DEFECT AND LOCAL UNIFORMIZATION

STEVEN DALE CUTKOSKY AND HUSSEIN MOURTADA

Dedicated to Professor Felipe Cano on the occasion of his 60th birthday

ABSTRACT. We give a simple algorithm showing that the reduction of the multiplicity of a characteristic $p > 0$ hypersurface singularity along a valuation is possible if there is a finite linear projection which is defectless. The method begins with the algorithm of Zariski to reduce multiplicity of hypersurface singularities in characteristic 0 along a valuation. This gives a simple demonstration that the only obstruction to local uniformization in positive characteristic is from defect arising in finite projections of singularities.

1. INTRODUCTION

In this paper, we give a simple algorithm showing that the reduction of the multiplicity of a characteristic $p > 0$ hypersurface singularity along a valuation is possible if there is a finite linear projection which is defectless. The method begins with the algorithm of Zariski to reduce multiplicity of hypersuface singularities in characteristic 0 along a valuation. This gives a simple demonstration that the only obstruction to local uniformization in positive characteristic is from defect arising in finite projections of singularities.

Definition 1.1. *Local uniformization holds in dimension m (LU holds in dimension m) if for every algebraic function field K over an algebraically closed field k of dimension m and for every valuation ν of K/k, there exists an algebraic local ring R of K such that R is regular and ν dominates R.*

An algebraic local ring R of a function field K/k is a local domain which is essentially of finite type over k with quotient field K. A valuation ν of an algebraic function field K/k is required to be trivial on k. A valuation ν of K dominates R if $R \subset V_\nu$, where V_ν is the valuation ring of ν, and $m_\nu \cap R = m_R$ where m_ν is the maximal ideal of V_ν and m_R is the maximal ideal of R. We will denote the value group of ν by Φ_ν. Foundational results on valuations can be found in Chapter VI of [35], [2], [14] and [27].

If X is a variety (which is assumed to be separated) with function field $k(X) = K$, then there is at most one point $p \in X$ such that ν dominates the local ring $\mathcal{O}_{X,p}$. We will say that ν dominates p, or that the center of ν on X is p. If $X_1 \to X$ is proper and birational and ν has a center on X, then ν has a center on X_1 (by the valuative criterion for properness).

Zariski, [34], found a clever patching argument (which has been extended to positive characteristic and to other situations by Abhyankar [3] and Piltant [26]) which proves that local uniformization in dimension ≤ 3 implies resolution of singularities in dimension ≤ 3. However, there still is not a direct proof (even in characteristic zero) that a set of local uniformizations can be birationally modified so that they patch together to form a global

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(proper) resolution of singularities, unless you start out with such a strong local version of resolution of singularities that patching becomes unnecessary.

Since k is assumed to be algebraically closed, K is a primitive extension of a rational function field, so we can assume initially in seeking to establish LU that there is a hypersurface singularity whose local ring has K as its quotient field and is dominated by ν.

Local uniformization has been proven in all dimensions over characteristic zero ground fields k by Zariski [33] and local uniformization has been proven in dimension ≤ 3 over ground fields k of characteristic $p > 0$ by Abhyankar [3] (in characteristic $p > 5$) and Cossart and Piltant [7] and [8] (in all characteristics). A reasonably short proof of Abhyankar's result can be found in [11].

The existence of resolutions of singularities implies local uniformization, so Hironaka's proof [15] of resolution of singularities in all dimensions over fields of characteristic zero implies local uniformization (in characteristic zero) in all dimensions.

All of the above proofs of local uniformization in dimension $m \geq 3$ (or resolution of singularities in dimension $m \geq 3$) require that embedded local uniformization (or embedded resolution of singularities) be true for hypersurfaces embedded in $\mathrm{Spec}(A)$ where A is a polynomial ring in m variables over k (or for hypersurfaces embedded in a nonsingular variety of dimension m).

Suppose ν is a valuation whose center is a nonsingular point p on a variety X, and $f \in \mathcal{O}_{X,p}$ is the germ of a hypersurace on X through p. Embedded local uniformization holds if there exists a birational morphism $X_1 \to X$ such that X_1 is nonsingular at the center p_1 of ν and f is a unit times a monomial in (suitable) regular parameters in $\mathcal{O}_{X_1, p_1}$.

There are proofs of resolution of singularities (or local uniformization) after taking a suitable finite extension of K (an alteration). Some of these proofs are by de Jong [13] (giving resolution of singularities after a finite extension of fields), Knaf and Kuhlmann [18], Temkin [30] and Gabber [17] (giving local uniformization after a finite extension). A situation where LU is known in positive characteristic is for Abhyankar valuations, [19] and [29].

Suppose that $L \to K$ is a finite field extension and $\omega = \nu|L$. An important invariant of this extension of valued fields is the defect, $\delta(\nu/\omega)$ (Section 3). If the defect $\delta(\nu/\omega)$ is zero, then the extension can be understood by knowledge of the quotient group Φ_ν/Φ_ω and the field extension V_ν/m_ν of V_ω/m_ω. The part of the field extension which comes from nontrivial defect is extremely complicated and not well understood. In characteristic zero, the defect is always zero, which is one explanation for why local uniformization is much simpler in characteristic zero.

The fact that defect is an obstruction to local uniformization in positive characteristic was observed by Kuhlmann in [20], and other papers such as [21]. The role of defect as an obstruction to local uniformization in characteristic $p > 0$ is not readily visible in the proofs of local uniformization or resolution in characteristic $p > 0$ in dimension ≤ 3 ([1], [3], [22], [6], [7], [8], [11]) and it does not appear in the approaches to resolution of singularities in higher dimension and positive characteristic in [16], [4] and [5].

In this paper, we show (as follows from Theorem 7.1) that if embedded local uniformization is true within nonsingular varieties of the dimension of K, and a suitable linear projection of a hypersurface singularity with function field K which is dominated by a given valuation ν can always be found such that if L is the function field of the linear projection and $\omega = \nu|L$ then the defect $\delta(\nu/\omega) = 0$, then local uniformization holds in K.

A valuation ν of an algebraic function field K/k is said to be zero dimensional if the transcendence degree of the residue field of the valuation ring of ν over k is zero. This is equivalent to the statement that for every projective variety X with algebraic function field k, if q is the center of ν on X, then $\dim \mathcal{O}_{X,q} = \dim X$ (q is a closed point of X). The essential case of local uniformization is for 0-dimensional valuations ([33], [25]). For simplicity, we will assume that this condition holds. Further, we will assume that we are in the essential case of a hypersurface singularity in a polynomial ring. We now state some definitions, within the context which we have just established.

An extension of domains $R \to S$ is said to be birational if R and S have the same quotient field.

Definition 1.2. *Embedded local uniformization holds in dimension m (ELU holds in dimension m) if the following is true: Suppose that $A = k[x_1, \ldots, x_m]$ is a polynomial ring in m variables over an algebraically closed field k and ν is a zero dimensional valuation of the quotient field K of A which dominates $A_{\mathfrak{m}}$ where $\mathfrak{m} = (x_1, \ldots, x_m)$. Then if $0 \neq f \in A$, there exists a birational extension $A \to A_1$ where A_1 is a polynomial ring $A_1 = k[x_1(1), \ldots, x_m(1)]$ such that ν dominates $(A_1)_{\mathfrak{m}_1}$ where $\mathfrak{m}_1 = (x_1(1), \ldots, x_m(1))$, there exists $n \leq m$ such that $\{\nu(x_1(1)), \ldots, \nu(x_n(1))\}$ is a rational basis of $\Phi_\nu \otimes \mathbb{Q}$ and $f = x_1(1)^{b_1} \cdots x_n(1)^{b_n} \overline{f}$ with $b_1, \ldots, b_n \in \mathbb{N}$ and $\overline{f} \in A_1 \setminus \mathfrak{m}_1$.*

Definition 1.3. *Local reduction of multiplicity holds in dimension m (LRM holds in dimension m) if the following is true: Suppose that $A = k[x_1, \ldots, x_m, x_{m+1}]$ is a polynomial ring in $m+1$ variables over an algebraically closed field k, $f \in A$ is irreducible and K is the quotient field of $A/(f)$, with $f \in \mathfrak{m} = (x_1, \ldots, x_{m+1})$ and $r = \mathrm{ord}(f(0, \ldots, 0, x_{m+1}))$ satisfies $1 < r < \infty$. Suppose that ν is a zero dimensional valuation of the quotient field of $A/(f)$ which dominates $(A/(f))_{\mathfrak{m}}$. Then there exists a birational extension $A \to A_1$ where A_1 is a polynomial ring $A_1 = k[x_1(1), \ldots, x_{m+1}(1)]$ such that ν dominates $(A_1/f_1)_{\mathfrak{m}_1}$ where f_1 is a strict transform of f in A_1, $\mathfrak{m}_1 = (x_1(1), \ldots, x_{m+1}(1))$, and $1 \leq r_1 = \mathrm{ord}(f_1(0, \ldots, 0, x_{m+1}(1))) < r$.*

Now ELU in dimension m immediately implies LU in dimension $m-1$ for hypersurfaces, which implies LU in dimension $m-1$. This is since there exists a separating transcendence basis $x_1, \ldots, x_{m-1}$ of K over k as k is perfect, and thus there exists a primitive element x_m of K over $k(x_1, \ldots, x_{m-1})$.

Consider the following statements:

(1)   LRM in dimension m implies ELU in dimension $m+1$.

(2)   ELU in dimension m implies LRM in dimension m.

If statements (1) and (2) are true, then we could immediately deduce that ELU in dimension m implies ELU in dimension $m+1$, and we would then know that LU holds in all dimensions.

All of the above cited proofs of local uniformization and resolution of singularities in characteristic zero involve proving the two statements (1) and (2). The proofs of resolution in dimension three and positive characteristic cited above involve tricks to obtain a proof that ELU in dimension 3 and LRM in dimension 3 in the special case $r = p = \mathrm{char}\ k$ implies LU in dimension 3. The problem is that we do not know ELU in dimension 4, so we are unable to proceed to LU in dimension 4.

Now (1) is not so difficult. In fact, induction on r in LRM in dimension m almost gives ELU in dimension $m+1$. So the really hard thing that needs to be proven (to obtain LU) is (2). Now there are methods, for instance in [33] and [25], to reduce LRM to the case of rank 1 valuations, so we see that the really essential problem is to prove (2) for rank 1 valuations (the value group Φ_ν of ν is order isomorphic to a subgroup of $\mathbb{R}$).

Zariski's original characteristic zero proof of (2) for rank 1 valuations from [33] is summarized after Theorem 6.1. The key statement that fails in positive characteristic is the validity of equation (20) in our summary of the characteristic zero proof, as we have that in characteristic $p > 0$,

$$\binom{r}{r-1} = 0 \text{ if } p \text{ divides } r.$$

In other words, the problem is the failure of the binomial theorem in positive characteristic, or put more positively, the fact that the Frobenius map is a homomorphism in positive characteristic.

We show in this paper, how starting with Zariski's analysis, a simple proof can be given of (2) in characteristic $p > 0$ for a valuation ν, if the field extension $k(x_1, \ldots, x_m) \to K$ where K is the quotient field of $k[x_1, \ldots, x_{m+1}]/(f)$ is without defect with respect to ν and the restriction of ν to $k(x_1, \ldots, x_m)$. The final statement of this is given in Theorem 7.1.

We thank a reviewer for bringing the paper [28] of San Saturnino to our attention, from which our Theorem 7.1 also follows.

2. ASYMPTOTIC PROPERTIES OF FINITE EXTENSIONS

Suppose that $K \to K^*$ is a finite field extension, R is a local domain with maximal ideal m_R and quotient field K and S is a local domain with maximal ideal m_S and quotient field K^*. We say that S dominates R if $R \subset S$ and $m_S \cap R = m_R$. We say that S lies over R if S dominates R and S is a localization of the integral closure of R in K^*.

An extension of local rings $R \subset R_1$ is said to be a birational extension if R_1 dominates R, R_1 is essentially of finite type over R and R and R_1 have the same quotient field.

Suppose that R and S are normal local rings such that R is excellent, S lies over R, ν^* is a valuation of the quotient field K^* of S which dominates S, and ν is the restriction of ν^* to the quotient field K of R. Suppose that K^* is finite separable over K.

We let V_ν be the valuation ring of ν, with maximal ideal m_ν and Φ_ν be the value group of ν. We write $e(\nu^*/\nu) = [\Phi_{\nu^*} : \Phi_\nu]$ and $f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$.

Lemma 2.1. *(Lemma 5.1 [10]) Suppose that S_0 is a local ring which is a birational extension of S and is dominated by ν^*. Then there exists a normal local ring R' which is a birational extension of R and is dominated by ν, which has the property that if R'' is a normal local ring which is a birational extension of R' and is dominated by ν, and if S'' is the normal local ring of K^* which lies over R'' and is dominated by ν^*, then S'' dominates S_0.*

If K^* is a Galois extension of K with Galois group $G(K^*/K)$ and A is a normal local ring whose quotient field is K and is dominated by ν and B is the normal local ring with quotient field K^* which lies over A and is dominated by ν^*, then the splitting group of B over A is defined as

$$G^s(B/A) = \{\sigma \in G(K^*/K) \mid \sigma(B) = B\}.$$

The splitting field of K^* over K is the fixed field

(3)
$$K^s = (K^*)^{G^s(V_{\nu^*}/V_\nu)}.$$

The following lemma follows from Lemma 3.4 [10].

Lemma 2.2. *Suppose that K^* is Galois over K. Then there exists a birational extension R' of R, where R' is a normal local ring which is dominated by ν, such that if R'' is a normal local ring which is a birational extension of R' which is dominated by ν, then*

$$G^s(C''/R'') = G^s(V_{\nu^*}/V_\nu)$$

where C'' is the normal local ring with quotient field K^* which lies over R'' and is dominated by ν^*.

Suppose that A is a subring of V_ν. Then the center of ν on A is the ideal $A \cap m_\nu$. An inclusion $A \to B$ of domains is said to be a birational extension if A and B have the same quotient field and B is essentially of finite type over A.

3. Defect

In this section we define the defect of an extension of valued fields, and discuss some basic properties. More details can be found in [35], [14], [20] and [27].

Let K be a field and ν be a valuation of K. Let K^* be a finite extension of K and ν^* be an extension of ν to K^*. The defect $\delta(\nu^*/\nu)$ is the natural number defined on pages 58 and 78 of [35].

If K^* is a finite extension of K and ν^* is the unique extension of ν to K^*, then we have Ostrowski's lemma (Theorem 2, page 236 [27]),

(4)
$$[K^* : K] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}.$$

The definition of the defect $\delta(\nu^*/\nu)$ for an arbitrary finite and separable extension K^* of K is obtained by taking a Galois closure L of K^* over K, choosing an extension $\tilde{\nu}$ of ν^* to L, and defining (formula (26) [12])

$$\delta(\nu^*/\nu) = \delta(\tilde{\nu}/\nu) - \delta(\tilde{\nu}/\nu^*) \geq 0.$$

From multiplicativity of the ramification index, residue degree and degree of field extensions, we have that the formula is well defined. By the formula before Definition 7.1 on page 36 of [12] and 3,

(5)
$$[(K^*)^s : K^s] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}.$$

Let ν^s be the restriction of $\tilde{\nu}$ to K^s and $(\nu^*)^s$ be the restriction of $\tilde{\nu}$ to $(K^*)^s$. Since $e(\tilde{\nu}/\nu^s) = e(\tilde{\nu}/\nu)$ and $f(\tilde{\nu}/\nu^s) = f(\tilde{\nu}/\nu)$ (Theorem 23, page 71 [35] and Theorem 22, page 70 [35]), we have that $\delta(\tilde{\nu}/\nu^s) = \delta(\tilde{\nu}/\nu)$, and

(6)
$$\begin{aligned} \delta((\nu^*)^s/\nu^s) &= \delta(\tilde{\nu}/(\nu^*)^s) - \delta(\tilde{\nu}/\nu^s) \\ &= \delta(\tilde{\nu}/\nu^*) - \delta(\tilde{\nu}/\nu) \\ &= \delta(\nu^*/\nu). \end{aligned}$$

4. Perron Transforms

Suppose that K^* is a field with a valuation ν^* and there is a field k which is contained in V_{ν^*} such that $V_{\nu^*}/m_{\nu^*} = k$ and ν^* has rank 1.

Suppose that we have a polynomial ring $k[x_1, \ldots, x_m]$ and an irreducible element $f \in k[x_1, \ldots, x_m]$ such that K^* is the quotient field of $k[x_1, \ldots, x_m]/(f)$, and the center of ν^* on $k[x_1, \ldots, x_m]/(f)$ is the maximal ideal $(\bar{x}_1, \ldots, \bar{x}_m)$ where $\bar{x}_i$ are the residues of x_i in $k[x_1, \ldots, x_m]/(f)$. We can extend ν^* to a pseudo valuation which dominates $k[x_1, \ldots, x_m]$ by prescribing that for $g \in k[x_1, \ldots, x_m]$,

$$\nu^*(g) = \begin{cases} \nu^*(\bar{g}) & \text{if the residue } \bar{g} \text{ of } g \text{ in } k[x_1, \ldots, x_m]/(f) \text{ is nonzero,} \\ \infty & \text{if } f \text{ divides } g \text{ in } k[x_1, \ldots, x_m]. \end{cases}$$

Suppose that the natural map $k[x_1, \ldots, x_{m-1}] \to k[x_1, \ldots, x_m]/(f)$ is an inclusion and $\bar{x}_m$ is nonzero. Further suppose that $\nu^*(x_1), \ldots, \nu^*(x_n)$ is a rational basis of $\Phi_{\nu^*} \otimes \mathbb{Q}$ (so that $n \leq \mathrm{trdeg}_k K^* = m - 1$).

In Section B of [33], Zariski uses the "Algorithm of Perron" to define two types of Cremona transforms, (7) and (8) below.

We construct a Perron transform

(7)
$$x_i = \begin{cases} \prod_{j=1}^n x_j(1)^{a_{ij}} & \text{if } 1 \leq i \leq n \\ x_i(1) & \text{if } n < i \leq m \end{cases}$$

where $a_{ij} \in \mathbb{N}$ satisfy $\mathrm{Det}(a_{ij}) = 1$ and $0 < \nu^*(x_i(1)) < \infty$ for $1 \leq i \leq m$

We now construct another type of Perron transform. We have that $\nu^*(x_m)$ is rationally dependent on $\nu^*(x_1), \ldots, \nu^*(x_n)$. There exists a Perron transform

(8)
$$x_i = \begin{cases} (\prod_{j=1}^n x_j(1)^{a_{ij}})(x_m(1) + c)^{a_{i,n+1}} & \text{if } 1 \leq i \leq n \\ (\prod_{j=1}^n x_j(1)^{a_{n+1,j}})(x_m(1) + c)^{a_{n+1,n+1}} & \text{if } i = m \\ x_i(1) & \text{if } n < i < m \end{cases}$$

with $c \in k$ nonzero, $a_{ij} \in \mathbb{N}$ and $\mathrm{Det}(a_{ij}) = 1$ such that $0 < \nu^*(x_i(1)) < \infty$ for $1 \leq i \leq m-1$ and $\nu^*(x_1(1)), \ldots, \nu^*(x_n(1))$ is a rational basis of $\Phi_{\nu^*} \otimes \mathbb{Q}$. Further, $0 < \nu^*(x_m(1))$. We have that $\nu^*(x_m(1)) < \infty$, unless $x_m(1)$ is a local equation of the strict transform of f in $k[x_1(1), \ldots, x_m(1)]$.

In (7) and (8), the a_{ij} are not random, but are constructed by Zariski's algorithm. For Perron transforms of type (7) or (8), $k[x_1(1), \ldots, x_m(1)]$ is the ring of regular functions of an affine neighborhood of the center of ν^* on a sequence of blow ups of nonsingular subvarieties above the affine m-space with regular functions $k[x_1, \ldots, x_m]$. This follows directly from Zariski's construction of Perron transforms. The maximal ideal of the center of ν^* on $k[x_1(1), \ldots, x_m(1)]$ is $(x_1(1), \ldots, x_m(1))$. We have an expression

$$f = x_1(1)^{c_1} \cdots x_m(1)^{c_m} f_1$$

where $f_1 \in k[x_1(1), \ldots, x_m(1)]$ is irreducible. The ring $k[x_1(1), \ldots, x_m(1)]/(f_1)$ is the ring of regular functions of an affine neighborhood of the center of ν^* on a birational transformation of the affine variety with regular functions $k[x_1, \ldots, x_m]/(f)$.

Lemma 4.1. *Suppose that* $M_1 = x_1^{d_1} \cdots x_n^{d_n}$ *and* $M_2 = x_1^{e_1} \cdots x_n^{e_n}$ *with*

$$d_1, \ldots, d_n, e_1, \ldots, e_n \geq 0$$

and $\nu(M_1) < \nu(M_2)$. *Then there exists a Perron transform of type (7) such that* M_1 *divides* M_2 *in* $k[x_1(1), \ldots, x_m(1)]$.

This lemma is proved in Theorem 2 [33] of Lemma 4.2 [9].

Suppose that we have a Perron transform of type (8). Let $A = (a_{ij})^t$, an $(n+1) \times (n+1)$ matrix. Suppose that $d_1, \ldots, d_{n+1}, e_1, \ldots, e_{n+1} \in \mathbb{N}$ and

$$(9) \qquad \nu^*(x_1^{d_1} \cdots x_n^{d_n} x_m^{d_{n+1}}) = \nu^*(x_1^{e_1} \cdots x_n^{e_n} x_m^{e_{n+1}}).$$

We have that

$$x_1^{d_1} \cdots x_n^{d_n} x_m^{d_{n+1}}$$
$$= \left[\prod_{i=1}^n [(\prod_{j=1}^n x_j(1)^{a_{ij}})(x_m(1)+c)^{a_{i,n+1}}]^{d_i} \right] \left[(\prod_{j=1}^n x_j(1)^{a_{n+1,j}})(x_m(1)+c)^{a_{n+1,n+1}} \right]^{d_{n+1}}$$
$$= \left(\prod_{j=1}^n x_j(1)^{\sum_{i=1}^{n+1} a_{ij} d_i} \right) (x_m(1)+c)^{\sum_{i=1}^{n+1} a_{i,n+1} d_i}.$$

Similarly, we have

$$x_1^{e_1} \cdots x_n^{e_n} x_m^{e_{n+1}}$$
$$= \left(\prod_{j=1}^n x_j(1)^{\sum_{i=1}^{n+1} a_{ij} e_i} \right) (x_m(1)+c)^{\sum_{i=1}^{n+1} a_{i,n+1} e_i}.$$

By (9), we have that

$$\sum_{i=1}^{n+1} a_{ij} d_i = \sum_{i=1}^{n+1} a_{ij} e_i \text{ for } 1 \le j \le n.$$

Let

$$\gamma = \sum_{i=1}^{n+1} a_{i,n+1} e_i - \sum_{i=1}^{n+1} a_{i,n+1} d_i.$$

We have that

$$(10) \qquad A \begin{pmatrix} d_1 - e_1 \\ d_2 - e_2 \\ \vdots \\ d_n - e_n \\ d_{n+1} - e_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \gamma \end{pmatrix}.$$

By Cramer's rule,

$$(11) \qquad d_i - e_i = (-1)^{n+1+i} \gamma \mathrm{Det}(A_{n+1,i}) \text{ for } 1 \le i \le n+1$$

where $A_{n+1,i}$ is the submatrix of A obtained by removing the $(n+1)$-st row and i-th column from A.

5. Admissible families of valuations and defect

Let K be a field and ν be a valuation on K. Let $K^* = K(z)$ be a finite primitive extension of K with $z \notin K$ and ν^* be an extension of ν to K^*. Let $f(x) \in K[x]$ be the minimal polynomial of z over K. As explained in [31] and [32] (which extend work of MacLane in [23] and [24]), we may represent ν^* by an admissible family of valuations $\mathcal{A} = (\mu_\alpha)_{l \in I}$ of $K[x]$ whose restriction to K is ν, and are successive approximations to ν^*, whose final element is a pseudo valuation which gives ν^* on $K^* \cong K[x]/(f)$. There is a decomposition (Section 2.1 [31])

$$\mathcal{A} = \mathcal{S}^{(1)} \cup \cdots \cup \mathcal{S}^{(N)}$$

where $N \ge 1$ and each $\mathcal{S}^{(i)}$ is a simple admissible family (Definition, page 3471 [31]). Here I is a totally ordered set, and except for the last element of I, which is a pseudo valuation, each μ_l is either an augmented valuation or a limit augmented valuation. In

the first case, of an augmented valuation (Definition on page 3443 [31]), we have that $\mu_l = [\mu_{l-1}; \mu_l(\varphi_l) = \gamma_l]$ where φ_l is a key polynomial (page 3442 [31]), defining μ_l in terms of μ_{l-1}, and in the second case, of a limit augmented valuation (Definition on page 3466 [31]), we have that $\mu_l = [(\mu_\alpha)_{\alpha \in A}; \mu_l(\varphi_l) = \gamma_l]$ where φ_l is a limit key polynomial (page 3465 [31]) defining the valuation ν_l in terms of the continuous family $(\mu_\alpha)_{\alpha \in A}$.

Each $\mathcal{S}^{(j)} = (\mu_1^{(j)}, \dots, \mu_{n_j}^{(j)}; (\mu_\alpha^{(j)})_{\alpha \in A^{(j)}})$ for $1 \leq j \leq N$. We have that $\deg \varphi_i^{(j)} > \deg \varphi_{i-1}^{(j)}$ if $2 \leq i \leq n_j$ and $\deg \varphi_\alpha^{(j)} = \deg \varphi_{n_j}^{(j)}$ for all $\alpha \in A^{(j)}$. Each $(\mu_\alpha^{(j)})_{\alpha \in A^{(j)}}$ is a continuous, exhaustive family of iterated augmented valuations (Definition, page 3464 [31]). In particular, $A^{(j)}$ does not have a maximum.

The first valuation $\mu_1^{(1)}$ of the family $\mathcal{S}^{(1)}$ is an augmented valuation of the form $\mu_1^{(1)} = [\mu_0; \mu_1^{(1)}(\varphi_1^{(1)}) = \gamma_1]$ where μ_0 is the valuation ν of K and $\varphi_1^{(1)}$ is a polynomial of degree 1 in $K[x]$. For $t \geq 2$, the first valuation $\mu_1^{(t)}$ of $\mathcal{S}^{(t)}$ is a limit augmented valuation for the family $(\mu_\alpha^{(t-1)})_{\alpha \in A^{(t-1)}}$. $\mathcal{S}^{(N)}$ has a largest element, which is either an augmented valuation, or a limit augmented valuation $\mu_{\bar{l}}$ associated to the polynomial $\varphi_{\bar{l}} = f$ with value $\gamma_{\bar{l}} = \infty$.

For $1 \leq j \leq N - 1$, we have associated rational numbers $S^{(j)}(\mathcal{A})$ called jumps, and the total jump of the extension is (page 861 [32])

$$S^{\mathrm{tot}}(\mathcal{A}) = \prod_{j=2}^{N} S^{(j-1)}(\mathcal{A}).$$

The jump $S^{(j-1)}(\mathcal{A})$ is defined (Definition 2.7, page 870 [32]) by

$$\deg \varphi_1^{(j)} = S^{(j-1)}(\mathcal{A}) \deg \varphi_\alpha^{(j-1)}$$

where $\varphi_1^{(j)}$ is the limit key polynomial defining the valuation $\mu_1^{(j)}$ and $\varphi_\alpha^{(j-1)}$ is a key polynomial associated to the continuous family $(\mu_\alpha^{(j-1)})_{\alpha \in A^{(j-1)}}$.

By Lemma 2.11 [32]

(12) $$S^{(j-1)}(\mathcal{A}) > 1 \text{ for } 2 \leq j \leq N$$

and by Corollary 2.10 [32],

(13) $$[K^* : K] = e(\nu^*/\nu) f(\nu^*/\nu) S^{\mathrm{tot}}(\mathcal{A}).$$

Thus if ν^* is the unique extension of ν to K^*, then by Ostrowski's lemma, equation (4),

(14) $$p^\delta(\nu^*/\nu) = \mathcal{S}^{\mathrm{tot}}(\mathcal{A}).$$

Proposition 5.1. *Suppose that*

$$\nu^*(z - K) = \{\nu^*(z - a) \mid a \in K\}$$

does not have a largest element. Then $N \geq 2$.

The set $\nu^*(z - K)$ determines the distance of z from K, investigated by Kuhlmann in [21].

Proof. Since $\nu^*(z - K)$ does not have a largest element, we calculate from Proposition 2.3 and Theorem 2.4 [31] that

$$S^{(1)} = \mathcal{D}^{(1)} \cup \mathcal{C}^{(1)}$$

where $\mathcal{D}^{(1)} = (\nu_1^{(1)})$ and $\mathcal{C}^{(1)} = (\nu_\alpha^{(1)})_{\alpha \in A}$, with $\nu_1^{(1)} = [\nu; \nu_1^{(1)}(\varphi_1^{(1)}) = \gamma_1^{(1)}]$ with $\varphi_1^{(1)} = z$ and $\gamma_1^{(1)} = \nu^*(z)$ and

$$A = \{\alpha \in \nu^*(z - K) \mid \alpha > \gamma_1^{(1)}\},$$

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and for $\alpha \in A$,
$$\nu_\alpha^{(1)} = [\nu_1^{(1)}; \nu_\alpha^{(1)}(\varphi_\alpha^{(1)}) = \alpha],$$
where $\varphi_\alpha^{(1)} = z - h_\alpha$ for some $h_\alpha \in K$ such that $\nu^*(\varphi_\alpha^{(1)}) = \alpha$. We have that $S^{(2)}$ begins with the augmented limit valuation
$$\nu_1^{(2)} = \lim_{\alpha \in A} \nu_\alpha^{(1)}.$$
Thus $N \geq 2$. $\qquad\square$

Lemma 5.2. *Suppose that K contains a field k such that $k \subset V_{\nu^*}$ and $V_{\nu^*}/m_{\nu^*} = k$. Further suppose that $\nu^*(z - K)$ has a largest element γ. Then $\gamma \notin \Phi_\nu$.*

Proof. Suppose that $\gamma \in \Phi_\nu$. Let $h \in K$ be such that $\nu^*(z - h) = \gamma$. Since $\gamma \in \Phi_\nu$, there exists $g \in K$ such that $\nu(g) = \gamma$. Let c be the class of $\frac{z-h}{g}$ in $V_{\nu^*}/m_{\nu^*} = k$. Then $\nu^*(z - h - cg) > \nu^*(z - h) = \gamma$, a contradiction. $\qquad\square$

6. An algorithm for reduction of multiplicity

Let k be an algebraically closed field, T be a polynomial ring $T = k[x_1, \ldots, x_m]$, $f \in T$ be irreducible and monic in x_m and $S = T/(f)$. Let K^* be the quotient field of S. Let z be the class of x_m in $S \subset K^*$. Let $R = k[x_1, \ldots, x_{m-1}]$ and K be the quotient field of R, so that K^* is a finite extension of K. Assume that K^* is separable over K. Let $r = \operatorname{ord} f(0, \ldots, 0, x_m)$. Let ν^* be a valuation of K^* with restriction ν to K. Suppose that ν^* has rank 1 and $V_{\nu^*}/m_{\nu^*} = k$. Suppose that $S \subset V_{\nu^*}$.

We can regard ν^* as a pseudo valuation on T, where for $g \in T$, $\nu^*(g) = \infty$ if f divides g. Suppose that the center of ν^* on T is $(x_1, \ldots, x_m)$.

The first part of the algorithm of Theorem 6.1 (through equation (19)) is the same as in [33].

Theorem 6.1. *Let assumptions be as above, and suppose that embedded local uniformization (Definition 1.2)) is true in dimension $m - 1$. Suppose that $r > 1$ and $\nu^*(z) \notin \Phi_\nu$. Then there exists a birational extension along $\overline{\nu}^*$, $T \to T' = k[x_1', \ldots, x_m']$, such that $T' \subset V_{\overline{\nu}^*}$ and $(x_1', \ldots, x_m')$ is the center of $\overline{\nu}^*$ on T', and if f' is the strict transform of f in T', so that $S' = T'/(f')$ is a birational extension $S \to S'$ along ν^*, we have that $\operatorname{ord} f'(0, \ldots, 0, x_m') < r$.*

Proof. Let n be the rational rank of ν^*. Since K^* is a finite extension of K, we have that rank $\nu = 1$ and rat rank $\nu^* = $ rat rank $\nu = n$. By embedded local uniformization in R, there exists a birational transform $R \to R_1 = k[x_1(1), \ldots, x_{m-1}(1)]$ such that the center of ν on R_1 is $(x_1(1), \ldots, x_{m-1}(1))$ and $\nu(x_1(1)), \ldots, \nu(x_n(1))$ is a rational basis of $\Phi_\nu \otimes \mathbb{Q}$. Let $x_m(1) = x_m$ and T_1 be the polynomial ring $T_1 = k[x_1(1), \ldots, x_m(1)]$. We have that the center of ν^* on T_1 is $(x_1(1), \ldots, x_m(1))$. Since f is monic in x_m, f is irreducible in T_1, and is monic in $x_m(1)$. Thus $S_1 = T_1/(f)$ is a domain with quotient field K^*, so $S \to S_1$ is a birational extension.

Expand

(15) $$f = x_m(1)^e + a_{e-1}x_m(1)^{e-1} + \cdots + a_0$$

with $a_i \in R_1$ for $0 \leq i \leq e-1$. Let $a_e = 1$. By embedded local uniformization in R_1, there exists a birational extension $R_1 \to R_2 = k[x_1(1), \ldots, x_{m-1}]$ along ν such that the center of ν on R_2 is $(x_1(2), \ldots, x_{m-1}(2))$,

$$\nu(x_1(2)), \ldots, \nu(x_n(2))$$

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is a rational basis of $\Phi_\nu \otimes \mathbb{Q}$ and if $a_i \neq 0$, then

$$a_i = x_1(2)^{d_1(i)} \cdots x_n(2)^{d_n(i)} \overline{a}_i$$

with $d_1(i), \ldots, d_n(i) \in \mathbb{N}$ and $\overline{a}_i \notin (x_1(2), \ldots, x_{m-1}(2))$. Let $x_m(2) = x_m(1)$ and $T_2 = k[x_1(2), \ldots, x_m(2)]$. We have that $d_1(e) = \cdots = d_n(e) = 0$. Let

$$\rho = \min\{\nu^*(a_i x_m(2)^i) \mid 0 \leq i \leq e\}.$$

There exist t natural numbers σ_i with

$$0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_t \leq r$$

such that $\nu^*(a_{\sigma_i} x_m(2)^{\sigma_i}) = \rho$ for $1 \leq i \leq t$ and $\nu^*(a_l x_m(2)^l) > \rho$ if $l \neq \sigma_i$ for some i with $1 \leq i \leq t$. Since $\nu^*(f) = \infty$, we have that

(16) $$1 < t.$$

We now perform a Perron transform $T_2 \to T_3 = k[x_1(3), \ldots, x_m(3)]$ of type (8) along ν^*,

$$x_i(2) = \begin{cases} (\prod_{j=1}^n x_j(3)^{a_{ij}})(x_m(3) + c)^{a_{i,n+1}} & \text{if } 1 \leq i \leq n \\ (\prod_{j=1}^n x_j(3)^{a_{n+1,j}})(x_m(3) + c)^{a_{n+1,n+1}} & \text{if } i = m \\ x_i(3) & \text{if } n < i < m. \end{cases}$$

Let $A = (a_{ij})^t$.

For $1 \leq l \leq r$ and $1 \leq j \leq n$, let

$$\tau_{j,l} = \left(\sum_{i=1}^n a_{ij} d_i(l) \right) + (a_{n+1,j})l$$

and for $1 \leq l \leq r$, let

$$\lambda_l = \sum_{i=1}^n a_{i,n+1} d_i(l) + (a_{n+1,n+1})l.$$

Then

$$a_l x_m(1)^l = \overline{a}_l \left(\prod_{j=1}^n x_j(3)^{\tau_{j,l}} \right) (x_m(3) + c)^{\lambda_l}$$

for $0 \leq l \leq e$ and if $a_l \neq 0$. We have

$$\tau_{j,\sigma_i} = \tau_{j,\sigma_1}$$

for $1 \leq j \leq n$ and $1 \leq i \leq t$ since $\nu(x_1(3)), \ldots, \nu(x_n(3))$ is a rational basis of $\Phi_\nu \otimes \mathbb{Q}$. By Lemma 4.1, after possibly performing a Perron transform of type (7) in $x_1(3), \ldots, x_n(3)$, we have an expression

$$f = \prod_{j=1}^n x_j(3)^{\tau_{j,\sigma_1}} f_1$$

where

(17) $$f_1 = \sum_{i=1}^t \overline{a}_{\sigma_i} (x_m(3) + c)^{\lambda_{\sigma_i}} + h$$

with $h \in (x_1(3), \ldots, x_n(3))T_3$ is a local equation of the strict transform of f in T_3. Let $d = \mathrm{Det}(A_{n+1,n+1})$. By (11),

(18) $$(\lambda_{\sigma_i} - \lambda_{\sigma_1})d = \sigma_i - \sigma_1 \text{ for } 1 \leq i \leq t.$$

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By (16), $t > 1$ so $d \neq 0$. Suppose $d > 0$. Then
$$
\begin{aligned}
f_1 &= (x_m(3) + c)^{\lambda_{\sigma_1}} \left(\sum_{i=1}^{t} \bar{a}_{\sigma_i}(x_m(3) + c)^{\lambda_{\sigma_i} - \lambda_{\sigma_1}} + h \right) \\
&= (x_m(3) + c)^{\lambda_{\sigma_1}} \left(\sum_{i=1}^{t} \bar{a}_{\sigma_i}(x_m(3) + c)^{\frac{\sigma_i - \sigma_1}{d}} + h \right).
\end{aligned}
$$

Let $r_1 = \text{ord}(f_1(0, \ldots, 0, x_m(3)))$. We have that
$$
r_1 \leq \frac{\sigma_t - \sigma_1}{d} \leq r.
$$

Suppose that $r_1 = r$. Then

(19)
$$
\sigma_t = r, \sigma_1 = 0 \text{ and } d = 1.
$$

So far, the proof has been as Zariski's in [33]. The remainder of his proof requires characteristic zero, and is not valid in characteristic $p > 0$. We provide a different analysis from here on. After this proof, we give an outline of the conclusion of Zariski's characteristic zero proof.

By (10),
$$
A \begin{pmatrix} d_1(\sigma_1) - d_1(\sigma_t) \\ \vdots \\ d_n(\sigma_1) - d_n(\sigma_t) \\ \sigma_1 - \sigma_t \end{pmatrix} = A \begin{pmatrix} d_1(\sigma_1) \\ \vdots \\ d_n(\sigma_1) \\ -r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \end{pmatrix}
$$
where $\gamma = \lambda_{\sigma_1} - \lambda_{\sigma_t}$. By (18) and since $d = 1$,
$$
\gamma = -r.
$$

Further, by (11), $d_i(\sigma_1) = (-1)^{n+i} r \text{Det}(A_{n+1,i})$ for $1 \leq i \leq n$. Thus r divides $d_i(\sigma_1)$ for $1 \leq i \leq n$. Since
$$
r\nu^*(x_m) = \nu^*(x_m^r) = \nu^*(x_m(2)^r) = \nu(x_1(2)^{d_1(\sigma_1)} \cdots x_n(2)^{d_n(\sigma_1)}) = \sum_{i=1}^{n} d_i(\sigma_1)\nu(x_i(2)),
$$
we have that $\nu^*(x_m) \in \Phi_\nu$, giving a contradiction.

If $d < 0$, we have a similar argument, writing
$$
f_1 = (x_m(3) + c)^{\lambda_{\sigma_t}} \left(\sum_{i=1}^{t} \bar{a}_{\sigma_i}(x_m(3) + c)^{\frac{\sigma_i - \sigma_t}{d}} + h \right).
$$

$\square$

Zariski's proof of reduction of singularities in characteristic zero in [33] (or Theorem 8.4 [3]), proceeds after (19) as follows. If we have $r_1 = r$ in the above algorithm, we have that
$$
\sum_{i=1}^{t} \bar{a}_{\sigma_i}(0, \ldots, 0)(x_m(3) + c)^{\sigma_i} = \bar{a}_r(0, \ldots, 0) x_m(3)^r
$$
in $k[x_m(3)]$. Set
$$
g(u) = \sum_{i=1}^{t} \bar{a}_{\sigma_i}(0, \ldots, 0) u^{\sigma_i} = \bar{a}_r(0, \ldots, 0)(u - c)^r.
$$

The binomial theorem tells us (since we are in characteristic zero) that

(20)
$$
\sigma_{t-1} = r - 1
$$

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so $\nu^*(x_m(1)) = \nu^*(a_{r-1})$. There thus exists $\omega \in k = V_{\nu^*}/m_{\nu^*}$ such that

$$\nu^*(x_m(1) - \omega a_{r-1}) > \nu^*(x_m(1)).$$

We now make a change of variables in T_1, replacing $x_m(1)$ with $x_m(1)' = x_m(1) - \omega a_{r-1}$. After a finite number of iterations of the algorithm from (15), we must obtain a reduction $r_1 < r$, since otherwise we construct an infinite increasing sequence of values in R_1 which is bounded above by $\nu^*\left(\frac{\partial f}{\partial x_m(1)}\right)$, which is impossible since R_1 is Noetherian,

7. REDUCTION OF MULTIPLICITY UNDER A DEFECTLESS PROJECTION

The following Theorem 7.1 is also a consequence of Theorem 6.5 [28].

Theorem 7.1. *Let assumptions be as in Section 6 and suppose that embedded local uniformization (Definition 1.2) is true in dimension $m-1$,*

$$r = \operatorname{ord} f(0, \ldots, x_m) > 1$$

and the defect $\delta(\nu^*/\nu) = 0$. Then there exists a birational transform along ν^*, $T \to T' = k[x_1', \ldots, x_m']$, such that $(x_1', \ldots, x_m')$ is the center of ν^* on T', and if f' is the strict transform of f in T', so that $S' = T'/(f')$ is a birational extension $S \to S'$ along ν^*, we have that $\operatorname{ord} f'(0, \ldots, 0, x_m') < r$.

Proof. Let L be a Galois closure of K^* over K and $\tilde\nu$ be an extension of ν^* to L. With notation as in formula (6), we have that $\delta((\nu^*)^s/\nu^s) = \delta(\nu^*/\nu) = 0$. Further, $(\nu^*)^s$ is the unique extension of ν^s to $(K^*)^s$ by Proposition 1.46 [2]. We have that $\Phi_{\nu^s} = \Phi_\nu$, $\Phi_{(\nu^*)^s} = \Phi_{\nu^*}$ by Theorem 23, page 71 [35] and $V_{\nu^s}/m_{\nu^s} = V_\nu/m_\nu = k$, $V_{(\nu^*)^s}/m_{(\nu^*)^s} = V_{\nu^*}/m_{\nu^*} = k$ by Theorem 22, page 70 [35], so by (5) or (4),

$$[(K^*)^s : K^s] = e((\nu^*)^s/\nu^s) = e(\nu^*/\nu).$$

First suppose that $(K^*)^s \neq K^s$. Recall that z is the class of x_m in $S \subset K^*$. Since $(K^*)^s = K^* K^s = K^s(z)$, we have that $(\nu^*)^s(z - K^s)$ has a largest element $\gamma \in \Phi_{(\nu^*)^s} = \Phi_{\nu^*}$ by (12), (14) and Proposition 5.1. Further,

$$\gamma \notin \Phi_{\nu^s} = \Phi_\nu \tag{21}$$

by Lemma 5.2. Let $h \in K^s$ be such that $(\nu^*)^s(z - h) = \gamma$. We have that $\gamma \geq \nu^*(z) > 0$ and $\nu^s(h) \geq \nu^*(z) > 0$.

Let U be the localization of the integral closure of R in K^s at the center of ν^s. There exists a birational extension $U \to U_0$ where U_0 is a normal local ring which is dominated by ν^s such that $h \in U_0$ (just take U_0 to be the localization of the integral closure of $U[h]$ at the center of ν^s). By Lemma 2.2 and Lemma 2.1 and by embedded local uniformization, there exists a birational extension $R \to R_1 = k[x_1(1), \ldots, x_{m-1}(1)]$ along ν such that the center of ν on R_1 is $\mathfrak{m}_1 = (x_1(1), \ldots, x_{m-1}(1))$ and

$$G^s(W_1/(R_1)_{\mathfrak{m}_1}) = G^s(V_{\tilde\nu}/V_\nu) \tag{22}$$

where W_1 is the localization of the integral closure of R_1 in L at the center of $\tilde\nu$ and $h \in U_1$ where U_1 is the localization of the integral closure of R_1 in K^s at the center of ν^s. By (22) and Theorem 1.47 [2], we have that $\mathfrak{m}_1 U_1$ is the maximal ideal of U_1 and the residue field of U_1 is $k = R_1/\mathfrak{m}_1$, so U_1 is a regular local ring with completion $\hat{U}_1 = \hat{R}_1$, where the completion of R_1 is at the maximal ideal $\mathfrak{m}_1$. Thus $h \in \hat{R}_1$.

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Now $\gamma < \nu((\mathfrak{m}_1 \hat{R}_1)^t)$ for some $t \in \mathbb{Z}_+$ since ν has rank 1, and thus there exists $h' \in R_1$ such that $h - h' \in (\mathfrak{m}_1 \hat{R}_1)^t$. Thus

$$\nu^*(z - h') = (\nu^*)^s((z - h) + (h - h')) = \nu^*(z - h) = \gamma.$$

We have that γ is the largest element of $\nu^*(z - K)$ since $\nu^*(z - K) \subset (\nu^*)^s(z - K^s)$. Thus $\nu^*(z - h') = \gamma \notin \Phi_\nu$ by (21).

Let $x_m(1) = x_m$ and $T_1 = k[x_1(1), \ldots, x_m(1)]$ so that the center of $\tilde{\nu}$ on T_1 is

$$(x_1(1), \ldots, x_m(1)).$$

Now $R_1[z] \cong T_1/(f)$, and regarding f as a polynomial in T_1, we have that

$$\text{ord } f(0, \ldots, 0, x_m(1)) = r.$$

Making the change of variable in T_1 replacing $x_m(1)$ with $x_m(1) - h'$ and z with $z - h'$, we have that $\nu^*(z) \notin \Phi_\nu$. Then the reduction of r in the conclusions of the theorem follows from Theorem 6.1.

Now suppose that $K^s = (K^*)^s$. Then $z \in K^s$. As in the above case, there exists a birational transformation $R \to R_1$ along ν such that $R_1 = k[x_1(1), \ldots, x_{m-1}(1)]$, the center of ν on R_1 is $m_{R_1} = (x_1(1), \ldots, x_{m-1}(1))$ and $z \in U_1$, where U_1 is the normal local ring of K^s which is dominated by ν^s, and U_1 is a regular local ring with $\hat{U}_1 = \hat{R}_1$ where the completion of R_1 is with respect to the maximal ideal $\mathfrak{m}_1 = (x_1(1), \ldots, x_{m-1}(1))$ of R_1. Letting $x = x_m$ and writing $f(x) = f(x_1(1), \ldots, x_{m-1}(1), x) \in U_1[x]$, we have a factorization $f(x) = (x - z)f_1(x)$ with $f_1(x) \in U_1[x]$. Since K^* is separable over K, we have that $f_1(z) \in K^s$ is nonzero, and has nonnegative value. Let $\tau = \nu^s(f_1(z)) \geq 0$. Since $\Phi_{\nu^*} = \Phi_\nu$ (by Theorem 23, page 71 [35]) and ν has rank 1, we may replace x with $x - h$ and z with $z - h$ for suitable $h \in R_1$ so that $\nu^s(z) > \tau$. Then writing

$$(23) \qquad\qquad f_1(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$$

with $a_i \in U_1$ and $d \in \mathbb{N}$, we have that $\nu^s(z^i a_i) > \tau$ for $i > 0$ and $\nu^s(a_0) = \tau$. Let $G = G(L/K)$ be the Galois group of L over K and for $0 \leq i \leq d - 1$ such that $a_i \neq 0$, let $b_i = \prod_{\sigma \in G} \sigma(a_i) \in R_1$ and let $c = \prod_{\sigma \in G} \sigma(z) \in R_1$.

By embedded local uniformization in R_1, applied to $c \prod b_i$, there is a birational extension $R_1 \to R_2$ along ν such that $R_2 = k[x_1(2), \ldots, x_{m-1}(2)]$, the center of ν on R_2 is $\mathfrak{m}_2 = (x_1(2), \ldots, x_{m-1}(2))$ and each b_i and c is a monomial in $x_1(2), \ldots, x_n(2)$ (where $\nu(x_1(2)), \ldots \nu(x_n(2))$ is a rational basis of $\Phi_\nu \otimes \mathbb{Q}$) times a unit in $(R_2)_{\mathfrak{m}_2}$. Let U_2 be the normal local ring of K^s which is dominated by ν^s and lies over $(R_2)_{\mathfrak{m}_2}$. Then U_2 is a regular local ring with $\hat{U}_2 = \hat{R}_2$ (by Lemma 2.2). In particular, $x_1(2), \ldots, x_{m-1}(2)$ are regular parameters in U_2 and each nonzero

$$(24) \qquad\qquad a_i = x_1(2)^{c_1(i)} \cdots x_n(2)^{c_n(i)} \varphi_i$$

is a monomial in $x_1(1), \ldots, x_n(2)$ times a unit φ_i in U_2 and $z = x_1(2)^{b_1} \cdots x_n(2)^{b_n} \lambda$ is a monomial in $x_1(2), \ldots, x_n(2)$ times a unit λ in U_2.

Let $T_2 = k[x_1(2), \ldots, x_{m-1}(2), x_m]$, a birational extension of T, such that the center of ν^* on T_2 is $(x_1(2), \ldots, x_{m-1}(2), x_m)$. Now perform the Perron transform $T_2 \to T_3$ along ν^* defined by $x_i(2) = x_i(3)$ for $1 \leq i \leq m - 1$ and

$$(25) \qquad\qquad x_m = x_1(3)^{b_1} \cdots x_n(3)^{b_n}(x_m(3) + \beta)$$

where β is the residue of λ in $V_{\nu^s}/m_{\nu^s} \cong k$. Thus

$$x_m - z = x_1(3)^{b_1} \cdots x_n(3)^{b_n}(x_m(3) - (\lambda - \beta))$$

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in $U_2[x_m(3)]$. After substituting $x = x_m$, (24) and (25) into (23), we have an expression

$$f_1 = x_1(3)^{e_1(d)} \cdots x_n(3)^{e_n(d)} \varphi_d(x_m(3) + \beta)^d + \cdots + x_1(3)^{e_1(0)} \cdots x_n(3)^{e_n(0)} \varphi_0$$

where each φ_i is a unit in U_2 or is zero (and $x_i(3) = x_i(2)$ for $1 \le i \le m-1$). Since

$$\nu^s(f_1) = \tau < \nu^s(a_i x^i) = \nu(x_1(3)^{e_1(i)} \cdots x_n(3)^{e_n(i)})$$

for $i > 0$, we have that

$$\nu(x_1(3)^{e_1(0)} \cdots x_n(3)^{e_n(0)}) < \nu(x_1(3)^{e_1(i)} \cdots x_n(3)^{e_n(i)})$$

for $1 < i$. Thus by Lemma 4.1, there exists a Perron transform along ν of type (7),

$$R_2 \to R_4 = k[x_1(4), \ldots, x_{m-1}(4)],$$

such that $x_1(3)^{e_1(0)} \cdots x_n(3)^{e_n(0)}$ divides $x_1(3)^{e_1(i)} \cdots x_n(3)^{e_n(i)}$ in R_4 for all $i > 0$. Now let U_4 be the normal local ring of K^s which lies over $(R_4)_{\mathfrak{m}_4}$ and is dominated by ν^s, where $\mathfrak{m}_4 = (x_1(4), \ldots, x_{m-1}(4))$, so that U_4 is a regular local ring with $\hat{U}_4 = \hat{R}_4$ where the completion of R_4 is with respect to the maximal ideal $\mathfrak{m}_4$ of R_4 (by Lemma 2.2). We then have that the strict transform $\overline{f}_1$ of f_1 in $U_4[x_m(3)]$ is $\overline{f}_1 = e_0 + e_1 x_m(3) + \cdots + e_d x_m(3)^d$ with $e_i \in U_4$ for $1 \le i$ and e_0 a unit in U_4. Letting $\overline{f}$ be the strict transform of $f(x_m)$ in $T_4 = R_4[x_m(3)]$, we have that $\overline{f} = (x_m(3) - (\lambda - \beta))\overline{f}_1$ in $U_4[x_m(3)]$, so that ord $\overline{f}(0, \ldots, 0, x_m(3)) = 1$, since $\lambda - \beta \in \mathfrak{m}_4 U_4$.

$\square$

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STEVEN DALE CUTKOSKY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

Email address: cutkoskys@missouri.edu

HUSSEIN MOURTADA, EQUIPE GÉOMÉTRIE ET DYNAMIQUE, INSTITUT MATHÉMATIQUE DE JUSSIEU-PARIS RIVE GAUCHE, UNIVERSITÉ PARIS 7, BÂTIMENT SOPHIE GERMAIN, CASE 7012, 75205, PARIS CEDEX 13, FRANCE

Email address: hussein.mourtada@imj-prg.fr