MIXED MULTIPLICITIES OF DIVISORIAL FILTRATIONS

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Abstract. Suppose that R is an excellent local domain with maximal ideal m_R. The theory of multiplicities and mixed multiplicities of m_R-primary ideals extends to (possibly non Noetherian) filtrations of R by m_R-primary ideals, and many of the classical theorems for m_R-primary ideals continue to hold for filtrations. The celebrated theorems involving inequalities continue to hold for filtrations, but the good conclusions that hold in the case of equality for m_R-primary ideals do not hold for filtrations.

In this article, we consider multiplicities and mixed multiplicities of R by m_R-primary divisorial filtrations. We show that some important theorems on equalities of multiplicities and mixed multiplicities of m_R-primary ideals, that are not true in general for filtrations, are true for divisorial filtrations. We prove that a theorem of Rees showing that if there is an inclusion of m_R-primary ideals $I \subset I'$ with the same multiplicity then I and I' have the same integral closure also holds for divisorial filtrations. This theorem does not hold for arbitrary filtrations. The classical Minkowski inequalities for m_R-primary ideals I_1 and I_2 hold quite generally for filtrations. If R has dimension two and there is equality in the Minkowski inequalities, then Teissier and Rees and Sharp have shown that there are powers I_1^a and I_2^b that have the same integral closure. This theorem does not hold for arbitrary filtrations. The Teissier-Rees-Sharp theorem has been extended by Katz to m_R-primary ideals in arbitrary dimension. We show that the Teissier-Rees-Sharp theorem does hold for divisorial filtrations in an excellent domain of dimension two.

We also show that the mixed multiplicities of divisorial filtrations are anti-positive intersection products on a suitable normal scheme X birationally dominating R, when R is an algebraic local domain (essentially of finite type over a field).

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Intro

1. Introduction

The study of mixed multiplicities of m_R-primary ideals in a Noetherian local ring R with maximal ideal m_R was initiated by Bhattacharya [Bh] [3], Rees [R] [34] and Teissier and Risler [T1] [42]. In [CSS] [14] the notion of mixed multiplicities is extended to arbitrary, not necessarily Noetherian, filtrations of R by m_R-primary ideals. It is shown in [CSS] [14] that many basic theorems for mixed multiplicities of m_R-primary ideals are true for filtrations.

The development of the subject of mixed multiplicities and its connection to Teissier's work on equisingularity [T1] [42] can be found in [GGV] [20]. A survey of the theory of mixed multiplicities of ideals can be found in [HS] [41, Chapter 17], including discussion of the results of the papers [R1] [35] of Rees and [S] [40] of Swanson, and the theory of Minkowski inequalities of Teissier [T1] [42], [T2] [43], Rees and Sharp [RS] [38] and Katz [Ka] [22]. Later, Katz and Verma [KV] [23], generalized mixed multiplicities to ideals that are not all m_R-primary. Trung and Verma [TV] [45] computed mixed multiplicities of monomial ideals from mixed volumes of suitable polytopes.

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We will be concerned with multiplicities and mixed multiplicities of (not necessarily Noetherian) filtrations, which are defined as follows.

Definition 1.1. *A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of a ring R is a descending chain*

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

of ideals such that $I_iI_j \subset I_{i+j}$ for all $i, j \in \mathbb{N}$. A filtration $\mathcal{I} = \{I_n\}$ of a local ring R by m_R-primary ideals is a filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of R such that I_n is m_R-primary for $n \geq 1$. A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of a ring R is said to be Noetherian if $\bigoplus_{n\geq 0} I_n$ is a finitely generated R-algebra.

The following theorem is the key result needed to define the multiplicity of a filtration of R by m_R-primary ideals. Let $\ell_R(M)$ denote the length of an R-module M.

TheoremI20 **Theorem 1.2.** *([C2, Theorem 1.1] and [C3, Theorem 4.2]) Suppose that R is a Noetherian local ring of dimension d, and $N(\hat{R})$ is the nilradical of the m_R-adic completion $\hat{R}$ of R. Then the limit*

I5 (1) $$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists for any filtration $\mathcal{I} = \{I_n\}$ of R by m_R-primary ideals, if and only if $\dim N(\hat{R}) < d$.

The problem of existence of such limits ([I5, I1]) has been considered by Ein, Lazarsfeld and Smith [ELS] and Mustaţă [Mus2]. When the ring R is a domain and is essentially of finite type over an algebraically closed field k with $R/m_R = k$, Lazarsfeld and Mustaţă [LM, C3] showed that the limit exists for all filtrations of R by m_R-primary ideals. Cutkosky [C3] proved it in the complete generality stated above in Theorem [TheoremI20, I1.2].

As can be seen from this theorem, one must impose the condition that the dimension of the nilradical of the completion $\hat{R}$ of R is less than the dimension of R. The nilradical $N(R)$ of a d-dimensional ring R is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

We have that $\dim N(R) = d$ if and only if there exists a minimal prime P of R such that $\dim R/P = d$ and R_P is not reduced. In particular, the condition $\dim N(\hat{R}) < d$ holds if R is analytically unramified; that is, $\hat{R}$ is reduced. We define the multiplicity of R with respect to the filtration $\mathcal{I} = \{I_n\}$ to be

$$e_R(\mathcal{I}; R) = \lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d/d!}.$$

The multiplicity of a ring with respect to a non Noetherian filtration can be an irrational number. A simple example on a regular local ring is given in [CSS, I14].

Mixed multiplicities of filtrations are defined in [CSS, I14]. Let M be a finitely generated R-module where R is a d-dimensional Noetherian local ring with $\dim N(\hat{R}) < d$. Let $\mathcal{I}(1) = \{I(1)_n\}, \ldots \mathcal{I}(r) = \{I(r)_n\}$ be filtrations of R by m_R-primary ideals. In [CSS, I14, Theorem 6.1] and [CSS, I14, Theorem 6.6], it is shown that the function

M2 (2) $$P(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(M/I(1)_{mn_1} \cdots I(r)_{mn_r}M)}{m^d}$$

is equal to a homogeneous polynomial $G(n_1, \ldots, n_r)$ of total degree d with real coefficients for all $n_1, \ldots, n_r \in \mathbb{N}$.

We define the mixed multiplicities of M from the coefficients of G, generalizing the definition of mixed multiplicities for m_R-primary ideals. Specifically, we write

eqV6 (3) $$G(n_1,\ldots,n_r)=\sum_{d_1+\cdots+d_r=d}\frac{1}{d_1!\cdots d_r!}e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)n_1^{d_1}\cdots n_r^{d_r}.$$

We say that $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)$ is the mixed multiplicity of M of type $(d_1,\ldots,d_r)$ with respect to the filtrations $\mathcal{I}(1),\ldots,\mathcal{I}(r)$. Here we are using the notation

eqI6 (4) $$e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)$$

to be consistent with the classical notation for mixed multiplicities of M with respect to m_R-primary ideals from [T1, 42]. The mixed multiplicity of M of type $(d_1,\ldots,d_r)$ with respect to m_R-primary ideals $I_1,\ldots,I_r$, denoted by $e_R(I_1^{[d_1]},\ldots,I_r^{[d_r]};M)$ ([T1, 42], [HS, 41, Definition 17.4.3]) is equal to the mixed multiplicity $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)$, where the Noetherian I-adic filtrations $\mathcal{I}(1),\ldots,\mathcal{I}(r)$ are defined by $\mathcal{I}(1)=\{I_1^i\}_{i\in\mathbb{N}},\ldots,\mathcal{I}(r)=\{I_r^i\}_{i\in\mathbb{N}}$.

We have that

eqX31 (5) $$e_R(\mathcal{I};M)=e_R(\mathcal{I}^{[d]};M)$$

if $r=1$, and $\mathcal{I}=\{I_i\}$ is a filtration of R by m_R-primary ideals. We have that

$$e_R(\mathcal{I};M)=\lim_{m\to\infty}d!\frac{\ell_R(M/I_mM)}{m^d}.$$

The multiplicities and mixed multiplicities of m_R-primary ideals are always positive ([T1, 42] or [HS, 41, Corollary 17.4.7]). The multiplicities and mixed multiplicities of filtrations are always nonnegative, as is clear for multiplicities, and is established for mixed multiplicities in [CSV, 15, Proposition 1.3]. However, they can be zero. If R is analytically irreducible, then all mixed multiplicities are positive if and only if the multiplicities $e_R(\mathcal{I}(j);R)$ are positive for $1\le j\le r$. This is established in [CSV, 15, Theorem 1.4].

Suppose that R is a d-dimensional excellent local domain, with quotient field K. A valuation ν of K is called an m_R-valuation if ν dominates R ($R\subset V_\nu$ and $m_\nu\cap R=m_R$ where V_ν is the valuation ring of ν with maximal ideal m_ν) and $\mathrm{trdeg}_{R/m_R}V_\nu/m_\nu=d-1$.

Suppose that I is an ideal in R. Let X be the normalization of the blowup of I, with projective birational morphism $\varphi:X\to\mathrm{Spec}(R)$. Let $E_1,\ldots,E_t$ be the irreducible components of $\varphi^{-1}(V(I))$ (which necessarily have dimension $d-1$). The Rees valuations of I are the discrete valuations ν_i for $1\le i\le t$ with valuation rings $V_{\nu_i}=\mathcal{O}_{X,E_i}$. If R is normal, then X is equal to the blowup of the integral closure $\overline{I^s}$ of an appropriate power I^s of I.

Every Rees valuation ν that dominates R is an m_R-valuation and every m_R-valuation is a Rees valuation of an m_R-primary ideal by [R3, 37, Statement (G)].

Associated to an m_R-valuation ν are valuation ideals

eqX2 (6) $$I(\nu)_n=\{f\in R\mid \nu(f)\ge n\}$$

for $n\in\mathbb{N}$. In general, the filtration $\mathcal{I}(\nu)=\{I(\nu)_n\}$ is not Noetherian. In a two-dimensional normal local ring R, the condition that the filtration of valuation ideals of R is Noetherian for all m_R-valuations dominating R is the condition (N) of Muhly and Sakuma [MS, 31]. It is proven in [C6, 7] that a complete normal local ring of dimension two satisfies condition (N) if and only if its divisor class group is a torsion group. An example is given in [CGP, 5] of an m_R-valuation of a 3-dimensional regular local ring R that is not Noetherian.

DefDF **Definition 1.3.** *Suppose that R is an excellent local domain. We say that a filtration $\mathcal{I}$ of R by m_R-primary ideals is a divisorial filtration if there exists a projective birational morphism $\varphi : X \rightarrow Spec(R)$ such that X is the normalization of the blowup of an m_R-primary ideal and there exists a nonzero effective Cartier divisor D on X with exceptional support for φ such that $\mathcal{I} = \{I(mD)\}_{m\in\mathbb{N}}$ where*

eqX22
$$I(mD) = I_R(mD) = \Gamma(X, \mathcal{O}_X(-mD)) \cap R. \tag{7}$$

If R is normal, then $I(mD) = \Gamma(X, \mathcal{O}_X(-mD))$. If $D = \sum_{i=1}^t a_i E_i$ where the $a_i \in \mathbb{N}$ and the E_i are prime exceptional divisors of φ, with associated m_R-valuations ν_i, then

$$I(mD) = I(\nu_1)_{a_1 m} \cap \cdots \cap I(\nu_t)_{a_t m}.$$

Suppose that $\mathcal{I}(1), \ldots, \mathcal{I}(r)$ are divisorial filtrations of an excellent local domain R. We then have associated mixed multiplicities

eqX20
$$e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; R) \tag{8}$$

for $d_1, \ldots, d_r \in \mathbb{N}$ with $d_1 + \cdots + d_r = d$.

If R is analytically irreducible, then all mixed multiplicities (8) are positive by Proposition 2.1.

We show in (54) and (53) of Section 5 that if R has dimension two, then the mixed multiplicities (8) are positive rational numbers. In Example 6 of [16], an example is given of an m_R-valuation ν dominating a normal excellent local domain of dimension three such that $e_R(\mathcal{I}(\nu); R)$ is an irrational number. Thus the mixed multiplicities (8) can be irrational if $d \geq 3$.

The following theorem in [14] generalizes [41, Proposition 11.2.1] for m_R-primary ideals to filtrations of R by m_R-primary ideals.

Theorem13 **Theorem 1.4.** *([14, Theorem 6.9]) Suppose that R is a Noetherian d-dimensional local ring such that*

$$\dim N(\hat{R}) < d$$

and M is a finitely generated R-module. Suppose that $\mathcal{I}' = \{I'_i\}$ and $\mathcal{I} = \{I_i\}$ are filtrations of R by m_R-primary ideals. Suppose that $\mathcal{I}' \subset \mathcal{I}$ ($I'_i \subset I_i$ for all i) and the ring $\bigoplus_{n\geq 0} I_n$ is integral over $\bigoplus_{n\geq 0} I'_n$. Then

$$e_R(\mathcal{I}; M) = e_R(\mathcal{I}'; M).$$

We give a proof of Theorem 1.4 in the Appendix.

Rees has shown in [34] that if R is a formally equidimensional Noetherian local ring and $I \subset I'$ are m_R-primary ideals such that $e_R(I; R) = e_R(I'; R)$, then $\bigoplus_{n\geq 0}(I')^n$ is integral over $\bigoplus_{n\geq 0} I^n$ (I and I' have the same integral closure). An exposition of this converse to the above cited [41, Proposition 11.2.1] is given in [41, Proposition 11.3.1], in the section entitled "Rees's Theorem". Rees's theorem is not true in general for filtrations of m_R-primary ideals (a simple example in a regular local ring is given in [14]) but it is true for divisorial filtrations. In Theorem 3.5, we show that Rees's theorem (the converse of Theorem 1.4) is true for divisorial filtrations of an excellent local domain.

An analogue of the Rees theorem for projective varieties is proven in Theorem 4.2.

We prove in [14, Theorem 6.3] that the Minkowski inequalities hold for filtrations of m_R-primary ideals.

TheoremMI **Theorem 1.5.** *(Minkowski Inequalities for filtrations)([14, Theorem 6.3]) Suppose that R is a Noetherian d-dimensional local ring with $\dim N(\hat{R}) < d$, M is a finitely generated*

R-module and $\mathcal{I}(1) = \{I(1)_j\}$ and $\mathcal{I}(2) = \{I(2)_j\}$ are filtrations of R by m_R-primary ideals. Then

1) $e_R(\mathcal{I}(1)^{[i]}, \mathcal{I}(2)^{[d-i]}; M)^2 \leq e_R(\mathcal{I}(1)^{[i+1]}, \mathcal{I}(2)^{[d-i-1]}; M) e_R(\mathcal{I}(1)^{[i-1]}, \mathcal{I}(2)^{[d-i+1]}; M)$ *for* $1 \leq i \leq d-1$.
2) *For* $0 \leq i \leq d$,

$$e_R(\mathcal{I}(1)^{[i]}, \mathcal{I}(2)^{[d-i]}; M) e_R(\mathcal{I}(1)^{[d-i]}, \mathcal{I}(2)^{[i]}; M) \leq e_R(\mathcal{I}(1); M) e_R(\mathcal{I}(2); M),$$

3) *For* $0 \leq i \leq d$, $e_R(\mathcal{I}(1)^{[d-i]}, \mathcal{I}(2)^{[i]}; M)^d \leq e_R(\mathcal{I}(1); M)^{d-i} e_R(\mathcal{I}(2); M)^i$ *and*
4) $e_R(\mathcal{I}(1)\mathcal{I}(2)); M)^{\frac{1}{d}} \leq e_R(\mathcal{I}(1); M)^{\frac{1}{d}} + e_R(\mathcal{I}(2); M)^{\frac{1}{d}}$,
where $\mathcal{I}(1)\mathcal{I}(2) = \{I(1)_j I(2)_j\}$.

The Minkowski inequalities were formulated and proven for m_R-primary ideals by Teissier [T1, 42], [T2, 43] and proven in full generality, for Noetherian local rings, by Rees and Sharp [RS, 38]. The fourth inequality 4) was proven for filtrations of R by m_R-primary ideals in a regular local ring with algebraically closed residue field by Mustaţă ([Mus, 32, Corollary 1.9]) and more recently by Kaveh and Khovanskii ([KK1, 24, Corollary 7.14]). The inequality 4) was proven with our assumption that $\dim N(\hat{R}) < d$ in [C3, 11, Theorem 3.1]. Inequalities 2) - 4) can be deduced directly from inequality 1), as explained in [T1, 42], [T2, 43], [RS, 38] and [HS, 41, Corollary 17.7.3].

Teissier [T3, 44] (for Cohen-Macaulay normal two-dimensional complex analytic R), Rees and Sharp [RS, 38] (in dimension 2) and Katz [Ka, 22] (in complete generality) have proven that if R is a d-dimensional formally equidimensional Noetherian local ring and $I(1)$, $I(2)$ are m_R-primary ideals such that the Minkowski equality

$$e_R((I(1)I(2)); R)^{\frac{1}{d}} = e_R(I(1); R)^{\frac{1}{d}} + e_R(I(2); R)^{\frac{1}{d}}$$

holds, then there exist positive integers r and s such that the integral closures $\overline{I(1)^r}$ and $\overline{I(2)^s}$ of the ideals $I(1)^r$ and $S(2)^s$ are equal, which is equivalent to the statement that the R-algebras $\bigoplus_{n\geq 0} I(1)^{rn}$ and $\bigoplus_{n\geq 0} I(2)^{sn}$ have the same integral closure.

The Teissier-Rees-Sharp-Katz theorem is not true for filtrations, even in a regular local ring, as is shown in a simple example in [CSS, 14].

In Theorem 5.9, PropR11 we show that the Teissier-Rees-Sharp theorem is true for divisorial filtrations of an excellent two-dimensional local domain.

In Section 8, SecAPM1 we interpret the mixed multiplicities of divisorial filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$ as intersection multiplicities. We assume that R is an algebraic local domain; that is, a domain that is essentially of finite type over an arbitrary field k (a localization of a finitely generated k-algebra), and that $\varphi : X \rightarrow \mathrm{Spec}(R)$ is the normalization of the blowup of an m_R-primary ideal. We define in Section 7 SecAPM3 anti-positive intersection products $\langle F_1, \ldots, F_d \rangle$ of anti-effective Cartier divisors $F_1, \ldots, F_d$ on X with exceptional support for φ, generalizing the positive intersection product of Cartier divisors defined on projective varieties in [BFJ, 4] over an algebraically closed field of characteristic zero and in [C4, 10] over an arbitrary field.

Suppose that $D(1), \ldots, D(r)$ are Cartier divisors on X with exceptional support. Let $\mathcal{I}(j) = \{I(nD(j))\}$ for $1 \leq i \leq r$ be divisorial filtrations of R, where the m_R-primary ideals $I(nD(j))$ are defined by (7). eqX22

In Theorem 8.3, TheoremA we show that, when R is normal, the mixed multiplicities

$$e_R((1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; R) = -\langle (-D(1))^{d_1}, \ldots, (-D(r))^{d_r} \rangle$$

are the negatives of the corresponding anti-positive intersection multiplicities for all

$$d_1, \ldots, d_r \in \mathbb{N}$$

such that $d_1 + \cdots + d_r = d$. A related formula is given in Theorem 8.4 if R is not normal.

When R has dimension 2, the anti-positive intersection product

$$\langle (-D(1))^{d_1}, (-D(2))^{d_2} \rangle = (\Delta_1^{d_1} \cdot \Delta_2^{d_2})$$

is the ordinary intersection product of the anti-nef parts Δ_1, Δ_2 of the respective Zariski decompositions of D_1 and D_2.

In Section 5, we develop the theory of mixed multiplicities of divisorial filtrations in a two-dimensional excellent local domain using the theory of Zariski decomposition. We give a proof of Theorem 3.5 in dimension 2 using this method in Proposition 5.8 and use this method to prove Proposition 5.9 on the Minkowski equality.

We use the method of volumes of convex bodies associated to appropriate semigroups introduced in [33], [28] and [25].

We will denote the nonnegative integers by $\mathbb{N}$ and the positive integers by $\mathbb{Z}_+$. We will denote the set of nonnegative rational numbers by $\mathbb{Q}_{\geq 0}$ and the positive rational numbers by $\mathbb{Q}_+$. We will denote the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$. For a real number x, $\lceil x \rceil$ will denote the smallest integer that is $\geq x$ and $\lfloor x \rfloor$ will denote the largest integer that is $\leq x$. If $E_1, \ldots, E_r$ are prime divisors on a normal scheme X and $a_1, \ldots, a_r \in \mathbb{R}$, then $\lfloor \sum a_i E_i \rfloor$ denotes the integral divisor $\sum \lfloor a_i \rfloor E_i$ and $\lceil a_i E_i \rceil$ denotes the integral divisor $\sum \lceil a_i \rceil E_i$.

The maximal ideal of a local ring R will be denoted by m_R. The quotient field of a domain R will be denoted by $\mathrm{QF}(R)$. We will denote the length of an R-module M by $\ell_R(M)$.

2. First Properties of Mixed multiplicities of divisorial filtrations

In this section we prove some basic facts about mixed multiplicities of valuation ideals an divisorial filtrations that will be useful.

Proposition 2.1. *Suppose that R is an excellent, analytically irreducible d-dimensional local domain and $\nu_1, \ldots, \nu_t$ are m_R-valuations of R.*

1) *Suppose that $a_1, \ldots, a_t \in \mathbb{N}$ are not all zero. Let $I_n = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_t)_{na_t}$ and $\mathcal{I} = \{I_n\}$. Then*

$$e_R(\mathcal{I}; R) > 0.$$

2) *Suppose that $r \in \mathbb{Z}_+$ and $a_i(j) \in \mathbb{N}$ for $1 \leq i \leq t$ and $1 \leq j \leq r$ and for each j, not all $a_i(j)$ are zero. Let $I(j)_n = I(\nu_1)_{na_1(j)} \cap \cdots \cap I(\nu_t)_{na_t(j)}$ for $1 \leq j \leq r$ and $\mathcal{I}(j) = \{I(j)_n\}$ for $1 \leq j \leq r$. Then*

$$e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; R) > 0$$

for all $d_1, \ldots, d_r \in \mathbb{N}$ with $d_1 + \cdots + d_r = d$.

Proof. We first prove 1). By statement (G) of [37], for each m_R-valuation ν_i of R, there exists an m_R-primary ideal J_i such that ν_i is a Rees valuation of J_i. Now letting $J = J_1 J_2 \cdots J_t$, we have that $\nu_1, \ldots, \nu_t$ are amongst the Rees valuations of J. We can if necessary increase the set $\nu_1, \ldots, \nu_t$ and set $a_i = 0$ for each new i to assume that $\nu_1, \ldots, \nu_t$ are the entirety of the Rees valuations for J. By Rees's Izumi theorem [37], the topologies of the

ν_i are linearly equivalent. Let $\overline{\nu}_J$ be the reduced order. By the Rees valuation theorem (recalled in [R3, 37]),

$$\overline{\nu}_J(x) = \min_i \left\{ \frac{\nu_i(x)}{\nu_i(J)} \right\}$$

for $x \in R$, so the topology induced by $\overline{\nu}_J$ is linearly equivalent to the topology induced by the ν_i. We have that $\overline{\nu}_J$ is linearly equivalent to the J-topology by [R2, 36] since R is analytically unramified.

Thus there exists $\alpha \in \mathbb{Z}_+$ such that

eqX5 (9) $$I(\nu_i)_{\alpha n} \subset J^n \subset m_R^n \text{ for all } n \in \mathbb{Z}_+.$$

Let $a = \max\{a_1, \ldots, a_t\}$. Then $I_{a\alpha n} \subset m_R^n$ for all n. So $\ell_R(R/m_R^n) \leq \ell_R(R/I_{n\alpha a})$ for all n and so

$$e_R(\mathcal{I}; R) \geq \frac{1}{(a\alpha)^d} e_R(m_R; R) > 0.$$

We now prove 2). Statement 1) implies that $e_R(\mathcal{I}(j); R) > 0$ for $1 \leq j \leq r$. Thus all mixed multiplicities are positive by [CSV, 15, Theorem 1.4].

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Not

2.1. **Divisors and sections on blowups.** Suppose that R is an excellent d-dimensional local domain. Let S be the normalization of R, which is a finitely generated R-module, and let $m_1, \ldots, m_t$ be the maximal ideals of S. Let $\varphi : X \to \mathrm{Spec}(R)$ be a birational projective morphism such that X is the normalization of the blowup of an m_R-primary ideal. Since X is normal, φ factors through $\mathrm{Spec}(S)$. Let $\varphi_i : X_i \to \mathrm{Spec}(S_{m_i})$ be the induced projective morphisms where $X_i = X \times_{\mathrm{Spec}(S)} \mathrm{Spec}(S_{m_i})$. For $1 \leq i \leq t$, let $\{E_{i,j}\}$ be the irreducible exceptional divisors in $\varphi_i^{-1}(m_i)$.

Suppose that D is an effective exceptional Weil divisor on X. Write $D = \sum_{i,j} a_{i,j} E_{i,j}$ with $a_{ij} \in \mathbb{N}$. Define $D_i = \sum_j a_{i,j} E_{i,j}$ for $1 \leq i \leq t$. The reflexive coherent sheaf $\mathcal{O}_X(-D)$ of $\mathcal{O}_X$-modules is defined by $\mathcal{O}_X(-D) = i_* \mathcal{O}_U(-D|U)$ where U is the open subset of regular points of X and $i : U \to X$ is the inclusion. We have that $\dim(X \setminus U) \leq d - 2$ since X is normal. The basic properties of this sheaf are developed for instance in [C5, 12, Section 13.2]. We have that $S \subset \mathcal{O}_{X,p}$ for all $p \in X$, since $\mathcal{O}_{X,p}$ is normal. Now $\Gamma(X, \mathcal{O}_X)$ is a domain with the same quotient field as R, and is a finitely generated R-module since φ is proper. Thus $\Gamma(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X(0)) = S$.

Let

eqX30 (10)
$$\begin{aligned} J(D) &= \Gamma(X, \mathcal{O}_X(-D)), \\ J(D_i) &= \Gamma(X_i, \mathcal{O}_{X_i}(-D_i)), \\ I(D) &= J(D) \cap R, \\ I(D_i) &= J(D_i) \cap R. \end{aligned}$$

We have that

eqR6 (11) $$S/J(D) \cong \bigoplus_{i=1}^t S_{m_i}/\Gamma(X_i, \mathcal{O}_{X_i}(-D_i)) \cong \bigoplus_{i=1}^t S_{m_i}/J(D_i)$$

and so

eqR15 (12) $$\ell_R(S/J(D)) = \sum_{i=1}^t \ell_R(S_{m_i}/J(D_i)) = \sum_{i=1}^t [S/m_i : R/m_R] \ell_{S_{m_i}}(S_{m_i}/J(D_i)).$$

We have that $[S/m_i : R/m_R] < \infty$ for all i since S is a finitely generated R-module.

Let $D(1), \ldots, D(r)$ be effective Weil divisors on X with exceptional support in $\varphi^{-1}(m_R)$.

LemmaR1 **Lemma 2.2.** *For $n_1, \ldots, n_r \in \mathbb{N}$,*

$$\lim_{n\to\infty} \frac{\ell_R(R/I(nn_1D(1))\cdots I(nn_rD(r)))}{n^d} = \lim_{n\to\infty} \frac{\ell_R(S/J(nn_1D(1))\cdots J(nn_rD(r)))}{n^d}.$$

Proof. Fix $n_1, \ldots, n_r \in \mathbb{N}$. Let $\mathcal{C}$ be the conductor of R (which is a nonzero ideal in both R and S), and choose $0 \neq x \in \mathcal{C}$. We then have short exact sequences of S-modules

$$0 \to A_n \to S/J(nn_1D(1))\cdots J(nn_rD(r)) \xrightarrow{x^r} S/J(nn_1D(1))\cdots J(nn_rD(r)) \to C_n \to 0$$

where A_n and C_n are the respective kernels and cokernels of multiplication of

$$S/J(nn_1D(1))\cdots J(nn_rD(r))$$

by x^r. We have that

$$C_n \cong S/(x^rS+J(nn_1D(1))\cdots J(nn_rD(r))) \cong (S/x^rS)/(J(nn_1D(1))\cdots J(nn_rD(r))(S/x^rS)).$$

Thus $\lim_{n\to\infty} \frac{\ell_S(C_n)}{n^d} = 0$ since $\dim S/x^rS = d-1$. Now

$$S/J(nn_1D(1))\cdots J(nn_rD(r)) \cong \bigoplus_{j=1}^{t} S_{m_j}/J(nn_1D(1)_j)\cdots J(nn_rD(r)_j)).$$

By Theorem TheoremI20 1.2, the limit

$$\lim_{n\to\infty} \frac{\ell_S(S/J(nn_1D(1))\cdots J(nn_rD(r)))}{n^d} = \sum_{j=1}^{t} \lim_{n\to\infty} \frac{\ell_{S_{m_j}}(S_{m_j}/J(nn_1D(1)_j)\cdots J(nn_rD(r)_j))}{n^d}$$

exists and so $\lim_{n\to\infty} \frac{\ell_S(A_n)}{n^d} = 0$. Let F_n and B_n be the respective kernels and cokernels of the homomorphisms of R-modules

$$S/J(nn_1D(1))\cdots J(nn_rD(r)) \xrightarrow{x^r} R/I(nn_1D(1))\cdots I(nn_rD(r))).$$

Then we have short exact sequences of R-modules

$$0 \to F_n \to S/J(nn_1D(1))\cdots J(nn_rD(r)) \xrightarrow{x^r} R/I(nn_1D(1))\cdots I(nn_rD(r))) \to B_n \to 0.$$

We have natural surjections of R-modules

$$(R/x^rR)/I(nn_1D(1))\cdots I(nn_rD(r))(R/x^rR) \cong R/(x^rR+I(nn_1D(1))\cdots I(nn_rD(r))) \to B_n.$$

Now $\dim R/x^rR = d-1$ so

$$\lim_{n\to\infty} \frac{\ell_R((R/x^rR)/I(nn_1D(1))\cdots I(nn_rD(r))(R/x^rR))}{n^d} = 0,$$

and so

$$\lim_{n\to\infty} \frac{\ell_R(B_n)}{n^d} = 0.$$

Since the support of the S-module A_n is contained in the set of maximal ideals $\{m_1, \ldots, m_t\}$, we have that $A_n \cong \bigoplus_{j=1}^{t} (A_n)_{m_j}$ and $\ell_S(A_n) = \sum_{j=1}^{t} \ell_{S_{m_j}}((A_n)_{m_j})$. Thus

$$\begin{aligned} \ell_R(A_n) &= \sum_{j=1}^{t} [S/m_j : R/m_R]\ell_{S_{m_j}}((A_n)_{m_j}) \\ &\leq \mu\ell_S(A_n) \end{aligned}$$

where $\mu = \max_j\{[S/m_j : R/m_R]\}$. We then have that

$$lim_{n\to\infty} \frac{\ell_R(A_n)}{n^d} \leq \mu \lim_{n\to\infty} \frac{\ell_S(A_n)}{n^d} = 0.$$

There are natural inclusions $F_n \subset A_n$ for all n, so

$$\lim_{n\to\infty} \frac{\ell_R(F_n)}{n^d} = 0$$

and thus

$$\lim_{n\to\infty} \frac{\ell_R(R/I(nn_1D(1))\cdots I(nn_rD(r)))}{n^d} = \lim_{n\to\infty} \frac{\ell_R(S/J(nn_1D(1))\cdots J(nn_rD(r)))}{n^d}.$$

□

SecR

3. Rees's theorem for divisorial filtrations

In this section, suppose that R is a d-dimensional normal excellent local ring. Let $\varphi : X \to \mathrm{Spec}(R)$ be a birational projective morphism that is the blowup of an m_R-primary ideal such that X is normal.

Let $E_1, \ldots, E_r$ be the prime exceptional divisors of φ (which all contract to m_R), and let μ_i be the discrete valuation with valuation ring $\mathcal{O}_{X,E_i}$ for $1 \le i \le r$. Let D be a nonzero effective Cartier divisor on X with exceptional support. For $1 \le i \le r$ and $m \in \mathbb{N}$, let

$$I(\mu_i)_m = \{f \in R \mid \mu_i(f) \ge m\},$$

as defined in (6), and define

$$\tau_{E_i,m}(D) = \min\{\mu_i(f) \mid f \in \Gamma(X, \mathcal{O}_X(-mD))\}.$$

Let $\tau_{m,i} = \tau_{E_i,m}(D)$. Then since $\tau_{mn,i} \le n\tau_{m,i}$, we have that

eqAR1

$$\frac{\tau_{mn,i}}{mn} \le \min\{\frac{\tau_{m,i}}{m}, \frac{\tau_{n,i}}{n}\}. \tag{13}$$

Now define

$$\gamma_{E_i}(D) = \inf_m \frac{\tau_{m,i}}{m}.$$

Expand $D = \sum_{i=1}^r a_i E_i$ with $a_i \in \mathbb{N}$. We have that

$$\Gamma(X, \mathcal{O}_X(-mD)) = \{f \in R \mid \mu_i(f) \ge ma_i \text{ for } 1 \le i \le r\}.$$

Thus $\tau_{E_i,m}(D) \ge ma_i$ for all $m \in \mathbb{N}$, and so

eqAR12

$$\gamma_{E_i}(D) \ge a_i \text{ for all } i. \tag{14}$$

LemmaAR1

Lemma 3.1. *We have that*

$$\Gamma(X, \mathcal{O}_X(-mD)) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r m\gamma_{E_i}(D)E_i\rceil))$$

for all $m \in \mathbb{N}$.

Proof. We have that

$$\Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r m\gamma_{E_i}(D)E_i\rceil)) \subset \Gamma(X, \mathcal{O}_X(-mD))$$

by (14).

Suppose that $f \in \Gamma(X, \mathcal{O}_X(-mD))$. Then $\mu_i(f) \ge \tau_{E_i,m}(D) \ge m\gamma_{E_i}(D)$ for all i, so that $\mu_i(f) \ge \lceil m\gamma_{E_i}(D)\rceil$ for all i since $\mu_i(f) \in \mathbb{N}$. □

We now define a valuation that we will use to compute volumes of Cartier divisors D, and that will allow us to extract some extra information that we need to prove Theorem 3.4 below.

Let i be any fixed index with $1 \leq i \leq r$. Suppose that $p \in E_i$ is a closed point that is nonsingular on X and E_i and that is not contained in E_j for $j \neq i$. Let

$$X = Y_0 \supset Y_1 = E_i \supset \cdots \supset Y_d = \{p\} \tag{15}$$

be a flag; that is, the Y_j are subvarieties of X of dimension $d-j$ such that there is a regular system of parameters $b_1, \ldots, b_d$ in $\mathcal{O}_{X,p}$ such that $b_1 = \cdots = b_j = 0$ are local equations of Y_j for $1 \leq j \leq d$.

The flag determines a valuation ν on the quotient field K of R as follows. We have a sequence of natural surjections of regular local rings

$$\mathcal{O}_{X,p} = \mathcal{O}_{Y_0,p} \stackrel{\sigma_1}{\to} \mathcal{O}_{Y_1,p} = \mathcal{O}_{Y_0,p}/(b_1) \stackrel{\sigma_2}{\to} \cdots \stackrel{\sigma_{d-1}}{\to} \mathcal{O}_{Y_{d-1},p} = \mathcal{O}_{Y_{d-2},p}/(b_{d-1}). \tag{16}$$

Define a rank-d discrete valuation ν on K (an Abhyankar valuation) by prescribing for $s \in \mathcal{O}_{X,p}$,

$$\nu(s) = (\mathrm{ord}_{Y_1}(s), \mathrm{ord}_{Y_2}(s_1), \cdots, \mathrm{ord}_{Y_d}(s_{d-1})) \in (\mathbb{Z}^d)_{\mathrm{lex}}$$

where

$$s_1 = \sigma_1\left(\frac{s}{b_1^{\mathrm{ord}_{Y_1}(s)}}\right), s_2 = \sigma_2\left(\frac{s_1}{b_2^{\mathrm{ord}_{Y_2}(s_1)}}\right), \ldots, s_{d-1} = \sigma_{d-1}\left(\frac{s_{d-2}}{b_{d-1}^{\mathrm{ord}_{Y_{d-1}}(s_{d-2})}}\right)$$

and $\mathrm{ord}_{Y_{j+1}}(s_j)$ is the highest power of b_{j+1} that divides s_j in $\mathcal{O}_{Y_j,p}$. We have that

$$\nu(s) = \left(\mu_i(s), \omega\left(\frac{s}{b_1^{\mu_i(s)}}\right)\right)$$

where ω is the rank-$(d-1)$ Abhyankar valuation on the function field $k(E_i)$ of E_i determined by the flag

$$E_i = Y_1 \supset \cdots \supset Y_d = \{p\}$$

on the projective k-variety E_i, where $k = R/m_R$.

Consider the graded linear series $L_n := \Gamma(E_i, \mathcal{O}_X(-nE_i) \otimes_{\mathcal{O}_X} \mathcal{O}_{E_i})$ on E_i. Let $g = b_1$, so that $g = 0$ is a local equation of E_i in $\mathcal{O}_{X,p}$. Then for $n \in \mathbb{N}$, we have natural commutative diagrams

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X(-nE_i)) & \to & \Gamma(E_i, \mathcal{O}_X(-nE_i) \otimes \mathcal{O}_{E_i}) \\ \downarrow & & \downarrow \\ O_X(-nE_i)_p & \to & \mathcal{O}_X(-nE_i)_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{E_i,p} \\ = O_{X,p}g^n & & \cong \mathcal{O}_{E_i,p} \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}g^n \end{array}$$

where we denote the rightmost vertical arrow by $s \mapsto \varepsilon_n(s) \otimes g^n$ and the bottom horizontal arrow is

$$f \mapsto \left[\frac{f}{g^n}\right] \otimes g^n,$$

where $\left[\frac{f}{g^n}\right]$ is the class of $\frac{f}{g^n}$ in $\mathcal{O}_{E_i,p}$.

Let Ξ be the semigroup defined for our fixed index i by

$$\Xi = \{(n, \omega(\varepsilon_n(s))) \mid n \in \mathbb{N} \text{ and } s \in \Gamma(E_i, \mathcal{O}_X(-nE_i) \otimes_{\mathcal{O}_X} \mathcal{O}_{E_i})\} \subset \mathbb{Z}^d,$$

and let $\Delta(\Xi)$ be the intersection of the closed convex cone generated by Ξ in $\mathbb{R}^d$ with $\{1\} \times \mathbb{R}^{d-1}$. By the proof of Theorem 8.1 [C2, 9] or the proof of [LM, 28, Theorem A], $\Delta(\Xi)$ is compact and convex. Let

$$\Xi_n := \{(n, \omega(\varepsilon_n(s))) \mid s \in \Gamma(E_i, \mathcal{O}_X(-nE_i) \otimes_{\mathcal{O}_X} \mathcal{O}_{E_i})\}$$

be the elements of Ξ at level n. Suppose that δ is a positive integer. Let $\Gamma_\delta(D)$ be the semigroup

$$\Gamma_\delta(D) = \{(\nu(f), n) \mid f \in I(nD) \text{ and } \mu_i(f) \leq n\delta\} \subset \mathbb{N}^{d+1}.$$

Let $\Delta_\delta(D)$ be the intersection of the closed convex cone generated by $\Gamma_\delta(D)$ in $\mathbb{R}^{d+1}$ with $\mathbb{R}^d \times \{1\}$.

We have that the elements of $\Gamma_\delta(D)$ at level m are

$$\Gamma_\delta(D)_m := \{(\nu(f), m) \mid f \in I(mD) \text{ and } \mu_i(f) \leq m\delta\} \subset (\cup_{0 \leq j \leq m\delta} \Xi_j) \times \{m\}.$$

For $t \in \mathbb{R}_+$, let $t\Delta(\Xi) = \{t\sigma \mid \sigma \in \Delta(\Xi)\}$. For $(\sigma, m) \in \Gamma_\delta(D)$, we have that

$$\frac{\sigma}{m} \in \cup_{0 \leq j \leq \delta m} \frac{j}{m} \Delta(\Xi) \subset \cup_{t \in [0,\delta]} t\Delta(\Xi).$$

The continuous map $[0, \delta] \times \Delta(\Xi) \to \mathbb{R}^d$ defined by $(t, x) \mapsto tx$ has image $\cup_{t \in [0,\delta]} t\Delta(\Xi)$ which is compact since $\Delta(\Xi)$ is. The closed convex set $\Delta_\delta(D)$ is thus compact since $\Delta_\delta(D)$ is contained in this image, and so $\Gamma_\delta(D)$ satisfies condition (5) of [C2, 9, Theorem 3.2].

Now we verify that condition (6) of [C2, 9, Theorem 3.2] is satisfied; that is, $\Gamma_\delta(D)$ generates $\mathbb{Z}^{d+1}$ as a group. Let $G(\Gamma_\delta(D))$ be the subgroup of $\mathbb{Z}^{d+1}$ generated by $\Gamma_\delta(D)$. We have that the value group of ν is $\mathbb{Z}^d$, and $e_j = \nu(b_j)$ for $1 \leq j \leq d$ is the natural basis of $\mathbb{Z}^d$. Write $b_j = \frac{f_j}{g_j}$ with $f_j, g_j \in R$ for $1 \leq j \leq d$. There exists $0 \neq h \in I(D)$. Thus $hf_j, hg_j \in I(D)$. There exists $c \in \mathbb{Z}_+$ such that $hf_j, hg_j \notin I(\mu_i)_c$ for $1 \leq j \leq d$. Possibly increasing δ in the definition of $\Gamma_\delta(D)$, we then have $(\nu(hf_j), 1), (\nu(hg_j), 1) \in \Gamma_\delta(D)$ for $1 \leq j \leq d$. Thus $(e_j, 0) = (\nu(hf_j) - \nu(hg_j), 0) \in G(\Gamma_\delta(D))$ for $1 \leq j \leq d$. Since $(\nu(hf_j), 1) \in \Gamma_\delta(D)$, we then have that $(0, 1) \in G(\Gamma_\delta(D))$. Thus we have that

$$\lim_{n \to \infty} \frac{\#\Gamma_\delta(D)_n}{n^d} = \mathrm{Vol}(\Delta_\delta(D))$$

by [C2, 9, Theorem 3.2] or [LM, 28, Proposition 2.1].

By Rees's Izumi theorem [R3, 37], we have that there exists $\lambda \in \mathbb{Z}_+$ such that if $f \in R$ and $\mu_i(f) \geq n\lambda$, then $\mu_j(f) \geq n$ for $1 \leq j \leq r$. Thus $I(\mu_i)_{n\lambda} \subset I(\mu_j)_n$ for all $n \in \mathbb{N}$, so that

$$I(\mu_i)_{na\lambda} \subset I(\mu_i)_{na_1} \cap \cdots \cap I(\mu_r)_{na_r} = \Gamma(X, \mathcal{O}_X(-nD))$$

where $a = \max\{a_1, \ldots, a_r\}$.

Take δ to be greater than or equal to $a\lambda$ in the definition of $\Gamma_\delta(D)$. Let

$$\mu = [\mathcal{O}_{X,p}/m_p : R/m_R].$$

Consider the Newton-Okounkov bodies $\Delta_\delta(0)$ and $\Delta_\delta(D)$ constructed from the semigroups $\Gamma_\delta(0)$ and $\Gamma_\delta(D)$ with this δ. Then, as in [C3, 11, Theorem 5.6],

eqAR10 (17)
$$\lim_{m \to \infty} \frac{\ell_R(R/I(mD))}{m^d} = \mu(\mathrm{Vol}(\Delta_\delta(0)) - \mathrm{Vol}(\Delta_\delta(D))).$$

In fact, we have that

eqAR11 (18)
$$\lim_{n \to \infty} \frac{\ell_R(I(nD)/I(\mu_i)_{\delta n})}{n^d} = \mu \mathrm{Vol}(\Delta_\delta(D)).$$

LemmaCCL **Lemma 3.2.** *Suppose that Δ_1 and Δ_2 are compact, convex subsets of $\mathbb{R}^d$, $\Delta_1 \subset \Delta_2$ and $\mathrm{Vol}(\Delta_1) = \mathrm{Vol}(\Delta_2) > 0$. Then $\Delta_1 = \Delta_2$.*

Proof. Suppose that $\Delta_1 \neq \Delta_2$. Then there exists $p \in \Delta_2 \setminus \Delta_1$. Since Δ_1 is closed in $\mathbb{R}^d$, there exists an epsilon ball $B_\varepsilon(p)$ centered at p in $\mathbb{R}^d$ such that $B_\varepsilon(p) \cap \Delta_1 = \emptyset$. Now Δ_2 has positive volume, so there exist $w_1, \ldots, w_d \in \Delta_2$ such that $v_1 = w_1 - p, \ldots, v_d = w_d - p$ is a real basis of $\mathbb{R}^d$. Since Δ_2 is convex, there exists $\delta > 0$ such that letting W be the hypercube

$$W = \{p + \alpha_1 v_1 + \cdots + \alpha_d v_d \mid 0 \leq \alpha_i \leq \delta \text{ for } 1 \leq i \leq d\},$$

we have that $W \subset \Delta_2 \cap B_\varepsilon(p)$. But then

$$\mathrm{Vol}(\Delta_2) - \mathrm{Vol}(\Delta_1) \geq \mathrm{Vol}(W) > 0.$$

a contradiction. Thus $\Delta_1 = \Delta_2$. □

LemmaAR2 **Lemma 3.3.** *For $\delta \gg 0$, we have that $\mathrm{Vol}(\Delta_\delta(D)) > 0$.*

Proof. By (9) in the proof of Proposition 2.1, there exists $\alpha \in \mathbb{Z}_+$ such that $I(\mu_i)_{\alpha n} \subset m_R^n$ for all $n \in \mathbb{Z}_+$ (since an excellent normal local ring is analytically ireducible). Further, there exists $c \in \mathbb{Z}_+$ such that $m_R^c \subset I(D)$, so that $m_R^{nc} \subset I(nD)$ for all n. Choosing $\delta > 2\alpha c$ so that $I(\mu_i)_{\delta n} \subset m_R^{2cn}$ for all n, we have that

$$\begin{aligned} \mathrm{Vol}(\Delta_\delta(D)) &= \tfrac{1}{\mu} \lim_{n\to\infty} \tfrac{\ell_R(nD)/I(\mu_i)_{\delta n})}{n^d} \\ &\geq \tfrac{1}{\mu} \lim_{n\to\infty} \tfrac{\ell_R(m_R^{cn}/m_R^{2cn})}{n^d} \\ &= \tfrac{1}{\mu} \tfrac{e_R(m_R;R)c^d(2^d-1)}{d!} > 0. \end{aligned}$$

□

TheoremAR1 **Theorem 3.4.** *Let D_1, D_2 be effective Cartier divisors on X with exceptional support, such that $D_1 \leq D_2$ and $e_R(\mathcal{I}_1, R) = e_R(\mathcal{I}_2, R)$, where $\mathcal{I}_1 = \{I(mD_1)\}$ and $\mathcal{I}_2 = \{I(mD_2)\}$. Then*

$$\Gamma(X, \mathcal{O}_X(-mD_1)) = \Gamma(X, \mathcal{O}_X(-mD_2))$$

for all $m \in \mathbb{N}$.

Proof. Write $D_1 = \sum_{i=1}^r a_i E_i$ and $D_2 = \sum_{i=1}^r b_i E_i$ with $a_i, b_i \geq 0$ for all i. For each i with $1 \leq i \leq r$ choose a flag (15) with $Y_1 = E_i$ and p a closed point such that p is nonsingular on X and E_i and $p \notin E_j$ for $j \neq i$. Let $\pi_1 : \mathbb{R}^{d+1} \to \mathbb{R}$ be the projection onto the first factor.

By the definition of $\gamma_{E_i}(D_2)$ and since (for δ sufficiently large) $\gamma_{E_i}(D_2)$ is in the closure of the compact set $\pi_1(\Delta_\delta(D_2))$,

$$\pi_1^{-1}(\gamma_{E_i}(D_2)) \cap \Delta_\delta(D_2) \neq \emptyset$$

and

$$\pi_1^{-1}(a) \cap \Delta_\delta(D_2) = \emptyset \text{ if } a < \gamma_{E_i}(D_2).$$

We have that $D_1 \leq D_2$ implies $\Delta_\delta(D_1) \subset \Delta_\delta(D_2)$. We have that $\mathrm{Vol}(\Delta_\delta(D_1)) > 0$ by Lemma 3.3 (taking δ sufficiently large). Since we are assuming that $e_R(\mathcal{I}_1; R) = e_R(\mathcal{I}_2; R)$, by (17), we have that $\mathrm{Vol}(D_1) = \mathrm{Vol}(D_2)$, and so $\Delta(D_1) = \Delta(D_2)$ by Lemma 3.2 (taking δ sufficiently large). Thus

$$\gamma_{E_i}(D_1) = \gamma_{E_i}(D_2)$$

for $1 \leq i \leq r$. We obtain that

$$-\sum_{i=1}^{r} \gamma_{E_i}(D_2)E_i = -\sum_{i=1}^{r} \gamma_{E_i}(D_1)E_i.$$

By Lemma LemmaAR1 3.1, for all $m \geq 0$,

$$\begin{aligned}\Gamma(X, \mathcal{O}_X(-mD_1)) &= \Gamma(X, \mathcal{O}_X(-\lceil \textstyle\sum m\gamma_{E_i}(D_1)E_i\rceil)) \\ &= \Gamma(X, \mathcal{O}_X(-\lceil \textstyle\sum m\gamma_{E_i}(D_2)E_i\rceil)) \\ &= \Gamma(X, \mathcal{O}_X(-mD_2)).\end{aligned}$$

□

We now show that Rees's theorem for m_R-primary ideals, R [34], HS [41, Proposition 11.3.1], generalizes to divisorial filtrations, giving a converse to Theorem Theorem13 1.4 for divisorial filtrations.

TheoremX1 **Theorem 3.5.** *Suppose that R is a d-dimensional excellent local domain. Let $\varphi : X \to Spec(R)$ be the normalization of the blowup of an m_R-primary ideal. Suppose that $D(1)$ and $D(2)$ are effective Cartier divisors on X with exceptional support such that $D(1) \leq D(2)$ and $e_R(\mathcal{I}(1); R) = e_R(\mathcal{I}(2); R)$, where $\mathcal{I}(1), \mathcal{I}(2)$ are the filtrations by m_R-primary ideals $\mathcal{I}(i) = \{I(nD(i))\}$ and $\mathcal{I}(2) = \{I(nD(2))\}$. Then*

$$I(mD(1)) = I(mD(2))$$

for all $m \in \mathbb{N}$.

Proof. We use the notation introduced in Subsection Not 2.1. Let $D(1)_i, D(2)_i$ be the divisors induced by $D(1)$ and $D(2)$ on X_i. Since $D(1) \leq D(2)$, we have that

eqX1 (19) $$D(1)_i \leq D(2)_i \text{ for all } i.$$

Thus

eqX2* (20) $$e_{S_{m_i}}(\{J(mD(1)_i)\}; S_{m_i}) \leq e_{S_{m_i}}(\{J(mD(2)_i); S_{m_i}) \text{ for all } i.$$

Now Lemma LemmaR1 2.2 and (eqR15 12) imply

eqX3 (21) $$e_R(\mathcal{I}(j); R) = e_R(\{I(mD(j))\}; R) = \sum_{i=1}^{t} [S/m_i : R/m_R] e_{S_{m_i}}(\{J(mD(j)_i)\}; S_{m_i})$$

for $j = 1, 2$.

Now the assumption $e_R(\mathcal{I}(1); R) = e_R(\mathcal{I}(2); R)$, (eqX2* 20) and (eqX3 21) imply

eqX4 (22) $$e_{S_{m_i}}(\{J(mD(1)_i)\}; S_{m_i}) = e_{S_{m_i}}(\{J(mD(2)_i)\}; S_{m_i})$$

for all i. Now (eqX1 19), (eqX4 22) and Theorem TheoremAR1 3.4 imply

$$J(mD(1)_i) = \Gamma(X_i, \mathcal{O}_{X_i}(-mD(1)_i)) = \Gamma(X_i, \mathcal{O}_{X_i}(-mD(2)_i)) = J(mD(2)_i)$$

for all $m \in \mathbb{N}$ and all i. Thus

$$J(mD(1)) = \Gamma(X, \mathcal{O}_X(-mD(1))) = \Gamma(X, \mathcal{O}_X(-mD(2))) = J(mD(2))$$

for all $m \in \mathbb{N}$ by (eqR6 11). Thus

$$I(mD(1)) = J(mD(1)) \cap R = J(mD(2)) \cap R = I(mD_2)$$

for all $m \in \mathbb{N}$. □

4. A Geometric Rees Theorem

Let X be a d-dimensional normal projective variety over a field k. Suppose that D is an effective Cartier divisor on X. The volume of D is

$$\mathrm{Vol}(D) = \lim_{m\to\infty} \frac{\dim_k \Gamma(X, \mathcal{O}_X(mD))}{m^d/d!}.$$

Let E be a codimension one prime divisor on X. We now define $\tau_{E,m}$ and $\gamma_E(D)$ analogously to our definitions at the beginning of Section 3. Here we use slightly different language, since we (following tradition) work with divisors of sections. For $m \in \mathbb{N}$, define

$$\tau_{m,E}(D) = \min\{\mathrm{ord}_E\Delta \mid \Delta \in |mD|\}.$$

Then since $\tau_{mn,E} \leq n\tau_{m,E}$, we have that

$$\frac{\tau_{mn,E}}{mn} \leq \min\{\frac{\tau_{m,E}}{m}, \frac{\tau_{n,E}}{n}\}. \tag{23}$$

Now define

$$\gamma_E(D) = \inf_m \frac{\tau_{m,E}}{m}.$$

Expand $D = \sum_{i=1}^r a_i E_i$ with E_i prime divisors and $a_i \in \mathbb{Z}_+$.

Lemma 4.1. *We have that*

$$\Gamma(X, \mathcal{O}_X(mD)) = \Gamma(X, \mathcal{O}_X(mD - \sum_{i=1}^r \lceil m\gamma_{E_i}(D)\rceil E_i)) = \Gamma(X, \mathcal{O}_X(\lfloor mD - \sum_{i=1}^r m\gamma_{E_i}(D)E_i\rfloor))$$

for all $m \in \mathbb{N}$.

Proof. Suppose that $\Delta \in |mD|$. Then $\Delta - \sum_i \tau_{E_i,m}(D)E_i \geq 0$ so that $\Delta - \sum m\gamma_{E_i}E_i \geq 0$. Thus $\Delta - \sum_{i=1}^r \lceil m\gamma_{E_i}(D)\rceil E_i \geq 0$. □

We now recall the method of [28] to compute volumes of Cartier divisors, as extended in [9] to arbitrary fields. Suppose that $p \in X$ is a nonsingular closed point and

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_d = \{p\} \tag{24}$$

is a flag; that is, the Y_i are subvarieties of X of dimension $d-i$ such that there is a regular system of parameters $b_1, \ldots, b_d$ in $\mathcal{O}_{X,p}$ such that $b_1 = \cdots = b_i = 0$ are local equations of Y_i in X for $1 \leq i \leq d$.

The flag determines a valuation ν on the function field $k(X)$ of X as follows. We have a sequence of natural surjections of regular local rings

$$\mathcal{O}_{X,p} = \mathcal{O}_{Y_0,p} \stackrel{\sigma_1}{\to} \mathcal{O}_{Y_1,p} = \mathcal{O}_{Y_0,p}/(b_1) \stackrel{\sigma_2}{\to} \cdots \stackrel{\sigma_{d-1}}{\to} \mathcal{O}_{Y_{d-1},p} = \mathcal{O}_{Y_{d-2},p}/(b_{d-1}). \tag{25}$$

Define a rank d discrete valuation ν on $k(X)$ by prescribing for $s \in \mathcal{O}_{X,p}$,

$$\nu(s) = (\mathrm{ord}_{Y_1}(s), \mathrm{ord}_{Y_2}(s_1), \cdots, \mathrm{ord}_{Y_d}(s_{d-1})) \in (\mathbb{Z}^d)_{\mathrm{lex}}$$

where

$$s_1 = \sigma_1\left(\frac{s}{b_1^{\mathrm{ord}_{Y_1}(s)}}\right), s_2 = \sigma_2\left(\frac{s_1}{b_2^{\mathrm{ord}_{Y_2}(s_1)}}\right), \ldots, s_{d-1} = \sigma_{d-1}\left(\frac{s_{d-2}}{b_{d-1}^{\mathrm{ord}_{Y_{d-1}}(s_{d-2})}}\right).$$

Let $g = 0$ be a local equation of D at p. For $m \in \mathbb{N}$, define

$$\Phi_{mD} : \Gamma(X, \mathcal{O}_X(mD)) = \{f \in k(X) \mid (f) + mD \geq 0\} \to \mathbb{Z}^d$$

by $\Phi_{mD}(f) = \nu(fg^m)$. The Newton-Okounkov body $\Delta(D)$ of D is the closure of the set

$$\cup_{m\in\mathbb{N}}\frac{1}{m}\Phi_{mD}(\Gamma(X,\mathcal{O}_X(mD)))$$

in $\mathbb{R}^d$. This is a compact and convex set by [LM, 28, Lemma 1.10] or the proof of Theorem 8.1 [C2, 9].

Modifying the proof of [C2, 9, Theorem 8.1] and of [C3, 11, Lemma 5.4] we see that

GR4 (26) $$\mathrm{Vol}(D) = \lim_{m\to\infty}\frac{\dim_k\Gamma(X,\mathcal{O}_X(mD))}{m^d/d!} = d![\mathcal{O}_{X,p}/m_p : k]\mathrm{Vol}(\Delta(D)).$$

Suppose that $D_1 \leq D_2$ are effective Cartier divisors on X. Let $g_1 = 0$ be a local equation of D_1 at p, $g_2 = 0$ be a local equation of D_2 at p, so that $h = \frac{g_2}{g_1}$ is a local equation of $D_2 - D_1$ at p. We have commutative diagrams

$$\begin{array}{ccc}\Gamma(X,\mathcal{O}_D(mD_1)) & \to & \Gamma(X,\mathcal{O}_X(mD_2))\\ \downarrow \Phi_{mD_1}\times\{m\} & & \downarrow \Phi_{mD_2}\times\{m\}\\ \mathbb{Z}^{d+1} & \to & \mathbb{Z}^{d+1}\end{array}$$

where the top horizontal arrow is the natural inclusion and the bottom horizontal arrow is the map

$$(\alpha, m)\mapsto(\alpha + m\nu(h), m).$$

These diagrams induce an inclusion $\Lambda : \Delta(D_1)\to\Delta(D_2)$ defined by $\alpha\mapsto\alpha+\nu(h)$.

TheoremGRT **Theorem 4.2.** *Suppose that X is a normal projective variety over a field k and D_1, D_2 are effective Cartier divisors on X such that D_1 is big, $D_1 \leq D_2$ and $\mathrm{Vol}(D_1) = \mathrm{Vol}(D_2)$. Then*

$$\Gamma(X,\mathcal{O}_X(nD_1)) = \Gamma(X,\mathcal{O}_X(nD_2))$$

for all $n\in\mathbb{N}$.

Proof. Write $D_1 = \sum_{i=1}^r a_iE_i$ and $D_2 = \sum_{i=1}^r b_iE_i$ with $a_i, b_i \geq 0$ for all i. For each i with $1\leq i\leq r$ choose a flag (eqGR2, 24) with $Y_1 = E_i$ and p a point such that $p\in X$ is a nonsingular closed point of X and E_i and $p\notin E_j$ for $j\neq i$. Let $\pi_1 : \mathbb{R}^d\to\mathbb{R}$ be the projection onto the first factor. Then with the notation introduced above, $\nu(h) = (b_i - a_i, 0, \ldots, 0)$. By the definition of $\gamma_{E_i}(D_2)$ and since $\gamma_{E_i}(D_2)$ is in the closure of the compact set $\pi_1(\Delta(D_2))$, we have that

$$\pi_1^{-1}(\gamma_{E_i}(D_2))\cap\Delta(D_2)\neq\emptyset$$

and

$$\pi_1^{-1}(a)\cap\Delta(D_2)) = \emptyset \text{ if } a < \gamma_{E_i}(D_2).$$

Further, $\Lambda(\Delta(D_1))\subset\Delta(D_2)$ and $\mathrm{Vol}(D_1) = \mathrm{Vol}(D_2)$, so $\Lambda(\Delta(D_1)) = \Delta(D_2)$ by Lemma 3.2. Thus

$$\gamma_{E_i}(D_1) = \gamma_{E_i}(D_2) - (b_i - a_i)$$

for $1\leq i\leq r$. We obtain that

$$D_2 - \sum_{i=1}^r\gamma_{E_i}(D_2)E_i = D_1 - \sum_{i=1}^r\gamma_{E_i}(D_1)E_i.$$

By Lemma 4.1, for all $m\geq 0$,

$$\begin{array}{rcl}\Gamma(X,\mathcal{O}_X(mD_1)) &=& \Gamma(X,\mathcal{O}_X(\lfloor mD_1 - \sum m\gamma_{E_i}(D_1)E_i\rfloor))\\ &=& \Gamma(X,\mathcal{O}_X(\lfloor mD_2 - \sum m\gamma_{E_i}(D_2)E_i\rfloor))\\ &=& \Gamma(X,\mathcal{O}_X(mD_2)).\end{array}$$

□

Sec5

5. Mixed Multiplicities of two dimensional Excellent local rings

5.1. **2-dimensional normal local rings.** In this subsection, suppose that R is an excellent, normal local ring of dimension two, so that R is analytically irreducible. Resolutions of singularities of $\mathrm{Spec}(R)$ exist by [L2, 30] or [CJS, 6]. Let $\varphi : X \to \mathrm{Spec}(R)$ be a resolution of singularities with prime (integral) exceptional curves $E_1, \ldots, E_s$. By [L1, 29, Lemma 14.1], the intersection matrix of $E_1, \ldots, E_s$ is negative definite. Thus there exists an effective (necessarily Cartier) divisor B on X with exceptional support such that $\mathcal{O}_X(-B)$ is very ample, and so φ is the blowup of the m_R-primary ideal $\varphi_*\mathcal{O}_X(-B)$.

We refer to [L1, 29] for background material for this section. A $\mathbb{Q}$-divisor on X with exceptional support is a formal linear combination of prime exceptional curves with rational coefficients. A $\mathbb{Q}$-divisor C is anti-nef if $(C \cdot E) \leq 0$ for all exceptional curves E on X. Suppose that $f \in \mathrm{QF}(R)$ is nonzero. Then (f) will denote the divisor of f on X.

LemmaV1 **Lemma 5.1.** *Let D be an effective divisor on X with exceptional support. Then there is a unique minimal effective anti-nef $\mathbb{Q}$-divisor Δ on X with exceptional support such that $D \leq \Delta$.*

The $\mathbb{Q}$-divisor Δ is the unique effective $\mathbb{Q}$-divisor Δ on X such that

1) *$\Delta = D + B$ is anti-nef and B is effective.*
2) *$(\Delta \cdot E) = 0$ if E is a component of B.*

The first conclusion of the lemma follows from the proof of the existence of Zariski decomposition in [B, 2]. The second conclusion is the local formulation [CHR, 13, Proposition 2.1] of the classical theorem of Zariski [Z, 47].

We will say that the expression 1) is the Zariski decomposition of D and that Δ is the anti-nef part of the Zariski decomposition of D.

Remarkv1 **Remark 5.2.** *From the first conclusion of the lemma, we deduce that if $D_1 \leq D_2$ are effective divisors with exceptional support and respective anti-nef parts of their Zariski decompositions Δ_1 and Δ_2, then $\Delta_1 \leq \Delta_2$.*

CorV1 **Corollary 5.3.** *Suppose that $D_1 \leq D_2$ are effective divisors with effective support, and respective anti-nef parts of their Zariski decompositions Δ_1 and Δ_2. Then $(\Delta_2^2) \leq (\Delta_1^2)$ with equality if and only if $\Delta_1 = \Delta_2$.*

Proof. If Δ is an anti-nef divisor with exceptional support, and E is a nonzero effective $\mathbb{Q}$-divisor with exceptional support, then

$$(\Delta + E)^2 = (\Delta^2) + 2(\Delta \cdot E) + (E^2) < (\Delta^2)$$

since $(E^2) < 0$ as the intersection form on exceptional divisors on X is negative definite. □

Let ν_i be the discrete valuation with valuation ring $\mathcal{O}_{X,E_i}$ for $1 \leq i \leq r$, and $I(\nu_i)_n$ be the associated valuation ideals (as defined in (6) for $n \in \mathbb{N}$ and $1 \leq i \leq r$.

For $D = a_1E_1 + \cdots + a_rE_r$ an effective integral divisor on X with exceptional support ($a_i \in \mathbb{N}$ for all i), define

$$I(D) = \Gamma(X, \mathcal{O}_X(-D)) = \{f \in \mathrm{QF}(R) \mid (f) - D \geq 0\}.$$

This is in agreement with the notation of (10). In fact, we have that $I(D) = J(D)$ since R is normal.

We have that $I(0) = \Gamma(X, \mathcal{O}_X) = R$ since the ring $\Gamma(X, \mathcal{O}_X)$ is a finitely generated R-module with the same quotient field as R and R is normal. Thus $I(D)$ is an m_R-primary ideal if $D \neq 0$. For $n \in \mathbb{N}$, we have that

$$I(nD) = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_r)_{na_r}$$

is an m_R-primary ideal in R, and $\{I(nD)\}$ is a filtration of m_R-primary ideals in R. By Theorem 1.2, the limit

$$\mathrm{Vol}(D) := \lim_{n\to\infty} \frac{\ell_R(R/I(nD))}{n^2/2!} = e_R(\{I(nD)\}; R)$$

exists. In fact, by formula (7) and Lemma 2.5 on page 6 of [13], we have

eqV1 (27) $$\mathrm{Vol}(D) = -(\Delta^2)$$

where Δ is the anti-nef part of the Zariski decomposition of D.

RemarkV2 **Remark 5.4.** *We deduce from Corollary 5.3 that if $D_1 \leq D_2$ are effective divisors with exceptional support on X and respective anti-nef parts of their Zariski decompositons Δ_1 and Δ_2, then*

$$\mathrm{Vol}(D_1) \leq \mathrm{Vol}(D_2)$$

with equality if and only if $\Delta_1 = \Delta_2$.

We recall some notation introduced at the end of Section 1. Let $\lceil a \rceil$ denote the smallest integer that is greater than or equal to a real number a. If $D = \sum a_i E_i$ with $a_i \in \mathbb{Q}$ is a $\mathbb{Q}$-divisor, let $\lceil D \rceil = \sum \lceil a_i \rceil E_i$.

LemmaV2 **Lemma 5.5.** *Suppose that D is an effective divisor on X with exceptional support and $\Delta = D + B$ is the Zariski decomposition of D. Then for all $n \in \mathbb{N}$, $I(nD) = I(\lceil n\Delta \rceil)$.*

Proof. Suppose that $f \in I(\lceil n\Delta \rceil) = \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil))$. Then $(f) - \lceil n\Delta \rceil \geq 0$. Writing $n\Delta = \lceil n\Delta \rceil - G$ with $G \geq 0$, we have $-n\Delta = G - \lceil n\Delta \rceil$. From

$$-nD = -n\Delta + nB = -\lceil n\Delta \rceil + (G + nB)$$

and the fact that $G + nB \geq 0$, we have that $(f) - nD \geq 0$ so that $f \in \Gamma(X, \mathcal{O}_X(-nD)) = I(nD)$.

Let S be the set of irreducible curves in the support of B. Suppose that $f \in I(nD) = \Gamma(X, \mathcal{O}_X(-nD))$. Then $(f) - nD \geq 0$. Write $(f) - nD = A + C$ where A and C are effective divisors on X, no components of A are in S and all components of C are in S. We have that $(f) - n\Delta = A + (C - nB)$. If $E \in S$ then

$$(E \cdot (A + (C - nB))) = (E \cdot ((f) - n\Delta)) = 0$$

which implies $(E \cdot (C - nB)) = -(E \cdot A) \leq 0$. The intersection matrix of the curves in S is negative definite since it is so for the set of all exceptional curves, so $C - nB \geq 0$ (for instance by [Ba, Lemma 14.0]). Thus $(f) - n\Delta \geq 0$ which implies $(f) - \lceil n\Delta \rceil \geq 0$ since (f) is an integral divisor (that is, has integral coefficients). Thus $f \in \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil)) = I(\lceil n\Delta \rceil)$. □

PropV2 **Proposition 5.6.** *Suppose that D_1 and D_2 are effective divisors with exceptional support on X. Let $\mathcal{I}(1) = \{I(nD_1)\}$ and $\mathcal{I}(2) = \{I(nD_2)\}$. Suppose that $D_1 \leq D_2$ and*

$$e_R(\mathcal{I}(1); R) = e_R(\mathcal{I}(2); R).$$

Then $I(nD_1) = I(nD_2)$ for all $n \in \mathbb{N}$.

Proof. Let Δ_1 and Δ_2 be the respective anti-nef parts of the Zariski decompositions of D_1 and D_2. By Remark 5.4, $D_1 \leq D_2$ and $\mathrm{Vol}(D_1) = \mathrm{Vol}(D_2)$ implies $\Delta_1 = \Delta_2$. Thus

$$I(nD_1) = I(\lceil n\Delta_1 \rceil) = I(\lceil n\Delta_2 \rceil) = I(nD_2)$$

for all $n \in \mathbb{N}$ by Lemma 5.5. □

PropV1 **Proposition 5.7.** *Suppose that $D_1, \ldots, D_r$ are effective divisors on X with exceptional support. For $n_1, \ldots, n_r \in \mathbb{N}$, let*

$$G(n_1, \ldots, n_r) = \lim_{n\to\infty} \frac{\ell_R(R/I(nn_1D_1)\cdots I(nn_rD_r))}{n^2}.$$

Then for $n_1, \ldots, n_r \in \mathbb{N}$,

$$G(n_1, \ldots, n_r) = -\frac{1}{2}((n_1\Delta_1 + n_2\Delta_2 + \cdots + n_r\Delta_r)^2)$$

where $\Delta_1, \ldots, \Delta_r$ are the respective anti-nef parts of the Zariski decompositions of $D_1, \ldots, D_r$.

Proof. Fix $n_1, \ldots, n_r \in \mathbb{N}$. Given $\varepsilon > 0$, there exist effective $\mathbb{Q}$-divisors $F_{1,\varepsilon}, \ldots, F_{r,\varepsilon}, A_{1,\varepsilon}, \ldots, A_{r,\varepsilon}$ with exceptional support such that $-A_{i,\varepsilon}$ are ample for $1 \leq i \leq r$ (that is, $(A_{i,\varepsilon} \cdot E) < 0$ for all exceptional curves E and $(A_{i,\varepsilon}^2) > 0$), $-n_i\Delta_i = -A_{i,\varepsilon} + F_{i,\varepsilon}$ for $1 \leq i \leq r$,

$$|((n_1\Delta_1 + \cdots + n_r\Delta_r)^2) - ((A_{1,\varepsilon} + \cdots + A_{r,\varepsilon})^2)| < \varepsilon$$

and

$$|(n_i\Delta_i^2) - (A_{i,\varepsilon}^2)| < \varepsilon \text{ for } 1 \leq i \leq r.$$

Let $A_\varepsilon = A_{1,\varepsilon} + \cdots + A_{r,\varepsilon}$, $F_\varepsilon = F_{1,\varepsilon} + \cdots + F_{r,\varepsilon}$ so that

$$-(n_1\Delta_1 + \cdots + n_r\Delta_r) = -A_\varepsilon + F_\varepsilon.$$

There exists $s_\varepsilon \in \mathbb{Z}_+$ such that $s_\varepsilon A_{i,\varepsilon}$ and $s_\varepsilon\Delta_i$ are effective integral divisors (that is, have integral coefficients) for $1 \leq i \leq r$. Since the $-s_\varepsilon A_{i,\varepsilon}$ are ample integral divisors on X, there exists $\alpha_\varepsilon \in \mathbb{Z}_+$ such that the invertible sheaves $\mathcal{O}_X(-\alpha_\varepsilon s_\varepsilon A_{i,\varepsilon})$ are generated by global sections for $1 \leq i \leq r$. Thus for $n \in \mathbb{N}$,

$$\begin{aligned} I(\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon})^n \cdots I(\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon})^n \mathcal{O}_X &= I(n\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon}) \cdots I(n\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon})\mathcal{O}_X \\ &= I(n\alpha_\varepsilon s_\varepsilon A_\varepsilon)\mathcal{O}_X = I(\alpha_\varepsilon s_\varepsilon A_\varepsilon)^n \mathcal{O}_X. \end{aligned}$$

Thus the ideals

$$I(\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon})^n \cdots I(\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon})^n, I(n\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon}) \cdots I(n\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon}), I(n\alpha_\varepsilon s_\varepsilon A_\varepsilon), I(\alpha_\varepsilon s_\varepsilon A_\varepsilon)^n$$

have the same integral closure which is $I(n\alpha_\varepsilon s_\varepsilon A_\varepsilon)$, and so the R-algebra

$$\bigoplus_{n\geq 0} I(n\alpha_\varepsilon s_\varepsilon A_\varepsilon)$$

is integral over

$$\bigoplus_{n>0} I(n\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon}) \cdots I(n\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon}).$$

Now by Theorem 1.4 and (27),

eqV2 (28)
$$\begin{aligned} \lim_{n\to\infty} \tfrac{\ell_R(R/I(n\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon})\cdots I(n\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon}))}{n^2} &= \lim_{n\to\infty} \tfrac{\ell_R(R/I(n\alpha_\varepsilon s_\varepsilon A_\varepsilon))}{n^2} \\ &= -\tfrac{1}{2}((\alpha_\varepsilon s_\varepsilon A_\varepsilon)^2) = -\tfrac{\alpha_\varepsilon^2 s_\varepsilon^2}{2}(A_\varepsilon^2). \end{aligned}$$

For all $n \in \mathbb{N}$, we have inclusions

$$I(n\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon}) \cdots I(n\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon}) \subset I(n\alpha_\varepsilon s_\varepsilon n_1\Delta_1) \cdots I(n\alpha_\varepsilon s_\varepsilon n_r\Delta_r)) \subset I(n\alpha_\varepsilon s_\varepsilon(n_1\Delta_1 + \cdots + n_r\Delta_r))$$

inducing surjections

$$R/I(n\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon})\cdots I(n\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon})) \to R/I(n\alpha_\varepsilon s_\varepsilon n_1\Delta_1)\cdots I(n\alpha_\varepsilon s_\varepsilon n_r\Delta_r)) \\ \to R/I(n\alpha_\varepsilon s_\varepsilon(n_1\Delta_1+\cdots+n_r\Delta_r))$$

so that

$$\begin{aligned} -\tfrac{1}{2}(A_\varepsilon^2) &= \tfrac{1}{\alpha_\varepsilon^2 s_\varepsilon^2}\lim_{n\to\infty}\tfrac{\ell_R(R/I(n\alpha_\varepsilon s_\varepsilon A_{1,\varepsilon})\cdots I(n\alpha_\varepsilon s_\varepsilon A_{r,\varepsilon}))}{n^2} \\ &\geq \tfrac{1}{\alpha_\varepsilon^2 s_\varepsilon^2}\lim_{n\to\infty}\tfrac{\ell_R(R/I(n\alpha_\varepsilon s_\varepsilon n_1\Delta_1)\cdots I(n\alpha_\varepsilon s_\varepsilon n_r\Delta_r))}{n^2} \\ &\geq \tfrac{1}{\alpha_\varepsilon^2 s_\varepsilon^2}\lim_{n\to\infty}\tfrac{\ell_R(R/I(n\alpha_\varepsilon s_\varepsilon(n_1\Delta_1+\cdots+n_r\Delta_r)))}{n^2} \\ &= \tfrac{1}{\alpha_\varepsilon^2 s_\varepsilon^2}\left[-\tfrac{1}{2}((\alpha_\varepsilon s_\varepsilon(n_1\Delta_1+\cdots+n_r\Delta_r))^2)\right] \\ &= -\tfrac{1}{2}((n_1\Delta_1+\cdots+n_r\Delta_r)^2). \end{aligned}$$

Now

$$\begin{aligned} \lim_{n\to\infty}\tfrac{\ell_R(R/I(n\alpha_\varepsilon s_\varepsilon n_1\Delta_1)\cdots I(n\alpha_\varepsilon s_\varepsilon n_r\Delta_r))}{n^2} &= \lim_{n\to\infty}\tfrac{\ell_R(R/I(n\alpha_\varepsilon s_\varepsilon n_1 D_1)\cdots I(n\alpha_\varepsilon s_\varepsilon n_r D_r))}{n^2} \\ &= (\alpha_\varepsilon^2 s_\varepsilon^2)\lim_{n\to\infty}\tfrac{\ell_R(R/I(nn_1 D_1)\cdots I(nn_r D_r))}{n^2}. \end{aligned}$$

Thus

$$\begin{aligned} -\tfrac{1}{2}((n_1\Delta_1+\cdots+n_r\Delta_r)^2) &= \lim_{\varepsilon\to 0} -\tfrac{1}{2}(A_\varepsilon^2) \\ &= \lim_{n\to\infty}\tfrac{\ell_R(R/I(nn_1 D_1)\cdots I(nn_r D_r))}{n^2} \\ &= G(n_1,\ldots,n_r). \end{aligned} \tag{29}$$

eqV3

□

From Proposition PropV1 5.7 and equation (3), with $\mathcal{I}(i)=\{I(nD_i)\}$, we deduce that the mixed multiplicities are

$$e_R(\mathcal{I}(j)^{[2]};R) = -(\Delta_j^2) \text{ for all } j \tag{30}$$

eqV7

and

$$e_R(\mathcal{I}(i)^{[1]},\mathcal{I}(j)^{[1]};R) = -(\Delta_i\cdot\Delta_j) \tag{31}$$

eqV8

for $i\neq j$.

We have by Proposition PropPos 2.1 (or since $-(\Delta_j^2)>0$ for all j since $\Delta_j\neq 0$ and the intersection form is negative definite) that all mixed multiplicities are positive. Further, the mixed multiplicities are all rational numbers since the Δ_i are $\mathbb{Q}$-divisors.

5.2. **Two-dimensional local domains.** We now assume that R has dimension two and X is nonsingular. We use the notation introduced in Subsetion Not 2.1.

For $1\leq l\leq r$, write $D(l)=\sum_{i,j}a_{i,j}(l)E_{i,j}$ with $a_{i,j}(l)\in\mathbb{N}$ and let $D(l)_i=\sum_j a_{i,j}(l)E_{i,j}$. Let $\Delta(l)_i$ be the anti-nef part of the Zariski decomposition of $D(l)_i$. For $n_1,\ldots,n_r\in\mathbb{N}$,

$$\begin{aligned} \lim_{n\to\infty}\tfrac{\ell_R(S/J(nn_1 D(1)\cdots J(nn_r D(r))}{n^2} &= \textstyle\sum_{i=1}^t \lim_{n\to\infty}\tfrac{\ell_R(S_{m_i}/J(nn_1 D(1)_i)\cdots J(nn_r D(r)_i))}{n^2} \\ &= \textstyle\sum_{i=1}^t -\tfrac{1}{2}[S/m_i:R/m_R]((n_1\Delta(1)_i+\cdots+n_r\Delta(r)_i)^2) \end{aligned}$$

by (12) eqR15 and Proposition PropV1 5.7. Now by Lemma LemmaR1 2.2 and the multinomial theorem,

$$\lim_{n\to\infty}\tfrac{\ell_R(R/I(nn_1 D)\cdots I(nn_r D_r))}{n^2} = \textstyle\sum_{i=1}^t -\tfrac{1}{2}[S/m_i:R/m_R]((n_1\Delta(1)_i+\cdots+n_r\Delta(r)_i)^2) \\ = \textstyle\sum_{k_1+\ldots+k_r=2}\tfrac{1}{k_1!\cdots k_r!}\left(\sum_{i=1}^t -[S/m_i:R/m_R](\Delta(1)_i^{k_1}\cdot\ldots\cdot\Delta(r)_i^{k_r})\right)n_1^{k_1}\cdots n_r^{k_r} \tag{32}$$

eqR2

Let $\mathcal{I}(i) = \{I(nD(i))\}$ be the filtrations of m_R-primary ideals. Then by (3), the mixed multiplicities are

eqR13 (33) $$e_R(\mathcal{I}(j)^{[2]}; R) = \sum_{i=1}^{t} -[S/m_i : R/m_R](\Delta(j)_i^2)$$

and for $j \neq k$,

eqR14 (34) $$e_R(\mathcal{I}(j)^{[1]}, \mathcal{I}(k)^{[1]}; R) = \sum_{i=1}^{t} -[S/m_i : R/m_R](\Delta(j)_i \cdot \Delta(k)_i).$$

PropR10 **Proposition 5.8.** *Suppose that R is a two-dimensional excellent local domain, $\varphi : X \to Spec(R)$ is a resolution of singularities and that $D(1)$ and $D(2)$ are effective divisors with exceptional support on X. Let $\mathcal{I}(1) = \{I(nD(1))\}$ and $\mathcal{I}(2) = \{I(nD(2))\}$ be the associated filtrations of m_R-primary ideals. Suppose that $D(1) \leq D(2)$ and*

$$e_R(\mathcal{I}(1); R) = e_R(\mathcal{I}(2); R).$$

Then $I(nD(1)) = I(nD(2))$ for all $n \in \mathbb{N}$.

Proof. Let $\Delta(1)_i$ and $\Delta(2)_i$ be the respective anti-nef parts of the Zariski decompositions of $D(1)_i$ and $D(2)_i$. Then $D(1)_i \leq D(2)_i$ and so $\Delta(1)_i \leq \Delta(2)_i$ for all i, by Remark 5.2. Thus by Corollary 5.3, for all i, $(\Delta(2)_i^2) \leq (\Delta(1)_i^2)$ with equality if and only if $\Delta(1)_i = \Delta(2)_i$. Since $e_R(\mathcal{I}(1); R) = e_R(\mathcal{I}(2); R)$, equation (33) and (10) imply that

$$\sum_{i=1}^{t} [S/m_i : R/m_R][(\Delta(2)_i^2) - (\Delta(1)_i^2)] = 0.$$

Thus $\Delta(2)_i = \Delta(1)_i$ for all i, which implies that $J(nD(1)_i) = J(nD(2)_i)$ for all $n \in \mathbb{N}$ by Lemma 5.5 and so $J(nD(1)) = J(nD(2))$ for all n by (11). Thus

$$I(nD(2)) = J(nD(2)) \cap R = J(nD(1)) \cap R = I(nD(1))$$

for all $n \in \mathbb{N}$. □

Theorem 3.5 in the case that $\dim R = 2$ is an immediate corollary of Proposition 5.8. The following theorem is a generalization to divisorial valuations of a theorem of Teissier [44] and Rees and Sharp [38] for m_R-primary ideals.

PropR11 **Theorem 5.9.** *Suppose that R is a two-dimensional excellent local domain, $\varphi : X \to Spec(R)$ is a resolution of singularities and that $D(1)$ and $D(2)$ are effective divisors with exceptional support on X. Let $\mathcal{I}(1) = \{I(nD(1))\}$ and $\mathcal{I}(2) = \{I(nD(2))\}$ be the associated filtrations of m_R-primary ideals. Suppose that the Minkowski equality*

eqR4 (35) $$e_R(\mathcal{I}(1)\mathcal{I}(2); R)^{\frac{1}{2}} = e_R(\mathcal{I}(1); R)^{\frac{1}{2}} + e_R(\mathcal{I}(2); R)^{\frac{1}{2}}$$

holds (there is equality in inequality 4) of Theorem 1.5). Then there exist relatively prime $a, b \in \mathbb{Z}_+$ such that

$$I(naD(1)) = I(nbD(2))$$

for all $n \in \mathbb{N}$.

Proof. We will use the notation introduced before the statement of Lemma 2.2. Let $e_0 = e_R(\mathcal{I}(1)^{[2]}; R)$, $e_1 = e_R(\mathcal{I}(1)^{[1]}, \mathcal{I}(2)^{[1]}; R)$ and $e_2 = e_R(\mathcal{I}(2)^{[2]}; R)$. Let $\Delta(1)_i$ and

$\Delta(2)_i$ be the respective anti-nef parts of the Zariski decompositions of $D(1)_i$ and $D(2)_i$. Let

$$G(n_1,n_2)=\lim_{n\to\infty}\frac{\ell_R(R/I(nn_1D(1))I(nn_2D(2)))}{n^2}.$$

Then

$$G(n_1,n_2)=\frac{1}{2}e_0n_1^2+e_1n_1n_2+\frac{1}{2}e_2n_2^2$$

by (3). Now by (33) and (34),

$$e_0=\sum_{i=1}^t-[S/m_i:R/m_R](\Delta(1)_i^2),e_1=\sum_{i=1}^t-[S/m_i:R/m_R](\Delta(1)_i\cdot\Delta(2)_i),$$

$$e_2=\sum_{i=1}^t-[S/m_i:R/m_R](\Delta(2)_i^2).$$

We have the Minkowski inequality (inequality 1) of Theorem 1.5)

eqR5 (36) $$e_1^2\leq e_0e_2.$$

We conclude that

$$e_R(\mathcal{I}(1)\mathcal{I}(2);R)=2G(1,1)=e_0+2e_1+e_2\leq e_0+2e_0^{\frac{1}{2}}e_2^{\frac{1}{2}}+e_2=(e_0^{\frac{1}{2}}+e_2^{\frac{1}{2}})^2.$$

We deduce that equality holds in (35) if and only if equality holds in (36). Since we assume equality in (35), we have equality in (36). Write

$$\frac{e_1}{e_0}=\frac{e_2}{e_1}=\frac{a}{b}$$

with $a,b\in\mathbb{Z}_+$ relatively prime. Replacing $D(1)$ with $aD(1)$ and $D(2)$ with $bD(2)$ we obtain $e_0=e_1=e_2$ so

$$\sum_{i=1}^t-[S/m_i:R/m_R](\Delta(1)_i^2)=\sum_{i=1}^t-[S/m_i:R/m_R](\Delta(1)_i\cdot\Delta(2)_i)=\sum_{i=1}^t-[S/m_i:R/m_R](\Delta(2)_i^2).$$

We have that

$$\sum_{i=1}^t[S/m_i:R/m_R]((\Delta(1)_i-\Delta(2)_i)^2)=\sum_{i=1}^t[S/m_i:R/m_R][(\Delta(1)_i^2)-2(\Delta(1)_i\cdot\Delta(2)_i)+(\Delta_2^2)]=0$$

which implies that $\Delta(1)_i=\Delta(2)_i$ for all i since the intersection product is negative definite, so $J(nbD(1)_i)=J(naD(2)_i)$ for all i and $n\in\mathbb{N}$ by Lemma 5.5, and thus $J(naD(1))=J(nbD(2))$ for all $n\in\mathbb{N}$ by (11). Now

$$I(naD(1))=J(naD(1))\cap R=J(nbD(2))\cap R=I(nbD(2))$$

for all $n\in\mathbb{N}$.

□

Prop12 **Corollary 5.10.** *Suppose that R is a two-dimensional excellent local domain and ν_1, ν_2 are m_R-valuations. If the Minkowski equality*

$$e_R(\mathcal{I}(\nu_1)\mathcal{I}(\nu_2);R)^{\frac{1}{2}}=e_R(\mathcal{I}(\nu_1);R)^{\frac{1}{2}}+e_R(\mathcal{I}(\nu_2);R)^{\frac{1}{2}}$$

holds then $\nu_1=\nu_2$.

Proof. We have by Theorem 5.9 that $I(\nu_1)_{an} = I(\nu_2)_{bn}$ for all n and some positive, relatively prime integers a and b.

Suppose that $0 \neq f \in I(\nu_1)_n$. Then $f^a \in I(\nu_1)_{an} = I(\nu_2)_{bn}$ so that $a\nu_2(f) \geq bn$. If $f^a \in I(\nu_2)_{bn+1}$ then $f^{ab} \in I(\nu_2)_{b(bn+1)} = I(\nu_1)_{a(bn+1)}$ so that $\nu_1(f) > n$. Thus

$$\nu_1(f) = n \text{ if and only if } \nu_2(f) = \frac{b}{a}n. \tag{37}$$

Further, (37) holds for every nonzero $f \in \mathrm{QF}(R)$ since f is a quotient of nonzero elements of R.

Now the maps $\nu_1 : \mathrm{QF}(R) \setminus \{0\} \to \mathbb{Z}$ and $\nu_2 : \mathrm{QF}(R) \setminus \{0\} \to \mathbb{Z}$ are surjective, so there exists $0 \neq f \in \mathrm{QF}(R)$ such that $\nu_1(f) = 1$ and there exists $0 \neq g \in \mathrm{QF}(R)$ such that $\nu_2(g) = 1$ which implies that $a = b = 1$ since a, b are relatively prime. Thus $\nu_1 = \nu_2$.

□

6. Geometry above algebraic local rings

6.1. **Intersection products and multiplicity on local rings.** Let K be an algebraic function field over a field k. An algebraic local ring of K is a local ring R that is a localization of a finitely generated k-algebra and is a domain whose quotient field is K. Let R be a d-dimensional algebraic normal local ring of K. Let $\mathrm{BirMod}(R)$ be the directed set of blowups $\varphi : X \to \mathrm{Spec}(R)$ of an m_R-primary ideal I of R such that X is normal.

Suppose that $\varphi : X \to \mathrm{Spec}(R)$ is in $\mathrm{BirMod}(R)$. Let $\{E_1, \ldots, E_t\}$ be the irreducible exceptional divisors of φ. We define $M^1(X)$ to be the subspace of the real vector space $E_1\mathbb{R} + \cdots + E_t\mathbb{R}$ that is generated by the Cartier divisors. An element of $M^1(X)$ will be called an $\mathbb{R}$-divisor on X. We will say that $D \in M^1(X)$ is a $\mathbb{Q}$-Cartier divisor if there exists $n \in \mathbb{Z}_+$ such that nD is a Cartier divisor.

We give $M^1(X)$ the Euclidean topology. We first define a natural intersection product $(D_1 \cdot D_2 \cdot \ldots \cdot D_d)$ on X for $D_1, \ldots, D_d \in M^1(X)$. The intersection product is a restriction of the one defined in [26]. We first define the intersection product for Cartier divisors $D_1, \ldots, D_d \in E_1\mathbb{Z} + \cdots + E_t\mathbb{Z}$. Since this product is multilinear, it extends naturally to a multilinear product on $M^1(X)^d$.

There exists a subfield k_1 of K with the two properties that $k \subset k_1 \subset R$ and R/m_R is a finite extension of k_1. Thus there exists a projective k_1-variety Y and a closed point $q \in Y$ such that $\mathcal{O}_{Y,q} = R$. The m_R-primary ideal I naturally extends to an ideal sheaf $\mathcal{I}$ in $\mathcal{O}_Y$, defined by

$$\mathcal{I}_a = \begin{cases} \mathcal{O}_{Y,a} & \text{if } q \neq a \in Y \\ I & \text{if } a = q. \end{cases}$$

Let $\Psi : Z \to Y$ be the projective, birational morphism that is the obtained by blowing up $\mathcal{I}$. Observe that base change of this map by $\mathcal{O}_{Y,q} = R$ gives the original map $\varphi : X \to \mathrm{Spec}(R)$. We can thus view $E_1, \ldots, E_t$ as closed projective subvarieties of the normal variety Z.

Suppose that $F_1, \ldots, F_s$ are Cartier divisors on Z and $\mathcal{F}$ is a coherent sheaf on Z, such that $\dim \mathrm{supp}\, \mathcal{F} \leq s$. By [26] (surveyed in Chapter 19 of [12]) we have an intersection product $I(F_1, \ldots, F_s, \mathcal{F})$ on Z which has the good properties explained in [26] and [12]. The Euler characteristic

$$\chi(\mathcal{O}_Z(n_1F_1 + \cdots + n_sF_s) \otimes \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(Z, \mathcal{O}_Z(n_1F_1 + \cdots + n_sF_s) \otimes \mathcal{F})$$

where $h^i(Z,\mathcal{G}) = \dim_{k_1} H^i(Z,\mathcal{G})$ for $\mathcal{G}$ a coherent sheaf on Z, is a polynomial in $n_1,\ldots,n_s$ ([K1, 26], [C5, 12, Theorem 19.1]). The intersection product $I(F_1,\ldots,F_s,\mathcal{F})$ is defined to be the coefficient of $n_1\cdots n_s$ in the Snapper polynomial $\chi(\mathcal{O}_Z(n_1F_1+\cdots+n_sF_s)\otimes\mathcal{F})$. We always have that $I(F_1,\ldots,F_s,\mathcal{F})\in\mathbb{Z}$.

If $D_1,\ldots,D_s$ are Cartier divisors in $E_1\mathbb{Z}+\cdots+E_t\mathbb{Z}$, and $\mathcal{F}$ is a coherent sheaf on X whose support is contained in $\varphi^{-1}(m_R)$ (so that $\mathcal{F}$ naturally extends to a coherent sheaf on Z with the same support) and $\dim \operatorname{supp}\mathcal{F}\leq s$, then we define an intersection product

$$(D_1\cdot\ldots\cdot D_s\cdot\mathcal{F}) = \frac{1}{[R/m_R:k_1]}I(D_1,\ldots,D_s,\mathcal{F})$$

on X. If W is a closed subscheme of $\varphi^{-1}(m_R)$, we define

$$(D_1\cdot\ldots\cdot D_s\cdot W) = (D_1\cdot\ldots\cdot D_s\cdot\mathcal{O}_W).$$

If $s=d$, then we define

$$(D_1\cdot\ldots\cdot D_d) = (D_1\cdot\ldots\cdot D_d\cdot X) = \frac{1}{[R/m_R:k_1]}I(D_1,\ldots,D_s,\mathcal{O}_Z).$$

This product is well defined (independent of any choices made in the construction), as follows from the good properties of the intersection product ([K1, 26], [C5, 12]). This product naturally extends to a multilinear product on $M^1(X)^d$.

We will say that a divisor $F=a_1E_1+\cdots+a_tE_t\in M^1(X)$ is effective if $a_i\geq 0$ for all i, and anti-effective if $a_i\leq 0$ for all i. This defines a partial order $\leq$ on $M^1(X)$ by $A\leq B$ if $B-A$ is effective. The effective cone $\mathrm{EF}(X)$ is the closed convex cone in $M^1(X)$ of effective $\mathbb{R}$-divisors. The anti-effective cone $\mathrm{AEF}(X)$ is the closed convex cone in $M^1(X)$ consisting of all anti-effective $\mathbb{R}$-divisors.

We will say that an anti-effective divisor $F\in M^1(X)$ is numerically effective (nef) if

$$(F\cdot C) = (F\cdot\mathcal{O}_C)\geq 0$$

for all closed curves C in $\varphi^{-1}(m_R)$. The nef cone $\mathrm{Nef}(X)$ is the closed convex cone in $M^1(X)$ of all nef $\mathbb{R}$-divisors on X.

Lemma 6.1. *There is an inclusion of cones* $\mathrm{Nef}(X)\subset\mathrm{AEF}(X)$.

Proof. Suppose there exists a nef divisor $D\in M^1(X)$ that is not anti-effective. Since X is the blowup of an m_R-primary ideal, there exists an anti-effective ample Cartier divisor $A=a_1E_1+\cdots+a_tE_t$, with $a_1,\ldots,a_t<0$. There exists a smallest $\lambda\in\mathbb{R}$ such that $D+\lambda A$ is anti-effective. Necessarily, $\lambda>0$ and $D+\lambda A$ is nef. Expand $D+\lambda A=\sum b_iE_i$. After possibly reindexing the E_i, we have that there exists a number s with $1\leq s<t$ such that $b_1=\cdots=b_s=0$ and $b_{s+1},\ldots,b_t<0$. Now $\varphi^{-1}(m_R)$ is connected by Zariski's connectedness theorem ([Z1, 46, Section 20] or [G1, 21, Corollary III.4.3.2]). After reindexing the $E_1,\ldots,E_s$ and the $E_{s+1},\ldots,E_t$, we may assume that $E_s\cap E_{s+1}\neq\emptyset$. Let C be a closed curve on the projective variety E_s that is not contained in E_i for $i\geq s+1$ but intersects E_{s+1}. Then $((D+\lambda A)\cdot C)<0$, a contradiction. □

We will say that an anti-effective Cartier divisor $F\in M^1(X)$ is ample on X if there exists an ample Cartier divisor H on Y such that $\Psi^{-1}(H)+F$ is ample on Z. This definition is independent of the choice of Y in the construction. We define a divisor $F\in M^1(X)$ to be ample if F is a formal sum $F=\sum a_iF_i$ where F_i are ample anti-effective Cartier divisors and a_i are positive real numbers. A divisor D is anti-ample if $-D$ is ample. We define the convex cone

$$\mathrm{Amp}(X) = \{F\in M^1(X)\mid F \text{ is ample}\}.$$

We have that $\mathrm{Amp}(X) \subset \mathrm{Nef}(X)$, the closure of $\mathrm{Amp}(X)$ is $\mathrm{Nef}(X)$, and the interior of $\mathrm{Nef}(X)$ is $\mathrm{Amp}(X)$, as in [26], [27, Theorem 1.4.23].

Remark2 **Remark 6.2.** *If $G \in M^1(X)$, then there exists an effective $\mathbb{Q}$-divisor $D \in M^1(X)$ such that $G - D \in Amp(X)$.*

For $F \in M^1(X)$ an effective Cartier divisor, define $I(F) = \Gamma(X, \mathcal{O}_X(-F))$, an m_R-primary ideal in R since R is normal. Let $\pi : Y \to \mathrm{Spec}(k_1)$ be the structure morphism.

Lemma1 **Lemma 6.3.** *Suppose that $A \in M^1(X)$ is an effective Cartier divisor such that $-A$ is nef. Then*

$$\lim_{m\to\infty} \frac{\ell_R(R/I(mA))}{m^d} = \frac{-((-A)^d)}{d!}.$$

Proof. Let H be an ample Cartier divisor on Y and $L = \Psi^*(H)$. There exists $a \in \mathbb{Z}_+$ such that $aL - A$ is nef and big on Z.

We have that $R^1\Psi_*\mathcal{O}_Z(m(aL - A)) \cong \mathcal{O}_Y(maH) \otimes R^1\Psi_*\mathcal{O}_Z(-mA)$ is a coherent sheaf of $\mathcal{O}_Y$-modules whose support is q and

eq6 (38) $$H^1(X, \mathcal{O}_X(-mA)) \cong \pi_*(R^1\Psi_*\mathcal{O}_Z(m(aL - A)))$$

as an $R = \mathcal{O}_{Y,q}$-module.

By [Fu, 19, Theorem 6.2],

eq1 (39) $$\lim_{m\to\infty} \frac{h^i(Z, \mathcal{O}_Z(mG))}{m^d} = 0 \text{ if } i > 0$$

if G is a nef Cartier divisor on Z.

Now tensor the short exact sequence

$$0 \to \mathcal{O}_Z(-mA) \to \mathcal{O}_Z \to \mathcal{O}_{mA} \to 0$$

with $\mathcal{O}_Z(maL)$ to get a short exact sequence

$$0 \to \mathcal{O}_Z(m(aL - A)) \to \mathcal{O}_Z(maL) \to \mathcal{O}_{mA} \otimes \mathcal{O}_Z(maL) \cong \mathcal{O}_{mA} \to 0.$$

Taking the long exact cohomology sequence, we have that

$$\lim_{m\to\infty} \frac{h^i(Z, \mathcal{O}_{mA})}{m^d} = 0$$

for $i > 0$ by (39), and so

eq2 (40)
$$\begin{aligned} \lim_{m\to\infty} \tfrac{h^0(Z,\mathcal{O}_{mA})}{m^d} &= \lim_{m\to\infty} \tfrac{\chi(\mathcal{O}_{mA})}{m^d} \\ &= \lim_{m\to\infty} \tfrac{\chi(\mathcal{O}_Z)-\chi(\mathcal{O}_Z(-mA))}{m^d} \\ &= \lim_{m\to\infty} \tfrac{-\chi(\mathcal{O}_Z(-mA))}{m^d} \\ &= \tfrac{-((-A)^d)}{d!}, \end{aligned}$$

for instance by [C5, 12, Theorem 19.16]. The end of the cohomology 5 term sequence (forinstance in [RM, 39, Theorem 11.2]) of the Leray spectral sequence

$$R^i\pi_*R^j\Psi_*\mathcal{O}_Z(m(aL - A)) \Rightarrow R^{i+j}(\pi \circ \Psi)_*\mathcal{O}_Z(m(aL - A))$$

is the exact sequence

eq3 (41) $$R^1(\pi\circ\Psi)_*\mathcal{O}_Z(m(aL-A)) \to \pi_*(R^1\Psi_*\mathcal{O}_Z(m(aL-A))) \to R^2\pi_*(\Psi_*\mathcal{O}_Z(m(aL-A))).$$

Now $R^1(\pi \circ \Psi)_*\mathcal{O}_Z(m(aL - A)) = H^1(Z, \mathcal{O}_Z(m(aL - A))$,

$R^2\pi_*(\Psi_*\mathcal{O}_Z(m(aL-A))) = H^2(Y, \Psi_*\mathcal{O}_Z(m(aL-A))) = H^2(Y, \mathcal{O}_Y(maL)\otimes\Psi_*\mathcal{O}_Z(-mA))$

and $\pi_*(R^1\Psi_*\mathcal{O}_Z(m(aL-A))) = H^0(Y, R^1\Psi_*\mathcal{O}_Z(m(aL-A)))$.

Let $\mathcal{I}_m = \Psi_*\mathcal{O}_Z(-mA)$. From the short exact sequences

$$0 \to \mathcal{I}_m \otimes \mathcal{O}_Y(maL) \to \mathcal{O}_Y(maL) \to \mathcal{O}_Y/\mathcal{I}_m \to 0,$$

we obtain the exact cohomology sequences

$$H^1(Y, \mathcal{O}_Y/\mathcal{I}_m) \to H^2(Y, \mathcal{I}_m \otimes \mathcal{O}_Z(maL)) \to H^2(Y, \mathcal{O}_Y(maL)).$$

Now $H^1(Y, \mathcal{O}_Y/\mathcal{I}_m) = 0$ since $\mathcal{O}_Y/\mathcal{I}_m$ has zero dimensional support and $H^2(Y, \mathcal{O}_Y(maL)) = 0$ for $m \gg 0$ since L is ample. Thus

eq4 (42) $$H^2(Y, \mathcal{O}_Y(maL) \otimes \Psi_*\mathcal{O}_Z(-mA)) = 0 \text{ for } m \gg 0.$$

We have

eq7 (43)
$$\begin{aligned} \lim_{m\to\infty} \frac{\ell_R(H^1(X,\mathcal{O}_X(-mA))}{m^d} &= \lim_{m\to\infty} \frac{1}{[R/m_R:k_1]} \frac{\dim_{k_1} H^1(X,\mathcal{O}_X(-mA))}{m^d} \\ &= \lim_{m\to\infty} \frac{1}{[R/m_R:k_1]} \frac{h^0(Y,R^1\Psi_*\mathcal{O}_Z(m(aL-A))}{m^d} \\ &= 0 \end{aligned}$$

by (eq6), (eq3), (eq4) and (eq1) with $G = aL - A$ in (eq1). We have that $R = H^0(X, \mathcal{O}_X)$ since R is normal. Now from the exact sequences of R-modules

$$0 \to R/I(mA) \to H^0(X, \mathcal{O}_X/\mathcal{O}_X(-mA)) \to H^1(X, \mathcal{O}_X(-mA)),$$

(eq2) and (eq7) we obtain the formula of the statement of the lemma. □

Lemma3 **Lemma 6.4.** *Suppose that $D_1, \ldots, D_r \in M^1(X)$ are effective Cartier divisors and $\mathcal{O}_X(-D_i)$ is generated by global sections for all i. Then for $n_1, \ldots, n_r \in \mathbb{N}$,*

$$\lim_{m\to\infty} \frac{\ell_R(R/I(mn_1D_1)\cdots I(mn_rD_r))}{m^d} = -\frac{((-n_1D_1 - \cdots - n_rD_r)^d)}{d!}.$$

Proof. We have that

$$I(mn_1D_1)\cdots I(mn_rD_r)\mathcal{O}_X = \mathcal{O}_X(-m(n_1D_1 + \cdots + n_rD_r)) = I(mn_1D_1 + \cdots + n_rD_r)\mathcal{O}_X$$

since the $\mathcal{O}_X(-mn_iD_i)$ are generated by global sections. Thus the integral closure of $I(mn_1D_1)\cdots I(mn_rD_r)$ is $I(m(n_1D_1 + \cdots + n_rD_r))$ for all $m \in \mathbb{N}$, and so the R-algebra $\bigoplus_{m\geq 0} I(m(n_1D_1+\cdots+_rD_r))$ is integral over the R-algebra $\bigoplus_{m\geq 0} I(mn_1D_1)\cdots I(mn_rD_r)$. Thus

$$\begin{aligned} \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1D_1)\cdots I(mn_rD_r))}{m^d} &= \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1D_1+\cdots+mn_rD_r)}{m^d} \\ &= -\frac{((-n_1D_1-\cdots-n_rD_r)^d)}{d!} \end{aligned}$$

by Theorem Theorem13 1.4 and Lemma Lemma1 6.3. □

6.2. **Finite dimensional vector spaces and cones.** Suppose that $X \in \mathrm{BirMod}(R)$. Let $E_1, \ldots, E_r$ be the exceptional components of X for the morphism $X \to \mathrm{Spec}(R)$. For $0 < p \leq d$, we define $M^p(X)$ to be the direct product of $M^1(X)$ p times, and we define $M^0(X) = \mathbb{R}$. For $1 < p \leq d$, we define $L^p(X)$ to be the vector space of p-multilinear forms from $M^p(X)$ to $\mathbb{R}$, and define $L^0(X) = \mathbb{R}$.

The intersection product gives us p-multilinear maps

eq4* (44) $$M^p(X) \to L^{d-p}(X)$$

for $0 \leq p \leq d$. In the special case when $p = 0$, the map is just the linear map taking 1 to the map

$$(\mathcal{L}_1, \ldots, \mathcal{L}_d) \mapsto (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_d) = (\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_d \cdot X).$$

We will denote the image of $(\mathcal{L}_1,\ldots,\mathcal{L}_p)$ by $\mathcal{L}_1\cdot\ldots\cdot\mathcal{L}_p$. We will sometimes write

$$\mathcal{L}_1\cdot\ldots\cdot\mathcal{L}_p(\beta_{p+1},\ldots,\beta_d)=(\mathcal{L}_1\cdot\ldots\cdot\mathcal{L}_p\cdot\beta_{p+1}\cdot\ldots\cdot\beta_d).$$

We give all the vector spaces just defined the Euclidean topology, so that all of the mappings considered above are continuous.

Let $|\mathcal{L}|$ be a norm on $M^1(X)$ giving the Euclidean topology. The Euclidean topology on $L^p(X)$ is given by the norm $||A||$, that is defined on a multilinear form $A\in L^p(X)$ to be the greatest lower bound of all real numbers c such that

$$|A(x_1,\ldots,x_p)|\leq c|x_1|\cdots|x_p|$$

for $x_1,\ldots,x_p\in M^1(X)$.

Suppose that V is a closed p-dimensional subvariety of some E_i with $1\leq p\leq d-1$. Define $\sigma_V\in L^p(X)$ by

$$\sigma_V(\mathcal{L}_1,\ldots,\mathcal{L}_p)=(\mathcal{L}_1\cdot\ldots\cdot\mathcal{L}_p\cdot V)$$

for $\mathcal{L}_1,\ldots,\mathcal{L}_p\in M^1(X)$. For $p=d$, define $\sigma_X\in L^d(X)$ by

$$\sigma_X(\mathcal{L}_1,\ldots,\mathcal{L}_d)=(\mathcal{L}_1\cdot\ldots\cdot\mathcal{L}_d)=(\mathcal{L}_1\cdot\ldots\cdot\mathcal{L}_d\cdot X).$$

The pseudoeffective cone $\mathrm{Psef}(L^p(X))$ in $L^p(X)$ is the closure of the cone generated by all such σ_V in $L^p(X)$. We define $\mathrm{Psef}(L^0(X))$ to be the nonnegative real numbers.

Let V be a vector space and $C\subset V$ be a pointed (containing the origin) convex cone that is strict ($C\cap(-C)=\{0\}$). Then we have a partial order on V defined by $x\leq y$ if $y-x\in C$.

Lemma3.1* **Lemma 6.5.** *Suppose that $X\in\mathrm{BirMod}(R)$ and $1\leq p\leq d$.*

1) *Suppose that $\alpha\in$ Psef$(L^p(X))$ and $\mathcal{L}_1,\ldots,\mathcal{L}_p\in M^1(X)$ are nef. Then*

$$\alpha(\mathcal{L}_1,\ldots,\mathcal{L}_p)\geq 0.$$

2) *Psef$(L^p(X))$ is a strict cone.*

The proof of Lemma 6.5 is as the proof of [C4, Lemma 3.1].

Since $\mathrm{Psef}(L^p(X))$ is a strict cone, we have a partial order on $L^p(X)$, defined by

$$\alpha\geq 0 \text{ if } \alpha\in\mathrm{Psef}(L^p(X)).$$

We have that $\geq$ is the usual order on $\mathbb{R}$ since $L^0(X)=\mathbb{R}$ and $\mathrm{Psef}(L^0(X))$ is the set of nonnegative real numbers. We also have the partial order on $M^1(X)$ defined by $\alpha\geq 0$ if α is effective.

Remark1 **Lemma 6.6.** *Suppose that $F_1,\ldots,F_p\in M^1(X)$ are such that F_1 is anti-effective and $F_2,\ldots,F_p$ are nef. Then $F_1\cdot\ldots\cdot F_p\leq 0$ in $L^{d-p}(X)$.*

Proof. We have that $-F_1\in M^1(X)$ is effective. Thus $(-F_1)\cdot F_2\cdot\ldots\cdot F_p\in\mathrm{Psef}(L^{d-p}(X))$ by Lemma 3.11 [C4]. □

Lemma3.2* **Lemma 6.7.** *Suppose that $\beta\in$ Psef$(L^p(X))$. Then the set*

$$\{\alpha\in\text{Psef}(L^p(X))\mid 0\leq\alpha\leq\beta\}$$

is compact.

The proof of Lemma 6.7 is the same as the proof of [C4, Lemma 3.2].

Suppose that $X,Y\in\mathrm{BirMod}(R)$ and $f:Y\to X$ is an R-morphism. Then f induces continuous linear maps $f^*:M^1(X)\to M^1(Y)$ (from f^* of a Cartier divisor),

$f^* : M^p(X) \to M^p(Y)$ and $f_* : L^p(Y) \to L^p(X)$. By Proposition I.2.6 [K1, 26], for $1 \leq t \leq d$, we have that

eq5* (45) $$f^*(\mathcal{L}_1) \cdot \ldots \cdot f^*(\mathcal{L}_d) = \mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_d$$

for $\mathcal{L}_1, \ldots, \mathcal{L}_d \in M^1(X)$. Thus for $0 \leq p \leq d$ we have commutative diagrams of linear maps

eq4** (46) $$\begin{array}{ccc} M^p(Y) & \to & L^{d-p}(Y) \\ f^* \uparrow & & f_* \downarrow \\ M^p(X) & \to & L^{d-p}(X). \end{array}$$

For $\alpha \in M^1(X)$, we have that

eq7* (47) $$f^*(\alpha) \in \mathrm{Nef}(Y) \text{ if and only if } \alpha \in \mathrm{Nef}(X)$$

and

eq8* (48) $$f^*(\alpha) \text{ is effective on } Y \text{ if and only if } \alpha \text{ is effective on } X.$$

Lemma3.3* **Lemma 6.8.** *Suppose that $X, Y \in \mathrm{BirMod}(R)$ and $f : Y \to X$ is an R-morphism. Then $f_*(Psef(L^p(Y))) \subset Psef(L^p(X))$.*

The proof of Lemma 6.8 is as the proof of [C4, 10, Lemma 3.3].

6.3. **Infinite dimensional topological spaces.** We have that $\mathrm{BirMod}(R)$ is a directed set by the R-morphisms $Y \to X$ for $X, Y \in \mathrm{BirMod}(R)$. There is at most one R-morphism $X \to Y$ for $X, Y \in \mathrm{BirMod}(X)$.

The set $\{M^p(Y_i) \mid Y_i \in \mathrm{BirMod}(R)\}$ is a directed system of real vector spaces, where we have a linear mapping $f_{ij}^* : M^p(Y_i) \to M^p(Y_j)$ if the natural birational map $f_{ij} : Y_j \to Y_i$ is an R-morphism. We define

$$M^p(R) = \lim_{\to} M^p(Y_i)$$

with the strong topology (the direct limit topology, c.f. Appendix 1. Section 1 [P, 17]). Let $\rho_{Y_i} : M^p(Y_i) \to M^p(R)$ be the natural mappings. A set $U \subset M^p(R)$ is open if and only if $\rho_{Y_i}^{-1}(U)$ is open in $M^p(Y_i)$ for all i.

We have that $M^p(R)$ is a real vector space. As a vector space, $M^p(R)$ is isomorphic to the p-fold product $M^1(R)^p$.

We define $\alpha \in M^1(R)$ to be $\mathbb{Q}$-Cartier (respectively nef or effective) if there exists a representative of α in $M^1(Y)$ that has this property for some $Y \in \mathrm{BirMod}(R)$. We define $\mathrm{Nef}^p(R)$ to be the subset of $M^p(R)$ of nef divisors. We define $\mathrm{EF}^p(R)$ to be the subset of $M^p(R)$ of effective divisors and define $\mathrm{AEF}^p(R)$ to be the subset of $M^p(R)$ of anti-efective divisors. Both of these sets are convex cones in the vector space $M^p(R)$.

By (47) and (48), $\{\mathrm{Nef}(Y)^p\}$, $\{\mathrm{EF}(Y)^p\}$ and $\{\mathrm{AEF}(Y)^p\}$ also form directed systems. As sets, we have that

$$\mathrm{Nef}^p(R) = \lim_{\to}(\mathrm{Nef}(Y)^p),\ \mathrm{EF}^p(R) = \lim_{\to}(\mathrm{EF}(Y)^p) \text{ and } \mathrm{AEF}^p(R) = \lim_{\to}(\mathrm{AEF}(Y)^p).$$

We give all of these sets their respective strong topologies.

Let $\rho_Y : M^p(Y) \to M^p(R)$ be the induced continuous linear maps for $Y \in \mathrm{BirMod}(R)$. We will also denote the induced continuous maps $\mathrm{Nef}(Y)^p \to \mathrm{Nef}^p(R)$, $\mathrm{EF}(Y)^p \to \mathrm{EF}^p(R)$ and $\mathrm{AEF}(Y)^p \to \mathrm{AEF}^p(R)$ by ρ_Y.

The set $\{L^p(Y_i)\}$ is an inverse system of topological vector spaces, where we have a linear map $(f_{ij})_* : L^p(Y_j) \to L^p(Y_i)$ if the birational map $f_{ij} : Y_j \to Y_i$ is a morphism. We define

$$L^p(R) = \lim_{\leftarrow} L^p(Y_i),$$

with the weak topology (the inverse limit topology). Thus the open subsets of $L^p(R)$ are the sets obtained by finite intersections and arbitrary unions of sets $\pi_{Y_i}^{-1}(U)$ where $\pi_{Y_i} : L^p(R) \to L^p(Y_i)$ is the natural projection and U is open in $L^p(Y_i)$.

In general, good topological properties on a directed system do not extend to the direct limit (c.f. Section 1 of Appendix 2 [17], especially the remark before 1.8). In particular, we cannot assume that $M^1(R)$ is a *topological* vector space. However, good topological properties on an inverse system do extend (c.f. Section 2 of Appendix 2 [17]). In particular, we have the following proposition.

Proposition 6.9. *$L^p(R)$ is a Hausdorff real topological vector space that is isomorphic (as a vector space) to the p-multilinear forms on $M^1(R)$.*

Let $\pi_Y : L^p(R) \to L^p(Y)$ be the induced continuous linear maps for $Y \in \mathrm{BirMod}(R)$.

The following lemma follows from the universal properties of the inverse limit and the direct limit (c.f. Theorems 2.5 and 1.5 [17]).

Lemma3.5* **Lemma 6.10.** *Suppose that $\mathcal{F}$ is M^p or Nef^p Then giving a continuous mapping*

$$\Phi : \mathcal{F}(R) \to L^{d-p}(R)$$

is equivalent to giving continuous maps $\varphi_Y : \mathcal{F}(Y) \to L^{d-p}(Y)$ for all $Y \in \mathrm{BirMod}(R)$, such that the diagram

$$\begin{array}{ccc} \mathcal{F}(Z) & \stackrel{\varphi_Z}{\to} & L^{d-p}(Z) \\ f^* \uparrow & & \downarrow f_* \\ \mathcal{F}(Y) & \stackrel{\varphi_Y}{\to} & L^{d-p}(Y) \end{array}$$

commutes, whenever $f : Z \to Y$ is in $\mathrm{BirMod}(R)$.

In the case when $\mathcal{F} = M^p$, if the φ_Y are all multilinear, then Φ is also multilinear (via the vector space isomorphism of $M^p(R)$ with p-fold product $M^1(R)^p$).

As an application, we have the following useful property.

3.6* **Lemma 6.11.** *The intersection product gives us a continuous map*

$$\mathcal{F}(R) \to L^{d-p}(R)$$

whenever $\mathcal{F}$ is M^p or Nef^p. The map is multilinear on $M^p(R)$.

We will denote the image of $(\alpha_1, \ldots, \alpha_p)$ by $\alpha_1 \cdot \ldots \cdot \alpha_p$. For $\beta_{p+1}, \ldots, \beta_d \in M^1(R)$, we will often write

$$\alpha_1 \cdot \ldots \cdot \alpha_p(\beta_{p+1}, \ldots, \beta_d) = (\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \beta_{p+1} \cdot \ldots \cdot \beta_d).$$

Given $\alpha \in M^1(R)$, there exists $X \in \mathrm{BirMod}(R)$ such that α is represented by an element D of $M^1(X)$. If $Y \in \mathrm{BirMod}(R)$ and $f : Y \to X$ is an R-morphism, then α is also represented by $f^*(D) \in M^1(Y)$. To simplify notation, we will often regard α as an element of $M^1(X)$ and of $M^1(Y)$, and write $\alpha \in M^1(X)$ and $\alpha \in M^1(Y)$.

6.4. **Pseudoeffective classes in $L^p(R)$.** We define a class $\alpha \in L^p(R)$ to be pseudoeffective if $\pi_Y(\alpha) \in L^p(Y)$ is pseudoeffective for all $Y \in \mathrm{BirMod}(R)$.

Lemma3.7*

Lemma 6.12. *The set of pseudoeffective classes Psef($L^p(R)$) in $L^p(R)$ is a strict closed convex cone in $L^p(R)$.*

The proof of Lemma 6.12 is as the proof of [C4, Lemma 3. 7].

By Lemma 6.12, we can define a partial order ≥ 0 on $L^p(R)$ by $\alpha \geq 0$ if $\alpha \in \mathrm{Psef}(L^p(R))$.

We have that $L^0(R) = \mathbb{R}$ and $\mathrm{Psef}(L^0(R))$ is the set of nonnegative real numbers (by the remark before Lemma 6.5), so $\geq$ is the usual order on $\mathbb{R}$.

Lemma3.8*

Lemma 6.13. *Suppose that $\mathcal{L}_1, \ldots, \mathcal{L}_p \in Nef(R)$ and $\alpha \in Psef(L^p(R))$. Then*

$$\alpha(\mathcal{L}_1, \ldots, \mathcal{L}_p) \geq 0.$$

The proof of Lemma 6.13 follows from Lemma 6.5 as in the proof of [C4, Lemma 3.8].

Lemma3.9*

Lemma 6.14. *Suppose that $Y \in \mathrm{BirMod}(R)$ and $E_1, \ldots, E_r$ are the irreducible exceptional divisors of $Y \to Spec(R)$. Suppose that $V \subset Y$ is a p-dimensional closed subvariety of some E_i. Then there exists $\alpha \in Psef(L^p(R))$ such that $\pi_Y(\alpha) = \sigma_V$.*

The proof of Lemma 6.14 is as the proof of [C4, Lemma 3.9].

Lemma3.10*

The proof of Lemma 6.15 below is as the proof of [C4, Lemma 3.10].

Lemma3.10*

Lemma 6.15. *Suppose that $\alpha \in Psef(L^p(R))$. Then the set*

$$\{\beta \in L^p(R) \mid 0 \leq \beta \leq \alpha\}$$

is compact.

Lemma3.11*

Lemma 6.16. *Suppose that $\alpha_i \in M^1(R)$ for $1 \leq i \leq p$, with $\alpha_1 \in \mathrm{EF}^1(R)$ and $\alpha_i \in \mathrm{Nef}^1(R)$ for $i \geq 2$. Then $\alpha_1 \cdot \ldots \cdot \alpha_p \in \mathrm{Psef}(L^{d-p}(R))$.*

The proof of Lemma 6.16 follows from the proof of [C4, Lemma 3.11], using Lemma 6.6.

Prop3.12*

Proposition 6.17. *Suppose that α_i and α_i' for $1 \leq i \leq p$ are nef classes in $M^1(R)$, and that $\alpha_i \geq \alpha_i'$ for $i = 1, \ldots, p$. Then*

$$\alpha_1 \cdot \ldots \cdot \alpha_p \geq \alpha_1' \cdot \ldots \cdot \alpha_p'$$

in $L^{d-p}(R)$.

The proof of Propositoin 6.17 is as the proof of [C4, Proposition 3.12].

7. anti-positive intersection products

SecAPM3

We continue in this section with the notation introduced in Section 6.

A partially ordered set is directed if any two elements of it can be dominated by a third. A partially ordered set is filtered if any two elements of it dominate a third.

We state Lemma 7.1 below for completeness. A proof can be found in [C4, Lemma 4.1].

Lemma4.1*

Lemma 7.1. *Let V be a Hausdorff topological vector space and K a strict closed convex cone in V with associated partial order relation $\leq$. Then any nonempty subset S of V that is directed with respect to $\leq$ and is contained in a compact subset of V has a least upper bound with respect to $\leq$ in V.*

Lemma4.2*

Lemma 7.2. *Suppose that $\alpha \in M^1(R)$ is anti-effective. Then the set $\mathcal{D}(\alpha)$ of effective $\mathbb{Q}$-divisors D in $M^1(R)$ such that $\alpha - D$ is nef is nonempty and filtered.*

The proof of Lemma 7.2, using Remark 6.2, is as the proof of [10, Lemma 4.2]. The following proposition generalizes [10, Proposition 4.3].

Proposition 7.3. *Suppose that $\alpha_1, \ldots, \alpha_p \in M^1(R)$ are anti-effective. Let*

$$\begin{aligned} S &= \{(\alpha_1 - D_1) \cdot \ldots \cdot (\alpha_p - D_p) \in L^{d-p}(R) \text{ such that} \\ &\quad D_1, \ldots, D_p \in M^1(R) \text{ are effective } \mathbb{Q}\text{-divisors and } \alpha_i - D_i \text{ are nef for } 1 \leq i \leq p\}. \end{aligned}$$

Then

1) *S is nonempty.*
2) *S is a directed set with respect to the partial order $\leq$ on $L^{d-p}(R)$.*
3) *S has a (unique) least upper bound with respect to $\leq$ in $L^{d-p}(R)$.*

Proof. There exists $\varphi : X \to \operatorname{Spec}(R)$ in $\operatorname{BirMod}(R)$ such that $\alpha_1, \ldots, \alpha_p \in M^1(X)$. Since X is the blowup of an m_R-primary ideal, there exists an effective $\mathbb{Q}$-divisor ω in $M^1(R)$ such that $-\omega$ is ample on X and $\alpha_i - \omega$ is nef for all i. Suppose $D_i \in M^1(R)$ are effective $\mathbb{Q}$-divisors such that $\alpha_i - D_i$ are nef for all i. Lemma 7.2 implies there exist effective $\mathbb{Q}$-divisors $D_i^* \in M^1(R)$ such that for all i, $\alpha_i - D_i$ are nef, $D_i^* \leq D_i$, $D_i^* \leq \omega$ and $\alpha_i - D_i^*$ are nef. Thus $\alpha_i - \omega \leq \alpha_i - D_i^* \leq 0$ and $\alpha_i - D_i \leq \alpha_i - D_i^*$. Proposition 6.17 implies

$$(\alpha_1 - \omega) \cdot (\alpha_2 - \omega) \cdot \ldots \cdot (\alpha_p - \omega) \leq (\alpha_1 - D_1^*) \cdot (\alpha_2 - D_2^*) \cdot \ldots \cdot (\alpha_p - D_p^*) \leq 0.$$

Thus $\gamma \in L^{d-p}(R)$ is an upper bound for S if and only if γ is an upper bound for $S \cap Z$ where

$$Z = \{x \in L^{d-p}(R) \mid (\alpha_1 - \omega) \cdot \ldots \cdot (\alpha_p - \omega) \leq x \leq 0\}.$$

The set $S \cap Z$ is nonempty since $(\alpha_1 - \omega) \cdot \ldots \cdot (\alpha_p - \omega) \in S \cap Z$. The set $S \cap Z$ is directed since S is and since whenever $\beta_1, \ldots, \beta_p \in M^1(R)$ are anti-effective and nef, $\beta_1 \cdot \ldots \cdot \beta_p \leq 0$ (by Lemma 6.6). The set Z is compact by Lemma 6.15. Thus by Lemma 7.1, $S \cap Z$ has a least upper bound with respect to $\leq$ in $L^{d-p}(R)$. □

The following definition is well defined by Proposition 7.3. Definition 7.4 gives a local version of the definition [10, Definition 4.4] of the positive intersection product on a proper variety.

Definition 7.4. *Suppose that $\alpha_1, \ldots, \alpha_p \in M^1(R)$ are anti-effective. Their anti-positive intersection product $\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \in L^{d-p}(R)$ is defined to be the least upper bound of the set of classes $(\alpha_1 - D_1) \cdot \ldots \cdot (\alpha_p - D_p) \in L^{d-p}(R)$ where $D_i \in M^1(R)$ are effective $\mathbb{Q}$-Cartier divisors in $M^1(R)$ such that $\alpha_i - D_i$ are nef.*

The proof of the following proposition is as the proof of Proposition 4.7 [10].

Proposition 7.5. *The map $AEF^p(R) \to L^{d-p}(R)$ defined by $(\alpha_1, \ldots, \alpha_p) \mapsto \langle \alpha_1, \cdot, \ldots, \alpha_p \rangle$ is continuous.*

8. Mixed multiplicities and anti-positive intersection products

We continue in this section with the notation of Sections 6 and 7. In this section, suppose that $\alpha_1, \ldots \alpha_r \in M^1(R)$ are effective Cartier divisors. For $n_1, \ldots, n_r \in \mathbb{N}$, define

$$F(n_1, \ldots, n_r) = \lim_{m \to \infty} \frac{\ell_R(R/I(mn_1\alpha_1) \cdots I(mn_r\alpha_r))}{m^d}.$$

We have that $F(n_1, \ldots, n_r)$ is a homogeneous polynomial of degree d by [14, Theorem 6.6].

We now describe a construction that we will use in this section. Let $X \in \mathrm{BirMod}(R)$ be such that $\alpha_1, \ldots, \alpha_r \in M^1(X)$. For $s \in \mathbb{Z}_+$, let

eq13 (49) $$Y[s] \to X$$

be in $\mathrm{BirMod}(X)$ and let $\pi_s : Y_s \to Y[s]$ be the normalization of the blowup of

$$I(s\alpha_1) \cdots I(s\alpha_r)\mathcal{O}_{Y[s]}.$$

Let $\psi_s : Y_s \to \mathrm{Spec}(R)$ be the induced morphism. Define effective Cartier divisors $F_{s,i}$ on Y_s by

$$I(s\alpha_i)\mathcal{O}_{Y_s} = \mathcal{O}_{Y_s}(-F_{s,i}) \subset \mathcal{O}_{Y_s}(\pi_s^*(-s\alpha_i)).$$

Let $D_{s,i} = F_{s,i} - \pi_s^*(s\alpha_i)$, that we will write as $F_{s,i} - s\alpha_i$. Then $D_{s,i}$ is an effective Cartier divisor on Y_s and $-\alpha_i - \frac{1}{s}D_{s,i} = -\frac{1}{s}F_{s,i}$ is anti-effective and nef. We have that

eq12 (50) $$I(s\alpha_1)^{mn_1} \cdots I(s\alpha_r)^{mn_r} \subset I(mn_1F_{s,1}) \cdots I(mn_rF_{s,r}) \subset I(msn_1\alpha_1) \cdots I(msn_r\alpha_r)$$ for all $m, n_1, \ldots, n_r \in \mathbb{N}$.

For $n_1, \ldots, n_r \in \mathbb{N}$, define

$$H_s(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1F_{s,1}) \cdots I(mn_rF_{s,r}))}{s^d m^d}.$$

We have that $H_s(n_1, \ldots, n_r)$ is a homogeneous polynomial of degree d in $n_1, \ldots, n_r$ by Theorem [CSS, Theorem 6.6].

Expand the polynomials

$$H_s(n_1, \ldots, n_r) = \sum b_{i_1,\ldots,i_r}(s) n_1^{i_1} \cdots n_r^{i_r}$$

and

$$F(n_1, \ldots, n_r) = \sum b_{i_1,\ldots,i_r} n_1^{i_1} \cdots n_r^{i_r}$$

with $b_{i_1,\ldots,i_r}(s), b_{i_1,\ldots,i_r} \in \mathbb{R}$.

Prop2 **Proposition 8.1.** *For all $n_1, \ldots, n_r \in \mathbb{N}$,*

$$\lim_{s\to\infty} H_s(n_1, \ldots, n_r) = F(n_1, \ldots, n_r)$$

and for all $i_1, \ldots, i_r$,

eq177 (51) $$\lim_{s\to\infty} b_{i_1,\ldots,i_r}(s) = b_{i_1,\ldots,i_r}.$$

Proof. For $s \in \mathbb{Z}_+$, let $\{I_s(j)_i\}$ be the s-th truncated filtration of $\{I(j)_i\}$ where $I(j)_i = I(i\alpha_j)$ is defined in [CSS, Definition 4.1]. That is, $I_s(j)_i = I(i\alpha_j)$ if $i \le s$ and if $i > s$, then $I_s(j)_i = \sum I_s(j)_a I_s(j)_b$ where the sum is over all $a, b > 0$ such that $a + b = i$. Let

$$F_s(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(R/I_s(1)_{mn_1} \cdots I_s(r)_{mn_r})}{m^d}$$

for $n_1, \ldots, n_r \in \mathbb{N}$. Now there exists $m(s) \in \mathbb{Z}_+$ such that

$$I_s(1)_{smn_1} \cdots I_s(r)_{smn_r} = I(s\alpha_1)^{mn_1} \cdots I(s\alpha_r)^{mn_r}$$

for $m \ge m(s)$. By (50), we have

$$F_s(n_1, \ldots, n_r) = \frac{F_s(sn_1, \ldots, sn_r)}{s^d} \ge H_s(n_1, \ldots, n_r) \ge \frac{F(sn_1, \ldots, sn_r)}{s^d} = F(n_1, \ldots, n_r)$$

for all $n_1, \ldots, n_r \in \mathbb{N}$. By [CSS, Proposition 4.3], for all $n_1, \ldots, n_r \in \mathbb{Z}_+$,

$$\lim_{s\to\infty} F_s(n_1, \ldots, n_r) = F(n_1, \ldots, n_r).$$

Thus for all $n_1, \dots, n_r \in \mathbb{Z}_+$,

eq11 (52) $$\lim_{s\to\infty} H_s(n_1, \dots, n_s) = F(n_1, \dots, n_r).$$

By [14, Lemma 3.2] and (52), we have that

$$\lim_{s\to\infty} b_{i_1,\dots,i_r}(s) = b_{i_1,\dots,i_r}$$

for all $i_1, \dots, i_r$. Thus

$$\lim_{s\to\infty} H_s(n_1, \dots, n_r) = F(n_1, \dots, n_r)$$

for all $n_1, \dots, n_r \in \mathbb{N}$. □

Theorem1 **Theorem 8.2.** *The coefficients of $F(n_1, \dots, n_r)$ are*

$$b_{i_1,\dots,i_r} = \frac{-1}{i_1! \cdots i_r!} \langle (-\alpha_1)^{i_1} \cdot \ldots \cdot (-\alpha_r)^{i_r} \rangle$$

for all $i_1, \dots, i_r$.

Proof. For $s \in \mathbb{Z}_+$, let $\varepsilon_s = \frac{1}{2^s}$. There exist effective $\mathbb{Q}$-Cartier divisors $D_1(s), \dots, D_r(s) \in M^1(R)$ such that $-\alpha_1 - D_1(s), \dots, -\alpha_r - D_r(s)$ are nef and $((-\alpha_1 - D_1(s))^{n_1} \cdot \ldots \cdot (-\alpha_r - D_r(s))^{n_r})$ is within ε_s of $\langle (-\alpha_1)^{n_1} \cdot \ldots \cdot (-\alpha_r)^{n_r} \rangle$ for all $n_1, \dots, n_r \in \mathbb{Z}_+$ with $n_1 + \cdots + n_r = d$. Let $Y(s) \to X \in \mathrm{BirMod}(R)$ be such that $\alpha_1, \dots, \alpha_r, D_1(s), \dots, D_r(s) \in M^1(Y(s))$. Let A_s be effective and anti-ample on $Y(s)$. Then by Proposition 7.5, for $t > 0$ sufficiently small, each product $((-\alpha_1 - D_1(s) - tA_s)^{n_1} \cdot \ldots \cdot (-\alpha_r - D_r(s) - tA_s)^{n_r})$ is within ε_s of $\langle (-\alpha_1)^{n_1} \cdot \cdots \cdot \ldots \cdot (-\alpha_r)^{n_r} \rangle$ for all $n_1, \dots, n_r \in \mathbb{Z}_+$ with $n_1 + \cdots + n_r = d$. Replacing $D_i(s)$ with $D_i(s) + tA_s$ for such a small rational t, we may assume that $-\alpha_i - D_i(s)$ are ample for all i.

There exist $m_i \in \mathbb{Z}_+$ for $i \in \mathbb{Z}_+$ such that $m_1 < m_2 < \cdots$, the $m_s\alpha_i$ are effective Cartier divisors on $Y(s)$, $m_sD_s(s)$ is an effective Cartier divisor on $Y(s)$ and $\mathcal{O}_{Y(s)}(-m_s\alpha_i - m_sD_i(s))$ is very ample on $Y(s)$ for all s and $1 \le i \le r$. In (49), let $Y[m_s] = Y(s)$ for $s \in \mathbb{Z}_+$ and $Y[t] = X$ for $t \notin \{m_1, m_2, \dots\}$.

With the notation introduced after (49), let $F_{m_s,i}$ be the Cartier divisor on Y_{m_s} defined by $\mathcal{O}_{Y_{m_s}}(-F_{m_s,i}) = I(m_s\alpha_i)\mathcal{O}_{Y_{m_s}}$. We have that

$$I(m_s(\alpha_i + D_i(s)) = \Gamma(Y(s), \mathcal{O}_{Y(s)}(-m_s\alpha_i - m_sD_i(s))) \subset \Gamma(Y(s), \mathcal{O}_{Y(s)}(-m_s\alpha_i)) = I(m_s\alpha_i).$$

Since $-m_s\alpha_i - m_sD_i(s)$ is very ample on $Y(s)$,

$$\mathcal{O}_{Y(s)}(-m_s\alpha_i - m_sD_i(s)) = I(-m_s\alpha_i - m_sD_i(s))\mathcal{O}_{Y(s)} \subset I(m_s\alpha_i)\mathcal{O}_{Y(s)}.$$

Thus

$$\mathcal{O}_{Y_{m_s}}(-m_s\alpha_i - m_sD_i(s)) \subset I(m_s\alpha_i)\mathcal{O}_{Y_{m_s}} = \mathcal{O}_{Y_{m_s}}(-F_{m_s,i}) \subset \mathcal{O}_{Y_{m_s}}(-m_s\alpha_i)$$

for all i, s. Thus

$$-\alpha_i - D_i(s) \le -\frac{F_{m_s,i}}{m_s} \le -\alpha_i.$$

Now $\frac{-F_{m_i,s}}{m_s}$ is nef and

$$\frac{-F_{m_s,i}}{m_s} = -\alpha_i - E_{m_s,i}$$

where $E_{m_s,i}$ is an effective $\mathbb{Q}$-Cartier divisor. We have that

$$\begin{aligned} ((-\alpha_1 - D_1(s))^{n_1} \cdot \cdots \cdot \ldots \cdot (\alpha_r - D_r(s))^{n_r}) &\le \left(\left(\tfrac{-F_{m_s,1}}{m_s} \right)^{n_1} \cdot \ldots \cdot \left(\tfrac{-F_{m_s,r}}{m_s} \right)^{n_r} \right) \\ &\le \langle (-\alpha_1)^{n_1} \cdot \ldots \cdot (-\alpha_r)^{n_r} \rangle \end{aligned}$$

for all s and $n_1, \dots, n_r \in \mathbb{N}$ with $n_1 + \cdots + n_r = d$. The first inequality is by Proposition 6.17 and the second inequality is by Definition 7.4. Thus

eq14 (53) $$\left(\left(\frac{-F_{m_s,1}}{m_s}\right)^{n_1} \cdot \ldots \cdot \left(\frac{-F_{m_s,r}}{m_s}\right)^{n_r}\right) \text{ is within } \varepsilon_s \text{ of } \langle(-\alpha_1)^{n_1} \cdot \ldots \cdot (-\alpha_r)^{n_r}\rangle$$

for all $n_1, \dots, n_r \in \mathbb{N}$ with $n_1 + \cdots + n_r = d$.

eq15 (54) Given $\varepsilon > 0$, for $s \gg 0$, the coefficients $b_{i_1,\dots,i_r}(m_s)$ of $H_{m_s}(n_1, \dots, n_r)$ are within ε of the coefficients $b_{i_1,\dots,i_r}$ of $F(n_1, \dots, n_r)$

by Proposition 8.1 and

eq16 (55) $\frac{1}{m_s^d}((-F_{m_s,1}))^{i_1} \cdot \ldots \cdot (-F_{m_s,r})^{i_r})$ is within ε of $\langle(-\alpha_1)^{i_1} \cdot \ldots \cdot (-\alpha_r)^{i_r}\rangle$ for all $i_1, \dots, i_r \in \mathbb{N}$ with $i_1 + \cdots + i_r = d$

by (53). Now

eq17 (56) $$\begin{aligned} H_{m_s}(n_1, \dots, n_r) &= \tfrac{1}{m_s^d}\left(\lim_{m\to\infty} \tfrac{\ell_R(R/I(mn_1F_{m_s,1})\cdots I(mn_rF_{m_s,r}))}{m^d}\right) \\ &= \tfrac{-1}{m_s^d d!}((-n_1F_{m_s,1} - \cdots - n_rF_{m_s,r})^d) \end{aligned}$$

by Lemma 6.4, since $F_{m_s,1}, \dots, F_{m_s,r}$ are effective Cartier divisors and $\mathcal{O}_{Y_{m_s}}(-F_{m_s,i})$ are generated by global sections for all i. Then expanding the last line of (56) by the multinomial theorem, we obtain

$$b_{i_1,\dots,i_r}(m_s) = \frac{-1}{m_s^d i_1! \cdots i_r!}((-F_{m_s,1})^{i_1} \cdot \ldots \cdot (-F_{m_s,r})^{i_r})$$

for all $i_1, \dots, i_r \in \mathbb{N}$ with $i_1 + \cdots + i_r = d$. By (54) and (55), we have that

$$b_{i_1,\dots,i_r} = \frac{-1}{i_1! \cdots i_r!}\langle(-\alpha_1)^{i_1} \cdot \ldots \cdot (-\alpha_r)^{i_r}\rangle$$

for all $i_1, \dots, i_r$.

□

The mixed mutiplicities $e_R(\mathcal{I}(1)^{[d_1]}, \dots, \mathcal{I}(r)^{[d_r]}; R)$ of the filtrations $\mathcal{I}(1), \dots, \mathcal{I}(r)$ of m_R-primary ideals are defined in CSS [14] from the coefficients $b_{d_1,\dots,d_r}$ of $F(n_1, \dots, n_r)$ by defining

$$b_{d_1,\dots,d_r} = \frac{1}{d_1! \cdots d_r!} e_R(\mathcal{I}(1)^{[d_1]}, \dots, \mathcal{I}(r)^{[d_r]}; R).$$

The following theorem follows immediately from Theorem 8.2.

TheoremA **Theorem 8.3.** *Let R be a normal algebraic local ring, $\alpha_1, \dots, \alpha_r \in M^1(R)$ be effective Cartier divisors and let $\mathcal{I}(j)$ be the filtration $\mathcal{I}(j) = \{I(n\alpha_j)\}$ for $1 \le j \le r$.*

Then the mixed multiplicities

$$e_R(\mathcal{I}(1)^{[d_1]}, \dots, \mathcal{I}(r)^{[d_r]}; R) = -\langle(-\alpha_1)^{d_1} \cdot \ldots \cdot (-\alpha_r)^{d_r}\rangle$$

for $d_1, \dots, d_r \in \mathbb{N}$ with $d_1 + \cdots + d_r = d$ are the negatives of the anti-positive intersection products of $-\alpha_1, \dots, -\alpha_r$.

From the case $r = 1$ of Theorem 8.3, we obtain the statement that

$$e_R(\mathcal{I}; R) = \langle(-\alpha)^d\rangle$$

if $\alpha \in M^1(R)$ is an effective Cartier divisor and $\mathcal{I} = \{I(m\alpha)\}$.

Theorem10 **Theorem 8.4.** *Suppose that R is a d-dimensional algebraic local domain, and $\mathcal{I}(j) = \{I(mD(j))\}$ are divisorial filtrations of R for $1 \leq j \leq r$. Then the mixed multiplicities*

$$e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) = \sum_{i=1}^{t} -[S/m_i : R/m_R]\langle(-D(1)_i)^{d_1} \cdot \ldots \cdot (-D(r)_i)^{d_r}\rangle$$

for $d_1, \ldots, d_r \in \mathbb{N}$ with $d_1 + \cdots + d_r = d$.

Proof. We use the notation introduced before the statement of Lemma LemmaR1 2.2. From Lemma LemmaR1 2.2 and (eqR15 12), we have that

$$\begin{aligned}&\lim_{n\to\infty} \frac{\ell_R(R/I(nn_1D(1)\cdots I(nn_rD(r)))}{n^d}\\ &= \sum_{i=1}^{t}[S/m_i : R/m_R]\left(\lim_{n\to\infty} \frac{\ell_{S_{m_i}}(S_{m_i}/J(nn_1D(1)_i)\cdots J(nn_rD(r)_i))}{n^d}\right).\end{aligned}$$

The theorem now follows from Theorem TheoremA 8.3. □

The following theorem follows from Theorem TheoremA 8.3 and CSS [14, Theorem 1.2]. It shows that the Minkowski inequalities hold for the absolute values of the anti-positive intersection products.

TheoremB **Theorem 8.5.** *(Minkowski Inequalities) Let assumptions be as in Theorem TheoremA 8.3, with $r = 2$. Then*

1) $(\langle(-\alpha_1)^i, (-\alpha_2)^{d-i}\rangle)^2 \leq \langle(-\alpha_1)^{i+1}, (-\alpha_2)^{d-i-1}\rangle\langle(-\alpha_1)^{i-1}, (-\alpha_2)^{d-i+1}\rangle$ *for* $1 \leq i \leq d-1$.
2) *For* $0 \leq i \leq d$,
$$\langle(-\alpha_1)^i, (-\alpha_2)^{d-i}\rangle\langle(-\alpha_1)^{d-i}, (-\alpha_2)^i\rangle \leq \langle(-\alpha_1)^d\rangle\langle(-\alpha_2)^d\rangle,$$
3) *For* $0 \leq i \leq d$, $(-\langle(-\alpha_1)^{d-i}, (-\alpha_2)^i\rangle)^d \leq (-\langle(-\alpha_1)^d\rangle)^{d-i}(-\langle(-\alpha_2)^d\rangle)^i$ *and*
4) $(-\langle(-\alpha_1 - \alpha_2)^d\rangle)^{\frac{1}{d}} \leq (-\langle(-\alpha_1)^d\rangle)^{\frac{1}{d}} + (-\langle(-\alpha_2)^d\rangle)^{\frac{1}{d}}$.

We mention a version of the Minkowski inequalities in terms of positive intersection numbers for pseudo effective divisors on a projective variety.

Theorem 8.6. *(Minkowski Inequalities) Suppose that X is a complete algebraic variety of dimension d over a field k and $\mathcal{L}_1$ and $\mathcal{L}_2$ are pseudo effective Cartier divisors on X. Then*

1) $(\langle\mathcal{L}_1^i, \mathcal{L}_2^{d-i}\rangle)^2 \geq \langle\mathcal{L}_1^{i+1}, \mathcal{L}_2^{d-i-1}\rangle\langle\mathcal{L}_1^{i-1}, \mathcal{L}_2^{d-i+1}\rangle$ *for* $1 \leq i \leq d-1$.
2) $\langle\mathcal{L}_1^i, \mathcal{L}_2^{d-i}\rangle\langle\mathcal{L}_1^{d-i}, \mathcal{L}_2^i >\rangle \geq \langle\mathcal{L}_1^d\rangle\langle\mathcal{L}_2^d\rangle$ *for* $1 \leq i \leq d-1$.
3) $(\langle\mathcal{L}_1^{d-i}, \mathcal{L}_2^i\rangle)^d \geq (\langle\mathcal{L}_1^d\rangle)^{d-i}(\langle\mathcal{L}_2^d\rangle)^i$ *for* $0 \leq i \leq d$.
4) $(\langle(\mathcal{L}_1 \otimes \mathcal{L}_2)^d\rangle)^{\frac{1}{d}} \geq (\langle\mathcal{L}_1^d\rangle)^{\frac{1}{d}} + (\langle\mathcal{L}_2^d\rangle)^{\frac{1}{d}}$.

Proof. Statements 1) - 3) follow from the inequality of Theorem 6.6 C4 [10]. Statement 4) follows from 3) and C4 [10, Lemma 4.13], which establishes the super additivity of the positive intersection product. □

Appendix: A proof of Theorem Theorem13 1.4

SecApp

In this appendix we give a proof of Theorem Theorem13 1.4. We fix a potentially confusing index error in the proof in CSS [14].

Step 1). We first observe that if $I' \subset I$ are m_R-primary ideals and $\bigoplus_{n\geq 0} I^n$ is integral over $\bigoplus_{n\geq 0}(I')^n$, then, by HS [41, Theorem 8.2.1, Corollary 1.2.5 and Proposition 11.2.1], $e_R(I; R) = e_R(I'; R)$.

Step 2). Suppose $\mathcal{I} = \{I_i\}$ and $\mathcal{I}' = \{I'_i\}$ are Noetherian filtrations of R by m_R-primary ideals and $\mathcal{I}' \subset \mathcal{I}$. Suppose $b \in \mathbb{Z}_+$. Define $\mathcal{I}^{(b)} = \{I_i^{(b)}\}$ where $I_i^{(b)} = I_{bi}$ and $(\mathcal{I}')^{(b)} = \{(I')_i^{(b)}\}$ where $(I')_i^{(b)} = (I')_{bi}$. Then from [CSS, Lemma 3.3] we deduce that

$$e_R(\mathcal{I}; R) = e_R(\mathcal{I}'; R) \text{ if and only if } e_R(\mathcal{I}^{(b)}; R) = e_R((\mathcal{I}')^{(b)}; R).$$

Step 3). Suppose $\mathcal{I}' \subset \mathcal{I}$ are filtrations of R by m_R-primary ideals. Suppose $a \in \mathbb{Z}_+$. Let $\mathcal{I}_a = \{I_{a,n}\}$ be the a-th truncated filtration of $\mathcal{I}$ defined in [CSS, Definition 4.1]. Then there exists $\overline{a} \in \mathbb{Z}$ such that every element of $\bigoplus_{n\geq 0} I_{a,n}$ (considered as a subring of $\bigoplus_{n\geq 0} I_n$) is integral over $\bigoplus_{n\geq 0} I'_{\overline{a},n}$, where $\mathcal{I}'_{\overline{a}} = \{I'_{\overline{a},i}\}$ is the $\overline{a}$-th truncated filtration of $\mathcal{I}'$ defined in [CSS, Definition 4.1]s.

Define a Noetherian filtration $\mathcal{A}_a = \{A_{a,i}\}$ of R by m_R-primary ideals by

$$A_{a,i} = \sum_{\alpha+\beta=i} I_{a,\alpha} I'_{\overline{a},\beta}.$$

Recall that $I_{a,0} = I'_{\overline{a},0} = R$. We restrict to $\alpha, \beta \geq 0$ in the sum. Thus we have inclusions of graded rings $\bigoplus_{n\geq 0} I'_{\overline{a},n} \subset \bigoplus_{n\geq 0} A_{a,n}$ and $\bigoplus_{n\geq 0} A_{a,n}$ is finite over $\bigoplus_{n\geq 0} I'_{\overline{a},n}$. By Steps 2) and 1),

$$e_R(\mathcal{I}'_{\overline{a}}; R) = e_R(\mathcal{A}_a; R).$$

By [CSS, Proposition 4.3],

$$\lim_{a\to\infty} e_R(\mathcal{I}'_{\overline{a}}; R) = e_R(\mathcal{I}'; R)$$

and thus

$$\lim_{a\to\infty} e_R(\mathcal{A}_a; R) = e_R(\mathcal{I}'; R).$$

Step 4) Let notation be as in the proof of [CSS, Proposition 4.3], but taking $J_i = I_i$ and $J(a)_i = I_{a,i}$. Define

$$\begin{aligned}\Gamma(\mathcal{A}_a)^{(t)} &= \{(m_1, \ldots, m_d, i) \in \mathbb{N}^{d+1} \mid \dim_k A_{a,i} \cap K_{m_1\lambda_1+\cdots+m_d\lambda_d}/A_{a,i} \cap K^+_{m_1\lambda_1+\cdots+m_d\lambda_d} \geq t \\ &\quad \text{and } m_1 + \cdots + m_d \leq \beta i\}.\end{aligned}$$

Now $\Gamma(a)^{(t)} \subset \Gamma(\mathcal{A}_a)^{(t)} \subset \Gamma^{(t)}$ for all t, so

$$\Delta(\Gamma(a)^{(t)}) \subset \Delta(\Gamma(\mathcal{A}_a)^{(t)}) \subset \Delta(\Gamma^{(t)})$$

for all a. By equation (14) [CSS],

$$\lim_{a\to\infty} \mathrm{Vol}(\Delta(\Gamma(a)^{(t)})) = \mathrm{Vol}(\Delta(\Gamma^{(t)})),$$

and so

$$\lim_{a\to\infty} \mathrm{Vol}(\Delta(\Gamma(\mathcal{A}_a)^{(t)})) = \mathrm{Vol}(\Delta(\Gamma^{(t)})).$$

Thus

$$\lim_{a\to\infty} e_R(\mathcal{A}_a; R) = e_R(\mathcal{I}; R)$$

by (12) of the proof of [CSS, Proposition 4.3] applied to $\mathcal{A}_a$.

Step 5). We have that $e_R(\mathcal{I}; R) = e_R(\mathcal{I}'; R)$ by Steps 3) and 4). Now $e_R(\mathcal{I}; M) = e_R(\mathcal{I}'; M)$ by [CSS, Theorem 6.8](with $r = 1$).

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