Distributions of Textbook Problems Predict Student Learning: Data from Decimal Arithmetic

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Abstract

This study investigated relations between the distribution of practice problems in textbooks and students' learning of decimal arithmetic. In Study 1, we analyzed the distributions of decimal arithmetic practice problems that appeared in three leading math textbook series in the US. Similar imbalances in the relative frequencies of decimal arithmetic problems were present across the three series: Addition and subtraction more often involved two decimals than a whole number and a decimal, but the opposite was true for multiplication and division. We expected children's learning of decimal arithmetic to reflect these distributional biases. In Studies 2, 3, and 4, we tested the prediction that children would have more difficulty solving types of problems that appeared less frequently in textbooks, regardless of the intrinsic complexity of solving the problems. We analyzed students' performance on decimal arithmetic from an experiment conducted in a different lab 35 years ago (Study 2), from a contemporary large-scale web-based learning platform (Study 3), and from a recent controlled experiment conducted in our own lab (Study 4). Despite many differences among the three studies, performance in all three was in accord with the predictions. These findings suggest that the distributions of practice problems in math textbooks may influence what children do and do not learn. Usefulness of analyzing textbook problem distributions, as well as educational implications of the current findings, are discussed.

Keywords: textbook analysis, decimal arithmetic, rational numbers, math learning, and practice problems.

Educational Impact and Implications Statement

Analysis of three leading math textbook series in the US revealed large imbalances in distributions of decimal arithmetic problems: Addition and subtraction problems more often had two decimal operands than a whole number operand and a decimal operand, but the reverse was true for multiplication and division. Children, tested over a wide range of time periods and in a wide range of contexts, consistently showed lower accuracy on problems that appeared less frequently in the textbooks. This finding suggests that distributions of practice problems in textbooks may influence children's performance and that more balanced distributions may lead to better learning. The results have the potential to improve math education, because in contrast to most factors influencing children's learning, changes in distributions of practice problems can be implemented relatively easily by textbook companies, teachers, and parents.

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Solving practice problems is a major part of learning mathematics. However, relatively little is known about what effects, if any, the specific characteristics of practice problems have on specific aspects of mathematics learning. The present research analyzes the distributions of decimal arithmetic practice problems that appear in three math textbook series and tests the hypothesis that the distributions influence children's learning.

The practice problems that we examined were those presented in mathematics textbooks. This focus does not reflect a belief that these are the only practice problems that children receive: They clearly are not. Instead, the focus reflects a belief that textbooks are one major source of practice problems in children's mathematics learning (Horsley & Sikorová, 2014) – a source that is publicly accessible and verifiable – and also a belief that other sources of practice problems, such as workbooks and websites, are likely to reflect biases similar to those that shape textbook problems (Siegler et al., under review; Tian et al., in prep).

An important potential advantage of focusing on the impact of practice problems on learning is that suboptimal distributions of practice problems could be changed relatively easily. This contrasts with many less tractable sources of difficulty in math learning, including socioeconomic disadvantages, societal values and beliefs, inadequate teacher knowledge, and weak student motivation. If the specific types of problems that children encounter create unnecessary difficulties for learning, changes could be made at scale, relatively quickly, to mitigate those difficulties.

The present study focused on relations between textbook problems and children's learning of rational number arithmetic, specifically decimal arithmetic. This focus was justified by the importance and difficulty of this domain. In a national survey of more than 1,000 US Algebra I teachers, failure to understand "rational numbers and operations involving fractions and decimals" was rated the second largest source of difficulty in their students' preparation for Algebra I, trailing only the amorphous category "word problems" (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Data on children's acquisition of mathematics knowledge also attests to the importance of understanding rational numbers: students' rational number arithmetic competence in Grade 5 uniquely predicted their mathematics achievement in Grade 10 in both the US and the UK, even after controlling for IQ, working memory, parental education, and family income (Siegler et al., 2012; see also Booth & Newton, 2012; Hurst & Cordes, 2018; Powell, Gilbert, & Fuchs, 2019; see Bush & Karp, 2013 for a review). Given the importance of this area, it is especially unfortunate that many students have little understanding of rational number arithmetic (Lortie-Forgues, Tian, & Siegler, 2015).

Below, we first describe previous findings on the effects of textbook problems on children's learning of mathematics. We then briefly review previous research on children's decimal arithmetic and discuss how textbook problem distributions might affect children's decimal arithmetic learning.

Effects of Textbook Problems on Mathematics Learning

Although diverse teaching resources are available online and from other sources, textbooks continue to be widely used by math teachers in classrooms (Elsaleh, 2010). On the 2011 Trends in International Mathematics and Science Study (TIMSS), more than 75% of the students in Grades 4 and 8 reported that their math teachers used textbooks as the “basis for instruction” (Horsley & Sikorová, 2014). Textbooks also often serve as a key instrument for translating educational reform ideas into classroom instruction. The first step that many school districts take in response to a reform is to adopt a new textbook series that at least claims to align with the proposed reform (Remillard, 2000).

Influences of math textbooks on children’s concepts of mathematical equality. Despite the widespread use of math textbooks, surprisingly, little is known about how distributions of textbook problems are related to children’s learning. However, one area in which the impact of textbook problem distributions has been examined involves understanding of mathematical equivalence. Children in elementary and middle school often interpret the equal sign as a signal to perform mathematical operations instead of as a relational symbol that represents mathematical equivalence (Baroody & Ginsburg, 1983; McNeil & Alibali, 2005; Renwick, 1932). Illustrating children’s interpretation of the equal sign as a “go signal,” most sixth graders in Falkner, Levi, and Carpenter (1999) answered “8+4=_+5” by writing “12” or “17,” answers that would emerge if children added the numbers to the left of the equal sign or added all numbers in the expression.

Analyses of math textbooks used in elementary (Powell, 2012) and middle school (McNeil et al., 2006) suggested that the distribution of problems involving the equal sign contributes to children's misconceptions about the equal sign. Problems in US math textbooks, especially those used in elementary school, usually present the arithmetic operation to the left of the equal sign and the unknown to the right of it (e.g., 2 + 5 = __). They less often present problems in non-standard forms, including those where operations appear on both sides of the equal sign (e.g., 2 + 5 = __ + 4) or only to the right of it (e.g., __ = 2 + 5) or problems where no operation appears on either side of the equal sign (e.g., 7 = __; McNeil et al., 2006; Powell, 2012). Virtually invariant presentation of the standard problem format in textbooks may encourage children to interpret the equal sign as a "go signal."

Consistent with this hypothesis, McNeil (2008) found that receiving math lessons in which problems were presented in non-standard formats immediately before test problems in the standard format weakened the bias toward interpreting the equal sign as a "go signal." In contrast, receiving lessons with problems in the standard format immediately before test problems in the standard format strengthened the bias. The improvement in understanding of mathematical equivalence among children who encountered problems in non-standard formats persisted 5 to 6 months later (McNeil, Fyfe, & Dunwiddie, 2015). Thus, infrequent presentation of certain types of problem may create or reinforce misconceptions, and more frequent presentation of rare problem types can help correct such misconceptions.

Influences of math textbooks on children's fraction arithmetic. The influence of textbook problem distributions has also been examined with fraction arithmetic. Braithwaite, Pyke, and Siegler (2017) found extreme imbalances in the distributions of fraction arithmetic problems in the fourth, fifth, and sixth grade math textbooks of three leading series. For example, multiplication and division problems almost never involved fractions with equal denominators (equal denominator multiplication and division problems each constituted only about 1% of items in each of the textbook series).

Braithwaite et al. (2017) also found that these imbalanced problem distributions were highly predictive of middle school children's fraction arithmetic performance. For example, despite correct multiplication and division procedures being identical for problems with equal and unequal denominators, two previous studies indicated that children were considerably less accurate on the infrequently presented multiplication and division problems with equal denominators than on the frequently presented multiplication and division problems with unequal denominators (Siegler & Pyke, 2013; Siegler et al., 2011). In Siegler and Pyke (2013), children were correct on 58% of multiplication items with unequal denominators (e.g., 3/5 × 1/4) but on only 36% of multiplication items with equal denominators (e.g., 3/5 × 1/5).

Braithwaite et al.'s (2017) results went beyond previous findings (e.g., McNeil et al., 2006; Powell, 2012) by showing that not only individual features of practice problems, but also interactions between problem features (in this case, interactions between arithmetic operations and characteristics of operands), predict students' problem-solving performance. However, this finding might be due to some idiosyncrasy of fraction arithmetic rather than reflecting a general property of math learning. Thus, a major purpose of the present study was to test whether interactions between problem features would predict students' performance in a different domain: decimal arithmetic.

Decimal Arithmetic

Despite its importance, decimal arithmetic has received much less attention from researchers than fraction arithmetic. Exploring relations between textbook problem distributions and decimal arithmetic performance allowed us to test the generality of Braithwaite et al.'s (2017) conclusions, which were based on fraction arithmetic findings, and also to enrich the currently-sparse literature on decimal arithmetic.

Many children experience difficulty with decimal arithmetic (Hiebert & Wearne, 1985, 1986). For example, on a National Assessment of Educational Progress (NAEP), less than half of 13-year-olds correctly solved the problem 4.5 - .53 (Kouba et al., 1988). In a more recent study, fourth- and fifth-grade children correctly answered only 77% of decimal addition problems involving decimals with one or two digits to the right of the decimal point (Hurst & Cordes, 2018).

Children's inaccurate decimal arithmetic performance may relate to their limited conceptual understanding of decimal place values and decimal arithmetic (Durkin & Rittle-Johnson, 2015; Hiebert & Wearne, 1985, 1986; Lortie-Forgues et al., 2015). In one study, sixth graders correctly answered only 39% of items assessing understanding of decimal place value, such as "3 tenths are worth how many hundredths?" (Rittle-Johnson & Koedinger, 2009). This poor conceptual understanding could open the door to arithmetic errors, such as adding digits with different place values, as when claiming that 6 + .32 = .38. Lortie-Forgues and Siegler (2017) reported that middle school students and even pre-service teachers usually judged that multiplying a number by a decimal would produce an answer larger than that number, even when both multiplicands were between

0 and 1. This misconception may prevent students from recognizing the implausibility of decimal multiplication and division answers, such as 5.5 * 0.6 = 33.

Possible Effects of Textbook Problem Distributions on Decimal Arithmetic Learning

Although decimal arithmetic is challenging in general for many children, our expectation was that children would encounter particular difficulty with types of problems that are rarely presented in textbooks, even if the problems could be solved easily via simple procedures. This expectation derives from previous findings that most US students have little conceptual understanding of decimal place value (Durkin & Rittle-Johnson, 2015; Rittle-Johnson & Koedinger, 2009) and decimal arithmetic (Lortie-Forgues & Siegler, 2017). As proposed by Braithwaite, et al. (2017), absence of conceptual knowledge opens the door to irrelevant associations in practice problems strongly influencing performance.

To investigate relations between textbook problem distributions and decimal arithmetic performance, we first conducted textbook analyses to generate predictions about performance; then, we tested the predictions empirically. In the textbook analysis, we examined the distributions of decimal arithmetic problems in three widely-used contemporary US textbook series to identify possible biases in the distributions. We also examined distributions in a textbook from more than 30 years ago, to determine the stability of the biases over time. In the empirical tests, we analyzed three datasets on children's decimal arithmetic to determine whether children's decimal arithmetic performance corresponds in detailed ways to the textbook problem distributions. The study procedures of Studies 1, 2, and 3 were approved under Teachers College, Columbia University, IRB protocol 19-150, "Improving Fraction Understanding - Existing Data".

Study 1: Distribution of Decimal Arithmetic Problems in US Textbooks

To obtain a comprehensive and representative database of decimal arithmetic problems in US textbooks, we coded decimal arithmetic problems from the current editions (as of 2016, when we began to create the textbook database) of three US math textbook series (the same ones examined in Braithwaite et al., 2017): Houghton Mifflin Harcourt's *GO Math!* (Dixon, Adams, Larson, & Leiva, 2012), Pearson's *enVision MATH* (Charles et al., 2012), and McGraw-Hill's *Everyday Mathematics* (University of Chicago School Mathematics Project, 2015a, 2015b). These series were among the most widely-used in the US (Opfer, Kaufman, Pane, & Thompson, 2018) and covered the entire period from Grade 4 (when introduction of decimals is recommended by the Common Core State Standard Initiatives, CCSSI; National Governors Association Center for Best Practices, 2015) to Grade 6 (last mention of understanding decimals as an explicit goal in CCSSI; National Governors Association Center for Best Practices, 2015). The database consisted of all decimal arithmetic problems that met three criteria: 1) the problem had two operands, 2) at least one operand was a decimal, and 3) the problem required children to generate an exact numeric answer (rather than being a worked example or requiring an estimate). The database did not include word problems, due to the complexity of categorizing such problems (Geary, 2004). A total 1441 problems were entered: 715 from *GO Math!*, 551 from *enVision MATH*, and 175 from *Everyday Mathematics*[1]. See the Supplementary Materials, Section A for more details about the problems.

[1] One reason that *Everyday Mathematics* has so few problems compared to the other two series is that it frequently uses games for practicing problem solving (University of Chicago School Mathematics Project, n.d.). Because numbers in most games of *Everyday Mathematics* are generated randomly each time a game is played, the specific problems in these games could not be analyzed.

Coding and Analyses

Problems were categorized by arithmetic operation and characteristics of operands. For each of the four arithmetic operations (addition, subtraction, multiplication, and division), we distinguished among three types of problems:

1) *Whole-decimal (WD) problems*: items with a whole number operand and a decimal operand, which inherently had unequal numbers of decimal digits (e.g., 5 × 1.2);
2) *Decimal-decimal equal decimal digits (DDE) problems*: items with two decimal operands that had an equal number of decimal digits (e.g., 1.23 + 4.56);
3) *Decimal-decimal unequal decimal digits (DDU) problems*: items with two decimal operands that had unequal numbers of decimal digits (e.g., 4.5 - 1.23).

Here and throughout this article, we use "decimal" to refer to numbers that include a decimal point (e.g., 1.52 or 0.6). We use the term "decimal digits" to refer to the digits to the right of the decimal point (e.g. 1.52 has two decimal digits; 1.527 has three). The features that we coded (whether operands in each problem were two decimals or a whole number and a decimal; whether number of decimal digits were equal) were chosen because they were analogous to features that had previously been found to influence the relatively difficulty of fraction arithmetic problems (Braithwaite et al., 2017) and were not confounded with whole number arithmetic complexity (e.g., the total number of digits in the problem).

Tables 1-3 show the frequencies of WD, DDE, and DDU problems for each of the four arithmetic operations in each textbook series. Because patterns of frequencies of

WD, DDE, and DDU problems in the textbooks were highly similar for addition and subtraction, we combined frequency of addition and subtraction items into a single operation group (addition/subtraction). For the same reason, we combined frequency of multiplication and division items into another operation group (multiplication/division). In the following analyses, we first examined the associations between operand type (WD vs. DD) and arithmetic operation group (addition/subtraction vs. multiplication/division). Within DD problems, we examined the associations between equality of decimal digits (DDE vs. DDU) and arithmetic operation group (addition/subtraction vs. multiplication/division).

==========================Tables 1 – 3 here ========================

Results

Operand type. Chi-square tests of independence revealed associations between operand type (WD vs. DD) and arithmetic operation group (addition/subtraction vs. multiplication/division) in all three textbook series: $\chi^2(1) = 208.21$, $p < .001$, $\Phi_{\text{cramer}} = .54$ for *GO Math!*; $\chi^2(1) = 137.38$, $p < .001$, $\Phi_{\text{cramer}} = .50$ for *enVision MATH*; and $\chi^2(1) = 29.35$, $p < .001$, $\Phi_{\text{cramer}} = .42$ for *Everyday Mathematics*. As shown in Tables 1-3, addition and subtraction problems far more frequently involved two decimal operands (DD) than a whole number and a decimal operand (95% DD operands, 5% WD operands), but multiplication and division problems showed the opposite pattern (39% DD operands, 61% WD operands).

Equality of decimal digits. Considering only DD problems, associations again were present in all textbooks between arithmetic operation group (addition/subtraction vs. multiplication/division) and equality of number of decimal digits in the operands (DDE vs. DDU): $\chi^2(1) = 25.15$, $p < .001$, $\Phi_{\text{cramer}} = .26$ for *GO Math!*; $\chi^2(1) = 5.18$, $p = .02$, $\Phi_{\text{cramer}} = .14$ for *enVision MATH*; and $\chi^2(1) = 5.78$, $p = .02$, $\Phi_{\text{cramer}} = .26$ for *Everyday Mathematics*. As shown in Tables 1-3, addition and subtraction problems more often involved DDE than DDU operands (71% vs. 29%), whereas multiplication and division problems equally often involved DDE and DDU operands (51% vs. 49%).

Discussion

In all three contemporary math textbook series that we examined, strong associations were present between operations and operands. Addition and subtraction more often involved two decimals than a whole number and a decimal, but the reverse held for multiplication and division. Among problems with two decimal operands, addition and subtraction more often involved operands with equal than unequal numbers of decimal digits, whereas multiplication and division equally often involved operands with equal and unequal numbers of decimal digits. Of special interest was the extremely low frequency of addition and subtraction WD problems, such as "4 + .56." Such problems were only 5% of textbook addition and subtraction problems involving at least one decimal operand.

There was no apparent mathematical basis for these relations between operations and operands. However, the low frequency of certain types of problems might have reflected textbook authors believing that children do not need much experience to succeed on the rarely presented problem types. For example, the infrequency of WD addition problems might have been due to textbook authors assuming that children can accurately answer such problems with minimal practice. After all, solving "4 + .56" only requires concatenating the two numbers, a procedure that might be learned with little practice. Alternatively, textbook authors may assume that children know that arithmetic procedures are independent of operands and therefore assume that practicing adding DD problems leads to children learning to solve both DD and WD addition problems.

However, because children's conceptual understanding of decimal arithmetic is very limited, we expected their performance to be influenced by these distributional biases in the practice problems. We expected that children would have difficulty solving types of problems that appear rarely in textbooks—including seemingly easy problems such as "4 + .56." Conversely, we expected that more frequent presentation of problems would lead to better performance, as children would have more opportunities to practice and receive feedback on them. Based on this logic and the findings of Study 1, we made three predictions regarding children's performance:

Prediction 1. The effect of operand type (WD vs. DD) on children's accuracy should interact with arithmetic operation group (addition/subtraction vs. multiplication/division).

Prediction 2. For addition and subtraction, children's accuracy should be higher on DD than WD problems.

Prediction 3. For multiplication and division, children's accuracy should be higher on WD than on DD problems.

We tested these predictions in Studies 2, 3, and 4. Study 2 examined data from an experiment from a different lab published 35 years ago (Hiebert & Wearne, 1985). Study 3 examined unpublished contemporary data from a large-scale web-based learning platform. Study 4 examined new data from a controlled experiment conducted in our lab. The goal was to examine the generality of the findings over labs (the Hiebert/Wearne lab versus our own), time of data collection (before 1985 versus 2013-2019), and data source (web-based platform versus controlled experiment).

Prediction 2 deserves special emphasis, because simply considering the complexity of the procedures required to solve the problems would lead to the opposite prediction, at least for addition. Adding a whole number and a decimal seems particularly easy, especially when the decimal does not have a whole number part (e.g., .7). Such problems have no computational demand, because they only require copying the whole number and concatenating the decimal operand to it (e.g., 5 + .7 = 5.7). In contrast, adding two decimals, especially when the decimals have unequal numbers of decimal digits, requires a more complex procedure involving aligning the addends' digits according to their place values, adding the decimal parts and the whole number parts of the operands, and sometimes carrying from the decimal parts to the whole number parts (e.g., 5.62 + .7 = (5) + (.62+.7) = 5 + 1.32 = (5+1) + 0.32 = 6.32)[2]. Nonetheless, the much lower frequency of WD than DDE or DDU addition problems in textbooks led us to predict that WD problems would elicit more errors despite their lesser complexity.

Study 1 also revealed that for addition and subtraction, DDE problems were more frequent than DDU problems, suggesting that children would solve the former more accurately than the latter. However, addition and subtraction are intrinsically easier with DDE than DDU operands, because aligning the rightmost decimal digits leads to correct answers on the former but not the latter problems. Therefore, testing such a prediction

[2] It might be argued that WD addition problems are intrinsically difficult because it is tempting to misalign the addends, as in 5 + .7 = 1.2, and avoiding such errors requires understanding of place value. However, the same is true of DDU addition problems, such as .5 + .73. As shown below, WD addition problems proved more difficult for children than both DDE and DDU addition problems, a finding that is difficult to explain in terms of intrinsic difficulty alone.

would not be very informative regarding how textbook problem distributions influence learning.

Study 2: Exploration with a Pre-Existing Dataset

The purpose of Study 2 was to examine relations between the textbook problem distributions and children's performance on decimal arithmetic problems. We wanted to test these relations first using an existing data set collected outside of our lab. To allow the required comparisons, such a data set needed to report separately WD, DDU, and DDE problems for each arithmetic operation.

The one data set that we found that met these requirements was reported in Hiebert and Wearne (1985). In that study, seventh graders[3] were presented decimal arithmetic problems that included WD, DDE, and DDU items for each operation. There were five addition problems (4.6 + 2.3, 5.3 + 2.42, 5.1 + .46, 6 + .32, 4 + .3), five subtraction problems (.78 - .35, .60 - .36, .86 - .3, 4.7 - .24, 7 - .4), six multiplication problems (.4 × .2, .34 × 2.1, .05 × .4, 6 × .4, 2 × 3.12, 8 × .06), and six division problems (.24 ÷ .03, .028 ÷ .4, 2.1 ÷ 3, .56 ÷ 7, 42 ÷ .6, and 3 ÷ .6). Children attempted to solve these problems with paper and pencil, once in their fall semester and once in their spring semester. In the fall, 279 students received all addition and subtraction problems, and 81 also received all multiplication and division problems. In the spring, 272 students received all addition and subtraction items, and 80 also received all multiplication and division problems.

[3] Hiebert and Wearne (1985) reported data on fifth, sixth, seventh, and ninth graders. Our analyses focused on the seventh graders because (1) the fifth and sixth graders did not receive problems involving all four arithmetic operations, and (2) the ninth graders would have completed formal instruction in decimal arithmetic more than two years before the study was conducted and would have received many unknown kinds of decimal arithmetic problems in math and science classes in the two years beyond those covered in our textbook analyses.

Predictions regarding children's performance were based on the assumption that the distributions of problems that children encountered 35 years ago approximated those in contemporary textbooks. To test this assumption, we coded and analyzed decimal arithmetic problems from the one volume of a math textbook series from the 1980s that we were able to find, *Addison-Wesley Mathematics* (1985)[4]. This textbook series was described by Stigler, Ham, Kim, and Fuson (1986) as "widely used." The volume we coded included 647 decimal arithmetic problems; the distribution of these problems is shown in Table 4.

Comparison of Tables 1-3 and Table 4 indicates that patterns of problem frequencies in the 1985 textbook closely resembled those in the contemporary textbooks. Particularly relevant to our predictions, addition and subtraction problems in the 1985 textbook never involved a whole number and a decimal operand (0% WD operands vs. 100% DD operands), but multiplication and division problems more often involved a whole number and a decimal operand than two decimal operands (58% WD operands vs. 42% DD operands), $\chi^2(1) = 150.97$, $p < .001$, $\Phi_{\text{cramer}} = .49$. This finding, in addition to demonstrating the stability over time of textbook problem distributions, strengthened our expectation that students' performance in Hiebert and Wearne (1985) would conform to our three predictions.

==========================Table 4 here ==========================

Hiebert and Wearne (1985) reported the overall percent correct and the most frequent error for each problem at each grade at each testing occasion. Data for individual

[4] Due to the difficulty of accessing math textbooks from more than 35 years ago, we were only able to locate one volume of a math textbook published about when the Hiebert and Wearne (1985) study was done. Fortunately, topics in this volume included all four decimal arithmetic operations.

children were not available, nor were data separately reported for the subset of students who received problems for all four operations. This precluded inferential statistics, including tests of the interaction posited in Prediction 1. Nonetheless, the data proved revealing.

Results

The accuracy data from Hiebert and Wearne (1985) are presented in Table 5, organized in terms of arithmetic operation; time of testing; whether the problem included a whole number; and if not, whether the two operands had equal numbers of decimal digits.

============================Table 5 here ========================

Prediction 2: Higher accuracy on DD than WD addition and subtraction problems. Consistent with Prediction 2, for addition and subtraction, students were less accurate on WD problems than on DD problems both in the fall (37% vs. 70% correct) and in the spring (38% vs. 73% correct). The poor performance on WD addition and subtraction problems, and the differences in accuracy between them and DD addition and subtraction problems, was highly consistent across items. Percent correct on individual WD addition and subtraction problems at the two times of testing ranged from 31% to 42%, whereas percent correct on individual DD addition and subtraction problems (including both DDE and DDU) ranged from 65% to 90%.

Prediction 3: Higher accuracy on WD than DD multiplication and division problems. Consistent with Prediction 3, for multiplication and division, students were more accurate on WD than DD problems in both the fall (44% vs. 26% correct) and the spring (72% vs. 64% correct). The superior performance on WD problems was highly consistent among multiplication items at each testing time. Percent correct on each WD multiplication problem ranged from 54% to 79% in the fall and 84% to 86% in the spring, whereas percent correct on each DD multiplication problem ranged from 17% to 41% in the fall and 60% to 79% in the spring. The trend was less consistent across division problems. Percent correct on individual WD division problems ranged from 2% to 68% in the fall and 44% to 91% in the spring, whereas percent correct on individual DD division problems ranged from 20% to 25% in the fall and 56% to 61% in the spring. The reasons for the high variability among WD division problems are unknown.

Discussion

As expected, textbook problem frequency was related to middle school students' decimal arithmetic performance. Accuracy on addition/subtraction problems was much higher on DD than WD items; on multiplication/division items, the opposite was true.

The findings were especially striking for addition of a whole and a decimal. For the seemingly simple problem "6 + .32," the 7th graders' percent correct was 42% in the spring and 39% in the fall; for the seemingly simple problem "4 + .3," their percent correct was 39% in the fall and 37% in the spring. Moreover, the most common errors children made on these problems were likely produced by overgeneralizing strategies from frequently presented problems. For example, the most common incorrect response to "6 + .32" was 0.38; this response accounted for 45% and 50% of the errors children made on this problem in the Fall and Spring, respectively. Similarly, the most common error for "4 + .3" was 0.7, which comprised 51% and 48% of all the errors in the Fall and Spring, respectively (see Hiebert & Wearne, 1985 for common errors on the other problems). These errors on WD addition problems were likely generated by aligning the rightmost digits of the operands, a strategy that would yield a correct answer on the frequently presented DDE addition problems (as well as for whole number addition). Given that WD addition problems make minimal computational demands, the inaccurate performance seems likely to reflect their infrequent presentation (combined with poor conceptual understanding of place value).

Study 3: Validation with a Large-Scale, Online, Contemporary Dataset

The results of Study 2 revealed that children's performance on decimal arithmetic problems 35 years ago was related to distributions of problems in math textbooks. However, due to the limitations of the available data from Study 2, we were unable to conduct inferential statistical analyses to confirm these relations or to examine the interaction implied by Prediction 1. Study 3 was an effort to address these issues and test the generality of the relations with performance data from a large number of contemporary students.

The data used in this study were obtained from the Skill Builder module on ASSISTments, a web-based platform that allows teachers to assign practice problems for several school subjects, including math (Heffernan & Heffernan, 2014). Skill Builder hosts a large bank of practice problems (for example, the *adding decimals* problem set included 77 problems) developed by researchers and teachers for topics aligned with the CCSSI (National Governors Association Center for Best Practices, 2015). The large number and variety of problems and the large number of active users of ASSISTments (over 50,000 students across 42 states in 2018, according to assistments.org) allowed us to test each of the three predictions with inferential statistical analyses. The ASSISTments data also offered a unique opportunity to test whether the Study 2 findings were due to idiosyncrasies of the problems, students, or time period in which the data were collected.

Dataset Description

We requested all available data (as of February, 2019) on students' decimal arithmetic performance in the ASSISTments Skill Builder module[5] from the ASSISTments team. We included in our analyses data from all trials that met the following criteria: 1) the problem fit the inclusion criteria of Study 1, 2) each data point was a student's first attempt to answer a problem on which the student had not asked for a hint[6], and 3) the data were from a student who completed addition/subtraction problems, multiplication/division problems, and at least one WD and one DD problem for each of the two arithmetic operation groups. The data analyzed were collected from 2013-2019 and included 3359 students' performance on 637 decimal arithmetic problems.

Participants

The 3359 students included in this study were in Grade 6 when tested. Among them, 98% (3303 students) indicated the school district they attended; they were from 73 different school districts. For 92% of these students (3081), data were available on the proportion of children eligible for free or reduced-price lunch (FRPL) in the school district. Among them, 54% attended school districts in which less than 25% of students

[5] Only data on accuracy were available; data on children's specific wrong answers were not.

[6] ASSISTments allows students to ask for a hint on how to solve a given problem without attempting to answer; it also allows students to make multiple attempts on a given problem. Students whose data were analyzed in this experiment asked for a hint on their first attempt on less than 2% of the problems. Analyses that included these attempts as incorrect responses yielded results similar to those reported.

were eligible for FRPL, 42% attended districts where between 25% and 75% were eligible, and 4% attended districts where more than 75% were eligible.

Results

On average, students answered 20 decimal arithmetic problems (*SD* = 14) that met the study's criteria. Because students worked on different problems, we predicted the accuracy of answers by fitting mixed-effects logistic regression models with R (R Core Team, 2018) and *lme4* (Bates, Mächler, Bolker, & Walker, 2014). *P*-values were obtained by likelihood ratio tests comparing the full model, including the effect in question, to the model without the effect in question. In each model, random intercepts for participant and problem were added. Because there is no agreed-on approach for calculating standard effect sizes for individual model terms in such models (see Rights & Sterba, 2019), none were reported. Analyses that examined each arithmetic operation separately yielded similar results to those reported (see Supplementary Materials, Section B). Table 6 indicates accuracy on each type of problem.

===========================Table 6 here ==========================

Prediction 1: An interaction between operand type and arithmetic operation group. Consistent with Prediction 1, operand type (WD vs. DD) interacted with arithmetic operation group (addition/subtraction vs. multiplication/division), χ^2 (1) = 98.16, $p < .001$, indicating that the effect of operand type on the odds of generating correct answers differed by arithmetic operation group.

Prediction 2: Higher accuracy on DD than WD addition and subtraction problems. Consistent with Prediction 2, the odds of students correctly answering addition/subtraction problems were higher on DD than WD problems (M = 84%, SD = 36%, versus M = 76%, SD = 43%), $\chi^2(1) = 57.43$, $p < .001$.

To test whether this effect was driven by DDE addition/subtraction problems being easier than WD problems, we ran two separate models, one comparing accuracy on WD and DDE problems, the other comparing accuracy on WD and DDU problems. The odds of giving correct answers on addition/subtraction problems were higher not only for DDE than WD problems, (M = 86%, SD = 35%, versus M = 76%, SD = 43%), $\chi^2(1) = 46.56$, $p < .001$, but also for DDU than WD problems (M = 83%, SD = 38% versus M = 76%, SD = 43%), $\chi^2(1) = 32.95$, $p < .001$.

To ensure that these results were not solely due to differences in performance on subtraction problems, we also fitted separate models for only addition problems. Results were similar to those when addition and subtraction problems were combined. The odds of giving correct answers were higher on DDE than WD addition problems (M = 85%, SD = 36%, versus M = 80%, SD = 40%), $\chi^2(1) = 3.92$, $p < .05$, and also on DDU than WD addition problems (M = 85%, SD = 35% versus M = 80%, SD = 40%), $\chi^2(1) = 9.91$, $p < .01$.

Prediction 3: Higher accuracy on WD than DD multiplication and division problems. Consistent with Prediction 3, the odds of giving correct answers for multiplication and division items were higher on WD than on DD problems (M = 86%, SD = 34%, versus M = 79%, SD = 41%), $\chi^2(1) = 44.31$, $p < .001$. To better understand this effect, we ran separate models on multiplication/division problems, one comparing accuracy on WD and DDE problems, the other comparing accuracy on WD and DDU problems. The odds of correct answers were higher on WD than DDE multiplication/division problems (*M* = 86%, *SD* = 34%, versus *M* = 78%, *SD* = 41%), $\chi^2(1) = 25.62$, $p < .001$, and also on WD than DDU problems (*M* = 86%, *SD* = 34%, versus *M* = 80%, *SD* = 40%), $\chi^2(1) = 40.56$, $p < .001$. Thus, the effect was general across the DD problems.

Discussion

As in Study 2, decimal arithmetic performance of middle school students mirrored biases in textbook problem distributions. The presence of a whole number operand influenced children's accuracy in opposite directions on addition/subtraction and multiplication/division problems. Students were more likely to answer correctly decimal addition and subtraction problems without a whole number operand than with one. In contrast, students were more likely to answer correctly decimal multiplication and division problems with a whole number operand than without one. Especially noteworthy, despite the seeming simplicity of adding a whole number and a decimal, children's accuracy on such problems was lower than when adding two decimals with equal or unequal number of decimal digits.

Compared to children in Study 2, children in Study 3 showed a smaller difference between accuracies on the frequently presented and the rarely presented problems. Their accuracy across all problems also was higher.

Nevertheless, the two studies demonstrated considerable stability of findings on relations between distributions of textbook problems and children's accuracy on different types of problems. In both studies, types of problems that were rarely presented in textbooks were more difficult than would have been expected from the complexity of solution procedures for those types of problems. The data in the two studies were collected decades apart, a period during which many changes in approaches to mathematics education and in student populations occurred. The studies also differed in how the data were collected. Students in Study 2 solved problems in a single session, in groups, with paper and pencil; students in Study 3 solved problems over multiple sessions, individually, on a web-based platform. The similar pattern of findings despite these differences, along with the sheer number of students from diverse backgrounds and geographical locations in Study 3, suggest that the predictions generalize to a broad range of US students.

Study 4: Validation with a Controlled Experiment

Studies 2 and 3 revealed stability over a 35-year period of the finding that students' decimal arithmetic performance paralleled the distributions of textbook problems. However, in neither study did all students receive a complete and balanced set of decimal arithmetic problems, including all four arithmetic operations and WD, DDE, and DDU problems within each operation. Study 4 was designed to address this limitation with a carefully controlled experiment that tested more precisely whether parallels were present between students' performance and textbook problem distributions. We asked sixth graders to complete sets of decimal arithmetic problems that included all types of problems needed to test predictions based on the textbook data described in Study 1. Sixth graders were selected because decimal arithmetic is recommended to be taught in fifth grade in CCSSI (National Governors Association Center for Best Practices, 2015), so that sixth graders would recently have received instruction in the subject. The same three predictions as in Study 3 were tested.

Method

Participants. Sixty-six sixth graders from a public middle school in Pittsburgh, PA participated in the study in the Fall semester of 2016. We aimed to have approximately 60 students, which was comparable to the sample size of sixth graders in previous studies using similar tasks that had sufficient power to reveal a number of effects of interest (e.g., Siegler & Pyke, 2013). The study was conducted in a whole-class format, and we included all students (i.e., 66) in the participating classes.

The participating school enrolled students in Grades 6 through 8. Of students at this school, 34% were eligible for FRPL. About 50% of the sixth graders in the participating school achieved “Advanced/Proficient” level on the math section of Pennsylvania System of School Assessments (the standardized test used in Pennsylvania) administered in the Spring of 2017, which was somewhat, but not greatly, higher than the state average of 40.3%. The study procedures were approved under Carnegie Mellon University, IRB protocol 2016_00000420, “Improving Understanding of Fractions”.

Task. Each child completed 12 decimal arithmetic problems from one of two problem sets. Each set included three problems for each arithmetic operation: one problem with a whole number and a decimal (WD), one problem with two decimals having an equal number of decimal digits (DDE), and one problem with two decimals with unequal numbers of decimal digits (DDU). To avoid differences among operations due to idiosyncratic differences among the operands, the same three operand pairs were used for all four arithmetic operations (Set A: 3 and 1.5, 3.6 and 1.5, and 4.74 and 1.5; Set B: 4.5 and 2, 4.5 and 1.8, and 4.5 and 1.25). The larger number in each operand pair was always the first operand.

Each problem set was presented equally often in a fixed quasi-random order or the reverse of that order. The quasi-random order was generated with the following constraints: 1) each operation appeared once in each block of four problems, 2) neither an operation nor an operand pair was repeated on successive trials, 3) each operand pair appeared equally often in the first and second halves of the trials, and 4) the first operation was either subtraction or multiplication.

Procedure. Children from each classroom participated in the study as a whole group during a regular class period. All children in each class were presented a printed booklet by a trained experimenter and asked to write answers in it at their own pace (see Supplementary Materials, Section C for an example booklet). Calculator use was not allowed.

Results

Table 7 indicates accuracy on each type of problem. Examining each arithmetic operation separately yielded similar results to those produced by pairing addition/subtraction and multiplication/division (see Supplementary Materials, Section D).

==========================Table 7 here ==========================

Prediction 1: An interaction between operand type and arithmetic operation group. A 2 (arithmetic operation group) * 2 (operand type) ANOVA on percent correct yielded the predicted interaction, $F(1, 65) = 40.92$, $p < .001$, $\eta_p^2 = 0.39$.

Prediction 2: Higher accuracy with DD than WD operands on addition and subtraction problems. Consistent with Prediction 2, for addition/subtraction, children were more accurate on DD (*M* = 80%, *SD* = 25%) than on WD problems (*M* = 68%, *SD* =

44%), *F*(1, 65)= 6.28, *p* = .01, η_p^2 = 0.09. Dividing the DD category into DDE and DDU items, paired *t*-tests revealed that children were more accurate on DDE than on WD addition/subtraction problems (*M* = 88%, *SD* = 25%, versus *M* = 68%, *SD* = 44%), *t*(65) = 3.90, *p* < .001, *d* = 0.55; but no more accurate on WD addition/subtraction problems than on the seemingly harder DDU addition/subtraction problems (*M* = 68%, *SD* = 44% versus *M* = 73%, *SD* = 34%), *t*(65) = 0.85, *p* = .40, *d* = 0.11.

As in Study 3, we performed similar paired *t*-tests for accuracy on addition problems alone. The results were very similar to those on addition and subtraction problems combined. Children were more accurate on DDE than on WD addition problems (*M* = 88%, *SD* = 33% versus *M* = 68%, *SD* = 47%), *t*(65) = 3.40, *p* = .001, *d* = 0.49, and their accuracy did not differ on DDU and WD addition problems (*M* = 68%, *SD* = 47% versus *M* = 68%, *SD* = 47%), *t*(65) = 0, *p* = 1, *d* = 0.

Prediction 3: Higher accuracy with WD than DD operands on multiplication and division problems. A one-way ANOVA with operand type as a within-subject variable was conducted on multiplication/division accuracy. Consistent with Prediction 3, for multiplication and division, children were more accurate on WD than DD problems (*M* = 57%, *SD* = 35% versus *M* = 28%, *SD* = 31%), *F*(1, 65) = 42.69, *p* < .001, η_p^2 = 0.40.

Also as predicted, paired *t*-tests revealed that for multiplication/division, children were more accurate on WD than on either DDU problems (*M* = 57%, *SD* = 35% versus *M* = 20%, *SD* = 32%), *t*(65) = 7.09, *p* < .001, *d* = 1.09, or DDE problems (*M* = 57%, *SD* = 35% versus *M* = 36%, *SD* = 39%), *t*(65) = 4.47, *p* < .001, *d* = 0.55.

Discussion

In a carefully controlled experiment, the hypothesized relations between textbook problem frequency and children's performance emerged again. The presence of a whole number operand affected children's accuracy differently on addition/subtraction and multiplication/division problems. On addition and subtraction items, accuracy with WD operands was lower than with DD operands, but on multiplication and division items, accuracy was higher on WD than on DD items.

As in Study 3, children's accuracy on the seemingly easy WD addition problems was lower than accuracy on the DDE addition problems. Accuracy on WD addition problems also did not differ from that on the seemingly more difficult DDU addition problems. Despite the seeming simplicity of WD addition problems, the sixth graders in Study 4 erred on more than 30% of trials with them.

As in Study 2, children's most common errors on the WD addition problems likely arose from overgeneralizing the strategy of aligning the rightmost digit, a correct strategy on DDE and whole number addition problems. On "3 + 1.5", the answer 1.8 accounted for 64% of errors; on "4.5 + 2", the answer 4.7 accounted for 54% of errors (see Supplementary Materials, Section E for common errors on the other problems). Again, students' minimal prior exposure to WD addition problems seems likely to have contributed to their relatively inaccurate performance on such problems.

General Discussion

In this concluding section, we first summarize our findings on relations between textbook problem distributions and children's decimal arithmetic performance. Then, we discuss how analyzing textbook problems can improve understanding of children's learning more generally. Following that, we consider how textbook biases might combine with limited conceptual knowledge to weaken children's learning of decimal arithmetic. Finally, we note educational implications and limitations of the current findings.

Textbook Problem Distributions Predict Children's Decimal Arithmetic

Our analysis of three popular contemporary US math textbook series and one older series revealed similar imbalances in the frequencies of decimal arithmetic problems across the four series. Problems involving addition and subtraction of a whole number and a decimal were rare; they made up less than 2% of all arithmetic problems involving any decimal and less than 5% of problems involving a whole number and a decimal. In contrast, multiplication and division problems with a whole and a decimal were common, more common than multiplication and division with two decimals.

Findings from Studies 2, 3, and 4 were consistent with the hypothesis that biases in textbook problem distributions influence children's decimal arithmetic. The infrequent presentation of problems was associated with relatively inaccurate decimal arithmetic performance among children. In particular, presence of a whole number was associated with lower accuracy on addition and subtraction problems, where WD problems were rare, but was associated with higher accuracy on multiplication and division problems, where WD problems were frequent.

The pattern was general over the three studies: Study 2 (conducted before 1985, with children solving problems with paper and pencil in a single session), Study 3 (conducted from 2013 to 2019, with children solving problems on a web-based platform in multiple sessions), and Study 4 (conducted in 2016, with children solving problems with paper and pencil in a single session). Even though the effects were very large in some studies and smaller in others, the consistency of the effects across the three studies showed that the relations were not idiosyncratic to a specific time period, testing format, or sample.

Some associations between children's performance and textbook problem distributions could have arisen from children's performance and textbook problem distributions both reflecting the inherent difficulty of the decimal arithmetic problems. For example, WD multiplication problems (e.g., 3×1.24) seem inherently easier than DDU multiplication problems (e.g., 3.1×1.24), because WD multiplication problems can be solved correctly by placing the decimal point in the answer according to the number of decimal digits in the decimal operand, whereas DDU problems cannot be solved in this way. From this perspective, multiplication should be more accurate on WD than DDU problems. Data in Studies 2, 3, and 4 were consistent with this prediction.

Although such differences in the inherent difficulty of decimal arithmetic problems probably contributed to the observed accuracy patterns, they could not account for the overall pattern of findings in any of the three studies of children's performance. In all three, children consistently exhibited poor performance on addition problems with a whole number and a decimal, even though such problems make minimal computational demands. For example, solving $4 + .3$, and $6 + .32$, two WD problems presented in Study 2, only requires concatenating the two operands, a procedure that can likely be learned with minimal practice.

Despite this seeming simplicity, children were less accurate on WD than DDE addition problems in all three studies. They also were less accurate on WD than DDU addition problems in two of the three studies, with no difference in the third. Children's most common errors suggested that they frequently overgeneralized the right alignment procedure（which is appropriate for DDE addition problems and whole number addition problems) to WD addition problems, where that strategy is inappropriate. Addition with DDE operands appearing as 18 times often as addition with WD operands in the textbooks (as shown in Tables 1-3) seems likely to have contributed to the overgeneralization of the right alignment procedure.

Braithwaite, Pyke, and Siegler (2017) proposed that, in the domain of fraction arithmetic, distributions of practice problems affect performance via reinforcement learning. Whether a student would correctly solve a problem mainly depended on choosing the correct strategy; the probability of choosing the correct strategy mainly depended on how often the student had solved similar problems with that strategy in the past. Thus, students were more likely to use correct strategies on types of problem that they had encountered frequently than on those they had encountered rarely. These assumptions were implemented in a computational model of fraction arithmetic learning, FARRA. After being trained on fraction arithmetic problems extracted from a popular math textbook, FARRA's pattern of accuracies on different types of problem paralleled the frequency distribution of problem types in the textbook; the same pattern emerged when FARRA was trained with problems from a different textbook. Moreover, children's pattern of accuracy in two previous studies closely resembled the pattern generated by the model. The present findings dovetail with those of Braithwaite et al. (2017) and suggest that FARRA's theoretical assumptions extend to decimal arithmetic.

Usefulness of Analyzing Textbook Problems for Understanding Children's Learning

The present findings support the proposition that analyzing textbook problem distributions is useful for understanding children's learning. The idea that textbooks influence children learning is widespread (Acosta-Tello, 2010; Mayer, Steinhoff, Bower, & Mars, 1995). For example, Acosta-Tello (2010) analyzed the readability of word problem narratives in math textbooks and found that the narratives were hard to comprehend for a substantial proportion of students. These findings suggest that levels of readability of word problems in the textbooks may contribute to many children's difficulty with solving these problems.

More similar to the current approach, analyses of how textbooks present the equal sign shed light on the development of children's misconceptions about mathematical equality (Li, Ding, Capraro, & Capraro, 2008; McNeil et al., 2006; Powell, 2012). The present findings go beyond these earlier ones in extending the findings to a new domain and in demonstrating that the specifics of textbook problems, at the level of interactions between operands and operations, are related to the specifics of what children learn about rational number arithmetic procedures.

Although the focus of the current study was on decimal arithmetic, we expect that analyzing textbook problems will be helpful in understanding children's learning in other areas of mathematics, as well as in other subjects where problems sets are common. Physics, chemistry, statistics, and computer science are some of the many such areas. In both fraction arithmetic and decimal arithmetic, the two areas where we have examined the specifics of textbook problems and relations of the problem distributions to children's learning, textbook analysis has revealed the same biases across different contemporary textbook series. Similar biases were present in the textbook series published in 1985. Moreover, the distribution of problems in textbooks showed similar relations to children's learning in both decimal and fraction arithmetic.

For example, just as problems with a whole number and a decimal constituted only about 2% of decimal addition and subtraction problems in the current study, problems with two fractions with equal denominators constituted only about 1% of fraction multiplication and division problems in the three mathematics textbooks examined by Braithwaite et al. (2017). In both cases, children's performance was less accurate on the rarely presented types of problems than would have been expected from the seeming computational difficulty of the problems. Such similarities across textbooks in fraction and decimal arithmetic suggest that biases in textbook input probably are present in other areas of mathematics and other school subjects. It also suggests that textbook problem distributions in those areas may influence learning in them as well. Determining whether these hypotheses are correct clearly requires additional research on relations between the specifics of textbook problem distributions and learning in a variety of areas. In addition to the types of experiments in this study, it would be useful to test experimentally whether more balanced problem presentation leads to greater learning of types of problems that are rarely presented in current textbooks.

The Potential Interaction of Conceptual Knowledge and Textbook Problem Input

Weak conceptual knowledge of place value and decimal arithmetic seems likely to increase the influence on math learning of distributional biases in textbook problems. In the context of decimal addition, understanding place value would lead students to align the decimal points of addends, so that tenths are added to tenths, hundredths to hundredths, and so on. The absence of such understanding, together with plenty of experience with DDE addition items (which constituted 79%, 61% and 62% of all decimal addition problems in *GO Math!*, *enVision MATH*, and *Everyday Mathematics*, respectively), may lead students to conclude that decimals can be added by aligning their rightmost digit. Prior experience with whole number addition would also support the right alignment strategy. Consistent with this view, in both Studies 2 and 4, the most common error when adding a whole number and a decimal was that produced by aligning the rightmost digits of the addends.

Children's minimal exposure to WD addition problems in textbooks – 0 problems in *GO Math!,* 0 problems in *Everyday Mathematics*, and 8 problems in *enVision MATH* - seems especially unfortunate because it deprives students of opportunities to learn about place value on problems that might facilitate such learning. That is, students' greater understanding of whole numbers than decimals might make WD addition problems especially useful for conveying how place value concepts apply to decimals. Students who would be unfazed if asked whether .6 + .32 could really equal .38, for example, might recognize the implausibility if asked whether 6 + .32 could really equal .38. Adding whole numbers to decimals also might improve understanding of place value through the act of aligning whole number digits with whole number digits and decimal digits with decimal digits. Current distributions of textbook problems deprive students of such opportunities to learn concepts relevant to place value and decimal addition.

Limitations

The present study was correlational in nature. The findings are consistent with, but do not demonstrate, a causal connection between textbook problem distributions and children's learning. Even if a causal connection does exist, its direction is not entirely clear. It could be children's difficulty with certain types of problems leading textbook writers to avoid them, rather than the limited textbook presentation of the problems causing children's difficulty with them. Causal evidence for the effect of problem presentation on children's learning could be obtained by manipulating problem distributions in textbooks or other contexts, such as homework problems, and then comparing effects on children's learning.

A second limitation is that students receive practice problems from many sources other than textbooks. Given that textbooks usually serve as the primary resource for instruction in math classrooms, and that there is no obvious reason to expect problem distributions from other sources to differ, we assumed that problems in textbooks were representative of the overall distribution of problems children encounter. However, further research is needed to verify whether the actual practice problems students are assigned by teachers are distributed similarly to those in textbooks. We are currently examining homework problems to test whether distributions of fraction and decimal arithmetic problems assigned by teachers parallel those in textbooks.

A third limitation is that although similar patterns of performance were present across Studies 2, 3, and 4, the data also differed in some important ways. In particular, accuracy on WD addition/subtraction problems was much lower in Study 2 than in Studies 3 and 4. One intriguing possibility involves a difference in the format of the WD problems in the three studies. In Study 2, but not in Studies 3 and 4, decimals smaller than 1 were presented without a zero to the left of the decimal point. Students might more often correctly add "4 + 0.3" than "4 + .3," because the presence of the two whole numbers ("4" and "0") could lure them to left adjust the operands and therefore to add whole to whole rather than whole to decimal digit. Contemporary students also might more often encounter WD problems that explicitly indicate that there are zero whole numbers in the decimal than problems that leave it implicit (e.g., 5 + 0.42 more frequent than 5 + .42). The small number of WD addition problems in the textbooks (8 WD addition problems across the three textbooks) precluded testing this possibility. Alternatively, because Study 2 was conducted 30 years before Studies 3 and 4, instruction might have improved over time on this type of problem. Whatever the explanation, the common pattern of findings across the three studies, despite the many differences among them, is encouraging.

Another limitation was that we lacked sufficient data from individual children to examine individual differences. Children probably are impacted by distributional biases in practice problems to different degrees. For example, a recent analysis of individual differences in fraction arithmetic identified four patterns of accuracy on different problem types (Braithwaite, Leib, Siegler, & McMullen, 2019). One of the four patterns paralleled textbook problem distributions much more strongly than the other three patterns. Children with greater conceptual understanding might be less influenced by the distributions than children with little or no conceptual understanding. The possibility of individual differences in the degree to which children's performance is influenced by textbook problem distributions should be evaluated in future studies.

Educational Implications

Many efforts have been made to improve children's knowledge of fractions and decimals. One approach has been to try to improve general factors related to math achievement, such as teachers' mathematical knowledge (Hill, Rowan, & Ball, 2005), time spent on homework (Singh, Granville, & Dika, 2002), and students' mathematics self-efficacy (Williams & Williams, 2010). Unfortunately, these factors have proven difficult to change. Another approach has been to improve the specifics of instruction (e.g., Fuchs et al., 2013; Moss & Case, 1999). However, successful interventions have required multiple instructional components and extensive training for instructors to implement them successfully. The multifaceted nature of these interventions has made it difficult to specify what differences between the interventions and typical curricula are responsible for improved learning. For these and other reasons, such as lack of incentives for making large changes in instruction, none of these successful interventions has been widely implemented in classrooms.

The present findings suggest an approach that could be implemented at scale quite easily: Change the distributions of problems that children receive in textbooks, and ideally in other instructional material as well, so underrepresented types of problems are presented more often. Regardless of whether poor performance on problems that appear infrequently in textbooks is due to children's lack of experience with the problems, to lack of experience combined with insufficient conceptual knowledge, or to some other source of difficulty, increasing the frequency of the underrepresented problems should improve performance on them.

Changes in textbook problem distributions could be made much more easily than many other approaches to improving learning. Moreover, even without changes in textbooks, teachers and parents could implement changes for themselves by providing worksheets with types of problems that rarely appear in textbooks.

Changes in distributions of textbook problems may be useful but insufficient to produce strong decimal arithmetic learning. Improved conceptual understanding of how and why correct procedures yield the results they do, why the results produced by common incorrect procedures are incorrect, and the approximate magnitudes produced on specific problems by correct arithmetic procedures, also may be necessary. In general, acquiring mathematical capabilities appears to be a hand-over-hand process, in which initial conceptual knowledge helps shape initial procedural knowledge, the initial procedural knowledge provides a base for more advanced conceptual understanding, the more advanced conceptual knowledge promotes yet more advanced procedural knowledge, and so on (Rittle-Johnson, Siegler, & Alibali, 2001). In the context of decimal arithmetic, improved initial conceptual understanding of the magnitudes yielded by decimal arithmetic operations might reduce the temptation to use a right adjustment procedure that yields “6+.32 = .38,” because the sum of positive addends must be greater than either addend. Conversely, acquiring procedural knowledge of problems such as “6 + .32 = 6.32” could deepen children’s understanding of the place value concept in the context of decimal addition. Of course, these potential gains require encountering relevant problems in textbooks or elsewhere.

Using findings on textbook problem distributions to improve students’ mathematics learning will require further research on textbook problem distributions; dissemination of the research to textbook companies, teachers, and parents; and demonstrations that presenting underrepresented problems improves children’s learning. We hope to contribute to this effort, and we hope many others also will.

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Table 1

Percentage of Each Type of Problem for Each Arithmetic Operation in GO Math!

Arithmetic Operation	Type of Problem		
	WD	DDE	DDU
Addition	0	11	3
Subtraction	1	10	3
Multiplication	24	7	8
Division	21	7	4

Note. Here and in Tables 2 – 7, WD = problems on which the operands are a Whole number and a Decimal; DDE = problems on which one operand is a Decimal, the other operand is a Decimal, and the two have Equal numbers of decimal digits; DDU = problems on which one operand is a Decimal, the other operand is a Decimal, and the two have Unequal numbers of decimal digits.

Table 2

Percentage of Each Type of Problem for Each Arithmetic Operation in enVision MATH

Arithmetic Operation	Type of Problem		
	WD	DDE	DDU
Addition	1	9	5
Subtraction	1	9	6
Multiplication	20	9	7
Division	23	5	6

Table 3

Percentage of Each Type of Problem for Each Arithmetic Operation in Everyday Mathematics

Arithmetic Operation	Type of Problem		
	WD	DDE	DDU
Addition	0	7	5
Subtraction	2	10	3
Multiplication	15	8	7
Division	25	6	12

Table 4

Percentage of Each Type of Problem for Each Arithmetic Operation in Addison-Wesley Mathematics (1985 Edition)

Arithmetic Operation	Type of Problem		
	WD	DDE	DDU
Addition	0	8	2
Subtraction	0	10	2
Multiplication	18	10	10
Division	28	2	10

Table 5

Seventh Graders' Percent Correct on Decimal Arithmetic Problems in Hiebert and Wearne (1985).

Arithmetic Operation	Time of Testing	Type of Problem		
		WD	DDE	DDU
Addition	Fall	40	89	66
	Spring	38	90	72
Subtraction	Fall	31	71	54
	Spring	37	74	53
Multiplication	Fall	70	17	34
	Spring	85	60	72
Division	Fall	24	20	25
	Spring	62	61	56

Table 6

Percent Correct (Means and Standard Deviations) on Each Type of Problem: Study 3

Arithmetic Operation	Type of Problem					
	WD		DDE		DDU	
	M	SD	M	SD	M	SD
Addition	80	40	85	36	85	35
Subtraction	75	43	86	34	82	38
Multiplication	78	42	78	42	73	45
Division	87	34	79	41	83	38

Table 7

Percent Correct (Means and Standard Deviations) on Each Type of Problem: Study 4

Arithmetic Operation	Type of Problem					
	WD		DDE		DDU	
	M	SD	M	SD	M	SD
Addition	68	47	88	33	68	47
Subtraction	68	47	88	33	77	42
Multiplication	71	46	48	50	26	44
Division	42	50	24	43	15	36