_1>\Lambda_v$, we obtain

$$-\left(2\times 10^7\Lambda_v C(v,\rho)\right)^{-1}<-\left(2\times 10^7(C_1+2)C(v,\rho)\right)^{-1}=-\tilde{c}(v,\rho).$$

Hence,

$$\operatorname{mes}\left\{x\in\mathbb{T}:\left|u_n(x)-L_n\right|>\frac{1}{20}L\right\}\le\exp\left(-\tilde{c}(v,\rho)nL^2\right),$$

as claimed.

3.3 Proof of Corollary 1.3

In general, a large uniform norm of v does not guarantee a positive Lyapunov exponent. However, if the potential function v is of the form λf, then the following well-known result by Sorets–Spencer [31] gives a lower bound of the Lyapunov exponent in the large coupling regime.

Theorem 3.4. For any non-constant real analytic potential f with an analytic extension on $\{|\Im z|<\rho\}$, there exist constants $\lambda_0=\lambda_0(f)>0$ and $h_0=h_0(f)$ depending only on f, such that for all E, ω, and $\lambda>\lambda_0$, the Lyapunov exponent $L(\omega,E)\ge\log\lambda+h_0$.

Let $\lambda_0=\lambda_0(f)$ be given as in Theorem 3.4. For $\lambda>\lambda_1(f):=\max\left(e^{-19h_0},3,\lambda_0\right)$, we have

$$L(\omega,E)>\log\lambda+h_0>\frac{18}{19}\log\lambda>1, \tag{3.43}$$

holds uniformly in ω and E. (λ_0 is in general large however for some concrete examples, e.g., $f=\cos$, $\lambda_0=2$, cf. [9].) Let $\Lambda_v=\Lambda_{\lambda f}$ be defined as in (2.1); we have

$$L(\omega,E)\le\Lambda_{\lambda f}=\log\left(3+2\lambda\|f\|_{L^\infty(\mathbb{T})}\right)\le\frac{20}{19}\log\lambda, \tag{3.44}$$

provided that $\lambda\ge\lambda_2(\|f\|_{L^\infty(\mathbb{T})})$.

Let $C(\lambda f, \rho), c(\lambda f, \rho)$ and $\tilde{c}(\lambda f, \rho)$ be defined as in (2.5) and (3.17). With the help of Lemma 2.3, we can make the dependence of the three constants on λ more explicit.

First, Lemma 2.3 yields that there exists $C_0 = C_0(f, \rho)$ such that

$$C(\lambda f, \rho) \leq C_0(f, \rho), \tag{3.45}$$

for any $\lambda \geq 0$.

Second, plugging (3.44) and (3.45) into our definition of C_1, see (2.11), we have,

$$C_1 + 2 = 4 + \Lambda_{\lambda f} + (8 + 4\pi C_3)C(\lambda f, \rho) \leq 4 + \frac{20}{19}\log\lambda + (8 + 4\pi C_3)C_0 \leq 2\log\lambda, \tag{3.46}$$

provided that $\lambda \geq \lambda_3(f, \rho) := \max\left(\lambda_2, \exp\left(\frac{19}{18}(4 + (8 + 4\pi C_3)C_0\right)\right)$. Thus, putting (3.45) and (3.46) together, we have that for $\lambda \geq \lambda_3$,

$$\tilde{c}(\lambda f, \rho) = \left(2 \times 10^7 (C_1 + 2)C(v, \rho)\right)^{-1} \geq \left(4 \times 10^7 C_0 \log\lambda\right)^{-1}. \tag{3.47}$$

Third, note that (3.45) also yields

$$c(\lambda f, \rho) = (36000C(\lambda f, \rho))^{-1} \geq (36000C_0)^{-1}. \tag{3.48}$$

Let us take

$$\tilde{\lambda}(f, \rho) := \max(\lambda_1, \lambda_3),$$

and $\lambda > \tilde{\lambda}$. We are in the place to apply Theorem 1.1. Let us note that by (3.43), we always have $L(\omega, E) > 1$; hence, we will only apply part (b). One condition of the theorem is $0 \leq \beta(\omega) < c(\lambda f, \rho)L(\omega, E)$. In view of (3.48) and $L(\omega, E) > \frac{18}{19}\log\lambda$, this condition will always be satisfied if

$$\beta(\omega) < (36000C_0)^{-1}\frac{18}{19}\log\lambda = (38000C_0)^{-1}\log\lambda =: B^{-1}\log\lambda. \tag{3.49}$$

Therefore, for $\lambda > \max(\tilde{\lambda}, \exp(B\beta(\omega)))$, part (b) of Theorem (1.1) implies

$$\text{mes}\left\{x \in \mathbb{T} :\mid u_n(\omega, E; x) - L_n(\omega, E) \mid > \frac{1}{20}L(\omega, E)\right\} \leq \exp\left(-\tilde{c}(\lambda f, \rho)L^2(\omega, E)n\right). \tag{3.50}$$

Using upper and lower bounds of $L(\omega, E)$ in (3.44) and (3.43), we obtain from (3.50) that

$$\begin{aligned}
\text{mes}\left\{x \in \mathbb{T} :| u_n(\omega, E; x) - L_n(\omega, E) |> \frac{1}{19}\log\lambda\right\} \\
&\leq \text{mes}\left\{x \in \mathbb{T} :| u_n(\omega, E; x) - L_n(\omega, E) |> \frac{1}{20}L(\omega, E)\right\} \\
&\leq \exp\left(-\tilde{c}(\lambda f, \rho)L^2(\omega, E)n\right) \qquad (3.51) \\
&\leq \exp\left(-\tilde{c}(\lambda f, \rho)\frac{18^2(\log\lambda)^2}{19^2}n\right) \\
&\leq \exp\left(-n\frac{\log\lambda}{5\times 10^7 C_0}\right) =: \exp\left(-nb\log\lambda\right),
\end{aligned}$$

in which we used (3.47) in the last inequality.

4 The Proofs of the Lemmas

4.1 Proof of Lemma 2.1

We need the following result.

Lemma 4.1. [21, Lemma 2.2] Let $u : \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Suppose that $\partial\Omega$ consists of finitely many piecewise C^1 curves. There exists a positive measure μ on Ω such that for any $\Omega_1 \Subset \Omega$ (i.e., Ω_1 is a compactly contained sub-region of Ω)

$$u(z) = \int_{\Omega_1} \log|z - \zeta|\, d\mu(\zeta) + h(z), \tag{4.1}$$

where h is harmonic on Ω_1 and μ is unique with this property. Moreover, μ and h satisfy the bounds

$$\mu(\Omega_1) \leq C(\Omega, \Omega_1)\left(\sup_{\Omega} u - \sup_{\Omega_1} u\right) \tag{4.2}$$

$$\left\| h - \sup_{\Omega_1} u \right\|_{L^\infty(\Omega_2)} \leq C(\Omega, \Omega_1, \Omega_2)\left(\sup_{\Omega} u - \sup_{\Omega_1} u\right) \tag{4.3}$$

for any $\Omega_2 \Subset \Omega_1$.

Note that $u_n(z)$ is a bounded subharmonic function on $\Omega := \{z : |\text{Re}z| < 1, |\text{Im}z| < \rho\}$. We consider the following nested domains $\Omega_0 \Subset \Omega_2 \Subset \Omega_1 \Subset \Omega$, where

$$\Omega_1 = \left\{z : |\mathrm{Re}z| \le \frac{5}{6},\ |\mathrm{Im}z| < \frac{\rho}{2}\right\}$$
$$\Omega_2 = \left\{z : |\mathrm{Re}z| \le \frac{4}{5},\ |\mathrm{Im}z| < \frac{\rho}{4}\right\} \tag{4.4}$$
$$\Omega_0 = \left\{z : |\mathrm{Re}z| \le \frac{3}{4},\ |\mathrm{Im}z| = 0\right\} = \left[-\frac{3}{4}, \frac{3}{4}\right].$$

Now we apply Lemma 4.1 to $u(z) = u_n(z)$ on Ω. We have then a positive measure μ and a harmonic function h on Ω_1 satisfying (4.1), (4.2), and (4.3).

Since $h - \sup_{\Omega_1} u$ is a harmonic function, by the Poission integral formula and (4.3), we have

$$\max\left(\|\partial_x h\|_{L^\infty(\Omega_0)}, \|\partial_x^2 h\|_{L^\infty(\Omega_0)}\right) \le C(\Omega, \Omega_1, \Omega_2, \Omega_0)(\sup_{\Omega} u - \sup_{\Omega_1} u). \tag{4.5}$$

We only need the bound for $\partial_x h$ here, we will use the one for $\partial_x^2 h$ in Section 4.3.

Combining (4.1) with the technique in [7], one can then show that for some absolute constant $C_2 > 0$, the following holds for any $k \neq 0$:

$$\left|\hat{u}_n(k)\right| \le \frac{C_2}{|k|}\left(\mu(\Omega_1) + \|\partial_x h\|_{L^\infty(\Omega_0)} + \left\|h - \sup_{\Omega_1} u_n\right\|_{L^\infty(\Omega_0)}\right). \tag{4.6}$$

Clearly, (2.4) and (2.5) follow directly from (4.2)–(4.6) by setting

$$\alpha(\rho) := C_2 \max\left(C(\Omega, \Omega_1), C(\Omega, \Omega_1, \Omega_2), C(\Omega, \Omega_1, \Omega_2, \Omega_0)\right). \tag{4.7}$$

This finishes the proof of Lemma 2.1. We will include the proof of (4.6) in Appendix A.

4.2 Proof of Lemma 2.3

On one hand, for any $E \in \mathcal{N}$, trivially we have

$$\sup_{|\mathrm{Im}z|<\rho} \|A_j(E, z)\| \le 2\lambda\|f\|_\rho + 2 \le 3\lambda\|f\|_\rho,\ \text{provided } \lambda > 2\|f\|_\rho^{-1}$$

and

$$\sup_{|\mathrm{Im}z|<\rho} u_n(z) \le \log\left(3\lambda\|f\|_\rho\right).$$

On the other hand, since f is non-constant analytic on $|\mathrm{Im}z| < \rho$, for $\delta = \rho/2$, there exists $\varepsilon_0 = \varepsilon_0(f) > 0$ such that

$$\inf_{E_1} \sup_{y\in(\delta/2,\delta)} \inf_x |f(x+iy) - E_1| > \varepsilon_0.$$

This implies that for any λ, E, there is $y_0 \in (\delta/2, \delta)$ such that $\forall x$

$$\left|f(x+iy_0) - E/\lambda\right| > \varepsilon_0.$$

The computation contained in [7, Appendix] shows that for $\lambda > 2\varepsilon_0^{-1}$,

$$\|M_n(iy_0, E)\| \geq \prod_{j=1}^{n} \Big(|\lambda f(j\omega + iy_0) - E| - 1\Big) \geq \Big(\lambda\epsilon_0 - 1\Big)^n \geq \Big(\frac{1}{2}\lambda\epsilon_0\Big)^n. \tag{4.8}$$

Therefore,

$$\sup_{|\mathrm{Im} z|<\rho/2} u_n(z) \geq u_n(iy_0) = \frac{1}{n}\log\|M_n(iy_0, E)\| \geq \log\left(\frac{1}{2}\lambda\varepsilon_0\right).$$

Clearly, we have that for $\lambda > \max\{2\|f\|_\rho^{-1}, 3\varepsilon_0^{-1}\}$,

$$\sup_{|\mathrm{Im} z|<\rho} u_n(z) - \sup_{|\mathrm{Im} z|<\rho/2} u_n(z) \leq \log\left(3\lambda\|f\|_\rho\right) - \log\left(\frac{1}{2}\lambda\varepsilon_0\right) = \log\left(\frac{6\|f\|_\rho}{\varepsilon_0}\right).$$

Therefore, by (2.5),

$$C(\lambda f, \rho) \leq \alpha(\rho)\log\left(\frac{6\|f\|_\rho}{\varepsilon_0}\right) =: C_0(f, \rho) \text{ independent of } \lambda,$$

as desired.

4.3 Proof of Lemma 2.4

We have that by (2.2),

$$\begin{aligned}
&\|u_n(\cdot+\omega) - u_n(\cdot)\|_{L^\infty(\mathbb{T})}\\
&\quad= \frac{1}{n}\left\|\log\|M_n(\cdot+\omega)\| - \log\|M_n(\cdot)\|\right\|_{L^\infty(\mathbb{T})}\\
&\quad\leq \frac{1}{n}\left\|\log\|M_1(\cdot+n\omega)\| + \log\|M_{n-1}(\cdot+\omega)\| + \log\|M_1(\cdot)\| - \log\|M_{n-1}(\cdot+\omega)\|\right\|_{L^\infty(\mathbb{T})}\\
&\quad\leq \frac{2\Lambda_v}{n}.
\end{aligned} \tag{4.9}$$

This implies

$$\begin{aligned} \left|\hat{u}_n(k)e^{2\pi ik\omega} - \hat{u}_n(k)\right| &= \left|\int_T u_n(x+\omega)e^{-2\pi ikx}\,\mathrm{d}x - \int_{\mathbb{T}} u(x)e^{-2\pi ix}\,\mathrm{k}x\right| \\ &\leq \left\|u_n(\cdot+\omega) - u_n(\cdot)\right\|_{L^\infty(\mathbb{T})} \leq \frac{2\Lambda_v}{n}; \end{aligned} \tag{4.10}$$

(4.10) implies

$$2|\hat{u}_n(k)|\sin\left(\pi\,\|k\omega\|_{\mathbb{T}}\right) \leq \frac{2\Lambda_v}{n};$$

hence, by $\sin(\pi x) \geq 2x$ for $0 \leq x \leq \frac{1}{2}$, we get that for $k \neq 0$,

$$|\hat{u}_n(k)| \leq \frac{\Lambda_v}{2n\|k\omega\|_{\mathbb{T}}},$$

as stated.

Before we move on, let us mention a simple consequence of (4.9):

$$\|u_n(\cdot+\omega) - u_n(\cdot)\|_{L^\infty(\mathbb{T})} \leq \frac{2\Lambda_v|j|}{n}; \tag{4.11}$$

this estimate will be used in several parts of the argument.

4.4 Proof of Lemma 2.5

Let $R \geq R_0(L)$ and $n \geq N_3(v,\omega,L)$, where

$$R_0 := 144L^{-5}, \tag{4.12}$$

and

$$N_3 := 2\Lambda_v L^{-2} \sup_{1\leq|k|\leq L^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}}. \tag{4.13}$$

Lemma 4.1 implies that u_n has a Riesz representation with a positive measure μ and a harmonic function h. Let us take

$$\delta = (LR)^{-1}, \tag{4.14}$$

and

$$u_{n,\delta}(x) = \int_{\Omega_1} \log\left(|x-w|+\delta\right)\mu(\mathrm{d}w) + h(x). \tag{4.15}$$

We then have, pointwisely,

$$u_n(x) \leq u_{n,\delta}(x). \tag{4.16}$$

It is clear from our definitions of R_0 and δ that,

$$\delta \leq \frac{L^4}{144} < \frac{1}{144}. \tag{4.17}$$

4.4.1 *Fourier coefficients decay for $u_{n,\delta}$*

The following two inequalities (4.18) and (4.19) are (2.4) and (2.3) of [7] (see also (8.12) of [5]). We include their proofs in Appendix B.

Lemma 4.2. Let $C(v,\rho)$ be defined as in (2.5). There exists an absolute constant C_3 such that for any $k \in \mathbb{Z}$, we have

$$|\hat{u}_{n,\delta}(k)| \leq |\hat{u}_n(k)| + 3\delta \log \delta^{-1}, \tag{4.18}$$

and for any $k \neq 0$,

$$|\hat{u}_{n,\delta}(k)| \leq C_3 C(v,\rho) \min\left(\frac{1}{|k|}, \frac{1}{k^2\delta}\right) \tag{4.19}$$

holds for $k \neq 0$.

Note that (4.18) together with Lemma 2.4 leads to the following corollary.

Corollary 4.3. For $k \neq 0$, we have

$$|\hat{u}_{n,\delta}(k)| \leq \frac{\Lambda_v}{2n\|k\omega\|_{\mathbb{T}}} + 3\delta \log \delta^{-1}. \tag{4.20}$$

4.4.2 *Proof of Lemma 2.5*

Let $s \in \mathbb{N}$ be such that $q_s \leq R < q_{s+1}$. Recall that our definition of $u^{(R)}$, see (2.8). (4.16) clearly yields

$$0 \leq u_n^{(R)}(x) \leq u_{n,\delta}^{(R)}(x).$$

Let $F_R(k)$ be as in (3.1), invoking (3.5); we have

$$0 \leq u_n^{(R)}(x) \leq u_{n,\delta}^{(R)}(x) = \hat{u}_{n,\delta}(0) + \sum_{k\neq 0} \hat{u}_{n,\delta}(k) F_R(k). \tag{4.21}$$

We now split the Fourier series in (4.21) into low/high-frequency parts,

$$u_{n,\delta}^{(R)}(x) = \hat{u}_{n,\delta}(0) + \sum_{1\le|k|\le q_{s+1}/4} \hat{u}_{n,\delta}(k)F_R(k) + \sum_{|k|>q_{s+1}/4} \hat{u}_{n,\delta}(k)F_R(k)$$
$$=: \hat{u}_{n,\delta}(0) + \mathcal{S}_1 + \mathcal{S}_2. \tag{4.22}$$

Using the $(k^2\delta)^{-1}$ bound of $|\hat{u}_{n,\delta}(k)|$ in (4.19) and $|F_R(k)| \le 1$ in (3.2), we have

$$|\mathcal{S}_2| \le \sum_{|k|>q_{s+1}/4} |\hat{u}_{n,\delta}(k)| \le \sum_{|k|>q_{s+1}/4} \frac{C(v,\rho)}{k^2\delta} \le \frac{8C(v,\rho)}{q_{s+1}\delta} \le \frac{8C(v,\rho)}{\delta R} = 8C(v,\rho)L, \tag{4.23}$$

in which we used $R < q_{s+1}$ and our choice of δ, see (4.14).

We further decompose $\mathcal{S}_1$ into

$$|\mathcal{S}_1| \le \left(\sum_{1\le|k|\le L^{-1}} + \sum_{L^{-1}<|k|<q_s/4} + \sum_{q_s/4\le|k|\le q_{s+1}/4}\right) |\hat{u}_{n,\delta}(k)| F_R(k)$$
$$=: \mathcal{S}_{1,1} + \mathcal{S}_{1,2} + \mathcal{S}_{1,3}. \tag{4.24}$$

By (4.20) and $|F_R(k)| \le 1$, see (3.2), we have

$$\mathcal{S}_{1,1} \le \sum_{1\le|k|\le L^{-1}} \left(\frac{\Lambda_v}{2n\|k\omega\|_{\mathbb{T}}} + 3\delta\log\delta^{-1}\right)$$
$$\le \frac{2}{L}\left(\frac{\Lambda_v}{2n} \sup_{1\le|k|\le L^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}} + \frac{3}{RL}\log(RL)\right).$$

Using a trivial estimate $\log x \le \sqrt{x}$ that holds for any $x > 0$, we obtain

$$\mathcal{S}_{1,1} \le \left(\frac{\Lambda_v}{nL} \sup_{1\le|k|\le L^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}} + \frac{6}{\sqrt{RL^3}}\right) \le L; \tag{4.25}$$

in the last step we used $R \ge R_0 = 144L^{-5}$ and $n \ge N_3$, see (4.12) and (4.13).

Using the $|k|^{-1}$ bound of $|\hat{u}_{n,\delta}(k)|$ in (4.19), and non-trivial bound of $|F_R(k)|$ in (3.2), we have

$$\mathcal{S}_{1,2} \le 2C_3C(v,\rho)L \sum_{L^{-1}<|k|<q_s/4} \frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} \le 2C_3C(v,\rho)L \sum_{1\le|k|<q_s/4} \frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2}.$$

Applying (3.3), we obtain

$$\mathcal{S}_{1,2} \leq 4\pi C_3 C(v,\rho) L \frac{q_s}{R} \leq 4\pi C_3 C(v,\rho) L, \tag{4.26}$$

in which we used $q_s \leq R$.

The estimate of $\mathcal{S}_{1,3}$ is similar to that of $\mathcal{S}_{1,2}$, except that we use (3.4) instead of (3.3). Indeed, by (4.19), (3.2), and (3.4), we have

$$\begin{aligned} \mathcal{S}_{1,3} &\leq \sum_{\ell=1}^{[q_{s+1}/q_s]+1} \sum_{|k|\in[\ell q_s/4,(\ell+1)q_s/4)} |\hat{u}_{n,\delta}(k) F_R(k)| \\ &\leq \sum_{\ell=1}^{[q_{s+1}/q_s]+1} \sum_{|k|\in[\ell q_s/4,(\ell+1)q_s/4)} \frac{2|\hat{u}_{n,\delta}(k)|}{1+R^2\|k\omega\|_{\mathbb{T}}^2} \\ &\leq \sum_{\ell=1}^{[q_{s+1}/q_s]+1} \frac{8C(v,\rho)}{\ell q_s} \left(2 + 4\pi \frac{q_s}{R}\right) \\ &\leq 120 C(v,\rho) \frac{\log q_{s+1}}{q_s}. \end{aligned} \tag{4.27}$$

Note that by (4.18) with $k=0$, we have

$$\hat{u}_{n,\delta}(0) \leq L_n + \frac{1}{RL} \log(RL).$$

Trivial estimate $\log x \leq \sqrt{x}$ for $x > 0$ implies

$$\hat{u}_{n,\delta}(0) \leq L_n + \frac{1}{\sqrt{RL}} \leq L_n + \frac{L^2}{12} < L_n + L, \tag{4.28}$$

in which we used $R \geq R_0 \geq 144 L^{-5}$ and $0 < L < 1$.

Combining (4.21), (4.22), (4.23), (4.24), (4.25), (4.26), and (4.27) with (4.28), we arrive at

$$0 \leq u_n^{(R)}(x) \leq L_n + (2 + 8C(v,\rho) + 4\pi C_3 C(v,\rho))L + 120 C(v,\rho) \frac{\log q_{s+1}}{q_s}$$

holds uniformly in x.

4.5 Proof of Lemma 2.6

We apply Lemma 2.5 to $R = \lfloor 3Ln \rfloor$. The conditions $R \geq R_0$ and $n \geq N_3$, see (4.12) and (4.13), can be reduced to

$$n \geq N_0(\omega, L, v, \rho) := L^{-2} \max\left(2\Lambda_v \sup_{1\leq|k|\leq L^{-1}} \frac{1}{\|k\omega\|_{\mathbb{T}}},\ 49L^{-4}\right). \tag{4.29}$$

Indeed, due to $0 < L < 1$, we have

$$R \geq 3Ln - 1 \geq 147L^{-5} - 1 > 144L^{-5}.$$

Now for $n \geq N_0$, Lemma 2.5 implies

$$\begin{aligned} 0 &\leq u_n(x) \leq |u_n(x) - u_n^{(R)}(x)| + u_n^{(R)}(x) \\ &\leq |u_n(x) - u_n^{(R)}(x)| + L_n + (2 + 8C(v,\rho) + 4\pi C_3 C(v,\rho))L + 120C(v,\rho)\frac{\log q_{s+1}}{q_s}. \end{aligned} \tag{4.30}$$

By (4.9), we have

$$\left|u_n(x) - u_n^{(R)}(x)\right| \leq \sum_{|j|<R} \frac{R-|j|}{R^2} \cdot \frac{2\Lambda_v |j|}{n} = \frac{(R^2-1)\Lambda_v}{3Rn} < \Lambda_v L. \tag{4.31}$$

Hence, combining (4.30) with (4.31), we get

$$0 \leq u_n(x) \leq L_n + (2 + \Lambda_v + 8C(v,\rho) + 4\pi C_3 C(v,\rho))L + 120C(v,\rho)\frac{\log q_{s+1}}{q_s}$$

holds uniformly in x.

4.6 Proof of Lemma 2.10

Let

$$N_2 = \max\left(150\Lambda_v N_1 L^{-1},\ 400(C_1+2)N_1 + 1\right)$$

be as in (2.17). Let $n \geq N_2$ and $R = \lfloor (400(C_1+2))^{-1} n \rfloor + 1$.

By (4.11), we have

$$\left\|u_n(\cdot) - u_n^{(R)}(\cdot)\right\|_{L^\infty(\mathbb{T})} \le \sum_{|j|<R} \frac{R-|j|}{R^2}\left\|u_n(\cdot) - u_n(\cdot + j\omega)\right\|_{L^\infty(\mathbb{T})}$$
$$\le \sum_{|j|<R} \frac{|j|(R-|j|)}{nR^2}\left\|u_j(\cdot + n\omega) + u_j(\cdot)\right\|_{L^\infty(\mathbb{T})} \tag{4.32}$$
$$\le \sum_{|j|<R} \frac{2|j|(R-|j|)}{nR^2}\left\|u_j(\cdot)\right\|_{L^\infty(\mathbb{T})}.$$

By our choice of R and $n \ge N_2 \ge 400(C_1+2)N_1 + 1$, we have

$$R \ge \frac{n}{400(C_1+2)} > N_1. \tag{4.33}$$

We could split the sum in (4.32) into

$$\left\|u_n(\cdot) - u_n^{(R)}(\cdot)\right\|_{L^\infty(\mathbb{T})} \le \sum_{|j|<N_1} \frac{2|j|(R-|j|)}{nR^2}\left\|u_j(\cdot)\right\|_{L^\infty(\mathbb{T})} + \sum_{N_1\le|j|<R} \frac{2|j|(R-|j|)}{nR^2}\left\|u_j(\cdot)\right\|_{L^\infty(\mathbb{T})}. \tag{4.34}$$

We will use trivial upper bound $\|u_j(\cdot)\|_{L^\infty(\mathbb{T})} \le \Lambda_v$, see (2.2), in the 1st summation of (4.34). Note that $j \ge N_1 \ge N_0$; hence, we can apply Lemma 2.6 to u_j in the 2nd sum. We have

$$\left\|u_n(\cdot) - u_n^{(R)}(\cdot)\right\|_{L^\infty(\mathbb{T})}$$
$$\le \sum_{|j|<N_1} \frac{2\Lambda_v|j|(R-|j|)}{nR^2} + \sum_{N_1\le|j|<R} \frac{2|j|(R-|j|)}{nR^2}\left(L_j + C_1 L + 120C(v,\rho)\frac{\log q_{s+1}}{q_s}\right). \tag{4.35}$$

For $j \ge N_1 \ge \widetilde{N}_0 + 1$, Lemma 2.9 implies $L_j \le 21L/20 < 2L$; hence,

$$\left\|u_n(\cdot) - u_n^{(R)}(\cdot)\right\|_{L^\infty(\mathbb{T})}$$
$$\le \sum_{|j|<N_1} \frac{2\Lambda_v|j|(R-|j|)}{nR^2} + \sum_{N_1\le|j|<R} \frac{2|j|(R-|j|)}{nR^2}\left((C_1+1)L + 120C(v,\rho)\frac{\log q_{s+1}}{q_s}\right). \tag{4.36}$$

Use that

$$\sum_{|j|<N_1} \frac{2|j|(R-|j|)}{R^2} = N_1\frac{2(N_1-1)(3R-2N_1+1)}{3R^2} \le \frac{3}{4}N_1,$$

and

$$\sum_{N_1\le|j|<R}\frac{2|j|(R-|j|)}{R^2}=(R+1-N_1)\frac{2\left(R(R-1)+(R+1)N_1-2N_1^2\right)}{3R^2}$$
$$\le R\frac{2(R^2-R)}{3R^2}=\frac{2}{3}(R-1). \tag{4.36}$$

We could control (4.36) by

$$\left\|u_n(\cdot)-u_n^{(R)}(\cdot)\right\|_{L^\infty(\mathbb{T})}\le\frac{3\Lambda_v N_1}{4n}+\frac{2(R-1)}{3n}\left((C_1+2)L+120C(v,\rho)\frac{\log q_{s+1}}{q_s}\right). \tag{4.37}$$

For the 1st term in (4.37), note that $n\ge N_2\ge 150\Lambda_v N_1 L^{-1}$ implies

$$\frac{3\Lambda_v N_1}{4n}\le\frac{1}{200}L. \tag{4.38}$$

For the 2nd term, we plug in $R=\lfloor 400^{-1}(C_1+2)^{-1}n\rfloor+1$, then we have

$$\begin{aligned}&\frac{2\left(C_1+2\right)(R-1)}{3n}L<\frac{1}{200}L,\quad\text{and}\\&\frac{2(R-1)}{3n}\cdot 120C(v,\rho)\frac{\log q_{s+1}}{q_s}\le\frac{4C(v,\rho)}{15(C_1+2)}\frac{\log q_{s+1}}{q_s}\le\frac{1}{5}C(v,\rho)\frac{\log q_{s+1}}{q_s}.\end{aligned} \tag{4.39}$$

Incorporating the estimates in (4.38) and (4.39) into (4.37), we have

$$\left\|u_n(\cdot)-u_n^{(R)}(\cdot)\right\|_{L^\infty(\mathbb{T})}\le\frac{1}{100}L+\frac{1}{5}C(v,\rho)\frac{\log q_{s+1}}{q_s},$$

as stated.

5 Refined Hölder Continuity

Hölder regularity of $L(E)$ follows from combing LDT with AP. This scheme was developed by Goldstein and Schlag in [19], and has shown to be not restricted to quasi-periodic cocycles, see for example, [8] for skew-shift. This scheme was extended to general cocycles in any dimension, in a recent monograph [15]. Recently, it was also used to study the one-dimensional Anderson model in [10]. We sketch the proof below in our setting, making the indenpendence of the Hölder exponent explicit.

5.1 Proof of Theorem 1.5

Fix $(\omega_0, E_0) \in (\mathbb{R}\setminus\mathbb{Q}) \times \mathcal{N}_v$ with $L(\omega_0, E_0) = \gamma > 0$. As we explained in Remark 1.6 that the neighborhood $U \times I$ as in Theorem 1.5 always exists. For any $(\omega, E) \in U \times I$:

$$\frac{18}{19}\gamma \le L(\omega, E) \le \frac{20}{19}\gamma. \tag{5.1}$$

Let $c(v, \rho)$ and $\tilde{c} = \tilde{c}(v, \rho)$ be the constants in Theorem 1.1. Define a subset $\tilde{U}$ of U as follows:

$$\tilde{U} := \{\omega \in \mathbb{R}\setminus\mathbb{Q} :\ 0 \le \beta(\omega) < c(v, \rho)\gamma/2\} \cap U. \tag{5.2}$$

In particular, $\tilde{U}$ contains all the Diophantine numbers in U; thus, $\mathrm{mes}(U \setminus \tilde{U}) = 0$.

We are going to apply Theorem 1.1 on interval $[a, b] = I$. Note that for any $\omega \in \tilde{U}$, by (5.1), we have

$$0 \le \beta(\omega) < \frac{1}{2}c(v, \rho)\gamma < c(v, \rho) \inf_{E\in I} L(\omega, E). \tag{5.3}$$

Hence, the condition of Theorem 1.1 is verified. Let $N = N(\omega, \inf_{E\in I} L(\omega, E), v, \rho)$ be as in (3.19), which is the constant in Theorem 1.1. Let $\tilde{N} = N(\omega, \frac{18}{19}\gamma, v, \rho)$ be the constant defined in (3.19) with $\underline{L} = \frac{18}{19}\gamma$. Then by (5.1) and Remark 3.3, we have $\tilde{N} \ge N$. Let

$$\Omega_n(\omega, E) := \left\{x \in \mathbb{T} :\mid u_n(\omega, E; x) - L_n(\omega, E) \mid > \frac{1}{20}L(\omega, E)\right\}.$$

Theorem 1.1 implies that for $n \ge \tilde{N} \ge N$ and any $(\omega, E) \in \tilde{U} \times I$, we have

$$\mathrm{mes}\left(\Omega_n(\omega, E)\right) \le e^{-\tilde{c}nL(\omega,E)} \le e^{-\tilde{c}n\gamma/2}, \tag{5.4}$$

in which we used $L(\omega, E) \ge \frac{18}{19}\gamma > \frac{1}{2}\gamma$, see (5.1).

In the rest of the section, we will fix $\omega \in \tilde{U}$ and denote $L(E) = L(\omega, E)$, $L_n(E) = L_n(\omega, E)$ for simplicity whenever it is clear. Apply Lemma 2.9 to the interval I. Let $\widetilde{N}_0(\omega, I, v)$ be given as in Lemma 2.9. Then for any $n > \widetilde{N}_0$ and $E \in I$, we have

$$L(E) \le L_n(E) < \left(1 + \frac{1}{20}\right)L(E). \tag{5.5}$$

Combining (5.5) with the fact that $L_{2n}(E) \le L_n(E)$, we have for all $n > \widetilde{N}_0$ and $E \in I$,

$$0 \le L_n(E) - L_{2n}(E) < \frac{1}{20}L(E). \tag{5.6}$$

After combining the large deviation estimate (5.4), the initial scale estimate (5.6), and the AP (Theorem C.1), we obtain the following convergence rate of $L_n(E)$ to $L(E)$.

Proposition 5.1. There exists $N_4 \in \mathbb{N}$ explicitly depends on $\tilde{N}, \widetilde{N}_0, \Lambda_v, \tilde{c}(v,\rho)$, and γ. For any $n > N_4$ and $(\omega, E) \in \tilde{U} \times I$,

$$| L(E) + L_n(E) - 2L_{2n}(E) | < e^{-\tilde{c}(v,\rho)n\gamma/5}. \tag{5.7}$$

Proposition 5.1 can be derived from an induction method developed by Goldstein and Schlag in [19] (see also in [5, 33]). For sake of completeness, we include the proof in Appendix C.

Another key ingredient for the proof of Theorem 1.5 is the following control on $\partial_E L_n(\omega, E)$ with respect to γ.

Proposition 5.2. There exists $N_5 \in \mathbb{N}$ explicitly depends on $\widetilde{N}_0, \Lambda_v, \tilde{c}(v,\rho)$ and γ. For any $n > N_5$ and $(\omega, E) \in \tilde{U} \times I$,

$$|\partial_E L_n(E)| \leq 2e^{2n\gamma}. \tag{5.8}$$

Proposition 5.2 is essentially contained in [5]; we include the proof in Appendix D with these specific parameters.

Now we are in the place to complete the proof of Theorem 1.5 by using (5.7) and (5.8). For short hand we will write $\tilde{c}(c,\rho)$ as $\tilde{c}$, and denote

$$\tilde{c}_0 := \tilde{c} + 20. \tag{5.9}$$

Let $N_6 = \max\{N_4, N_5\}$ and

$$\eta := \min\left(e^{-2\gamma N_6 \tilde{c}_0/5},\ 8^{-4\tilde{c}_0/\tilde{c}}\right) < 1. \tag{5.10}$$

Now for any $E, E' \in I$ such that $|E - E'| < \eta$, let

$$n = \left\lfloor -\frac{5\log|E-E'|}{\gamma\tilde{c}_0} \right\rfloor. \tag{5.11}$$

Using the 1st term in (5.10), it is easy to check that

$$\frac{-5\log|E-E'|}{\gamma\tilde{c}_0} \geq n \geq \frac{-5\log|E-E'|}{2\gamma\tilde{c}_0} \geq N_6 = \max(N_4, N_5). \tag{5.12}$$

Now we can apply Propositions 5.1 and 5.2 to the above n, E, E' to obtain

$$\begin{aligned}|L(E) - L(E')| &\le |L(E) + L_n(E) - 2L_{2n}(E)| + |L(E') + L_n(E') - 2L_{2n}(E')| \\ &\quad + |L_n(E) - L_n(E')| + 2|L_{2n}(E) - L_{2n}(E')| \\ &\le 2e^{-\tilde{c}n\gamma/5} + 4e^{2n\gamma}|E - E'| + 2e^{4n\gamma}|E - E'| \\ &\le 2e^{-\tilde{c}n\gamma/5} + 6e^{4n\gamma}|E - E'|. \end{aligned} \tag{5.13}$$

In view of the upper and lower bound of n in (5.12), we have

$$e^{n\gamma} < |E - E'|^{-5/\tilde{c}_0}, \tag{5.14}$$

and

$$e^{-n\gamma} < |E - E'|^{5/(2\tilde{c}_0)}. \tag{5.15}$$

By (5.13), (5.14), and (5.15), we have that for all $\omega \in \tilde{U}$, $E, E' \in I$ and $|E - E'| < \eta < 1$,

$$\begin{aligned}|L(E) - L(E')| &\le 2|E - E'|^{\tilde{c}/(2\tilde{c}_0)} + 6|E - E'|^{1-20/\tilde{c}_0} \\ &= 2|E - E'|^{\tilde{c}/(2\tilde{c}_0)} + 6|E - E'|^{\tilde{c}/\tilde{c}_0} \\ &\le 8|E - E'|^{\tilde{c}/(2\tilde{c}_0)}. \end{aligned} \tag{5.16}$$

Using the 2nd term in (5.10), we have

$$8 \le \eta^{-\tilde{c}/(4\tilde{c}_0)} < |E - E'|^{-\tilde{c}/(4\tilde{c}_0)}.$$

Plugging it into (5.16), we obtain

$$|L(E) - L(E')| \le |E - E'|^{\tilde{c}/(4\tilde{c}_0)} =: |E - E'|^{\tau}. \tag{5.17}$$

This proves Theorem 1.5.

5.2 Proof of Theorem 1.10

Let $\tilde{\lambda}$, b, B and $N = N(\omega, \lambda, f, \rho)$ be given as in Corollary 1.3. Assume that $\lambda > \max\{\tilde{\lambda}, e^{B\beta(\omega)}\}$; Corollary 1.3 implies that for any $n \ge N$, we have

$$\text{mes}\left\{x \in \mathbb{T} : \left| u_n(\omega, E; x) - L_n(\omega, E) \right| > \frac{1}{19}\log\lambda\right\} \le e^{-n\, b \log\lambda}. \tag{5.18}$$

In view of (3.43) and (3.44), we have that for $n \geq N$,

$$\frac{18}{19}\log\lambda \leq L_n(E) \leq \frac{20}{19}\log\lambda, \quad 0 \leq L_n(E) - L_{2n}(E) \leq \frac{2}{19}\log\lambda. \tag{5.19}$$

By (5.18), (5.19), and the same reasoning for Proposition 5.1, we have the following:

Proposition 5.3. Assume that $\beta(\omega) < \infty$ and $\lambda > \max\{\tilde{\lambda}, e^{B\beta(\omega)}\}$. There exists $N_7 \in \mathbb{N}$ depending explicitly on λ and b such that for any $n > N_7$ and $E \in \mathcal{N}_{\lambda f}$,

$$|\, L(E) + L_n(E) - 2L_{2n}(E)\,| < e^{-\frac{1}{3}n\,b\log\lambda}. \tag{5.20}$$

By the trivial bound $\sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{T}}\sup_{E\in\mathcal{N}_{\lambda f}} u_n(x) \leq \Lambda_v \leq 2\log\lambda$, we have for any n, x and $E \in \mathcal{N}_{\lambda f}$,

$$\Big|\partial_E \log\|M_n(\omega,E;x)\|\Big| \leq \|\partial_E M_n(\omega,E;x)\| \leq \sum_{j=1}^{n}\|M_{n-j}(x+j\omega;E)\|\cdot\|M_{j-1}(\omega,E;x)\| \leq ne^{2n\log\lambda},$$

which implies

$$|\partial_E L_n(\omega,E)| \leq e^{2n\log\lambda}. \tag{5.21}$$

Clearly, by (5.20) and (5.21) and the same argument from (5.10) to (5.17), we can prove (1.22). More precisely, for all $E, E' \in \mathcal{N}_{\lambda f}$ satisfying

$$|E-E'| < \widetilde{\eta} := \min\{e^{-2(12+b)N_7(\log\lambda)/3}, 5^{-4(12+b)/b}\}, \tag{5.22}$$

set $n = \lfloor\frac{3\log|E-E'|^{-1}}{\log\lambda(12+b)}\rfloor$. Then we have

$$\begin{aligned} |L(E)-L(E')| &< 2e^{-\frac{1}{3}n\,b\log\lambda} + 3e^{4n\log\lambda}|E-E'| \\ &\leq 5|E-E'|^{\frac{b}{2(12+b)}} \\ &\leq |E-E'|^{\frac{b}{4(12+b)}} =: |E-E'|^{\tilde{\tau}}. \end{aligned} \tag{5.23}$$

This completes the proof of Theorem 1.10.

A Proof of (4.6)

The proof is essentially contained in [7, Section II]; we include a proof here for completeness.

Proof of (4.6). Let us pick a bump function $\eta(x)$ defined as follows:

$$\eta(x) = \begin{cases} 32\left(x+\frac{3}{4}\right)^3, & -\frac{3}{4} \le x < -\frac{1}{2}, \\ 1-32\left(x+\frac{1}{4}\right)^3, & -\frac{1}{2} \le x < -\frac{1}{4}, \\ 1, & -\frac{1}{4} \le x < \frac{1}{4}, \\ 1-32\left(x-\frac{1}{4}\right)^3, & \frac{1}{4} \le x < \frac{1}{2}, \\ 32\left(x-\frac{3}{4}\right)^3, & \frac{1}{2} \le x < \frac{3}{4}. \end{cases} \tag{A.1}$$

Then it is easy to see that

$$\begin{aligned} &\mathrm{supp}\eta \subset \left[-\frac{3}{4}, \frac{3}{4}\right], \quad \sum_{s\in\mathbb{Z}} \eta(x+s) = 1, \text{ and} \\ &0 \le \eta(x) \le 1, \quad |\eta'(x)| \le 6, \quad |\eta''(x)| \le 48 \quad \text{for all } x \in \mathbb{R}. \end{aligned} \tag{A.2}$$

Let $w(x) := \int_{\Omega_1} \log|x-\zeta|\,\mathrm{d}\mu(\zeta)$ and $t := \sup_{\Omega_1} u_n(z)$. Since $u_n(x)$ is 1-periodic on $\mathbb{R}$, we have

$$\begin{aligned} \hat{u}_n(k) &= \widehat{(u_n - t)}(k) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (u_n(x)-t)e^{-2\pi i k x}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} (u_n(x)-t)\eta(x)e^{-2\pi i k x}\,\mathrm{d}x \\ &= \frac{i}{2\pi k}\int_{\mathbb{R}} \partial_x\Big((w(x)+h(x)-t)\eta(x)\Big)e^{-2\pi i k x}\,\mathrm{d}x \\ &= \frac{i}{2\pi k}\int_{\mathbb{R}} \partial_x(w\eta)e^{-2\pi i k x}\,\mathrm{d}x + \frac{i}{2\pi k}\int_{\mathbb{R}} \partial_x\Big((h-t)\eta\Big)e^{-2\pi i k x}\,\mathrm{d}x \\ &= \frac{i}{2\pi k}\int_{\mathbb{R}} \eta(x)\partial_x w(x)e^{-2\pi i k x}\,\mathrm{d}x && \text{(A.3)} \\ &\quad + \frac{i}{2\pi k}\int_{\mathbb{R}} w(x)\partial_x\eta(x)e^{-2\pi i k x}\,\mathrm{d}x && \text{(A.4)} \\ &\quad + \frac{i}{2\pi k}\int_{\mathbb{R}} \eta(x)\partial_x h(x)e^{-2\pi i k x}\,\mathrm{d}x && \text{(A.5)} \\ &\quad + \frac{i}{2\pi k}\int_{\mathbb{R}} (h(x)-t)\partial_x\eta(x)e^{-2\pi i k x}\,\mathrm{d}x. && \text{(A.6)} \end{aligned}$$

Clearly, (A.5) and (A.6) can be bounded by

$$|(A.5)| + |(A.6)| \leq \frac{1}{2\pi|k|}\Big(\|\partial_x h\|_{L^\infty(\Omega_0)} + 6\|h - \sup_{\Omega_1} u_n\|_{L^\infty(\Omega_0)}\Big). \tag{A.7}$$

It is enough to estimate (A.3) and (A.4) by (4.1). The bound for (A.4) is trivial since

$$\begin{aligned} \Big|\int_{\mathbb{R}} w(x)\partial_x\eta(x)e^{-2\pi ikx}\,\mathrm{d}x\Big| &\leq 6\int_{\Omega_1}\int_{-1}^{1}\Big|\log|x-\zeta|\Big|\mathrm{d}x\,\mathrm{d}\mu(\zeta) \\ &\leq 6\int_{\Omega_1} d\mu(\zeta)\sup_{\zeta\in\Omega_1}\int_{-1}^{1}\Big|\log|x-\zeta|\Big|\mathrm{d}x \\ &\leq 6\mu(\Omega_1)\int_{-2}^{2}\Big|\log|x|\Big|\mathrm{d}x \\ &= (24\log 2)\mu(\Omega_1). \end{aligned}$$

The bound for (A.4) follows from the direct computation in [10]:

$$\begin{aligned} \Big|\int_{\mathbb{R}}\eta(x)\partial_x w(x)e^{-2\pi ikx}\,\mathrm{d}x\Big| &= \Big|\int_{\Omega_1}\int_{\mathbb{R}}\frac{x-\mathrm{Re}\zeta}{|x-\zeta|^2}e^{-2\pi ikx}\eta(x)\mathrm{d}x\,\mathrm{d}\mu(\zeta)\Big| \\ &\leq \int_{\Omega_1}\Big|\int_{\mathbb{R}}\frac{x-\mathrm{Re}\zeta}{|x-\zeta|^2}e^{-2\pi ikx}\eta(x)\mathrm{d}x\Big|\,\mathrm{d}\mu(\zeta) \\ &\leq \mu(\Omega_1)\sup_{\zeta\in\Omega_1}\Big|\int_{\mathbb{R}}\frac{x-\mathrm{Re}\zeta}{|x-\zeta|^2}e^{-2\pi ikx}\eta(x)\mathrm{d}x\Big| \leq C_4\mu(\Omega_1), \end{aligned}$$

where $C_4 > 0$ is some absolute constant given as in [10] such that

$$\sup_{\zeta\in\Omega_1}\Big|\int_{\mathbb{R}}\frac{x-\mathrm{Re}\zeta}{|x-\zeta|^2}e^{-2\pi ikx}\eta(x)\mathrm{d}x\Big| \leq C_4.$$

This finishes the proof. ■

B Proof of Lemma 4.2

Let $\eta(x)$ be the bump function defined as in (A.1). Then

$$\begin{aligned} \big|\hat{u}_{n,\delta}(k) - \hat{u}_n(k)\big| &= \Big|\int_{\mathbb{R}}\int_{\Omega_1}\log\Big(\frac{|x-w|+\delta}{|x-w|}\Big)e^{-2\pi ikx}\eta(x)\,\mu(\mathrm{d}w)\,\mathrm{d}x\Big| \\ &\leq \int_{\Omega_1}\Big|\int_{\mathbb{R}}\log\Big(\frac{|x-w|+\delta}{|x-w|}\Big)e^{-2\pi ikx}\eta(x)\,\mathrm{d}x\Big|\,\mu(\mathrm{d}w) \\ &\leq \mu(\Omega_1)\sup_{w\in\Omega_1}\Big|\int_{\mathbb{R}}\log\Big(\frac{|x-w|+\delta}{|x-w|}\Big)e^{-2\pi ikx}\eta(x)\,\mathrm{d}x\Big|. \end{aligned} \tag{B.1}$$

By Lemma 4.1, we already have control of $\mu(\Omega_1)$; thus, it suffices to estimate the following term for $w = w_1 + iw_2 \in \Omega_1$:

$$\begin{aligned} &\left|\int_{\mathbb{R}} \log\left(\frac{|x-w|+\delta}{|x-w|}\right) e^{-2\pi ikx}\eta(x)\, dx\right| \\ &\quad = \left|\int_{-3/4+w_1}^{3/4+w_1} \log\left(1+\frac{\delta}{\sqrt{x^2+w_2^2}}\right) e^{-2\pi ikx}\eta(x+w_1)\, dx\right|, \end{aligned} \tag{B.2}$$

in which we used $\mathrm{supp}(\eta) \subset [-3/4, 3/4]$. Next use the fact that $|\eta(x)| \le 1$ for any $x \in \mathbb{R}$ and the integrand is monotone decreasing in x; we have

$$\begin{aligned} &\left|\int_{-3/4+w_1}^{3/4+w_1} \log\left(1+\frac{\delta}{\sqrt{x^2+w_2^2}}\right) e^{-2\pi ikx}\eta(x+w_1)\, dx\right| \\ &\quad \le \int_{-3/4}^{3/4} \log\left(1+\frac{\delta}{\sqrt{x^2+w_2^2}}\right) dx \\ &\quad \le 2\int_0^{3/4} \log\left(1+\frac{\delta}{x}\right) dx \\ &\quad = 2\delta\log\delta^{-1} + \frac{3}{2}\log\left(1+\frac{4\delta}{3}\right) + 2\delta\log\left(\frac{3}{4}+\delta\right). \end{aligned}$$

Use that $\delta < \frac{1}{144}$, see (4.17), and that the following holds:

$$\frac{3}{2}\log\left(1+\frac{4\delta}{3}\right) + 2\delta\log\left(\frac{3}{4}+\delta\right) < \delta\log\delta^{-1}, \quad \text{for } 0<\delta<0.15;$$

we obtain that

$$\left|\int_{-3/4+w_1}^{3/4+w_1} \log\left(1+\frac{\delta}{\sqrt{x^2+w_2^2}}\right) e^{-2\pi ikx}\eta(x+w_1)\, dx\right| \le 3\delta\log\delta^{-1}; \tag{B.3}$$

(4.18) follows from combining (B.1) and (B.2) with (B.3).

The proof of (4.19) follows from a similar idea to that of (2.1); the difference is that we need to do integration by parts twice in order to get $(k^2\delta)^{-1}$ Fourier decay. Let us mention that one needs the control of $\|\partial_x^2 h\|_{L^\infty(\Omega_0)}$, which is provided in (4.5), as well as $|\eta''(x)| \le 48$ as in (A.2).

C Proof of Proposition 5.1

Theorem C1 (AP, [23]). Let $B_1, \cdots, B_m$ be a sequence of unimodular 2×2-matrices. Suppose that

$$\min_{1 \le j \le m} \|B_j\| \ge \mu > m \quad \text{and} \tag{C.1}$$

$$\max_{1 \le j < m} [\log \|B_{j+1}\| + \log \|B_j\| - \log \|B_{j+1} B_j\|] < \frac{1}{2} \log \mu. \tag{C.2}$$

Then

$$| \log \|B_m \cdots B_1\| + \sum_{j=2}^{m-1} \log \|B_j\| - \sum_{j=1}^{m-1} \log \|B_{j+1} B_j\| \,| < C_A \frac{m}{\mu}, \tag{C.3}$$

where C_A is an absolute constant.

For any $n \ge N(\omega, \frac{18}{19}\gamma, v, \rho)$ and $E \in I$, set

$$\Omega_n(j) = \left\{ x \in \mathbb{T} : \left| u_n\Big(x + (j-1)n\omega\Big) - L_n(E) \right| > \frac{1}{20} L(E) \right\}$$

$$\Omega_{2n}(j) = \left\{ x \in \mathbb{T} : \left| u_{2n}\Big(x + (j-1)n\omega\Big) - L_{2n}(E) \right| > \frac{1}{20} L(E) \right\}$$

$$\Omega = \cup_{j=1}^{m} \Omega_n(j) \bigcup \cup_{j=1}^{m-1} \Omega_{2n}(j);$$

(5.4) implies that $\mathrm{mes}\Omega_n(j) \le e^{-\frac{1}{2}\tilde{c}(v,\rho)n\gamma}$, $\mathrm{mes}\Omega_{2n}(j) \le e^{-\tilde{c}(v,\rho)n\gamma}$. Take $m = [n^{-1} \exp(\frac{1}{4}\tilde{c}(v,\rho)n\gamma)]$ and $n_1 = mn$, then $(2n)^{-1} \exp(\frac{1}{4}\tilde{c}(v,\rho)n\gamma) < m < n_1 < e^{\frac{1}{4}\tilde{c}(v,\rho)n\gamma}$. Therefore,

$$\mathrm{mes}\Omega < 2m e^{-\frac{1}{2}\tilde{c}(v,\rho)n\gamma} < 2e^{-\frac{1}{4}\tilde{c}(v,\rho)n\gamma} \tag{C.4}$$

provided $\exp(\frac{1}{4}\tilde{c}(v,\rho)n\gamma) > 2n$.

For any $x \notin \Omega$,

$$\left| u_n\Big(x + (j-1)n\omega\Big) - L_n(E) \right| < \frac{1}{20} L(E) < \frac{1}{20} L_n(E), \quad j = 1, \cdots, m, \tag{C.5}$$

$$|u_{2n}\Big(x + (j-1)n\omega\Big) - L_{2n}(E)| < \frac{1}{20} L(E) < \frac{1}{20} L_{2n}(E), \quad j = 1, \cdots, m-1. \tag{C.6}$$

Thus,

$$\frac{19}{20} L_n(E) < u_n(x + (j-1)n\omega) < \frac{21}{20} L_n(E), \tag{C.7}$$

$$\frac{19}{20} L_{2n}(E) < u_{2n}(x + (j-1)n\omega) < \frac{21}{20} L_{2n}(E). \tag{C.8}$$

Denote $B_j = M_n(x + (j-1)n\omega)$, then

$$u_n(x + (j-1)n\omega) = \frac{1}{n}\log \|M_n(x + (j-1)n\omega)\| = \frac{1}{n}\log \|B_j\|,$$
$$u_{2n}(x + (j-1)n\omega) = \frac{1}{2n}\log \|M_{2n}(x + (j-1)n\omega)\| = \frac{1}{2n}\log \|B_{j+1}B_j\|.$$

Notice that $\tilde{c}(v,\rho) < 1$; by (C.7) and the choice of m,

$$\|B_j\| > e^{\frac{19}{20}nL_n(E)} > e^{\frac{19}{20}nL(E)} := \mu > e^{\frac{18}{20}n\gamma} > e^{\frac{1}{4}\tilde{c}(v,\rho)n\gamma} > m, \quad j = 1, \cdots, m. \tag{C.9}$$

By (5.6), (C.5), and (C.6),

$$\begin{aligned}
&\Big|\log \|B_{j+1}\| + \log \|B_j\| - \log \|B_{j+1}B_j\|\Big| \qquad \text{(C.10)}\\
&< |\log \|B_{j+1}\| - nL_n(E)| + |\log \|B_j\| - nL_n(E)| \\
&\quad + |2nL_n(E) - 2nL_{2n}(E)| + |2nL_{2n}(E) - \log \|B_{j+1}B_j\|\ | \\
&< \frac{n}{20}L(E) + \frac{n}{20}L(E) + \frac{2n}{20}L(E) + \frac{2n}{20}L(E) \\
&= \frac{6}{20}nL(E) = \frac{6}{20}\cdot\frac{20}{19}\log\mu < \frac{1}{2}\log\mu. \qquad \text{(C.10)}
\end{aligned}$$

Now (C.1) and (C.2) required by AP are fulfilled. Apply Theorem C1 to $B_j, j = 1, \cdots, m$; we have

$$|\log \|B_m \cdots B_1\| + \sum_{j=2}^{m-1} \log \|B_j\| - \sum_{j=1}^{m-1} \log \|B_{j+1}B_j\|\ | < C_A \frac{m}{\mu}.$$

Recall $n_1 = mn$; clearly

$$\begin{aligned}
&\Big|\frac{1}{n_1}\log \|M_{n_1}(x + (j-1)n\omega)\| + \frac{1}{m}\sum_{j=2}^{m-1}\frac{1}{n}\log \|M_n(x + (j-1)n\omega)\| \\
&\quad - \frac{2}{m}\sum_{j=1}^{m-1}\frac{1}{2n}\log \|M_{2n}(x + (j-1)n\omega)\|\Big| < C_A \frac{m}{n_1\mu} < \frac{C_A}{\mu}.
\end{aligned} \tag{C.11}$$

Denote the sum of the left side of (C.11) by $F(x)$; we have got the above bound of $|F(x)|$ outside the set Ω. For those $x \in \Omega$, we use the upper bound (2.2) such that

$$\sup_{\Omega} |F(x)| < 4\Lambda_v. \tag{C.12}$$

Integrate $F(x)$ over $\mathbb{T}$, by (C.4) and (C.9); for $n > \max\{\widetilde{N}_0(\omega, I, v), N(\omega, \gamma/2, v, \rho)\}$, and $E \in I$, we have

$$\begin{aligned}\left|L_{n_1}(E) + \frac{m-2}{m}L_n(E) - \frac{2(m-1)}{m}L_{2n}(E)\right| &= \left|\int_{\mathbb{T}} F(x)\mathrm{d}x\right| \\ &< \frac{C_A}{\mu} + 4\Lambda_v \cdot mes\Omega \\ &< \frac{1}{20}e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma},\end{aligned} \tag{C.13}$$

provided

$$n > \frac{10}{7\tilde{c}(v,\rho)\gamma}\log(40C_A) + \frac{20}{\tilde{c}(v,\rho)\gamma}\log(320\Lambda_v).$$

By (C.13), (5.5), and (5.6) and the choice of m,

$$\begin{aligned}\mid L_{n_1}(E) + L_n(E) - 2L_{2n}(E) \mid &< \frac{2}{m} \mid L_n(E) - L_{2n}(E) \mid + \frac{1}{20}e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma} \\ &< \frac{1}{10}e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma}\end{aligned} \tag{C.14}$$

provided

$$\tilde{c}(v,\rho)n\gamma > 20\log(80n\gamma).$$

Take $\tilde{n} = 2n_1 = 2mn$; the above argument also shows that

$$\mid L_{2n_1}(E) + L_n(E) - 2L_{2n}(E) \mid < \frac{1}{10}e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma}. \tag{C.15}$$

Therefore,

$$\mid L_{2n_1}(E) - L_{n_1}(E) \mid < \frac{2}{10}e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma} < \frac{1}{40}\gamma < \frac{1}{20}L(E), \tag{C.16}$$

provided $n > 5(\tilde{c}(v,\rho)\gamma)^{-1}\log(8\gamma^{-1})$.

Let $n_0 = n$ and for $s = 0, 1, \cdots$, let

$$n_{s+1} = n_s[n_s^{-1}e^{\frac{1}{4}\tilde{c}(v,\rho)n_s\gamma}]. \tag{C.17}$$

Inductively, we can prove that:

Proposition C.2 (Iteration of $L_n(E)$).

1^s

$$\begin{aligned}&\mid L_{n_{s+1}}(E) + L_{n_s}(E) - 2L_{2n_s}(E) \mid < \frac{1}{10}e^{-\frac{1}{5}\tilde{c}(v,\rho)n_s\gamma}, \\ &\mid L_{2n_{s+1}}(E) + L_{n_s}(E) - 2L_{2n_s}(E) \mid < \frac{1}{10}e^{-\frac{1}{5}\tilde{c}(v,\rho)n_s\gamma}.\end{aligned} \tag{C.18}$$

2^s

$$\mid L_{2n_{s+1}}(E) - L_{n_{s+1}}(E) \mid < \frac{2}{10}e^{-\frac{1}{5}\tilde{c}(v,\rho)n_s\gamma} < \frac{1}{40}\gamma < \frac{1}{20}L(E) \tag{C.19}$$

3^s

$$| L_{n_{s+1}}(E) - L_{n_s}(E) |< \frac{1}{2} e^{-\frac{1}{5}\tilde{c}(v,\rho)n_{s-1}\gamma}, \quad n_0 = n. \tag{C.20}$$

Once we have $1^{s-1}, 2^{s-1}$, we prove 1^s first as (C.14) and (C.15). Then 2^s directly follows from 1^s as (C.16). By 1^s and 2^{s-1}, we get 3^s as follows:

$$\begin{aligned} | L_{n_{s+1}}(E) - L_{n_s}(E) <\ & | L_{n_{s+1}}(E) + L_{n_s}(E) - 2L_{2n_s}(E) | +2 \,| L_{n_s}(E) - L_{2n_s}(E) | \\ &< \frac{1}{10} e^{-\frac{1}{5}\tilde{c}(v,\rho)n_s\gamma} + \frac{4}{10} e^{-\frac{1}{5}\tilde{c}(v,\rho)n_{s-1}\gamma} \\ &< \frac{1}{2} e^{-\frac{1}{5}\tilde{c}(v,\rho)n_{s-1}\gamma}. \end{aligned}$$

□

When the iteration is established for all $s \geq 1$, it is easy to check $n_{s-1} > sn$ by (C.17); we have then

$$\begin{aligned} | L(E) - L_{n_1}(E) | &\leq \sum_{s=1}^{\infty} | L_{n_{s+1}}(E) - L_{n_s}(E) | \\ &\leq \frac{1}{2} \sum_{s=1}^{\infty} e^{-\frac{1}{5}\tilde{c}(v,\rho)n_{s-1}\gamma} \\ &\leq \frac{1}{2} \frac{e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma}}{1 - e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma}} \\ &\leq \frac{9}{10} e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma}, \end{aligned} \tag{C.21}$$

provided $e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma} < \frac{4}{9}$.

By (C.14), we have

$$| L(E) + L_n(E) - 2L_{2n}(E) |< e^{-\frac{1}{5}\tilde{c}(v,\rho)n\gamma} \tag{C.22}$$

D Proof Proposition 5.2

It is enough to show that for n large

$$\sup_{x\in\mathbb{T}} \left| \partial_E \log \|M_n(\omega, E; x)\| \right| \leq 2ne^{2n\gamma}. \tag{D.1}$$

Lemma 2.9 and (5.1) imply that for $n > \tilde{N}_0$, for any $x \in \mathbb{T}$ and $E \in I$,

$$u_n(\omega, E; x) \leq 2\gamma, \tag{D.2}$$

that is, $\|M_j(\omega, E; x)\| \leq e^{2n\gamma}$ for $j > \tilde{N}_0$. For $j \leq \tilde{N}_0$, we use the trivial bound

$$\|M_j(\omega, E; x)\| \leq e^{j\Lambda_v} \leq e^{\tilde{N}_0\Lambda_v} := C_5. \tag{D3}$$

Direct computation shows that for any $x \in \mathbb{T}$, and $n > 2C_5\tilde{N}_0 > 2\tilde{N}_0$,

$$\begin{aligned}\left|\partial_E \log \|M_n(\omega, E; x)\|\right| &\leq \|\partial_E M_n(\omega, E; x)\| \\ &\leq \sum_{j=1}^{n} \|M_{n-j}(x + j\omega; E)\| \cdot \|M_{j-1}(\omega, E; x)\| \\ &= \sum_{j=1}^{\tilde{N}_0} + \sum_{j=\tilde{N}_0+1}^{n-\tilde{N}_0} + \sum_{j=n-\tilde{N}_0+1}^{n} \\ &\leq \sum_{j=1}^{\tilde{N}_0} C_5 e^{2(n-j)\gamma} + \sum_{j=\tilde{N}_0+1}^{n-\tilde{N}_0} e^{2(n-j)\gamma} \cdot e^{2(j-1)\gamma} + \sum_{j=n-\tilde{N}_0+1}^{n} C_5 e^{2(j-1)\gamma} \\ &\leq 2ne^{2n\gamma}.\end{aligned}$$

E Proofs of (3.2),(3.3), and (3.4)

Proof of (3.2). First, trivially we have $F_R(k) \leq 1$. Direct computation shows

$$0 \leq F_R(k) = \frac{\sin^2(\pi Rk\omega)}{R^2 \sin^2(\pi k\omega)} = \frac{\sin^2(\pi R\|k\omega\|_{\mathbb{T}})}{R^2 \sin^2(\pi \|k\omega\|_{\mathbb{T}})} \leq \frac{\sin^2(\pi R\|k\omega\|_{\mathbb{T}})}{4R^2\|k\omega\|^2},$$

in which we used $\sin(\pi x) \geq 2x$ for $0 \leq x \leq 1/2$.

Distinguishing the cases $R\|k\omega\|_{\mathbb{T}} \geq 1$ and $R\|k\omega\|_{\mathbb{T}} < 1$, one can easily prove the stated bound. ■

Proofs of (3.3) and (3.4):

Since $\frac{p}{q}$ is a continued fraction approximant of ω, we have $|\omega - \frac{p}{q}| < \frac{1}{q^2}$. This implies that for any $0 \neq |k| < \frac{q}{2}$, $\left|k\omega - \frac{kp}{q}\right| < \frac{k}{q^2} < \frac{1}{2q}$, and hence

$$\|k\omega\|_{\mathbb{T}} \geq \|kp/q\|_{\mathbb{T}} - \left|k\omega - \frac{kp}{q}\right| \geq \frac{1}{2q}. \tag{E.1}$$

If we take $j_1 \neq j_2 \in (0, \frac{q}{4}] \in \mathbb{Z}$, then clearly $|j_1 \pm j_2| < \frac{1}{2q}$. Thus, by (E.1), $\left|\|j_1\omega\|_{\mathbb{T}} - \|j_2\omega\|_{\mathbb{T}}\right| \geq \min(\|(j_1+j_2)\omega\|_{\mathbb{T}}, \|(j_1+j_2)\omega\|_{\mathbb{T}}) \geq \frac{1}{2q}$. This implies that $\{\|k\omega\|_{\mathbb{T}}\}_{k=1}^{[\frac{q}{4}]}$ are $\frac{1}{2q}$ departed, and by (E.1) the smallest one is $\geq \frac{1}{2q}$. If we rearrange them in the increasing order and label them as $\|k_1\omega\|_{\mathbb{T}} < \|k_2\omega\|_{\mathbb{T}} < \cdots < \|k_{[q/4]}\omega\|_{\mathbb{T}}$, then $\|k_s\omega\|_{\mathbb{T}} \geq \frac{s}{2q}$. Hence,

$$\sum_{1\leq|k|<\frac{q}{4}} \frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} = 2\sum_{1\leq k<\frac{q}{4}} \frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} \leq 2\sum_{s=1}^{[q/4]} \frac{1}{1+R^2(\frac{s}{2q})^2} \leq \frac{4q}{R}\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = 2\pi\frac{q}{R};$$

this proved (3.3).

For $\ell \geq 1$, let $I_\ell := [\frac{q}{4}\ell, \frac{q}{4}(\ell+1)) \cap \mathbb{Z}, \ell \geq 1$. We divide I_ℓ into two disjoint sets, $S_1 = \{k \in I_\ell, |k\omega - [k\omega]| < 0.5\}$, $S_2 = \{k \in I_\ell, |k\omega - [k\omega]| > 0.5\}$. Then for $j_1 \neq j_2 \in I_\ell$ belonging to the same subset (either S_1 or S_2), we have $\left| \|j_1\omega\|_{\mathbb{T}} - \|j_2\omega\|_{\mathbb{T}} \right| = \|(j_1 - j_2)\omega\|_{\mathbb{T}}$. Since clearly $|j_1 - j_2| < \frac{q}{4}$, by (E.1), we have $\|(j_1-j_2)\omega\|_{\mathbb{T}} \geq \frac{1}{2q}$. This implies that $\{\|k\omega\|_{\mathbb{T}}\}_{k\in S_1}$ are $\frac{1}{2q}$ apart from each other, and the same holds for S_2. Thus, we could arrange the terms $\{\|k\omega\|_{\mathbb{T}}\}_{k\in S_1\ (\text{or} S_2)}$ in the increasing order and label them as $\|k_1\omega\|_{\mathbb{T}} < \|k_2\omega\|_{\mathbb{T}} < \cdots \|k_{[q/4]}\omega\|_{\mathbb{T}}$, and we have $\|k_s\omega\|_{\mathbb{T}} \geq \frac{s-1}{2q}$. Hence,

$$\sum_{|k|\in[\frac{q}{4}l, \frac{q}{4}(l+1))} \frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} = 2\sum_{k\in I_\ell} \frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2} = 2\left(\sum_{k\in S_1} + \sum_{k\in S_2}\right) \frac{1}{1+R^2\|k\omega\|_{\mathbb{T}}^2}$$
$$\leq 2\sum_{s=1}^{[q/4]} \frac{1}{1+R^2(\frac{s-1}{2q})^2} \leq 2 + 2\frac{4q}{R}\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = 2 + 4\pi\frac{q}{R};$$

this proves (3.4).

Acknowledgments

R.H. is grateful to Wilhelm Schlag for introducing the Riesz representation and LDT to him. Both authors would like to thank Wilhelm Schlag for useful discussions. R.H. would like to thank the Institute for Advanced Study, Princeton, for its hospitality during the 2017–2018 academic year.

Funding

This work was supported by the National Science Foundation [DMS-1638352, DMS-1600065 to S.Z., DMS-1758326 to S.Z., DMS-1638352 to R.H., DMS-1800689 to R.H].

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