

**The Sleep of Reason Produces Monsters:
How and When Biased Input Shapes Mathematics Learning**

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Abstract

The central argument of this chapter is that in mathematical development, children's failure to reason often leads to their performance being shaped by spurious associations from problem input and overgeneralization of inapplicable procedures rather than by whether answers are plausible or procedures make sense. We review research demonstrating how imbalanced distributions of problems, particularly in textbooks, lead children to create spurious associations between arithmetic operations and the numbers they combine. When conceptual knowledge is absent, these spurious associations contribute to the implausible answers, flawed strategies, and violations of principles characteristic of children's mathematics in many areas. To illustrate mechanisms that create flawed answers and strategies in some areas but not others, we contrast computer simulations of fraction and whole number arithmetic. Most of their mechanisms are similar, but the model of whole number arithmetic, unlike the model of fraction arithmetic, possesses conceptual knowledge that precludes strategies that violate basic mathematical principles. Presenting balanced problem distributions and inculcating conceptual knowledge for distinguishing flawed from legitimate strategies are promising means for improving children's learning.

Keywords: mathematics, fractions, decimals, textbooks, fraction arithmetic, decimal arithmetic

INTRODUCTION

In 1799, Francisco de Goya y Lucientes created the painting, “El sueño de la razon produce monstruos,” a title that roughly translates to, “The sleep of reason produces monsters” (Figure 1). Although Goya’s painting has been interpreted in many ways, a common theme is that failure to reason produces nightmarish consequences (Huxley 1960).

===== Insert Figure 1 about here =====

This disquieting painting might seem far removed from children’s mathematics learning. However, we believe that the origins of the owls, bats, and malign cats in the sleeper’s dream are not so different from the origins of common errors in children’s mathematics. In both, failure to reason opens the door to irrational thoughts.

The central argument of this chapter is that children’s failure to reason often leads to their mathematics performance being shaped by factors other than the plausibility of answers and procedures. We find that mathematically-irrelevant aspects of learning environments, in particular distributions of problems in textbooks, contribute to children’s weak performance and shape the errors they make and the flawed strategies they use. We illustrate these points primarily in the context of fraction arithmetic, though we more-briefly describe how other subtle features of the learning environment influence other areas of mathematics learning as well. To explain *how* biased problem input exercises its effects, we present a computer simulation of the way that problem distributions in textbooks, together with standard learning mechanisms of association and generalization, give rise to the specifics of children’s fraction arithmetic. We conclude by discussing how improved conceptual understanding can promote better mathematics learning and minimize the influence of mathematically-irrelevant factors.

BACKGROUND

Research on rational number arithmetic presents endless examples of children making errors that are implausible, violate basic mathematical principles, or both. For example, errors such as “ $1/2 + 3/4 = 4/6$ ” are common (Mack 1995; Ni & Zhou 2005). The answer “ $4/6$ ” is implausible, because it is much too small to be correct; it also violates the mathematical principle that adding positive numbers must yield answers larger than any of the addends.

Such rational number arithmetic errors take many forms. When thousands of US 8th graders were asked on the 1978 National Assessment of Educational Progress (NAEP) whether $12/13 + 7/8$ was closest to 1, 2, 19 or 21, only 24% answered “2” (Carpenter et al. 1980). The most common answers were “19” and “21.” This and similar findings triggered a variety of reform efforts to improve mathematics education, culminating in the Common Core State Standards. However, Lortie-Forgues et al. (2015) found that when the same problem was presented to 8th graders in 2014, percent correct had increased only from 24% to 27% over the more than three decades.

Lack of understanding of rational number arithmetic is not limited to fractions but rather is general across rational number notations. When 7th graders were presented the seemingly simple decimal arithmetic problem $6 + .32$, more than half answered incorrectly, with the most common error being .38 (Hiebert & Wearne 1985). Understanding of arithmetic with percentages is similarly flawed: for example, 55% of 7th and 8th graders in Gay & Aichele (1997) claimed that 87% of 10 was “greater than 10.”

Such errors might be interpreted as implying that children failed to learn correct rational number arithmetic procedures, but that interpretation is only partially correct. The same children who use flawed strategies and generate implausible errors on some trials use correct strategies and answer correctly on other trials. For example, in Siegler & Pyke (2013), most 6th and 8th

graders who were presented pairs of virtually identical fraction arithmetic problems (e.g., $3/5 \times 1/5$ and $3/5 \times 4/5$) used different strategies on at least one pair of the highly similar problems; 65% of such differing pairs of strategies included both a correct strategy and an incorrect one. Equally striking, children were not much more confident in their correct than in their incorrect answers. Together, these findings suggest that children learn both correct and incorrect strategies but are unable to identify through reasoning which are correct, leading to a competitive retrieval process without a reliable filter for rejecting incorrect strategies when they are retrieved.

The weak understanding of rational numbers extends beyond arithmetic. For example, when asked on the 2004 NAEP to order the three fractions $5/9$, $2/7$, and $1/2$, 50% of 8th graders failed to do so (Martin et al. 2007). Similarly, few elementary, middle, and high school students know that there are an infinite number of numbers between pairs of decimals such as .7 and .8 and pairs of fractions such as $1/3$ and $1/4$ (Hansen et al. 2017, Vamvakoussi & Vosniadou 2010). Unsurprising, given this weak understanding of rational numbers, majorities of both children and adults report far more negative attitudes toward dealing with fractions than whole numbers (Sidney et al. 2019).

Poor understanding of rational numbers matters, because good understanding of them is crucial for later success in and out of school. Consistent with the view that such knowledge is important for success in school, 5th graders' knowledge of fractions predicts 10th graders' overall math achievement in both the U.S. and the U.K., even after statistically controlling for IQ, reading comprehension, working memory, whole number knowledge, socioeconomic status, race, and other variables (Siegler et al. 2012). Consistent with the view that such knowledge is important beyond school, 68% of adults working in upper- and lower-level blue-collar and

white-collar jobs report using rational numbers at work (Handel 2016), and many employees fail at their jobs due to poor knowledge of rational numbers (McCloskey 2007).

Children's (and adults') difficulty understanding rational numbers and rational number arithmetic has many sources. Here, we focus on one source that has only been recognized recently – spurious associations between problems and procedures, with the associations formed largely in response to biased distributions of problems in textbooks. One reason for focusing on the role of textbook problem distributions in children's difficulty is that this source of difficulty could be remedied far more easily than many others, such as socioeconomic inequalities, uneven societal valuation of the importance of learning math, limited understanding of math by teachers, and weak motivation among many students. Indeed, changing from less to more effective textbooks has been found to be more cost-effective for improving student achievement than alternatives such as teacher professional development and class-size reductions (Chingos & Whitehurst 2012, Koedel & Polikoff 2017).

Several other considerations also recommend studying textbooks to better understand children's mathematics learning. Textbooks are an ecologically valid part of the learning environment, used by millions of children each year. Not only do textbooks indicate which problems are presented, they also indicate the order in which problems are presented. A third advantage of studying textbooks is that the raw data are widely available; this makes it easy to replicate analyses of textbook problems and perform new analyses to test alternative interpretations. A fourth advantage is that parallel analyses of textbooks can easily be done cross-nationally; textbooks are used throughout the world, and many features are easy to compare. In a survey of fourth and eighth graders from more than 20 countries who were surveyed as part of the 2011 TIMSS (Trends in International Mathematics and Science Study),

75% of students reported that their teachers primarily used textbooks for mathematics instruction (Horsley & Sikorová 2014). Thus, analyzing textbook content is a promising means for assessing the environments within which children learn math and therefore for understanding the learning process itself.

The chapter is organized into five main sections:

- 1) Descriptions of main phenomena in children's fraction arithmetic
- 2) Characteristics of fraction arithmetic problem distributions in textbooks and classroom assignments
- 3) A computer simulation of fraction arithmetic
- 4) Analyses of relations between input problems and children's performance in other mathematical domains: decimal arithmetic, the measurement interpretation of fractions, geometric shapes, counting, and mathematical equality
- 5) Conclusions regarding the roles of input problems and conceptual understanding in determining when and how spurious associations influence mathematical performance, and how instruction can reduce their influence

MAIN PHENOMENA IN CHILDREN'S FRACTION ARITHMETIC

At least eight consistent phenomena have emerged from studies of children's fraction arithmetic (Byrnes & Wasik 1991, Hecht & Vagi 2010, Jordan et al. 2013, Ni & Zhou 2005). All eight were present in Siegler & Pyke (2013); we illustrate the phenomena with data from that study to demonstrate that all of the phenomena can be observed in a single study and cite converging findings from other studies that used different procedures and problems to illustrate the generality of the phenomena.

The children in Siegler & Pyke (2013) were sixth and eighth graders, half from schools in a predominantly low-income school district and half from schools in a predominantly middle-income district. They were presented eight types of problems – four arithmetic operations, each with equal or unequal denominators, and two items of each problem type, for a total of 16 items. To maximize the comparability of items across operations, the same four pairs of operands – $3/5$ and $1/5$, $3/5$ and $1/4$, $3/5$ and $2/3$, and $4/5$ and $3/5$ – were presented with each of the four arithmetic operations. Children were given pencil and paper, but not calculators, to solve the problems.

Eight Basic Phenomena of Fraction Arithmetic

The eight phenomena observed in Siegler & Pyke (2013) and other studies were:

- 1) **Low overall accuracy:** Numerous studies in Europe and North America have found that 4th to 8th graders' fraction arithmetic is highly inaccurate (e.g., Hecht & Vagi 2010, Newton et al. 2014, Torbeyns et al. 2015). Accuracy improves beyond 8th grade, but slowly and to a low asymptotic level; both high school and community college students are quite inaccurate (Brown & Quinn 2006, Richland et al. 2012). Consistent with these findings, the 6th and 8th graders in Siegler & Pyke (2013) correctly answered only 52% of items.
- 2) **Especially low accuracy for division:** Accuracy tends to be especially low on fraction division problems (Siegler et al. 2011). In Siegler & Pyke (2013), only 20% of division answers were correct.
- 3) **Variable responses within individual problems:** Children generate multiple answers on each problem (Hecht 1998, Newton et al. 2014). For example, presented $4/5 \div 3/5$, children in Siegler and Pyke advanced on at least 4% of trials these answers:

“1/5” (21% of trials), “20/15” or “4/3” (20%), “15/20” or “3/4” (7%), “1.3/5” or “1.33/5” (7%), “1” (7%), “1.3” or “1.33” (6%), and “6/5” (4%).

- 4) **Variable strategy use by individual children:** Variable strategy use is not solely due to different children using different strategies; rather, the same child often uses different strategies on closely similar problems. This strategic variability is a widespread phenomenon (Siegler 2006), and rational numbers are no exception: As noted earlier, 65% of children in Siegler & Pyke (2013) used different strategies on at least one pair of closely similar problems (e.g., $3/5 \times 1/5$ and $3/5 \times 4/5$).
- 5) **More strategy errors than execution errors:** Mathematical errors are of two types: ones where the intended strategy is incorrect and ones where the intended strategy is correct but executed incorrectly. In fraction arithmetic, strategy errors are far more common than execution errors (Gabriel et al. 2012, 2013; Hecht 1998): 91% of errors in Siegler & Pyke (2013) were strategy errors.
- 6) **The most common errors are independent-whole-number errors and wrong-fraction-operation errors:** The best documented type of fraction arithmetic error involves treating numerators and denominators as independent whole numbers (e.g., Gelman 1991, Ni & Zhou 2005). These errors involve applying the arithmetic operation independently to numerators and denominators, as when claiming that $3/5 + 2/3 = 5/8$. However, Siegler & Pyke (2013) found that wrong-fraction-operation errors are at least as common. These errors involve overgeneralization of procedures for solving other fraction arithmetic operations. For example, on a fraction multiplication problem, a child might apply the fraction addition procedure of performing the operation on the numerators and passing through the denominator,

resulting in errors such as $3/5 \times 4/5 = 12/5$. Failure to detect wrong-fraction-operation errors in previous studies seems due to the problems in those studies not including equal denominator multiplication and division items, where such errors are most common.

7) Equal denominators increase addition/subtraction accuracy but decrease

multiplication/division accuracy: Problems with equal denominators elicit more accurate addition and subtraction performance but less accurate multiplication and division performance (Gabriel et al. 2013, Siegler et al. 2011). In Siegler & Pyke (2013), relative to unequal denominators, equal denominators elicited more accurate addition and subtraction answers (80% versus 55% correct), but less accurate multiplication and division answers (37% versus 58% correct).

8) Frequency of different types of errors on each operation varies with whether

denominators are equal: On addition and subtraction problems, children make the most common error, independent-whole-number errors, more often on problems with unequal than equal denominators (e.g., responding that $3/5 + 2/3 = 5/8$ is more common than responding that $3/5 + 4/5 = 7/10$) (Gabriel et al. 2013, Newton et al. 2014, Siegler & Pyke 2013). In contrast, on multiplication problems, the most common error, wrong-fraction-operation errors, are more common on equal than on unequal denominator problems (e.g., children more often respond that $3/5 \times 4/5 = 12/5$ than that $3/5 \times 2/3 = 60/20$ or $3/1$).

To understand the genesis of these phenomena, we examined the problems that children encounter while learning fraction arithmetic.

PROBLEM INPUT

A Basic Assumption: Textbooks Are a Major Source of Input

Understanding any aspect of development requires understanding the input that shapes development in that domain. In arithmetic instruction, textbooks provide a major part of that input (Cai 2014, Moseley et al. 2007, Valverde et al. 2002). In an international survey of 20 countries, the number of 8th grade textbook pages devoted to a given topic and the number of class periods that 8th grade teachers reported teaching the topic were strongly correlated; in the U.S., the correlation was $r=0.95$ (Schmidt, 2002).

Textbooks also provide the majority of examples that teachers assign (e.g., Horsley & Sikorová 2014). For example, a recent large-scale survey of math teachers found that 93% of teachers reported using textbooks in more than half of their lessons for purposes such as selecting examples (Blazar et al. 2019). The present chapter focuses primarily on textbook input in the context of fraction and decimal arithmetic, but similar analyses of input are possible in all areas of mathematics learning (e.g., Geary 1996, Hamann & Ashcraft 1986).

This section presents research on the problems that children receive in learning rational number arithmetic. The primary focus is on problems from textbooks, which have the advantage of being used by millions of students and of being publicly available. Some attention is also given to problems assigned in classrooms, which have the advantage of more directly measuring the problems that children encounter. In our analyses of input problems, we only coded items that were presented without a word problem context, due to the impossibility of knowing which operation children would use to solve problems for which the operation was not specified.

Textbook Problems

To assess the fraction arithmetic problems in textbooks, Braithwaite et al. (2017) coded all symbolic rational number arithmetic problems presented in the fourth through sixth grade

volumes of three popular U.S. mathematics textbook series: Pearson Education's *enVisionMATH* (Charles et al. 2012), Houghton Mifflin Harcourt's *GO MATH!* (Dixon et al. 2012a, 2012b), and McGraw Hill Education's *Everyday Mathematics* (University of Chicago School Mathematics Project 2015a, 2015b, 2015c). The problems were all those that a) had two operands, at least one of which was a fraction or mixed number, b) were in symbolic form (i.e., not word problems), and c) required exact answers (i.e., not estimates). Problems with these characteristics constituted the large majority of problems in all three textbooks that we analyzed, as well as in three other textbook series analyzed by Cady et al. (2015).

The analyses revealed strikingly nonrandom relations between arithmetic operations and the operands (numbers) in the problems. First, consider fraction arithmetic problems involving two fractions. As shown in Table 1A, in the 4th-6th grade volumes of the three textbook series cited above, only 4% of multiplication and division problems had equal denominators (e.g., $\frac{3}{5} \times \frac{4}{5}$). In contrast, in the same textbooks, 50% of addition and subtraction problems had equal denominators (e.g., $\frac{3}{5} + \frac{4}{5}$).

===== Insert Tables 1A and 1B about here =====

Other types of imbalances were also present in the distributions of fraction arithmetic problems in the textbooks. Consider the distribution of problems having one fraction and one whole number (Table 1B). Only 4% of addition and subtraction problems in the textbooks with at least one fraction operand also included a whole number operand (e.g., $6 - \frac{3}{5}$). In contrast, 59% of multiplication and division problems with at least one fraction operand also had a whole number operand (e.g., $6 \times \frac{3}{5}$).

These imbalanced problem distributions do not have any apparent mathematical justification. Learners need to be able to multiply fractions with identical denominators, just as

they need to be able to multiply fractions with unequal denominators. Learners also need to be able to add and subtract whole numbers and fractions, just as they need to be able to multiply and divide them.

Problems Used in Instruction

The fact that problems appear in textbooks does not guarantee that children encounter them. Teachers do not typically present all problems in textbooks; they also might compensate for the paucity of certain types of problems in textbooks by emphasizing them in class or homework assignments.

To test the assumption that textbook problems reflect the input children receive, J. Tian et al. (manuscript in preparation) asked fourteen 4th, 5th, and 6th grade math teachers from five school districts in the greater Pittsburgh area to provide all problems that they presented to students in math class or as homework during the 2017 – 2018 school year. The problems were coded as in Braithwaite et al. (2017).

One main finding was that 73% of the in-class and homework assignments came from textbooks; most of the other 27% of problems were teacher-created. Another significant finding was that the fraction arithmetic problems that teachers presented and assigned showed very similar distributions as those in the math textbooks in Table 1. This was true both for the textbook problems that teachers assigned and for the problems from other sources. These findings supported our assumption that textbook problems are a good proxy for the problems that children encounter.

Do Children Learn Characteristics of Problem Input?

The fact that distributions of textbook and homework problems are biased does not mean that children learn the biases. Indeed, there was reason to believe that they would not.

Mathematics instruction emphasizes general principles and procedures, not distributions of problems; there also would be no obvious reason for textbooks or teachers to call students' attention to imbalanced distributions of problems.

To determine whether children learned the distributions of problems in their textbooks, Braithwaite & Siegler (2018) presented 6th and 8th graders with two complementary types of problems. Choose-operation problems specified operands and asked children to choose an arithmetic operation that was likely to accompany them (e.g., $3/5 \square 2/5$). Generate-operand problems specified an arithmetic operation and asked children to choose two numbers that were likely to accompany it (e.g., $\square \times \square$). Children were told that the two numbers should be two fractions on half of the problems and a fraction and a whole number on the other half.

Children clearly learned the spurious operator-operand associations that were present in textbooks. On the generate-operands task, when the specified operation was addition or subtraction, they usually generated pairs of fractions with equal denominators. When the specified operation was multiplication or division, they usually generated operand pairs with a whole number and a fraction. Similarly, on the choose-operation task, when presented two fractions with equal denominators, children chose addition or subtraction more often than multiplication or division; when presented a whole number and a fraction, they chose multiplication or division more often than addition or subtraction. Children even learned the particular fractions (e.g., $3/4$, $7/8$) that were most likely to appear. The frequency with which each fraction appeared in textbooks and the frequency with which children generated that fraction on the generate-operand problems was $r = .78$. Thus, children are exceptionally good at learning mathematically irrelevant characteristics of instructional input, such as relations

between operations and operands and frequencies of particular fractions. Unfortunately, they are much less apt at learning desired procedures and concepts.

A COMPUTATIONAL MODEL OF FRACTION ARITHMETIC

To illustrate the mechanisms through which textbook input could give rise to children's fraction arithmetic performance, Braithwaite et al. (2017) generated a computer simulation, FARRA (Fraction Arithmetic Reflects Rules and Associations). As input, FARRA received all fraction arithmetic items from *enVision Math* in the order in which the problems appeared in the 4th, 5th, and 6th grade volumes of the series. As output, FARRA produced patterns of strategy choices, accuracies, and particular errors for all four fraction arithmetic operations on problems with equal and unequal denominators. FARRA reflected three main hypotheses:

- 1) Imbalances in the distribution of input problems that children receive from textbooks impair their learning of fraction arithmetic, particularly on the underrepresented problems.
- 2) Children use statistical associations between problem features and solution procedures to guide their strategy choices. Such associative learning is beneficial in many situations, but can be harmful in mathematics learning, where correct performance usually depends on explicit rules rather than statistical associations. In particular, if the practice problems that children receive are biased, the children's choices of strategies will reflect the biases.
- 3) Conceptual knowledge plays little, if any, role in most children's learning of fraction arithmetic. Because most children lack a conceptual basis for determining which procedures to use for which problems, they often commit overgeneralization errors—use of procedures that are correct for some types of problems to solve problems for which

those procedures are inappropriate.

Relevant to the third hypothesis, FARRA provides a test of whether a model devoid of conceptual knowledge can generate and explain the development of fraction arithmetic. FARRA lacks conceptual knowledge not because we believe that no children have such knowledge – some clearly do – but rather because the data indicate that most children have little or no conceptual understanding of fraction arithmetic, or at minimum do not use any conceptual knowledge that they have, leading them to routinely violate basic mathematical principles when solving fraction arithmetic problems (e.g., Siegler & Lortie-Forgues 2015).

How the Simulation Operates

FARRA is a production system that includes both correct and flawed strategy rules, as well as rules for implementing the strategies (execution rules). As with other production systems, each rule is a condition-action pair that includes both a set of conditions under which it can fire and a set of actions that are taken when it fires. Correct rules are standard fraction arithmetic procedures; all but one of FARRA's flawed strategy rules are overgeneralized versions of the correct procedures in which the arithmetic operation is not specified. The flawed rules lead to some correct answers (when the rule happens to be used on a problem for which it is appropriate) but also to overgeneralization errors (when the rule is used on a problem for which it is not appropriate). For example, the correct rule for adding fractions with equal denominators involves executing the operation on the numerators and passing through the denominator (e.g., $3/5 + 4/5 = 7/5$). However, this rule is often overgeneralized to multiplication, resulting in errors such as $3/5 \times 4/5 = 12/5$. FARRA learns the strong association present in textbooks between operands having equal denominators and the addition/subtraction rule being appropriate. This leads to frequent overgeneralization of the addition/subtraction rule to multiplication and

division on items involving equal denominators. The blocked presentation of fraction arithmetic problems of a given type (e.g., in the textbook chapter on equal denominator addition) may contribute to the problem by reducing attention to the operation, because if the last N problems could be solved by executing the operation on the numerators and passing through the denominator, the next problem almost certainly can be solved in the same way (Rohrer et al. 2020).

FARRA also includes execution rules, procedures for implementing the strategies. The execution rules involve whole-number arithmetic operations, such as doing the multiplication needed to create common denominators on fraction addition and subtraction problems that do not initially have them. Most execution rules produce correct answers, but three do not: incomplete execution (e.g., leaving numerators unchanged when multiplying to establish a common denominator), changing the operation to multiplication but not inverting either operand on division problems, and inverting a random operand rather than the correct one on division problems.

During the problem-solving process, FARRA often needs to choose between two or more applicable rules. To choose among them, the model assumes stochastic rule selection combined with a reinforcement learning mechanism, in which increases in the strength of rules are greater when the rule is part of a sequence leading to a correct answer rather than an incorrect one.

Input to the Simulation

In Braithwaite et al. (2017; Study 1), FARRA received 659 input problems in the order in which the problems appeared in the 4th to 6th grade volumes of *enVision MATH* (Charles et al. 2012). That series was chosen as the learning set, because it was intermediate between the other two series in the number of problems it included.

After FARRA was presented the *learning set* of 659 textbook problems, it was presented a *test set* of the problems presented to the 6th and 8th graders in Siegler & Pyke (2013). The test set included 16 items: two items each for the four arithmetic operations with equal and unequal denominators. A set of 1,000 simulated students was created by randomly choosing values for FARRA's three free parameters (learning rate, error discount, and decision noise) and presenting the learning set to FARRA using the values for each "student." The learning rate parameter determined the amount of reinforcement (increase in strength) that a correct answer produced in the productions that fired on the way to generating it. The error discount parameter specified how much less reinforcement the productions receive when the answer was wrong than when it was right. The decision noise parameter introduced random variability from trial to trial. To better understand the impact of these parameters and the simulation more generally, see Braithwaite et al. 2017 and (especially) Braithwaite et al. 2019.

FARRA's Performance and Its Relation to Children's Performance

FARRA generated all eight phenomena of children's performance noted above, with values quite close to those of the children in Siegler & Pyke (2013) on the same problems:

Low overall accuracy. FARRA's percent correct was 52%, exactly equal to the 52% accuracy of children in Siegler & Pyke (2013).

Especially low accuracy on division problems. Like children, FARRA was far less accurate on division than on the other arithmetic operations (20% correct for children; 26% for FARRA). This lower accuracy reflected less practice with division, interference from overgeneralized procedures used on earlier-presented operations, and frequent incorrect executions of the correct rule.

Variable responses within individual problems. Like children, FARRA generated a range of responses for each fraction arithmetic problem. Table 2 illustrates this phenomenon by showing the most common answers generated by FARRA and by children on one problem (see Braithwaite et al. 2017; Table 6, for similar data from other problems). Most answers generated by children were also generated by the simulation, with frequencies approximating those in the children's data. Over the entire set of problem-answer pairs ($N=391$), answer frequencies between the experimental and model datasets correlated $r=.96$. The strength of this correlation partially reflected correct answers being relatively frequent in both the children's and the simulation's data. However, the correlation remained strong when only the frequency of errors was considered ($N=354$, $r=.88$). Thus, children's and FARRA's frequencies of different answers were closely related.

===== Insert Table 2 about here =====

Variable strategy use by individual children and simulation runs. FARRA, like children, generated variable strategies on virtually identical problems. Nearly all runs of the simulation (99%) used different strategies on at least one of the eight pairs of virtually identical problems in the test set, such as $3/5+1/5$ and $4/5+3/5$.

Greater frequency of strategy errors than execution errors. Strategy errors comprised 91% of children's errors and 93% of FARRA's errors.

The most common errors were wrong-fraction-operation and independent-whole-numbers errors. As among children, almost all of FARRA's strategy errors (93%) were wrong-fraction-operation or independent-whole-number errors. Wrong-fraction-operation errors accounted for 64% of FARRA's strategy errors; independent-whole-number errors accounted for 29%. Each type of error also was most common on the same types of problems in the children's

and FARRA's performance. Wrong-fraction-operation errors (e.g., $3/5 \times 4/5 = 12/5$) were most common on multiplication and division problems; independent-whole-number errors ($3/5 + 1/4 = 4/9$) were most common on addition and subtraction problems.

Equal denominators increase addition and subtraction accuracy but decrease multiplication accuracy. Like children, FARRA was more accurate on addition/subtraction problems when operands on a problem had equal denominators (81% correct) than unequal ones (46% correct). Also as with children, FARRA showed the opposite pattern on multiplication problems: lower accuracy when operands had equal denominators (40% correct) than unequal ones (62% correct). Division did not follow the anticipated pattern for either children or FARRA, because performance on the rarely-presented equal denominator division problems was better than expected. The reason appeared to be use of an incorrect strategy that happened to yield a correct answer on one of the two equal-denominator division problems (see Braithwaite et al. 2017 for details).

The most frequent type of error on each operation varies with denominator equality. FARRA overgeneralized the addition/subtraction strategy more often on equal than on unequal denominator multiplication and division problems (40% vs. 18% of trials). In contrast, it overgeneralized the multiplication strategy more often on unequal than on equal denominator addition and subtraction problems (24% vs. 10% of trials). Children showed the same pattern (41% vs. 25%, and 26% vs. 14% of trials, respectively). Again, the phenomena appeared to stem from children and FARRA learning the statistical relations between denominator equality and arithmetic operation in the input problems.

Subsequent analyses of FARRA's performance in Braithwaite et al. (2017; Study 2) demonstrated that the simulation's success in modeling children's performance was equally

apparent with learning set problems from a different textbook series (*GO MATH!*) that had more problems and with a test set of problems from a different study that included the same eight types of problems (Siegler et al. 2011).

Of particular interest were the results of Braithwaite et al. (2017; Study 5) in which we tried to optimize FARRA's performance (within the bounds of plausibility). Tripling the number of learning set problems, presenting each of the eight types of problems equally often in the learning set, and improving the three free parameter values of the simulation led to substantial improvements in FARRA's accuracy (from 52% to 80% correct). Note, however, that the improved learning was still well short of 100% accuracy.

EFFECTS OF PROBLEM INPUT ON LEARNING IN OTHER AREAS OF MATHEMATICS

Similar relations between children's performance and distribution of problems in textbooks and other printed material have been found in a number of other areas of mathematics. Among them are decimal arithmetic, the measurement interpretation of fractions, geometric shapes, counting, mathematical equality, and order of operations.

Decimal Arithmetic

Textbook problems. To obtain a comprehensive and representative database of decimal arithmetic problems in US textbooks, Tian et al. (in press) coded all decimal arithmetic problems from the same textbook series as in Braithwaite et al. (2017). This study distinguished between problems that had two decimal operands (e.g., $.12 \times .34$) and problems that had a whole number and a decimal operand (e.g., $5 \times .6$); it also distinguished between problems with two decimal operands that had equal numbers of decimal digits (e.g., $1.23 + 4.56$) and ones that had two decimal operands with unequal numbers of decimal digits (e.g., $4.5 - 1.23$).

Across the three textbook series, 98% of items with a whole number operand and a decimal operand involved multiplication or division. In contrast, addition and subtraction problems far more frequently involved two decimal operands than a whole number and a decimal (95% versus 5%). Among problems that had two decimal operands, addition/subtraction items more often had equal than unequal numbers of decimal digits (71% vs. 29%), whereas operands on multiplication/division problems equally often involved equal and unequal numbers of decimal digits (51% vs. 49%).

Problems used in instruction. As with fractions, the distribution of decimal arithmetic problems that teachers presented in class or as homework paralleled the distribution in the textbooks (J. Tian et al. manuscript in preparation). The addition and subtraction problems assigned by teachers far more often involved two decimals than a whole number and a decimal, whereas with multiplication and division, the difference was in the opposite direction. Moreover, addition and subtraction problems with two decimal operands more often had equal than unequal numbers of decimal digits, whereas there was no difference on multiplication and division problems. These findings again supported the assumption that textbook problem distributions are a good proxy for the problems children encounter in school.

Relations of textbook input to children's performance. Based on the textbook input, Tian et al. (in press) predicted that the textbook distributions of decimal arithmetic problems would predict children's decimal arithmetic performance. They tested this prediction against children's performance in 1) an experiment published more than 30 years ago by researchers who had never been affiliated with our lab (Hiebert & Wearne 1985), 2) an unpublished data set obtained in 2019 from a large-scale web-based learning platform (ASSISTments, Heffernan & Heffernan 2014), and 3) data from a controlled recent experiment in our lab. The goal was to

examine the generality of the findings over labs (the Hiebert/Wearne lab versus our own), time of data collection (before 1985 versus 2013-2019), and data source (web-based platform versus controlled experiment).

Biased textbook distributions of problems predicted children's decimal arithmetic performance in all three data sets. For example, in children's performance as in the textbooks, presence of a whole number operand was associated with lower accuracy on addition and subtraction problems but with higher accuracy on multiplication and division problems. Thus, in decimal arithmetic as in fraction arithmetic, differences in presentation frequency of various types of problems in textbooks predict corresponding differences in children's accuracy.

The Measurement Interpretation of Fractions

Both correlational and causal evidence indicate that textbook problem input is related to learning of the measurement interpretation of fractions--that is, the interpretation that fractions are a measure of magnitude that can be placed and ordered on number lines. On the input side, textbooks emphasize the part-whole interpretation of fractions far more than the measurement interpretation (Cady et al. 2015, Charalambous et al. 2010, Hansen et al. 2019). On the output side, children are far more accurate on fractions problems that can be solved via the part-whole interpretation (e.g., problems on which units corresponding to the numerator and denominator can be counted) than on problems that require a measurement interpretation (e.g., estimation on a number line with only the endpoints marked; Charalambous & Pitta-Pantazi 2007, Hannula 2003, Tunc-Pekkan 2015).

These findings are correlational, but results of interventions in which children were randomly assigned to conditions suggest that causal relations are also present. Interventions that emphasized the measurement interpretation have yielded greater improvement in children's

fraction knowledge than conditions that emphasized the part-whole interpretation (e.g., Braithwaite & Siegler 2020, Fuchs et al. 2013, Hamdan & Gunderson 2017, Gunderson et al. 2019, Moss & Case 1999). For example, Barbieri et al. (2020) found that relative to instruction emphasizing the part-whole interpretation of fraction concepts, instruction emphasizing the measurement interpretation led to greater improvement in number line estimation and magnitude comparison among at-risk students. Similar findings have emerged with typical students (Saxe et al. 2013).

Geometric Shapes

Resnick et al. (2016) analyzed geometric input from preschoolers' books, games, and apps. They found that circles appeared in 93% of books, 85% of games, and 95% of apps, whereas rectangles appeared in 72% of books, 20% of games, and 65% of apps. Canonical versions of the shapes (e.g., equilateral triangles) were consistently more common than non-canonical versions. Parallel to these relative input frequencies, preschooler's shape identification was considerably more accurate for circles than rectangles (Clements et al. 1999) and for canonical than non-canonical shapes (Satlow & Newcombe 1998).

Counting

Similar parallels between imbalances in the input to which children are exposed and children's performance have been found for counting. Counting competence includes knowing how to recite the numbers in order and understanding the *cardinality principle*, which states that when counting from one, the last item in the count is the number of items in the set. Two studies of children's counting books (Powell & Nurnberger-Haag 2015, Ward et al. 2017) indicated that more than 70% of the books analyzed presented numbers in order, starting with one. In contrast, the cardinality principle was included in fewer than 10% of the books in both studies. Moreover,

in a study of parents reading counting books to their preschoolers, parents rarely provide cardinal labels after the count (Mix et al. 2012). Thus, the protracted development of the cardinality principle (Geary & vanMarle, 2018) may reflect a lack of input that calls attention to the principle.

Mix et al. (2012) provided causal evidence for this conclusion. Training randomly chosen preschoolers in labeling the set size of a display and then counting the objects in it (i.e., saying “Three crackers, count them, 1, 2, and 3”) led to better understanding of the cardinality principle than training randomly chosen peers in only labeling the set size or only counting the objects.

Mathematical Equality

Textbooks rarely present equations with operations on both sides of the equal sign. For example, McNeil et al. (2006) found that the large majority of problems in four middle-school textbooks presented the operation and operands to the left of the equal sign and a blank for the answer to its right (e.g., $4+5 = \underline{\quad}$). Only 5% of problems had operations on both sides of the equation (e.g., $4+5 = 2 + \underline{\quad}$). A similar pattern is present in elementary school textbooks (Powell, 2012).

Lack of experience with problems that depart from the usual format leaves openings for children to misinterpret the equal sign. For example, as late as fourth grade, most children answer incorrectly when presented problems with operations on both sides of the equal sign, such as by answering “12” or “17” to “ $8+4 = \square + 5$ ” (Falkner et al. 1999). These incorrect answers appear to reflect misinterpreting the equal sign as a signal to “add all numbers to the left of the equal sign” or “add all numbers in the problem,” rather than as expressing a relation of equality between the left and right sides of an equation.

McNeil et al. (2015) tested whether a modified workbook that included a broader range of problems than standard workbooks helped second graders form a relational understanding of the equal sign. The modified workbook had the same total number of problems as the control workbook, but included items that were absent from typical workbooks, for example, problems with operations on the right side of the equal sign (e.g., $__ = 4 + 3$) and problems that replaced the equal sign with the words “is the same amount as.” Children who were randomly assigned to use the modified workbook displayed greater understanding of mathematical equivalence than peers who used a standard workbook on both an immediate posttest and a delayed posttest 5-6 months later.

Order of Operations

Biased distributions of problems are not the only mathematically-irrelevant feature of input that influences mathematics performance. Even typographical features, such as internal spacing on a page or screen of problems involving both addition and multiplication (e.g., $2 + 3 \times 4$), influence speed and accuracy in solving problems (Landy & Goldstone 2007a, 2007b, 2010). In particular, narrower spacing between the operation and the surrounding operands increases the probability of performing that operation first, regardless of the formal rules for ordering operations. In the above problem, narrower spacing between “ $2 + 3$ ” than between “ 3×4 ” increases the likelihood of answering “20,” due to the narrower spacing leading people to add “ $2 + 3$ ” and then multiply “ 5×4 .” One reason for such errors may be the spacing that learners previously encountered in textbooks. In textbook presentations of arithmetic and algebra, multiplication problems tend to be written closer together than addition problems (Landy & Goldstone 2007a, 2010).

Thus, although mathematics involves abstraction over irrelevant features, this does not mean that learners abstract over those features. Rather, mathematically irrelevant characteristics of input influence learning in a wide range of contexts.

CONCLUSIONS

The Role of Textbook Problems

Distributions of textbook problems shape children's mathematical performance. Across many areas, including fraction arithmetic, decimal arithmetic, counting, and geometry, performance on rarely encountered types of problems lags behind that on frequently presented types of problems.

FARRA demonstrates that presenting fraction arithmetic problems from textbooks to a computer simulation with standard correct fraction arithmetic procedures, overgeneralized versions of those procedures, stochastic strategy choice mechanisms, and reinforcement learning mechanisms produces performance that closely resembles children's performance. Presenting FARRA a greater proportion of underrepresented problems improves the model's performance. Similarly, presenting greater numbers of rarely presented problems to randomly selected children produces gains in their understanding of other mathematical concepts. Balancing the distribution of textbook problems would be far simpler than addressing other sources of poor math achievement, such as socioeconomic inequality, racism, inconsistent values among U.S. families on math learning, and inconsistent knowledge of mathematics among U.S. teachers. Thus, presenting more balanced distributions of problems in mathematics textbooks is a promising way to improve children's mathematics learning.

The Importance of Conceptual Knowledge

A major reason why textbook problem distributions can strongly influence rational number arithmetic and numerous other areas of mathematics is that many children lack conceptual understanding of these areas. If children possessed such understanding, it could shield them from the influence of spurious associations, but they don't, so it doesn't. The impact of this absence can be seen by contrasting children's performance in whole number and rational number arithmetic.

Whole number versus rational number arithmetic. Children almost never make errors such as $3 \times 5 = 3$, but they often make errors such as $3/5 \times 1/5 = 3/5$: Why is it that implausible errors are rare in some contexts, such as whole number multiplication, but common in others, such as fraction multiplication?

A major difference between whole number and rational number arithmetic is that in at least some areas of whole number arithmetic, children employ a goal sketch that reduces use of flawed strategies. Goal sketches are domain-specific mechanisms for evaluating the plausibility and potential usefulness of strategies in that domain. They include requirements for legitimate strategies and principles, as well as estimation processes for evaluating the plausibility of answers. A goal sketch for fraction multiplication, for example, would include the information that multiplying two positive fractions below one must result in an answer less than either multiplicand; any strategy that violated that principle would be rejected. Thus, such a goal sketch would allow children to reject $3/5$ as a potential answer to $3/5 \times 1/5$, because that answer would be larger than one of the operands and equal to the other. Such evaluations could lead children to turn to the other main fraction multiplication strategy they know, the correct strategy, and thereafter choose it increasingly, because it produced answers that met the requirements of the goal sketch and received reinforcement.

The functions served by goal sketches resemble those of the System 2 reasoning described by Stanovich & West (2000) and Kahneman (2011), among others. However, the quick and seemingly effortless evaluations of both familiar and unfamiliar strategies by the kindergartners in Siegler & Crowley (1994) suggest a process more like System 1 reasoning. Perhaps when goal sketches are first formed, their application is slow and effortful, but with use they become automatic.

In this concluding section, we review evidence that young children possess considerable conceptual understanding of whole-number addition, describe the SCADS (Strategy Choice and Discovery Simulation) computer simulation and how its goal sketch prevents use of flawed whole-number arithmetic strategies, compare the empirical data and model for whole number arithmetic to those for rational number arithmetic, and explore how helping children form goal sketches for rational number arithmetic could improve their learning.

Children's understanding of whole number addition. Preschoolers have considerable understanding of whole number arithmetic (Gilmore et al. 2018). For example, they choose adaptively among the varied addition strategies they use, in the sense of using each approach most often on problems on which it yields favorable combinations of accuracy and speed (Siegler & Shrager 1984). In particular, preschoolers predominantly use retrieval, the fastest strategy, when they can execute it accurately; they predominantly use slower strategies, such as counting from one, on problems where they are necessary for accurate performance. Such adaptive strategy choices, along with the almost total absence of implausible answers such as $3+4=2$ or $3+4=22$, reflects a kind of implicit understanding of basic addition.

Preschoolers' understanding of whole number addition extends to discovery of new strategies. Siegler & Jenkins (1989) identified 4- and 5-year-olds who, on a pretest, solved

problems by counting from 1 but never counted-on from the larger addend, even on problems such as $2+9$ where counting-on could have been advantageous. The children were presented large numbers of addition problems, with feedback about the answer's correctness following each problem. Solving problems led almost all of the preschoolers to discover the counting-on strategy, though some took more than 200 problems to do so. Most children also discovered another correct strategy that was intermediate between counting from one and counting-on from the larger addend. Perhaps most striking, no preschooler ever tried a conceptually-flawed strategy, such as counting the first addend twice or only counting the second addend.

Beyond this implicit understanding, young children also possess some explicit understanding of whole number addition. On the trial on which they discovered the counting-on strategy, some preschoolers in Siegler & Jenkins (1989) explicitly noted its superiority to counting-from-one because, as one child put it, when you count-on, "You don't have to count a very long way." Moreover, when kindergartners in another study were asked to judge whether a strategy that an experimenter demonstrated was "very smart," "kind of smart," or "not smart," they judged counting-on, which they had not used on the pretest, to be much smarter than the conceptually flawed strategy of counting the first addend twice, which they also had not used (Siegler & Crowley 1994).

A computer simulation of preschoolers' whole number addition. The cognitive processes that generate preschoolers' adaptive strategy choices and discovery of useful new whole number addition strategies without use of flawed approaches were modeled in Shrager & Siegler's (1998) computer simulation, SCADS. Like FARRA, SCADS generated numerous changes in performance that closely resembled those of children. The learning mechanisms and strategy choice procedures in the two simulations were also highly similar.

What SCADS possessed that FARRA lacked, however, was a goal sketch that guided strategy discovery toward useful new strategies and away from flawed ones. SCADS generated between 15 and 21 strategies on different runs. However, many of these strategies were rejected without being tried, because they violated the requirements of the goal sketch that legitimate strategies must quantify each addend once and only once. In terms of the metaphor with which we began this chapter, the goal sketch protected children from the monsters.

The potential value of goal sketches for rational number arithmetic. FARRA does not include a goal sketch, because there is no evidence that children evaluate the plausibility of rational number arithmetic strategies or the answers they yield. Indeed, there is considerable evidence that children do not use goal sketches for rational number arithmetic. If children evaluated the plausibility of answers and the strategies that generated them, they would not claim that 19 was the closest answer to $12/13 + 7/8$, that $6 + .32 = .38$, or that $3/5 \times 4/5 = 12/5$.

More frequently presenting underrepresented types of problems improved FARRA's performance, and it probably would improve children's performance as well. However, the improvement would almost certainly be greater if balanced textbook presentation of problems were supplemented by goal sketches. For example, a fraction multiplication goal sketch would include the requirement that multiplying two positive fractions below one, such as $3/5 \times 4/5$, must result in an answer less than either multiplicand. This knowledge would allow children to reject $12/5$ as a potential answer, because it is larger than both $3/5$ and $4/5$. Such evaluations could lead children to turn to the other main strategy they know, the correct strategy, and choose it increasingly, because it would produce answers that meet the requirements of the goal sketch and elicit positive reinforcement. Similarly, a goal sketch for fraction addition would allow children to reject answers such as $1/2 + 1/2 = 2/4$, because they violate the requirement that adding positive

numbers must produce answers that exceed all addends. Children would again likely turn to the correct strategy, which most also know, and use it increasingly for the same reasons.

This analysis raises the issue of why, after years of extensive experience with rational number arithmetic, children do not form goal sketches for it. Weak knowledge of the magnitudes of individual rational numbers, and weak understanding of the meaning of arithmetic operations with rational numbers, probably interfere with formation of such goal sketches. Both weaknesses were evident in Braithwaite et al. (2018), in which 6th and 7th graders were asked to estimate the positions of individual fractions and sums of fractions on a 0-1 number line and to estimate the positions of individual whole numbers and sums of whole numbers on a 0-1000 number line. As expected, estimation accuracy was greater for individual whole numbers than individual fractions, thus demonstrating greater knowledge of whole number magnitudes. More striking, however, were the much larger differences in estimation accuracy between fraction and whole number sums. Estimation of fraction sums was so inaccurate that it would have improved if children had placed every estimate at the middle of the number line, regardless of the sum being estimated. This and other findings indicate that helping children create goal sketches for fraction arithmetic will require improving their understanding of how arithmetic operations work in the context of fractions, as well as improving children's understanding of the magnitudes of individual fractions. (See Braithwaite & Siegler 2020 for an effective intervention based on these ideas).

There probably is no way to prevent spurious associations in textbooks from influencing children's mathematics. Adults at high quality universities and even professional mathematicians are influenced by them under some circumstances (Obersteiner et al. 2013). However, more balanced presentation of textbook problems can mitigate the difficulty to an extent, inculcating

conceptual understanding like that in goal sketches can mitigate the difficulty further, and the two together can help keep the monsters at bay.

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SUMMARY POINTS

1. Imbalanced distributions of problems in math textbooks contribute to children's difficulty learning mathematics.
2. Children learn spurious associations from the statistical relations present in textbooks; these associations lead children to choose inappropriate strategies.
3. The negative influence of imbalanced problem distributions extends to many areas, including fraction and decimal arithmetic, counting, geometric shapes, and the concept of mathematical equality.
4. Presenting balanced problem distributions improves mathematics learning in many areas.
5. In domains where they lack conceptual understanding, children are especially vulnerable to the negative influences of imbalanced problem distributions.
6. Children have particularly little understanding of fraction and decimal arithmetic.
7. The FARRA computer simulation, which is totally devoid of conceptual knowledge, closely approximates children's fraction arithmetic performance.
8. Improving children's understanding of the difference between legitimate and flawed strategies can help children avoid irrational errors and reduce the influence of irrelevant problem features on performance.

Figure Caption

Figure 1

Francisco Goya (Francisco José de Goya y Lucientes), *Plate 43 from 'Los Caprichos': The Sleep of Reason Produces Monsters* (*El sueño de la razon produce monstruos*), 1799, The Metropolitan Museum of Art, New York, USA (Gift of M. Knoedler & Co., 1918), <https://www.metmuseum.org/art/collection/search/338473>.

Table 1A. Percent of US textbook problems classified by arithmetic operation and denominator equality

Denominator equality	Arithmetic Operation			
	Addition	Subtraction	Multiplication	Division
Equal denominators	12	13	1	1
Unequal denominators	13	12	29	19

Table 1B. Percent of US textbook problems classified by arithmetic operation and operand number type

Operand number type	Arithmetic Operation			
	Addition	Subtraction	Multiplication	Division
Fraction-fraction	25	23	13	8
Whole-fraction	0	2	17	13

Note. Percentages may not sum up to 100% because of rounding.

Table 2 Frequencies of common answers on $4/5 \times 3/5$ from children and FARRA

Problem	Answer	Frequency (% of responses)	
		Children ^a	FARRA ^b
$4/5 \times 3/5$	12/25^c	40.0	38.7
	12/5	36.7	40.9
	15/20	4.2	5.5
	20/15	3.3	9.6

^a Children denotes data from Siegler & Pyke (2013); ^b Abbreviation: FARRA, fraction arithmetic reflects rules and associations. FARRA denotes simulation data from Braithwaite et al. (2017); ^c The correct answer is bolded.



Figure 1. Francisco Goya (Francisco José de Goya y Lucientes), *Plate 43 from 'Los Caprichos': The Sleep of Reason Produces Monsters (El sueño de la razón produce monstruos)*, 1799, The Metropolitan Museum of Art, New York, USA (Gift of M. Knoedler & Co., 1918), <https://www.metmuseum.org/art/collection/search/338473>.