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Spacetime equilibrium at negative temperature and the attraction of gravity*

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We derive the Einstein equation from the condition that every small causal diamond is a variation of a flat empty diamond with the same free conformal energy, as would be expected for a near-equilibrium state. The attractiveness of gravity hinges on the negativity of the absolute temperature of these diamonds, a property we infer from the generalized entropy.

Keywords: Emergent gravity; Einstein equation; horizon thermodynamics; Unruh effect; causal diamond; negative temperature.

The discovery of the Unruh effect[1] revealed that the distinction between vacuum fluctuations and thermal fluctuations is not as great as previously thought.[2] Indeed, the simplest and most general statement of this relation is that, for all observables localized in a Rindler wedge, the Minkowski vacuum of a relativistic quantum field is a thermal state with respect to the Lorentz boost Hamiltonian. Since every point in any spacetime has an approximately Minkowskian neighborhood, one is led to the idea that spacetime can be viewed as a medium, everywhere near local thermodynamic equilibrium, somewhat like a fluid with local temperature, density, pressure, etc.[3]

The entanglement entropy of a quantum field vacuum across a Rindler horizon is UV-divergent, and scales with the horizon area A.[4–6] In quantum gravity, the UV divergent area law for entropy is presumably replaced by the finite,

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Bekenstein–Hawking area law, $S_{\rm BH} = A/4\hbar G$,[7,8] thus establishing a link between gravitation and thermodynamics of the vacuum. In the context of black hole thermodynamics, Bekenstein introduced the "generalized entropy", $S_{\rm gen} := S_{\rm BH} + S_{\rm m}$, where $S_{\rm m}$ is the ordinary matter entropy outside the horizon, and he argued that $S_{\rm gen}$ should satisfy the "generalized second law" (GSL) of thermodynamics, thanks to the Einstein equation. All evidence suggests that he was right, and that the GSL holds not only globally for black holes, but also locally for Rindler horizons.[9,10]

In Ref. 11, reversing the logic, the Einstein equation was derived from the equilibrium assumption that the generalized entropy of small causal diamonds is stationary at fixed spatial volume. Sorely lacking, however, was a prior rationale for holding fixed the volume. In this essay (see also Ref. 12) we reformulate the derivation of Ref. 11, replacing the stationarity of entropy at fixed volume by the stationarity of a free energy. The volume appears in the free energy, playing the role of the gravitational energy. That the volume should play this role can be derived from general relativity, and is related[12] to the observation of York[13] that the Hamiltonian of general relativity in the extrinsic curvature time gauge is proportional to the spatial volume of constant mean curvature slices. Here, however, since we aim to *derive* the Einstein equation, we cannot use results from general relativity. Instead, we infer from diffeomorphism invariance the need for a volume term in the free energy. Ultimately, perhaps a microscopic interpretation of the volume as some kind of energy can be found, in the same way that the area represents entanglement entropy.

Our key postulate is that any small causal diamond in any spacetime is "close" to being flat (Minkowski), in the sense that, to first-order in metric and matter variations, it is the variation of a flat reference causal diamond with the same free energy in the canonical ensemble. To qualify as "small" for this purpose, a diamond should be much smaller than both the shortest local curvature scale and the scale of any quantum field excitations present. Since all metrics are flat to first-order around any point, a small diamond can be regarded as a slight deformation of a flat diamond, i.e. the intersection of the future of one point with the past of another in Minkowski spacetime. Such diamonds admit a conformal isometry generated by a conformal Killing vector ζ,[14,15] satisfying $\mathcal{L}_\zeta g_{ab} \propto g_{ab}$, which is null at the past and future null boundaries of the diamond, so those boundaries are conformal Killing horizons. The horizon surface gravity κ (defined through $\nabla_a \zeta^2 = -2\kappa\zeta_a$[16]) is constant on the edge of the diamond by spherical symmetry, as well as along each generator of the null boundaries, as for the horizon of a stationary black hole. We normalize the conformal Killing vector below such that $\kappa = 1$.

We define the Helmholtz-like free (conformal) energy of flat diamonds as

$$F = H_\zeta - TS_{\rm UV}, \tag{1}$$

where H_ζ is the Hamiltonian generating evolution along the flow of the conformal Killing field ζ, T is the temperature of the diamond, and $S_{\rm UV}$ is the entanglement entropy of the diamond associated with the UV degrees of freedom. One might have thought that a true Killing vector is needed for gravitational thermal equilibrium, however we find that for causal diamonds a conformal Killing vector suffices. We assume there exists a UV-cutoff in quantum spacetime which renders the entanglement entropy finite,[17–19] and proportional to the area,

$$S_{\mathrm{UV}} = \eta A, \tag{2}$$

where η is a universal positive constant of dimension $[\mathrm{length}]^{2-d}$. As to the temperature, one might think it should be the Unruh temperature associated with the conformal Killing horizon, $T_{\mathrm{U}} = \hbar/2\pi$, since, for conformal matter on a background flat diamond, the variation of the matter Hamiltonian away from the conformal vacuum is equal to T_{U} times the variation of the matter entropy, $\delta H^{\mathrm{m}}_{\zeta} = T_{\mathrm{U}}\delta S^{\mathrm{m}}$. (This is the conformal generalization of the Unruh effect.[14]) When inserted into the variation (at fixed T) of the free energy (1) this yields

$$\delta F \supset T_{\mathrm{U}}\delta S^{\mathrm{m}} - T\delta S_{\mathrm{UV}}. \tag{3}$$

The two terms on the right should combine to form $-T\delta S_{\mathrm{gen}}$, but this happens only if

$$T = -T_{\mathrm{U}} = -\frac{\hbar}{2\pi}. \tag{4}$$

Thus, quantum field thermodynamics in a *fixed* diamond background is quite different from the self-gravitating case. In the former, the temperature of the vacuum is positive, whereas in *gravitational* thermodynamics, as we see here, the temperature of a causal diamond is negative.

The conformal Killing energy H_{ζ} has contributions from the metric and from matter fields. We work in the semiclassical regime, i.e. we consider *quantum* matter fields on a *classical* background spacetime. For the stationarity of free energy, we only need to know the *variation* of the conformal Killing energy, denoted by: $\delta H_{\zeta} = \delta\langle H^{\mathrm{m}}_{\zeta}\rangle + \delta H^{\mathrm{g}}_{\zeta}$. The variations we consider are arbitrary variations of the dynamical fields away from the flat diamond to nearby states. The variation of the expectation value of the matter Hamiltonian is given by an integral over the maximal slice Σ (which is a spherical ball) of the reference diamond,[a]

$$\delta\langle H^{\mathrm{m}}_{\zeta}\rangle = \int_{\Sigma} \delta\langle T_a{}^b\rangle \zeta^a u_b dV. \tag{5}$$

We take the reference configuration to be one with vanishing stress tensor; in effect, $\delta\langle T_a{}^b\rangle = \langle T_a{}^b\rangle$. Note that since we have not converted the matter stress-energy into an equivalent entanglement entropy, nonconformal invariant matter presents no extra complications in our derivation, unlike in Ref. 11 where an extension of the first law of entanglement entropy was required.

The gravitational contribution $\delta H^{\mathrm{g}}_{\zeta}$ to the Hamiltonian variation can be inferred, without assuming the Einstein–Hilbert action, from the requirement of

[a] For a traceless and divergence-free stress tensor the integral would be independent of the slice.

diffeomorphism invariance. Consider a variation induced by a diffeomorphism, denoted by $\hat{\delta}$. In that case $\hat{\delta}\langle H_\zeta^{\mathrm{m}}\rangle$ is zero, since the background value is taken to vanish. Stationarity of the free conformal energy (1) at fixed temperature then implies: $\hat{\delta}H_\zeta^{\mathrm{g}} = T\eta\hat{\delta}A$, where we used (2). In Sec. 3.3.2 of Ref. 12 we showed that the diffeo-induced area variation is equal to: $\hat{\delta}A = k\hat{\delta}V$, where $\hat{\delta}V$ is the variation of the volume of the maximal slice in the original diamond and k is the trace of the extrinsic curvature of the edge, as embedded in the maximal slice. The minimal choice for $\delta H_\zeta^{\mathrm{g}}$ consistent with diffeomorphism invariance is thus given just by the volume variation. Hence, we postulate that the gravitational Hamiltonian variation is equal to

$$\delta H_\zeta^{\mathrm{g}} = T\eta k\delta V. \tag{6}$$

The variation of the free energy (1) at fixed temperature is therefore,

$$\delta F = \delta\langle H_\zeta^{\mathrm{m}}\rangle - T\eta(\delta A - k\delta V). \tag{7}$$

Note that the definitions (2)–(6) of the terms in the free energy (variation) apply in principle to flat diamonds of *any* size.

Next, we evaluate δF for *small* diamonds. For such diamonds, to leading order in $\ell/L_{\mathrm{excitation}}$, the matter contribution to the Hamiltonian variation is[11,12]

$$\delta\langle H_\zeta^{\mathrm{m}}\rangle = \frac{\Omega_{d-2}\ell^d}{d^2-1}\langle T_{ab}\rangle u^a u^b, \tag{8}$$

where Ω_{d-2} is the area of a unit $(d-2)$-sphere, d is the spacetime dimension and $\langle T_{ab}\rangle u^a u^b$ is constant to leading-order on the maximal slice.[b] Meanwhile, to lowest-order in $\ell/L_{\mathrm{curvature}}$, we have

$$\delta A - k\delta V = -\frac{\Omega_{d-2}\ell^d}{d^2-1}G_{ab}u^a u^b, \tag{9}$$

where the Einstein curvature tensor G_{ab} is evaluated at the center of Σ. This purely geometric result was obtained in Ref. 11 using a Riemann normal coordinate expansion. Finally, inserting (8) and (9) into the expression (7) for the free energy variation yields

$$\delta F = \frac{\Omega_{d-2}\ell^d}{d^2-1}u^a u^b(\langle T_{ab}\rangle + T\eta G_{ab}). \tag{10}$$

Note that the *same* fraction appears in the combination of variations (9) as in the matter Hamiltonian variation (8), which is crucial for the agreement of the entropy area-density η with the Bekenstein–Hawking value to follow from the Einstein equation (11).

[b] The fraction on the right-hand side of (8) originates from the integral of the norm of the conformal Killing vector over de volume of the ball: $\int_\Sigma |\zeta| dV$. This integral is known as the "thermodynamic volume" in the case where ζ^a is a true Killing vector.[20]

The requirement $\delta F = 0$ for all time-like unit normals u^a and at every point in spacetime now implies the equation

$$G_{ab} = -\frac{1}{T\eta}\langle T_{ab}\rangle. \tag{11}$$

Recall that we previously inferred the diamond temperature $T = -\hbar/2\pi$. Thus, Eq. (11) is the semiclassical Einstein equation provided we make the identification $\eta = 1/4\hbar G$ (where G is Newton's constant), which agrees with the Bekenstein–Hawking entropy area-density. Note that the emergent gravity is attractive since the temperature is negative, and would have been repulsive had the temperature been positive! A cosmological constant is of course permitted as a piece of the stress tensor equal to a constant times the metric.

The negative temperature is a surprising feature of our derivation. That it must be negative is already evident from classical Einstein gravity, since the addition of energy to a diamond results in the *decrease* of its Bekenstein–Hawking entropy at fixed volume.[11,12] Negative temperature typically requires of a system that (i) the energy spectrum is bounded from above, and (ii) the Hilbert space is finite-dimensional. As argued by Klemm and Vanzo[21] for the de Sitter static patch, causal diamonds indeed satisfy these properties: (i) there is an upper bound on the energy, equal to the mass of the largest black hole that fits inside a diamond given its bounding area, and (ii) the holographic principle implies that the entropy of a causal diamond is bounded by the Bekenstein–Hawking entropy associated to the area of the edge. Thus, despite the positive value of the Unruh temperature, we must conclude that the temperature of a self-gravitating causal diamond is negative.

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