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ON THE QUADRATIC DUAL OF THE FOMIN–KIRILLOV ALGEBRAS

CHELSEA WALTON AND JAMES J. ZHANG

Abstract. We study ring-theoretic and homological properties of the quadratic dual (or Koszul dual) $\mathcal{E}_n^!$ of the Fomin–Kirillov algebras $\mathcal{E}_n$; these algebras are connected $\mathbb{N}$-graded and are defined for $n \geq 2$. We establish that the algebra $\mathcal{E}_n^!$ is module finite over its center (and thus satisfies a polynomial identity), is Noetherian, and has Gelfand–Kirillov dimension $\lfloor n/2 \rfloor$ for each $n \geq 2$. We also observe that $\mathcal{E}_n^!$ is not prime for $n \geq 3$. By a result of Roos, $\mathcal{E}_n$ is not Koszul for $n \geq 3$, so neither is $\mathcal{E}_n^!$ for $n \geq 3$. Nevertheless, we prove that $\mathcal{E}_n^!$ is Artin–Schelter (AS-)regular if and only if $n = 2$, and that $\mathcal{E}_n^!$ is both AS-Gorenstein and AS-Cohen–Macaulay if and only if $n = 2, 3$. We also show that the depth of $\mathcal{E}_n^!$ is ≤ 1 for each $n \geq 2$, conjecture that we have equality, and show that this claim holds for $n = 2, 3$. Several other directions for further examination of $\mathcal{E}_n^!$ are suggested at the end of this article.

1. Introduction

Throughout this work, we let $\Bbbk$ denote an algebraically closed field of characteristic zero, and we consider all algebraic structures to be $\Bbbk$-linear.

In 1999, Fomin and Kirillov introduced a family of quadratic algebras for the study of the quantum and the ordinary cohomology of flag manifolds in [14]. Since then these algebras, now referred to as *Fomin–Kirillov algebras* $\mathcal{E}_n$ (Definition 1.1), have been studied extensively by many authors with connections to algebraic combinatorics [7, 16, 35, 38], algebraic geometry, and cohomology theory [23, 27], Hopf algebras and Nichols algebras [1, 15, 18, 36], noncommutative geometry [31, 32], and number theory [33], and generalizations of these algebras have been examined in several works [5, 26, 36]. The cohomology algebras $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ of the Fomin–Kirillov algebras have also been of interest: They are related to the cohomology ring of quantum shuffle algebras in [13], for example, and is studied in detail by Ştefan and Vay [43] when $n = 3$. Moreover, several questions asked by Fomin and Kirillov

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in their 1999 work are still open—most notably, it is unknown whether $\mathcal{E}_n$ is finite dimensional for $n \geq 6$ [14, Problem 2.2]. (The algebra $\mathcal{E}_n$ is finite dimensional when $n = 2, 3, 4, 5$ [14, (2.8)].) Many of these questions can be recast in terms of $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$—for instance, the regularity of $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ as an A_∞-algebra is in a certain sense equivalent to the finite-dimensionality of $\mathcal{E}_n$ [30].

With the aim of providing insight into the structure of both $\mathcal{E}_n$ and its cohomology algebra, we examine in this article the subalgebra of $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ generated by $\mathrm{Ext}^1_{\mathcal{E}_n}(\Bbbk, \Bbbk)$. Namely, we study the structure of the *quadratic dual* (or *Koszul dual*) $\mathcal{E}_n^!$ of $\mathcal{E}_n$, also known as the *diagonal subalgebra* $\bigoplus \mathrm{Ext}^{i,i}_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ of $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$.

Note 1. Due to the scope of fields in which the Fomin–Kirillov algebras appear, the exposition of this article is intended for an audience broader than experts in noncommutative graded algebras. As discussed below, much of the theory of commutative local algebras has been generalized to the noncommutative setting by working with algebras that are connected graded. An $\mathbb{N}$-graded $\Bbbk$-algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$ is, by definition, *connected* if $A_0 = \Bbbk$.

To begin, the presentations of $\mathcal{E}_n$ and of its quadratic dual $\mathcal{E}_n^!$ are provided below.

Definition 1.1 ($\mathcal{E}_n$, $x_{i,j}$ [14, Definition 2.1])**.** For $n \geq 2$, the *Fomin–Kirillov algebra*, $\mathcal{E}_n$, is an associative $\Bbbk$-algebra generated by indeterminates $\{x_{i,j} \mid 1 \leq i < j \leq n\}$ of degree 1, subject to the following quadratic relations:

$$\begin{array}{ll} x_{i,j}^2 = 0, & i<j, \\ x_{i,j}x_{j,k} - x_{j,k}x_{i,k} - x_{i,k}x_{i,j} = 0, & i<j<k, \\ x_{j,k}x_{i,j} - x_{i,k}x_{j,k} - x_{i,j}x_{i,k} = 0, & i<j<k, \\ x_{i,j}x_{k,l} - x_{k,l}x_{i,j} = 0, & \{i,j\}\cap\{k,l\} = \emptyset,\ i<j,\ k<l. \end{array}$$

The presentation of $\mathcal{E}_n^!$ is obtained in a straightforward manner from the definition above and is recorded below. For a graded quadratic $\Bbbk$-algebra $A = T(V)/(R)$ with $T(V)$ as the tensor algebra on a $\Bbbk$-vector space V and (R) the two-sided ideal generated by a space $R \subset V \otimes V$, the quadratic dual $A^!$ of A is $T(V^*)/(R^\perp)$, where V^* is the linear dual of V and $R^\perp$ is the orthogonal complement of R in $V \otimes V$.

Definition-Lemma 1.2 ($\mathcal{E}_n^!$, $y_{i,j}$)**.** *The quadratic dual $\mathcal{E}_n^!$ of $\mathcal{E}_n$ is an associative $\Bbbk$-algebra generated by the indeterminates $\{y_{i,j} := x_{i,j}^* \mid 1 \leq i < j \leq n\}$ of degree* 1*, subject to the following quadratic relations:*

$$\begin{array}{ll} y_{i,j}y_{j,k} + y_{j,k}y_{i,k} = 0, & i,j,k \text{ distinct, where } y_{j,i} = -y_{i,j} \text{ for } i<j; \\ y_{i,j}y_{k,l} + y_{k,l}y_{i,j} = 0, & \{i,j\}\cap\{k,l\} = \emptyset,\ i<j,\ k<l. \end{array}$$ □

Our first result pertains to ring-theoretic results on $\mathcal{E}_n^!$: one on the *polynomial identity* (*PI*) property [Definition 2.6, Remark 2.7], i.e., a condition for a ring being "measurably" close to being commutative (e.g., via its *PI degree*); and another on the *Gelfand–Kirillov* (*GK-*)*dimension* (Definition 2.1, Remark 2.2), i.e., a growth measure on a ring that serves as a noncommutative version of the Krull dimension.

Theorem 1.3. *The algebra $\mathcal{E}_n^!$ satisfies the following properties.*

(1) *$\mathcal{E}_n^!$ is Noetherian and is module finite over its center* (*so it satisfies a polynomial identity*) *when $n \geq 2$.*

(2) $\mathcal{E}_n^!$ *has GK-dimension* $\lfloor n/2 \rfloor$ *when* $n \geq 2$.
(3) $\mathcal{E}_n^!$ *is not prime when* $n \geq 3$.

For Theorem 1.3(3), recall that a ring A is *prime* if for all nonzero elements a, b we get $aAb \neq 0$ (a weaker version of the domain property), and a ring is *semiprime* if it contains no nilpotent ideals (a weaker version of the prime property). Semiprime rings are still quite useful in their own right (see, e.g., [17, Chapter 6]). So, we ask the following.

Question 1.4.
(1) Are the algebras $\mathcal{E}_n^!$ semiprime?
(2) What is the PI degree of $\mathcal{E}_n^!$?

In any case, we obtain the following immediate results on the cohomology algebra $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ since $\mathcal{E}_n^!$ is a subalgebra of $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$.

Corollary 1.5. *We find that* $\mathrm{GKdim}(\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)) \geq \lfloor n/2 \rfloor$ *when* $n \geq 2$, *and that* $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ *is not a domain when* $n \geq 3$. ☐

The GKdim bound in Corollary 1.5 is not sharp: Indeed, $\mathrm{GKdim}\,\mathrm{Ext}^*_{\mathcal{E}_3}(\Bbbk, \Bbbk) = 2$ by a result of Ştefan and Vay [43, Theorem 4.17]. Nonetheless, it was shown recently by Ellenberg, Tran, and Westerlan that the growth of $\mathcal{E}_n^!$ and of $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ are related to the growth of the homology of Hurwitz spaces [13].

Now we consider various homological conditions on $\mathcal{E}_n^!$. To start, note that $\mathcal{E}_n$ is not Koszul for $n \geq 3$, due to a result of Roos [39]; thus, $\mathcal{E}_n^!$ is not Koszul for $n \geq 3$. Thus, $\mathcal{E}_n^!$ is a proper subalgebra of $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ (see [37, Section 1.3]). In any case, we examine $\mathcal{E}_n^!$ in view of the hierarchy of *Artin–Schelter* (*AS*) versions of desirable homological properties for connected graded algebras that are not necessarily commutative; see, e.g., [24, Introduction] for more details. In short, one has

AS-regular [Definition 4.1(2)] $\Longrightarrow$ *AS-Gorenstein* (Definition 4.1(1))
$\Longrightarrow$ *AS-Cohen–Macaulay* (Definition 4.1(4)).

A version of the *classical complete intersection* condition (Definition 4.2(3)) is also related; see Figure 1 in Section 4 for more details. Our second result determines whether $\mathcal{E}_n^!$ satisfies the conditions above.

Theorem 1.6. *We have the following statements for* $\mathcal{E}_n^!$ (*and for* $\mathcal{E}_2$, $\mathcal{E}_3$).
(1) $\mathcal{E}_n^!$ *is AS-regular if and only if* $n = 2$.
(2) $\mathcal{E}_n^!$ *is AS-Gorenstein if and only if* $n = 2, 3$.
(3) $\mathcal{E}_n^!$ *is AS-Cohen–Macaulay if and only if* $n = 2, 3$.
(4) $\mathcal{E}_2^!$, $\mathcal{E}_3^!$, $\mathcal{E}_2$, *and* $\mathcal{E}_3$ *are classical completion intersections.*

Since $\mathcal{E}_n^!$ is a Noetherian algebra that satisfies a polynomial identity (Theorem 1.3), one can measure the failure of the AS-Cohen–Macaulay property of $\mathcal{E}_n^!$ by considering its *depth* (Definition 4.1(11)). Namely, the depth of a connected graded, Noetherian algebra A that satisfies a PI is bounded above by its GK dimension, and we have equality of these values if and only if A is AS-Cohen–Macaulay due to the results of Jørgensen [20, Theorem 2.2]. In fact, we verify that $\mathcal{E}_n^!$ has depth ≤ 1 for $n \geq 2$ (Theorem 4.12; cf., Theorem 1.3(2)), and computational evidence yields the following conjecture.

Conjecture 1.7. *The algebras $\mathcal{E}_n^!$ have depth 1 for all $n \geq 2$.*

Note that if Question 1.4(1) has an affirmative answer, then Conjecture 1.7 holds by Theorem 4.12; see Remark 5.1.

There is a similar hierarchy of *Auslander* versions of the regularity (Definition 4.1(8)) and the Gorenstein (Definition 4.1(7)) conditions, in comparison with a version of the Cohen–Macaulay condition (Definition 4.1(9)) considered frequently in noncommutative algebra; see Figure 1. From Theorem 1.6, we obtain the following consequences.

Corollary 1.8. *The algebra $\mathcal{E}_n^!$ is Auslander-regular if and only if $n = 2$, and is both Auslander–Gorenstein and Cohen–Macaulay if and only if $n = 2, 3$.*

The proof of Theorem 1.3 is provided in Section 2, and Theorem 1.6 and Corollary 1.8 are established in Section 4. Two important commutative subalgebras $\mathcal{C}_n$ and $\mathcal{D}_n$ of $\mathcal{E}_n^!$ (Notation 2.5) that play a key role in Section 2 are examined further in Section 3. In addition to Question 1.4 and Conjecture 1.7, more questions and directions for further investigation are presented in Section 5.

2. Ring-theoretic preliminaries and proof of Theorem 1.3

Here, we recall the Gelfand–Kirillov (GK-)dimension and PI property of rings, along with providing examples of and remarks on these notions. Toward studying the GK-dimension and PI property of $\mathcal{E}_n^!$, commutative subalgebras $\mathcal{C}_n$ and $\mathcal{D}_n$ of $\mathcal{E}_n^!$ are constructed below. After preliminary results on $\mathcal{E}_n^!$, and on its relationship to $\mathcal{C}_n$ and $\mathcal{D}_n$, are verified, we prove Theorem 1.3 at the end of the section.

To begin, recall that an $\mathbb{N}$-graded $\Bbbk$-algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ is *connected graded* (c.g.) if $A_0 = \Bbbk$, and is *locally finite* if $\dim_{\Bbbk} A_i < \infty$ for all i. Moreover, the *Hilbert series* of an $\mathbb{N}$-graded, locally finite algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ is, by definition,

$$H_A(t) = \textstyle\sum_{i \in \mathbb{N}} (\dim_{\Bbbk} A_i) t^i.$$

Definition 2.1 ([25], [34, Chapter 8])**.** The *Gelfand–Kirillov dimension* (or *GK-dimension*) of a connected $\mathbb{N}$-graded, locally finite $\Bbbk$-algebra A is defined to be

$$\mathrm{GKdim}(A) = \limsup_{n \to \infty} \frac{\log(\sum_{i=0}^n \dim_{\Bbbk} A_i)}{\log(n)}.$$

Remark 2.2.

(1) Suppose A is finitely generated. Then $\mathrm{GKdim}(A) = 0$ if and only if $\dim_{\Bbbk} A < \infty$.

(2) Even though the definition of GK-dimension is technical, the dimension is computable in many cases. For instance, the GK-dimension of a polynomial ring $\Bbbk[z_1, \ldots, z_m]$ is m, and so is any noncommutative $\Bbbk$-algebra with a $\Bbbk$-vector space basis of monomials

$$\bigoplus \Bbbk\, z_{i_1}^{e_{i_1}} z_{i_2}^{e_{i_2}} \cdots z_{i_t}^{e_{i_t}} z_1^{r_1} z_2^{r_2} \cdots z_m^{r_m},$$

where the sum runs over finitely many values of e_{i_j} and $r_j \geq 0$. For $q \in \Bbbk^{\times}$, the q-polynomial ring

$$\Bbbk_q[z_1, \ldots, z_m] := \Bbbk\langle z_1, \ldots, z_m\rangle / (z_i z_j - q z_j z_i)_{i<j}$$

has such a monomial basis (with $e_{i_j} = 0$), so its GK-dimension is m. In fact, *polynomial growth* is the same condition as finite GK-dimension.

(3) The GK-dimension of a commutative finitely generated $\Bbbk$-algebra is its Krull dimension [34, Theorem 8.2.14].

(4) If B is either a subalgebra or a homomorphic image of a $\Bbbk$-algebra A, then we get GKdim(B)$\leq$ GKdim(A) [34, Proposition 8.2.2]. Moreover, if A is module finite over a $\Bbbk$-subalgebra C, then GKdim(A)= GKdim(C) [34, Proposition 8.2.9(ii)].

Toward computing the GK-dimension of $\mathcal{E}_n^!$, we present a set of monomials whose $\Bbbk$-span is $\mathcal{E}_n^!$.

Lemma 2.3. *Order the generators $\{y_{i,j}\}_{1\leq i<j\leq n}$ of $\mathcal{E}_n^!$ such that $y_{i,j} < y_{k,l}$ if $j < l$, or if $j = l$ and $i < k$. That is,*

$$y_{1,2} < y_{1,3} < y_{2,3} < y_{1,4} < y_{2,4} < y_{3,4} < y_{1,5} < \cdots < y_{n-2,n} < y_{n-1,n}.$$

Then every element in $\mathcal{E}_n^!$ is a linear combination of the monomials of the form

$$y_{1,2}^{r_{1,2}} y_{1,3}^{r_{1,3}} y_{2,3}^{r_{2,3}} y_{1,4}^{r_{1,4}} \cdots y_{n-2,n}^{r_{n-2,n}} y_{n-1,n}^{r_{n-1,n}} \qquad \text{for } \ r_{i,j} \geq 0.$$

Proof. First, we show that if $y_{k,l} > y_{i,j}$, then $y_{k,l}y_{i,j}$ can be written as a lower order term by using relations of Definition-Lemma 1.2. If $\{k,l\} \cap \{i,j\} = \emptyset$, then we use the last relation of $\mathcal{E}_n^!$. Otherwise, for $k > j > i$ we have

$$y_{i,k}y_{i,j} = -y_{i,j}y_{j,k}, \quad y_{j,k}y_{i,j} = -y_{i,k}y_{j,k}, \quad y_{j,k}y_{i,k} = -y_{i,j}y_{j,k}.$$

Now by Bergman's diamond lemma [6], every element in $\mathcal{E}_n^!$ is a linear combination of the monomials $y_{1,2}^{r_{1,2}} y_{1,3}^{r_{1,3}} y_{2,3}^{r_{2,3}} y_{1,4}^{r_{1,4}} \cdots y_{n-2,n}^{r_{n-2,n}} y_{n-1,n}^{r_{n-1,n}}$. □

The following is a preliminary result needed for constructing a central subalgebra of $\mathcal{E}_n^!$ to aid in determining its GK-dimension (and PI property, introduced later).

Lemma 2.4. *Let $a_{i,j} := y_{i,j}^2$. We find that $a_{i,j}$ is central in $\mathcal{E}_n^!$ for all $i < j$, and that the following relations hold in $\mathcal{E}_n^!$:*

$$a_{i,j}y_{j,k} = y_{j,k}a_{i,k} \quad \text{and} \quad a_{i,j}a_{j,k} = a_{i,j}a_{i,k} \quad \forall \text{ distinct } i,j,k.$$

Proof. If $\{i,j\} \cap \{s,t\} = \emptyset$, then by Definition-Lemma 1.2 we get

$$y_{s,t}a_{i,j} \;=\; y_{s,t}y_{i,j}^2 \;=\; -y_{i,j}y_{s,t}y_{i,j} \;=\; y_{i,j}^2 y_{s,t} \;=\; a_{i,j}y_{s,t}.$$

On the other hand, we will show that $a_{i,j}$ and $y_{j,k}$ commute for any $i < j$ with $k = i$ or j. Recall that $y_{j,i} = -y_{i,j}$ for all $i < j$. Indeed,

$$a_{i,j}y_{j,k} \;=\; -y_{i,j}^2 y_{k,j} \;=\; y_{i,j}y_{k,i}y_{i,j} \;=\; -y_{j,i}y_{k,i}y_{i,j} \;=\; y_{k,j}y_{j,i}y_{i,j} \;=\; y_{j,k}a_{i,j}.$$

Hence, $a_{i,j}$ is central.

By Definition-Lemma 1.2, we also obtain that

$$\begin{aligned} a_{i,j}y_{j,k} &= -y_{i,j}y_{j,k}y_{i,k} &&= y_{j,k}y_{i,k}y_{i,k} &&= y_{j,k}a_{i,k}, \\ a_{i,j}a_{j,k} &= -y_{i,j}y_{j,k}y_{i,k}y_{j,k} &&= y_{i,j}y_{j,k}y_{j,i}y_{i,k} &&= -y_{i,j}y_{k,j}y_{j,i}y_{i,k} \\ &= y_{k,i}y_{i,j}y_{j,i}y_{i,k} &&= a_{i,j}a_{i,k}, \end{aligned}$$

as claimed. □

Notation 2.5 ($a_{i,j}$, $\mathcal{C}_n$, $\mathcal{D}_n$). From now on, we let $a_{i,j}$ denote the central elements $y_{i,j}^2$ (see Lemma 2.4), and we use the notation $a_{i,j}$ for $a_{j,i}$ when $i > j$.

Let $\mathcal{C}_n$ be the commutative subalgebra of $\mathcal{E}_n^!$ generated by $\{a_{i,j} \mid 1 \leq i < j \leq n\}$. Let $\mathcal{D}_n$ be the commutative algebra generated by $\{a_{i,j} \mid 1 \leq i < j \leq n\}$, subject to the relations

$$a_{i,j}a_{j,k} = a_{i,j}a_{i,k} \qquad \forall \text{ distinct } i, j, k.$$

By Lemma 2.4, there is a natural algebra surjection from $\mathcal{D}_n$ to $\mathcal{C}_n$. Later, in Proposition 3.10, we will show that $\mathcal{D}_n$ is indeed isomorphic to $\mathcal{C}_n$. Note that the relations in the algebra $\mathcal{D}_n$ are similar to those of the algebra A_n that appear in [3, Section 4.1].

We use the commutative subalgebras $\mathcal{C}_n$ and $\mathcal{D}_n$ to study the GK-dimension and the PI property (defined below) of $\mathcal{E}_n^!$ as follows.

Definition 2.6 ([34, Chapter 13])**.** A *polynomial identity* (*PI*) for a ring A is a monic multilinear polynomial $f \in \mathbb{Z}\langle z_1, \dots, z_d\rangle$ such that $f(a_1, \dots, a_d) = 0$ for all $a_i \in A$. A ring A is called a *PI ring*, or PI for short, if such a polynomial f exists for A.

If a PI ring A is prime, then the *PI degree* of A is half of the minimal degree of a PI for A.

Remark 2.7.

(1) The PI property is preserved by taking it under subalgebra and homomorphic image, and algebras that are module finite over a commutative subalgebra are PI [34, Lemma 13.1.7, Corollary 13.1.13].
(2) If A is a commutative domain (and thus prime), then its PI degree is 1 since the commutator is a PI of minimal degree 2. On the other hand, the q-polynomial ring $\Bbbk_q[z_1, \dots, z_m]$ from Remark 2.2(2) is prime, and it is PI if and only if q is a root of unity. When q is a primitive dth root of unity, the PI degree of $\Bbbk_q[z_1, \dots, z_m]$ is $d^{\lfloor m/2 \rfloor}$ [8, I.14].
(3) See [41, p. 98] for the (technical) definition of PI degree for nonprime PI rings.

Lemma 2.8. *Recall Notation* 2.5. *The following statements hold.*

(1) *There is a natural surjective map $\mathcal{D}_n \twoheadrightarrow \mathcal{C}_n$, and there is a natural injective map $\mathcal{C}_n \hookrightarrow \mathcal{E}_n^!$. As a consequence, $\mathcal{E}_n^!$ is a module over both $\mathcal{C}_n$ and $\mathcal{D}_n$.*
(2) *$\mathcal{E}_n^!$ is a finitely generated module over both $\mathcal{C}_n$ and $\mathcal{D}_n$.*
(3) *$\mathcal{D}_n$ satisfies the relations*

$$a_{i,j}^2 a_{i,k} = a_{i,j} a_{i,k}^2 \tag{E1.8.1}$$

for all distinct i, j, k.

(4) $\operatorname{GKdim} \mathcal{D}_n = \lfloor n/2 \rfloor$.

Proof.

(1) This follows from Lemma 2.4.
(2) By Lemmas 2.3 and 2.4, we find that as a $\Bbbk$-vector space,

$$\mathcal{E}_n^! = \sum_{e_{i,j}=0,1} \Bbbk\, y_{1,2}^{e_{1,2}} y_{1,3}^{e_{1,3}} \cdots y_{n-1,n}^{e_{n-1,n}}\, \mathcal{C}_n.$$

Therefore, $\mathcal{E}_n^!$ is a finitely generated $\mathcal{C}_n$-module. The last claim holds as $\mathcal{D}_n \twoheadrightarrow \mathcal{C}_n$.

(3) In $\mathcal{D}_n$, we have $a_{i,j}(a_{i,k} - a_{j,k}) = 0$ for all distinct i, j, k. Then we get

$$a_{i,j}a_{i,k}(a_{i,j} - a_{i,k}) = a_{i,j}a_{j,k}(a_{i,j} - a_{i,k}) = 0,$$

as desired.

(4) Let $m := \lfloor n/2 \rfloor$, and let $I := \{(1, 2), (3, 4), \dots, (2m-1, 2m)\}$. Let B be the quotient algebra $\mathcal{D}_n/(a_{i,j})_{i<j,(i,j)\notin I}$. Observe that B is the commutative polynomial ring generated by the set $\{a_{i,j}\}_{(i,j)\in I}$. So,

$$\operatorname{GKdim} \mathcal{D}_n \geq \operatorname{GKdim} B = m$$

by Remark 2.2(2,4).

Next, we will show that $\operatorname{GKdim} \mathcal{D}_n \leq m$. Order the generators $\{a_{i,j}\}$ of $\mathcal{D}_n$ by $a_{i,j} < a_{k,l}$ if either $j < l$, or $j = l$ and $i < k$ (we assume that $i < j$ and $k < l$ here). Using the diamond lemma [6] and the relations in Lemmas 2.4 and 2.8(3), we obtain that if we have a nonzero monomial of the form

$$a_{i_1,j_1}^{r_{i_1,j_1}} \cdots a_{i_t,j_t}^{r_{i_t,j_t}}$$

that is not a linear combination of lower degree terms with $r_{i_s,j_s} \geq 2$ for all s, then $\{i_{s_1}, j_{s_1}\} \cap \{i_{s_2}, j_{s_2}\} = \emptyset$. Therefore, $t \leq m$. Let V be the graded subspace of $\mathcal{D}_n$ defined by

$$V = \sum \Bbbk a_{i_1,j_1}^{r_{i_1,j_1}} \cdots a_{i_t,j_t}^{r_{i_t,j_t}},$$

where the sum runs over all possible (i_s, j_s) such that $\{i_{s_1}, j_{s_1}\} \cap \{i_{s_2}, j_{s_2}\} = \emptyset$ for any pairs with $r_{i_{s_1},j_{s_1}}, r_{i_{s_2},j_{s_2}} \neq 0$. By the description of the monomials above, we get

$$\mathcal{D}_n = \sum_{e_{i,j}=0,1} \Bbbk\, a_{1,2}^{e_{1,2}} a_{1,3}^{e_{1,3}} \cdots a_{n-1,n}^{e_{n-1,n}}\, V.$$

Since $\operatorname{GKdim} V \leq m$, and since $\mathcal{D}_n$ is a quotient space of a finite direct sum of V, we obtain that $\operatorname{GKdim} \mathcal{D}_n \leq m$, as desired. □

Proof of Theorem 1.3.

(1) By Lemma 2.8(2), $\mathcal{E}_n^!$ is a finitely generated module over the finitely generated commutative (and thus Noetherian) $\Bbbk$-subalgebra $\mathcal{C}_n$ from Notation 2.5. Thus, $\mathcal{E}_n^!$ is a finitely generated Noetherian PI algebra and is module finite over its center. See [17, Proposition 1.6] and Remark 2.7(1).

(2) First, we construct a factor algebra of $\mathcal{E}_n^!$ to bound its GK-dimension from below. Let $m := \lfloor n/2 \rfloor$, and let $I := \{(1, 2), (3, 4), \dots, (2m - 1, 2m)\}$. Now take

$$B := \mathcal{E}_n^!/(y_{i,j})_{i<j,(i,j)\notin I},$$

a factor algebra of $\mathcal{E}_n^!$. By Definition-Lemma 1.2, we get B to be isomorphic to the (-1)-skew polynomial ring $\Bbbk_{-1}[y_{1,2}, y_{3,4}, \dots, y_{2m-1,2m}]$. So, by Remark 2.2(2,4),

$$\operatorname{GKdim} \mathcal{E}_n^! \geq \operatorname{GKdim} B = m.$$

On the other hand, since $\mathcal{E}_n^!$ is a finite module over $\mathcal{C}_n$ and $\mathcal{C}_n$ is a homomorphic image of $\mathcal{D}_n$ (Lemma 2.8(1,2)), we obtain that

$$\operatorname{GKdim} \mathcal{E}_n^! = \operatorname{GKdim} \mathcal{C}_n \leq \operatorname{GKdim} \mathcal{D}_n = m$$

by Remark 2.2(4) and Lemma 2.8(4). Thus, GKdim $\mathcal{E}_n^! = m = \lfloor n/2 \rfloor$.

(3) This follows from Lemma 2.4: Namely, $y_{j,k}(a_{i,j} - a_{i,k}) = 0$ for all distinct i, j, k, yet $y_{j,k}$ and $a_{i,j} - a_{i,k}$ are nonzero in $\mathcal{E}_n^!$ (by the diamond lemma [6]). □

3. Results of Etingof on the algebras $\mathcal{C}_n$ and $\mathcal{D}_n$

Recall that n is an integer ≥ 2. This section contains some results due to Pavel Etingof on the commutative algebras $\mathcal{C}_n$ and $\mathcal{D}_n$ from Notation 2.5. We thank him for suggesting these results and providing us with the key ideas for the proofs.

To begin, we need to set some notation on set partitions and corresponding graphs and monomials.

Notation 3.1 ($[n]$, π, Π_n, B_r, $\text{rk}(\pi)$, $\#_1(\pi)$, $\#_{\geq 2}(\pi)$). We fix some notation on set partitions.

- Let $[n]$ denote the set $\{1, \dots, n\}$.
- Let Π_n denote the collection of set partitions of $[n]$. Namely, $\pi \in \Pi_n$ if π is a set of nonempty subsets $B_1, \dots, B_d$ of $[n]$ such that each positive integer between 1 and n lies in exactly one of the B_i. We refer to the subsets B_i as the *blocks* of π, and we denote the number of blocks of π by $\#\pi$.
- The *rank* of $\pi \in \Pi_n$, denoted by $\text{rk}(\pi)$, is the value $n - \#\pi$.
- We denote by $\#_1(\pi)$ (resp., $\#_{\geq 2}(\pi)$) the number of blocks of π that are singletons (resp., have cardinality ≥ 2).
- We call $\pi \in \Pi_n$ *trivial* if $\#\pi = \#_1(\pi) = n$, that is, if $\pi = \{\{1\}, \{2\}, \dots, \{n\}\}$.

Notation 3.2 (G, G_r, m_r, $V(G_r)$, $S(G)$, $T(G)$). Let G be a graph with the vertex set $[n]$ subject to the following conditions. It is loopless and we allow for multiple edges between two vertices. Moreover, corresponding to $\pi \in \Pi_n$ as in Notation 3.1:

- The graph G has $\#\pi$ connected components of G denoted by $\{G_r\}_{r=1}^{\#\pi}$.
- Let $m_r = m_r(G_r)$ be the total number of edges in each connected component G_r for each r.
- We refer to π as the *support* of G, sometimes denoted by $S(G)$, in the sense that the block B_r is the vertex set $V(G_r)$ of G_r.
- The *type* of G, denoted by $T(G)$, is the collection $\{(B_r := V(G_r), m_r)\}_{r=1}^{\#\pi}$.

Notation 3.3 ($G(\underline{f})$, $G_r(\underline{f})$, $m_r(\underline{f})$, $V(G_r(\underline{f}))$, $S(\underline{f})$, $T(\underline{f})$). Let F be a not necessarily commutative algebra generated by $\{f_{i,j} \mid 1 \leq i < j \leq n\}$.

- For each monomial $\underline{f} := f_{i_1,j_1} f_{i_2,j_2} \cdots f_{i_w,j_w}$, we define a graph, denoted by $G(\underline{f})$, with vertex set $[n]$, and with edges i_s—j_s for each f_{i_s,j_s} in $\underline{f}$.

Then $G(\underline{f})$ is a disjoint union of connected components $\{G_r(\underline{f})\}_{r=1}^d$ for some $d \in \mathbb{N}$.

- Let $m_r(\underline{f})$ be the total number of edges in $G_r(\underline{f})$ for each r.
- The type of $\underline{f}$ is defined to be the collection of pairs

$$T(\underline{f}) := T(G(\underline{f})) = \left\{ \left(V(G_r(\underline{f})), m_r(\underline{f}) \right) \right\}_{r=1}^d ,$$

and the support of $\underline{f}$ is the set partition of the vertex set of $G(\underline{f})$,

$$S(\underline{f}) := S(G(\underline{f})) = \left\{ V(G_r(\underline{f})) \right\}_{r=1}^d .$$

We define a partial ordering on the types of graphs defined above.

Definition 3.4. Choose two graphs G and H on n vertices as in Notation 3.2; the same will apply for the graphs defined on monomials of the same degree as in Notation 3.3. We write $T(G) < T(H)$ if either of the following apply:

(a) The set partition $\pi_G \in \Pi_n$ corresponding to $T(G)$ is a *proper refinement* of $\pi_H \in \Pi_n$ of $T(H)$ (i.e., each block of π_G is a subset of a block of π_H, with one such being a proper subset).

(b) We have $\pi_G = \pi_H$ (i.e., G and H have the same support) and the sequence $\{m_r(G_r)\}_{r=1}^d$ is less than $\{m_r(H_r)\}_{r=1}^d$ in the lexicographic order.

We also write $S(G) < S(H)$ if condition (a) holds.

Next, we turn our attention to the commutative algebra $\mathcal{D}_n$ from Notation 2.5 (which is studied in Lemma 2.8). Recall that $\mathcal{D}_n$ is generated by commuting elements $\{a_{i,j} \mid 1 \leq i < j \leq n\}$ and is subject to the relations

$$a_{i,j}a_{j,k} = a_{i,j}a_{i,k} \qquad \forall \text{ distinct } i,j,k. \tag{E2.4.1}$$

We will use Bergman's diamond lemma [6] in the next proof. Let $\mathcal{S}$ be a reduction system for the set of relations $\mathcal{R}$ of a given finitely presented algebra. If all ambiguities of $\mathcal{R}$ can be resolved using $\mathcal{S}$, then we call $\mathcal{S}$ a *Gröbner basis* of $\mathcal{R}$.

Lemma 3.5. *Recall the commutative algebra $\mathcal{D}_n$ from Notation 2.5. Every monomial of $\mathcal{D}_n$ (in the variables $a_{i,j}$) is nonzero.*

Proof. This basically follows from Bergman's diamond lemma [6]. Rewrite the set of relations of $\mathcal{D}_n$ as

$$a_{i_1,j_1}a_{i_2,j_2} = a_{k_1,l_1}a_{k_2,l_2}$$

for some indices (i_1,j_1), (i_2,j_2), (k_1,l_1), and (k_2,l_2). By induction, every relation in the Gröbner basis derived from overlap ambiguity is of the form

$$a_{i_1,j_1}\cdots a_{i_w,j_w} = a_{k_1,l_1}\cdots a_{k_w,l_w} \tag{E2.5.1}$$

for some pairs of indices (i_s,j_s), (k_s,l_s) for $s = 1,\dots,w$. It is not necessary to specify what these indices are. In other words, the reduction in the diamond lemma uses the relations only of the form (E2.5.1). This implies that every monomial is equal to a reduced monomial. Therefore, every monomial of $\mathcal{D}_n$ is nonzero. □

We now discuss the growth of the algebra $\mathcal{D}_n$.

Proposition 3.6. *The Hilbert series of $\mathcal{D}_n$ is*

$$H_{\mathcal{D}_n}(t) \;=\; \sum_{\pi\in\Pi_n} \frac{t^{\mathrm{rk}(\pi)}}{(1-t)^{\#_{\geq 2}(\pi)}} \;=\; 1 + \sum_{\text{nontriv } \pi\in\Pi_n} \frac{t^{\mathrm{rk}(\pi)}}{(1-t)^{\#_{\geq 2}(\pi)}}.$$

Proof. Take $\underline{a} := \prod_{s=1}^{w} a_{i_s,j_s}$, a monomial of $\mathcal{D}_n$; it is fine to use $\prod$ since $\mathcal{D}_n$ is commutative. For $B = \{i_1,\dots,i_q\} \subseteq [n]$ and $m \geq q-1$, let

$$\underline{a}(B,m) := a_{i_1,i_2}^{m-q+2} a_{i_1,i_3}\cdots a_{i_1,i_q}$$

be a certain monomial of $\mathcal{D}_n$ of degree m. By convention, $\underline{a}(B,m) = 1$ if B is a singleton and, consequently, $m = 0$. By using the relations (E2.4.1), one sees that $\underline{a}$ is equal to $\underline{a}(B,m)$ in $\mathcal{D}_n$ if and only if $G(\underline{a})$ is connected except for singletons.

Further, let $\underline{a}$ be a monomial with $T(\underline{a}) = \{(B_r := V(G_r), m_r)\}_{r=1}^d$; then we claim that

$$\underline{a} = \prod_{m_r \geq 1} \underline{a}(B_r, m_r) = \prod_r \underline{a}(B_r, m_r) \tag{E2.6.1}$$

in the algebra $\mathcal{D}_n$. In other words, a monomial is determined by its type uniquely. In equation (E2.6.1), B_r's are disjoint subsets of $[n]$.

To show the above claim, we first show that every element of the form

$$\underline{a}(B_1, m_1) \cdots \underline{a}(B_d, m_d) \tag{E2.6.2}$$

is reduced. Every step in the reduction of the diamond lemma [6] uses one of the commuting relations of $\mathcal{D}_n$ or one of the relations in (E2.4.1) such that (i, j, k) includes one of the connected components of $G(\underline{a})$. This means that in every step $\underline{f} = \underline{f}'$ in the reduction process, we have $T(\underline{f}) = T(\underline{f}')$. Therefore, $T(\underline{a}) = T(\underline{a}')$ if $\underline{a} = \underline{a}'$ for two monomials in $\mathcal{D}_n$. Clearly, $\underline{a}(B_1, m_1) \cdots \underline{a}(B_d, m_d)$ is the smallest with respect to the lexicographic order. Therefore, $\underline{a}(B_1, m_1) \cdots \underline{a}(B_d, m_d)$ is reduced. Second, if $\underline{a}$ is a monomial that is not in the form of (E2.6.2), it can be reduced to another monomial of the form of (E2.6.2) by induction on the degree of the monomial. Thus, we proved the claim.

Now the assertion follows by counting monomials of the form in (E2.6.1). Namely, fix $\pi \in \Pi_n$. Then the monomials of positive degree of the form in (E2.6.1) corresponding to $\pi = \{B_1, \dots, B_{\#\pi}\}$ are $\prod_{r \text{ with } |B_r| \geq 2} \underline{a}(B_r, m_r)$; this contributes to the Hilbert series of $\mathcal{D}_n$ the term

$$\prod_{r \text{ with } |B_r| \geq 2} (t^{|B_r|-1} + t^{|B_r|} + t^{|B_r|+1} + \cdots) = \prod_{r \text{ with } |B_r| \geq 2} \frac{t^{|B_r|-1}}{1-t}$$

since the degree of $\underline{a}(B_r, m_r)$ is m_r, which has the lowest value $|B_r| - 1$. Since

$$\begin{aligned} \textstyle\sum_{r \text{ with } |B_r| \geq 2} (|B_r| - 1) &= \left(\textstyle\sum_{r \text{ with } |B_r| \geq 2} |B_r|\right) - \#_{\geq 2}(\pi) \\ &= (n - \#_1(\pi)) - \#_{\geq 2}(\pi) \\ &= \operatorname{rk}(\pi), \end{aligned}$$

we find that the monomials of positive degree corresponding to a partition $\pi \in \Pi_n$ contribute

$$\prod_{r \text{ with } |B_r| \geq 2} \frac{t^{|B_r|-1}}{1-t} = \frac{t^{\operatorname{rk}(\pi)}}{(1-t)^{\#_{\geq 2}(\pi)}}$$

to the Hilbert series of $\mathcal{D}_n$. □

Naturally, we then inquire as follows.

Question 3.7. What is the Hilbert series of $\mathcal{E}_n^!$?

Proposition 3.8 (Etingof)**.** *The algebra $\mathcal{D}_n$ is reduced; that is, $\mathcal{D}_n$ has no nonzero nilpotent elements. As a consequence, $\mathcal{D}_n$ is semiprime.*

Proof. Let $0 \neq f \in \mathcal{D}_n$ be a linear combination of monomials $\sum_i c_i \underline{a}_i$ for $c_i \in \Bbbk$. Here, $\underline{a}_i \in \mathcal{D}_n$ are linearly independent monomials in $a_{i,j}$s. We want to show that $f^m \neq 0$ for all $m \geq 1$. We pick a summand $c_{i_0} \underline{a}_{i_0} \neq 0$ of f such that $T(\underline{a}_{i_0})$ is one of minimal type according to the partial ordering of Definition 3.4. We want to show that f^m is a linear combination of $c_{i_1}\cdots c_{i_m}\underline{a}_{i_1}\cdots\underline{a}_{i_m}$, with its summand $c_{i_0}^m\underline{a}_{i_0}^m \neq 0$ being of minimal type. This is enough to conclude that $f^m \neq 0$ by Lemma 3.5.

Consider the term $\underline{a}_{i_1}\cdots\underline{a}_{i_m}$ that is not $\underline{a}_{i_0}^m$, and suppose by way of contradiction that $T(\underline{a}_{i_1}\cdots\underline{a}_{i_m}) < T(\underline{a}_{i_0}^m)$. Then $S(\underline{a}_{i_1}\cdots\underline{a}_{i_m}) < S(\underline{a}_{i_0}^m) = S(\underline{a}_{i_0})$ by condition (a) of Definition 3.4. This implies that $S(\underline{a}_{i_s}) \leq S(\underline{a}_{i_0})$ for all s. Since $T(\underline{a}_{i_0})$ is minimal, we obtain that $S(\underline{a}_{i_s}) = S(\underline{a}_{i_0})$. When all $\underline{a}_{i_s}$ have the same support, we use condition (b) of Definition 3.4 to obtain $T(\underline{a}_{i_s}) < T(\underline{a}_{i_0})$ for some $\underline{a}_{i_s} \neq \underline{a}_{i_0}$, which is a contradiction. Thus, f^m is a linear combination of monomials with one summand $c_{i_0}^m\underline{a}_{i_0}^m$ of a summand with minimal type. The assertion follows.

The semiprimeness of $\mathcal{D}_n$ now holds since $\mathcal{D}_n$ is commutative. □

We need the following construction for the next result.

Definition-Lemma 3.9. *Let $\{R_i\}_{i=1}^d$ be some $\mathbb{Z}_2$-graded algebras. The super tensor product between R_i and R_j is a $\mathbb{Z}_2$-graded algebra,*

$$R_i \otimes_{\text{super}} R_j,$$

which is equal to $R_i \otimes R_j$ as $\mathbb{Z}_2$-graded algebras, where for homogeneous elements $f, f' \in R_i$ and $g, g' \in R_j$ we have

$$(f\otimes g)(f'\otimes g') = (-1)^{|g||f'|}(ff'\otimes gg').$$

The $\mathbb{Z}_2$-grading of $R_i \otimes_{\text{super}} R_j$ is $(R_i \otimes_{\text{super}} R_j)_0 = (R_i)_0 \otimes (R_j)_0 + (R_i)_1 \otimes (R_j)_1$ and $(R_i \otimes_{\text{super}} R_j)_1 = (R_i)_1 \otimes (R_j)_0 + (R_i)_0 \otimes (R_j)_1$. Moreover, $R_i \otimes_{\text{super}} R_j$ is a graded twist of $R_i \otimes R_j$ in the sense of [45]. □

Recall that $\mathcal{C}_n$ is the central subalgebra of $\mathcal{E}_n^!$ generated by $a_{i,j} := y_{i,j}^2$ for all i, j.

Proposition 3.10 (Etingof). *There is a natural algebra isomorphism $\mathcal{D}_n \cong \mathcal{C}_n$.*

Proof. We achieve the result by studying the points of

$$X_n := \operatorname{Spec}\mathcal{D}_n \subseteq \Bbbk^{\binom{n}{2}}.$$

Let us set some notation used in this proof. Given a point $z = (z_{i,j})_{1\leq i<j\leq n} \in X_n$, let $G(z)$ be the graph on the vertex set $[n]$ where i and j are connected if and only if $z_{i,j} \neq 0$. Let $\{G_r(z)\}_{r=1}^d$ be the set of connected components of $G(z)$. The relations

$$z_{i,j}(z_{i,k} - z_{j,k}) = 0$$

(derived from (E2.4.1)) imply that, for some $x_r \in \Bbbk^\times$,

$$z_{i,j} = \begin{cases} x_r & \text{if } i,j \text{ belong to the same component } G_r(z), \\ 0 & \text{otherwise} \end{cases} \tag{E2.10.1}$$

by definition. Thus, each connected component $G_r(z)$ is a complete graph.

On the other hand, for each set partition $\pi = \{B_1, \ldots, B_d\} \in \Pi_n$, we can construct a stratum

$$X_n(\pi) := \{z \in X_n \mid G_r(z) \text{ is the complete graph } K(B_r)\ \forall r\}.$$

It is clear that $X_n(\pi)$ is a torus of dimension equal to $\#_2(\pi)$.

For the rest of the proof, recall that by Lemma 2.8(1), there is a natural algebra surjection from $\phi : \mathcal{D}_n \to \mathcal{C}_n$. So, the assertion is equivalent to the statement that the map ϕ is injective. Let I be the kernel of ϕ, and let $\mathcal{Z}(I) \subseteq X_n$ be the zero set of I. Since $\mathcal{D}_n$ is reduced (Proposition 3.8(1)), it suffices to show that $\mathcal{Z}(I) = X_n$.

Note that I is also equal to the kernel of the composition $(\mathcal{C}_n \hookrightarrow \mathcal{E}_n^!) \circ \phi$, where the inclusion holds by Lemma 2.8(1). Take a point $z \in X_n$ and consider the corresponding maximal ideal $\mathfrak{m}_z$ of $\mathcal{D}_n$ and form the algebra

$$\mathcal{E}_n^!(z) := \mathcal{E}_n^! \otimes_{\mathcal{D}_n} \mathcal{D}_n/(\mathfrak{m}_z).$$

Then it suffices to show that $\mathcal{E}_n^!(z) \neq 0$ for all $z \in X_n$. Namely, this condition is equivalent to the condition that $\mathfrak{m}_z$ contains I for all $z \in X_n$, which in turn is equivalent to $I \subset \bigcap_{z \in X_n} \mathfrak{m}_z$; here, $\bigcap_{z \in X_n} \mathfrak{m}_z = 0$ since $\mathcal{D}_n$ is reduced.

Let z be any point in X_n. Then it belongs to a stratum $X_n(\pi)$ for some set partition $\pi = \{B_1, \dots, B_{\#\pi}\}$ of $[n]$, where $B_r = G_r(z)$. Then $\mathcal{E}_n^!(z)$ is defined as the quotient algebra

$$\mathcal{E}_n^!(z) = \mathcal{E}_n^! / \left(\{y_{i,j}^2 - x_r\}_{i,j \in G_r(z)},\ \{y_{i,j}^2\}_{i \text{ or } j \notin G_r(z)}\right) \quad \text{for } x_r \text{ in (E2.10.1)}.$$

Let us replace the last relations by an even stronger relation: Take the ideal

$$J = (y_{i,j})_{i,\, j \text{ in different blocks of } \pi}.$$

Then the resulting quotient algebra $\mathcal{E}_n^!(z)/J$ is a super tensor product of algebras, denoted by R_r, associated with components B_r with $|B_r| \geq 2$. Now by Definition-Lemma 3.9, it suffices to show that each R_r is nonzero, or equivalently, to show that $\mathcal{E}_n^!(z)$ is nonzero when $\#\pi = 1$.

Without loss of generality, we can first assume that $\#\pi = 1$ and, by replacing $y_{i,j}$ with $x_1^{-1/2} y_{i,j}$, we may further assume that $x_1 = 1$. Now $z_{i,j} = 1$ for all $i \neq j$ by (E2.10.1). So it suffices to show that the algebra $\mathcal{E}_n^!(1)$ is nonzero.

For $n = 3$, one can check directly or by using a computer algebra program that the algebra

$$\mathcal{E}_3^!(1) = \frac{\Bbbk\langle y_{1,2},\ y_{1,3},\ y_{2,3}\rangle}{\left(\begin{array}{cc} \multicolumn{2}{c}{y_{1,2}^2 - 1, \quad y_{1,3}^2 - 1, \quad y_{2,3}^2 - 1,} \\ y_{1,2}y_{2,3} + y_{2,3}y_{1,3}, & y_{1,2}y_{2,3} + y_{1,3}y_{1,2}, \\ y_{2,3}y_{1,2} + y_{1,3}y_{2,3}, & y_{2,3}y_{1,2} + y_{1,2}y_{1,3} \end{array}\right)}$$

is nonzero. For $n \geq 4$, we can check that the algebra $\mathcal{E}_n^!(1)$ is isomorphic to a factor of the group algebra of a Schur cover $2 \cdot S_n^+$ of the symmetric group S_n. Namely, the group $2 \cdot S_n^+$ has a presentation with generators $w, s_1, \dots, s_{n-1}$ and relations

$$\begin{array}{ll} w^2 = 1, & \\ ws_i = s_i w \quad \text{and} \quad s_i^2 = 1 & \forall 1 \leq i \leq n-1, \\ s_{i+1}s_i s_{i+1} = s_i s_{i+1} s_i, & \forall 1 \leq i \leq n-2, \\ s_j s_i = s_i s_j w, & \forall 1 \leq i < j \leq n-1,\ |i-j| \geq 2; \end{array}$$

see [22, Chapter 12]. Now by identifying $w = -1$ and $s_i = y_{i,i+1}$, we get

$$\mathcal{E}_n^!(1) \cong \Bbbk(2 \cdot S_n^+)/(w+1).$$

Since $|2 \cdot S_n^+| = 2n!$ and the order of w is 2, we find that $\Bbbk(2 \cdot S_n^+)/(w+1)$ has dimension $n!$, so $\mathcal{E}_n^!(1)$ is not 0, as desired. $\square$

One quick consequence of the proposition above is the following.

Corollary 3.11. *Every monomial in $\mathcal{E}_n^!$ is nonzero.*

Proof. Since the natural map $\mathcal{D}_n \to \mathcal{E}_n^!$ is injective (Lemma 2.8(1) and Proposition 3.10), every element of the form $y_{i_1,j_1}^2 y_{i_2,j_2}^2 \cdots y_{i_w,j_w}^2$ is nonzero by Lemma 3.5. Starting from any monomial $y_{i_1,j_1} y_{i_2,j_2} \cdots y_{i_w,j_w}$ in $\mathcal{E}_n^!$, one can see that

$$(y_{i_1,j_1} y_{i_2,j_2} \cdots y_{i_w,j_w})(y_{i_w,j_w} \cdots y_{i_2,j_2} y_{i_1,j_1}) = y_{i_1,j_1}^2 y_{i_2,j_2}^2 \cdots y_{i_w,j_w}^2 \neq 0$$

in $\mathcal{E}_n^!$. Therefore, $y_{i_1,j_1} y_{i_2,j_2} \cdots y_{i_w,j_w} \neq 0$. □

Finally, we recover a bound on the GK-dimension of $\mathcal{E}_n^!$ via the proof of Proposition 3.10.

Remark 3.12 (Etingof). We refer to the notation of the proof of Proposition 3.10. By taking limits, one sees that one stratum, $X_n(\pi')$, is in the closure of another stratum, $X_n(\pi)$, if and only if π' is obtained from π by cutting some blocks of π into single points. Thus, the irreducible components of X_n are closures $\overline{X_n(\pi)}$ of strata with maximal set partitions π, i.e., those with at most one 1-point block. So, the dimension of the stratum $X_n(\pi)$ is $\#_{\geq 2}(\pi)$, which is at most $\lfloor n/2 \rfloor$. We then recover $\mathrm{GKdim}\mathcal{E}_n^! = \mathrm{GKdim}\mathcal{D}_n \leq \lfloor n/2 \rfloor$; see Lemma 2.8(4) and Theorem 1.3(2).

4. Homological preliminaries and proof of Theorem 1.6 and Corollary 1.8

The goal of this section is to establish Theorem 1.6 and Corollary 1.8 on various homological properties of $\mathcal{E}_n^!$. We begin by recalling these homological notions for general connected $\mathbb{N}$-graded algebras, and we show how these conditions are related in Figure 1. Then we present preliminary results for $\mathcal{E}_n^!$, and we end the section by establishing the proofs of Theorem 1.6 and Corollary 1.8.

We begin by presenting homological conditions on connected $\mathbb{N}$-graded (c.g.), locally finite $\Bbbk$-algebras that generalize the regularity, Gorenstein, Cohen–Macaulay, and other favorable conditions on commutative local algebras. Some of these homological conditions can be defined for $\Bbbk$-algebras that are not necessarily connected $\mathbb{N}$-graded nor locally finite, and we refer the reader to the references provided below for more information.

Definition 4.1 ([2, Intro.], [44, p. 674, Section 4], [28, Intro., Defs. 2.1 and 5.8])**.** Let A be a connected $\mathbb{N}$-graded (c.g.), locally finite $\Bbbk$-algebra. All A-modules and Ext-groups below will be graded.

(1) A is called *Artin–Schelter–Gorenstein* (or *AS-Gorenstein*) (of dimension d) if the following conditions hold:
   (a) A has finite injective dimension d as both a left and a right A-module,
   (b) $\mathrm{Ext}^i_A(\Bbbk, A) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\Bbbk, A) = 0$ for all $i \neq d$, and
   (c) $\mathrm{Ext}^d_A(\Bbbk, A) \cong \Bbbk(l)$ and $\mathrm{Ext}^d_{A^{\mathrm{op}}}(\Bbbk, A) \cong \Bbbk(l)$ for some integer l.
(2) A is called *Artin–Schelter-regular* (or *AS-regular*) of dimension d if A is AS-Gorenstein of finite global dimension d. (We do not assume finite GK-dimension, as was introduced originally in [2]. See [21], for instance.)

(3) For $\mathfrak{m} := A_{\geq 1}$, a graded maximal ideal of A, the *ith local cohomology module* for a left (or right) A-module M is defined to be

$$H^i_{\mathfrak{m}}(M) := \varinjlim_{m} \operatorname{Ext}^i_A(A/A_{\geq m},\ M).$$

(4) A is *Artin–Schelter-Cohen–Macaulay* (or *AS-Cohen–Macaulay*, *AS-CM*) if there exists an integer d such that $H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{m}^{\mathrm{op}}}(A) = 0$ for all $i \neq d$.

(5) The *grade number* of a left (or right) A-module M is defined to be

$$j_A(M) := \inf\{i \mid \operatorname{Ext}^i_A(M, A) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

Write $j(M)$ for $j_A(M)$ if A is understood. Note that $j_A(0) = +\infty$.

(6) In the case in which A is Noetherian, a left (right) A-module M satisfies the *Auslander condition* if for any $q \geq 0$, we get $j_A(N) \geq q$ for all right (left) A-submodules N of $\operatorname{Ext}^q_A(M, A)$.

(7) A Noetherian algebra A is *Auslander–Gorenstein* of dimension d if A has finite injective dimension d as both a left and a right A-module, and if every finitely generated left and right A-module satisfies the Auslander condition.

(8) A Noetherian algebra A is *Auslander-regular* of dimension d if A is Auslander–Gorenstein of finite global dimension d.

(9) A Noetherian algebra A is *Cohen–Macaulay* (*CM*) if $\operatorname{GKdim}(A) = d \in \mathbb{N}$, and if

$$j(M) + \operatorname{GKdim}(M) = d$$

for every finitely generated nonzero left (or right) A-module M.

(10) Continuing (9), we have *inequality of the grade* (*IG*) if the weaker condition that $j(M) + \operatorname{GKdim}(M) \geq d$ is satisfied.

(11) The *depth* of a left (or right) A-module M is defined to be

$$\operatorname{depth} M := \inf\{i \mid \operatorname{Ext}^i_A(\Bbbk, M) \neq 0\}.$$

If $\operatorname{Ext}^i_A(\Bbbk, M) = 0$ for all i, then $\operatorname{depth} M = \infty$.

The hierarchy of homological conditions on connected $\mathbb{N}$-graded, locally finite $\Bbbk$-algebras also involve certain factors of regular algebras, namely, the complete intersections discussed below. This is motivated by the fact that, in the context of commutative local rings, for a Noetherian ring R, we have the following:

- If R is regular, then R is a complete intersection.
- If R is a complete intersection, then R is Gorenstein, and hence, in turn, is CM.

See, for instance, the work of Bass [4] for the details of the commutative terminology; Definitions 4.1 and 4.2 are a noncommutative generalization of these concepts.

Definition 4.2 ([17], [24, Definition 1.3])**.**

(1) An element Ω of a ring A is called *normal* if $\Omega A = A\Omega$, and is called *regular* if Ω is a nonzero divisor in A.

(2) We say a collection of elements $\{\Omega_1, \ldots, \Omega_t\}$ of a ring A is a *normal sequence* in A if $\deg \Omega_i > 0$ for all i, and if Ω_i is a normal element in the factor ring $A/(\Omega_1, \ldots, \Omega_{i-1})$ for all i. If, further, each Ω_i is a regular element in the factor ring $A/(\Omega_1, \ldots, \Omega_{i-1})$, then we say that $\{\Omega_1, \ldots, \Omega_t\}$ is a *regular normal sequence* in A.

(3) A c.g. finitely generated $\Bbbk$-algebra A is called a classical complete intersection (cci) if there is a c.g. Noetherian AS-regular algebra C and a regular sequence of normalizing elements $\{\Omega_1, \dots, \Omega_t\}$ of positive degree such that

$$A \cong C/(\Omega_1, \dots, \Omega_t).$$

(4) In the case in which A is a cci, the *cci number* of A is defined to be

$$\text{cci}(A) = \min\{t \mid A \cong C/(\Omega_1, \dots, \Omega_t)\}.$$

Now we illustrate the connections amongst the terminology above in the case in which A is a connected $\mathbb{N}$-graded, locally finite $\Bbbk$-algebra. One fact that is used in the hypotheses of some of the references below is that a Noetherian PI algebra is *fully bounded Noetherian* [34, Definition 6.4.7, Corollary 13.6.6].

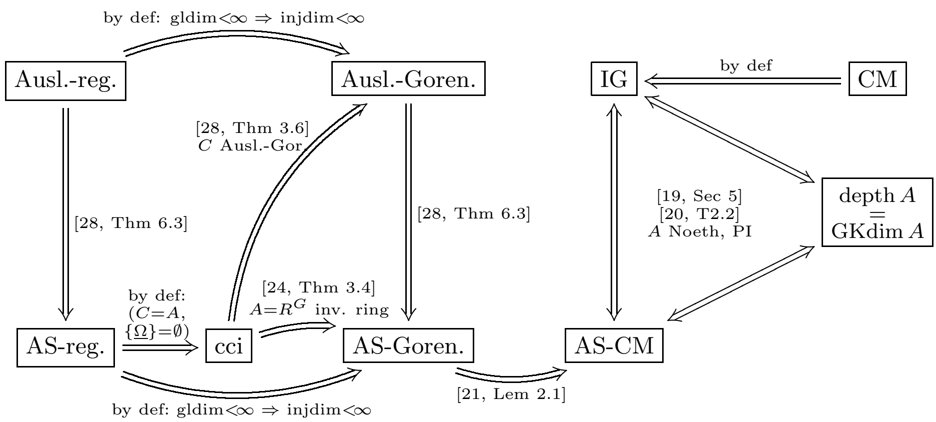

FIGURE 1. Homological conditions for c.g. locally finite algebras A (Definition 4.1)

The following result will be of use.

Proposition 4.3 ([28, Lemma 5.7], [10, Theorem 2.5])**.** *Take C as a connected $\mathbb{N}$-graded Noetherian algebra with finite GK-dimension.*

(1) *Let Ω be a regular element of C of positive degree. Then* $\text{GKdim}(C/(\Omega)) = \text{GKdim}(C) - 1$.

(2) *Let $m = \text{GKdim}(C)$. Suppose that C is Auslander–Gorenstein and CM, and that there is a normalizing sequence $\{\Omega_1, \dots, \Omega_m\}$ of C with each element Ω_i homogeneous of positive degree. Then $\{\Omega_1, \dots, \Omega_m\}$ is a regular sequence in C if and only if $C/(\Omega_1, \dots, \Omega_m)$ is finite dimensional (has GK-dimension 0). In this case, for each $t = 1, \dots, m$, we get $\{\Omega_1, \dots, \Omega_t\}$ as a regular sequence of C if and only if*

$$\text{GKdim}(C/(\Omega_1, \dots, \Omega_t)) = m - t. \qquad \square$$

Now we study the Fomin–Kirillov algebra $\mathcal{E}_3$ from Definition 1.1 and show that it is a cci. Recall that $\mathcal{E}_3$ is the $\Bbbk$-algebra generated by x_{12}, x_{13}, x_{23} (where we suppress the comma in the subscript of the indices) subject to the following relations:

$$x_{12}^2 = x_{13}^2 = x_{23}^2 = 0,$$
$$x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12} = 0,$$
$$x_{23}x_{12} - x_{13}x_{23} - x_{12}x_{13} = 0.$$

Notation 4.4 (C', S, Ω'_1, Ω'_2, Ω'_3). Let C' be the $\Bbbk$-algebra generated by x_{12}, x_{13}, x_{23} subject to the relations

$$x_{12}^2 + x_{13}^2 = 0,$$
$$x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12} = 0,$$
$$x_{23}x_{12} - x_{13}x_{23} - x_{12}x_{13} = 0.$$

Let S be the $\Bbbk$-algebra generated by x_{12} and x_{13} subject to the relation

$$x_{12}^2 + x_{13}^2 = 0.$$

Moreover, put $\Omega'_1 := x_{23}^2$, put $\Omega'_2 := x_{12}x_{23}x_{13} - x_{13}x_{23}x_{12}$, and put $\Omega'_3 := x_{13}^2$.

For the result below, we refer the reader to [17, Chapter 2] for details about σ-derivations and Ore extensions.

Lemma 4.5. *Retain Notation* 4.4. *Then the following statements hold.*

(1) *S is connected $\mathbb{N}$-graded, Noetherian, Auslander-regular, CM of global dimension* 2.
(2) *The map $\sigma : x_{12} \mapsto x_{13}$ and $x_{13} \mapsto x_{12}$ defines an algebra automorphism of S, and $\delta : x_{12} \mapsto x_{12}x_{13}$ and $x_{13} \mapsto -x_{13}x_{12}$ defines a σ-derivation of S.*
(3) *C' is an Ore extension $S[x_{23}; \sigma, \delta]$. Hence, C' is a connected $\mathbb{N}$-graded, Noetherian, Auslander-regular, CM $\Bbbk$-algebra of global dimension* 3.
(4) *$\{\Omega'_1, \Omega'_2, \Omega'_3\}$ is a normal sequence in C'.*
(5) *$C'/(\Omega'_1, \Omega'_2, \Omega'_3) = \mathcal{E}_3$.*

Proof.

(1) This holds as S is isomorphic to the (-1)-skew polynomial ring $\Bbbk_{-1}[z_1, z_2]$, which is well known to possess the desired properties.
(2) It is clear that σ is an algebra automorphism. To check that δ is a σ-derivation, we calculate

$$\begin{aligned}\delta(x_{12}^2 + x_{13}^2) &= \delta(x_{12})x_{12} + \sigma(x_{12})\delta(x_{12}) + \delta(x_{13})x_{13} + \sigma(x_{13})\delta(x_{13}) \\ &= (x_{12}x_{13})x_{12} + x_{13}(x_{12}x_{13}) - (x_{13}x_{12})x_{13} - x_{12}(x_{13}x_{12}) = 0.\end{aligned}$$

(3) By part (2) and the definition of C', we see that C' is an Ore extension $S[x_{23}; \sigma, \delta]$. Now the second statement holds by (1) and several standard results including [12, Theorem 4.2], [29, p. 184].
(4) First, we claim that $\Omega'_1 := x_{23}^2$ is central (and thus normal) in C'. We check

$$\begin{aligned}x_{23}^2 x_{12} &= x_{23}(x_{13}x_{23} + x_{12}x_{13}) \\ &= (x_{12}x_{23} - x_{13}x_{12})x_{23} + (x_{13}x_{23} + x_{12}x_{13})x_{13} \\ &= x_{12}x_{23}^2 - x_{13}(x_{12}x_{23} - x_{23}x_{13}) + x_{12}x_{13}x_{13} \\ &= x_{12}x_{23}^2 - x_{13}(x_{13}x_{12}) + x_{12}x_{13}x_{13} \\ &= x_{12}x_{23}^2.\end{aligned}$$

Similarly, $x_{23}^2 x_{13} = x_{13} x_{23}^2$ (and x_{23}^2 commutes with x_{23}). So, the first claim holds.

Second, we claim that Ω_2' is normal in $C'/(\Omega_1')$. We calculate

$$\begin{aligned}
\Omega_2' x_{12} &= x_{12}x_{23}(x_{13}x_{12}) - x_{13}(x_{23}x_{12})x_{12} \\
&= x_{12}x_{23}(x_{12}x_{23} - x_{23}x_{13}) - x_{13}(x_{13}x_{23} + x_{12}x_{13})x_{12} \\
&= x_{12}(x_{23}x_{12})x_{23} - x_{13}^2 x_{23}x_{12} - x_{13}x_{12}(x_{13}x_{12}) \\
&= x_{12}(x_{13}x_{23} + x_{12}x_{13})x_{23} - x_{13}^2 x_{23}x_{12} - x_{13}x_{12}(x_{12}x_{23} - x_{23}x_{13}) \\
&= x_{12}^2 x_{13}x_{23} - x_{13}^2 x_{23}x_{12} - x_{13}x_{12}(x_{12}x_{23} - x_{23}x_{13}) \\
&= x_{13}[-x_{13}^2 x_{23} - x_{13}x_{23}x_{12} - x_{12}(x_{12}x_{23} - x_{23}x_{13})] \\
&= x_{13}(-x_{13}x_{23}x_{12} + x_{12}x_{23}x_{13}) \\
&= x_{13}\Omega_2'.
\end{aligned}$$

Similarly, $\Omega_2' x_{13} = x_{12}\Omega_2'$ and $\Omega_2' x_{23} = -x_{23}\Omega_2'$. Thus, Ω_2' is normal in $C'/(\Omega_1')$.

Finally, we claim that $\Omega_3' := x_{13}^2$ is central in $C'/(\Omega_1', \Omega_2')$. We calculate

$$\begin{aligned}
x_{13}^2 x_{12} &= x_{13}(x_{12}x_{23} - x_{23}x_{13}) \\
&= (x_{12}x_{23} - x_{23}x_{13})x_{23} - (x_{23}x_{12} - x_{12}x_{13})x_{13} \\
&= -x_{23}x_{13}x_{23} - x_{23}x_{12}x_{13} + x_{12}x_{13}^2 \\
&= -x_{23}(x_{23}x_{12} - x_{12}x_{13}) - x_{23}x_{12}x_{13} + x_{12}x_{13}^2 \\
&= x_{12}x_{13}^2
\end{aligned}$$

and

$$\begin{aligned}
x_{23}x_{13}^2 &= (x_{12}x_{23} - x_{13}x_{12})x_{13} \\
&= x_{12}x_{23}x_{13} - x_{13}(x_{23}x_{12} - x_{13}x_{23}) \\
&= x_{13}^2 x_{23}.
\end{aligned}$$

Since x_{13}^2 commutes with x_{13}, we have x_{13}^2 being central in $C'/(\Omega_1', \Omega_2')$.

(5) By comparing the generators and relations, one sees that $C'/(\Omega_1', \Omega_3') = \mathcal{E}_3$. It remains to show that $\Omega_2' = 0$ in $C'/(\Omega_1', \Omega_3')$. We check, in $C'/(\Omega_1', \Omega_3')$, that

$$\Omega_2' = x_{12}x_{23}x_{13} - x_{13}x_{12}x_{13} - x_{13}x_{23}x_{12} + x_{13}x_{12}x_{13} \;=\; x_{23}x_{13}^2 - x_{13}^2 x_{23} = 0.$$

Thus, $\Omega_2' = 0$ in $C'/(\Omega_1', \Omega_3')$, as desired. □

Theorem 4.6. *We obtain that $\mathcal{E}_3$ is a cci, and that $cci(\mathcal{E}_3) = 3$. As a consequence, $\mathcal{E}_3$ is Auslander–Gorenstein, CM, and Frobenius.*

Proof. Recall Notation 4.4. By Lemma 4.5(3)–(5), we get $\{\Omega_1', \Omega_2', \Omega_3'\}$ as a normal sequence of an Auslander-regular and CM algebra C' such that $\mathcal{E}_3 = C'/(\Omega_1', \Omega_2', \Omega_3')$. By the definition of C' in Lemma 4.5(3), one sees that $\operatorname{GKdim} C' = 3$ (Remark 2.2(2)). Now by [14, (2.8)], $\operatorname{GKdim} \mathcal{E}_3 = 0$. Hence, $\operatorname{GKdim} C'/(\Omega_1', \Omega_2', \Omega_3') = 0$, and with Proposition 4.3(2) we obtain that $\{\Omega_1', \Omega_2', \Omega_3'\}$ is a regular normal sequence of C'. By definition, $\mathcal{E}_3$ is a cci.

Since $\mathcal{E}_3$ is finite dimensional and a cci, it is Auslander–Gorenstein and CM, and, consequently, Frobenius.

For the cci number of $\mathcal{E}_3$, note that by the argument above we have $\operatorname{cci}(\mathcal{E}_3) \leq 3$. If $\operatorname{cci}(\mathcal{E}_3) \leq 2$, then there is a Noetherian AS-regular algebra B and a regular normal sequence $\{f_1, f_2\}$ of B such that $\mathcal{E}_3 \cong B/(f_1, f_2)$. Recall that $\mathcal{E}_3$ is Auslander–Gorenstein and CM, so by [28, Theorem 3.6 and Remark 5.10], B is Auslander-regular and CM of GK-dimension and global dimension 2. Such a B must be generated by two elements [2, Introduction]. This contradicts the fact that $\mathcal{E}_3$ is generated by three elements. Thus, $\operatorname{cci}(\mathcal{E}_3) \geq 3$. □

Now we consider the quadratic dual $\mathcal{E}_3^!$ of $\mathcal{E}_3$. Recall that $\mathcal{E}_3^!$ is generated by y_{12}, y_{13}, y_{23} (where we suppress the comma in the subscript of the indices) subject to the relations

$$\begin{aligned} y_{12}y_{23} + y_{23}y_{13} &= y_{12}y_{23} + y_{13}y_{12} = 0, \\ y_{23}y_{12} + y_{13}y_{23} &= y_{23}y_{12} + y_{12}y_{13} = 0. \end{aligned}$$

Definition-Lemma 4.7 (a, b, c, C'')**.** *Let $a := y_{13} + y_{23}$, $b := y_{13} - y_{23}$, $c := y_{12}$. After a linear transformation, $\mathcal{E}_3^!$ is generated by a, b, c, subject to the following relations:*

$$ca + ac = cb - bc = -2bc + a^2 - b^2 = -2ac + (ab - ba) = 0.$$

Moreover, let C'' be the algebra generated by a, b, c, subject to the relations

$$ca + ac = cb - bc = a^2b - ba^2 = ab^2 - b^2a = 0.$$ □

Lemma 4.8. *The algebra C'' is Noetherian, AS-regular, Auslander-regular, CM, of global dimension 4, of GK-dimension 4, and it has Hilbert series $(1-t)^{-3}(1-t^2)^{-1}$.*

Proof. Let B be the $\Bbbk$-algebra $\Bbbk\langle a, b\rangle/(a^2b - ba^2, ab^2 - b^2a)$. This algebra is a Noetherian AS-regular algebra of global dimension 3 [2, (8.5)], with Hilbert series $(1-t)^{-2}(1-t^2)^{-1}$ [2, (1.15)], and is also Auslander-regular and CM (e.g., via [28, Corollary 5.10]) and has GK-dimension 3 (e.g., via Proposition 4.3(2)). Note that C'' is an Ore extension $B[c, \sigma]$, where $\sigma : a \mapsto -a,\ b \mapsto b$. So, C'' has Hilbert series $H_B(t)/(1 - t^{\deg c}) = (1-t)^{-3}(1-t^2)^{-1}$. The rest of the result follows from several standard results including [12, Theorem 4.2], [29, p. 184]. □

Lemma 4.9 (Ω_1'', Ω_2'', Ω_3'')**.** *Retain the notation of Definition-Lemma 4.7.*

(1) *Let $\Omega_1'' := (ab + ba)c$, $\Omega_2'' := -2bc + a^2 - b^2$, and $\Omega_3'' := -2ac + (ab - ba)$. Then $\{\Omega_1'', \Omega_2'', \Omega_3''\}$ is a normal sequence in C''.*
(2) *$C''/(\Omega_1'', \Omega_2'', \Omega_3'') = \mathcal{E}_3^!$.*

Proof.

(1) Note that $(ab+ba)c = -c(ab+ba)$ in C''. It is also easy to check that $(ab+ba)$ is central in the AS-regular algebra B from the proof of Lemma 4.8. Since C'' is an Ore extension $B[c; \sigma]$ (from the proof of Lemma 4.8), $(ab + ba)$ is normal in C'', and so is c. Hence, $\Omega_1'' = (ab + ba)c$ is normal in C''.

To show that Ω_2'' is normal in $C''/(\Omega_1'')$, note that $[\Omega_2'', c] = [\Omega_2'', b] = 0$ and

$$[\Omega_2'', a] = \Omega_2''a - a\Omega_2'' = 2(ab + ba)c = 0$$

in $C''/(\Omega_1'')$.

Now, to see that Ω_3'' is normal in $C''/(\Omega_1'', \Omega_2'')$, note that $\Omega_3''c + c\Omega_3'' = \Omega_3''a + a\Omega_3'' = 0$. Lastly, we have

$$\Omega_3''b + b\Omega_3'' \;=\; (-2ac + (ab - ba))b + b(-2ac + (ab - ba)) \;=\; -2acb - 2bac = 0$$

in $C''/(\Omega_1'', \Omega_2'')$. Therefore, $\{\Omega_1'', \Omega_2'', \Omega_3''\}$ is a normal sequence in C''.

(2) In the algebra $\mathcal{E}_3^! \cong \Bbbk\langle a, b, c\rangle/(ca + ac,\ cb - bc,\ \Omega_2'',\ \Omega_3'')$, we get

$$[a^2, b] = \Omega_2''b - b\Omega_2''$$

and

$$[a, b^2] = \Omega_3''b + b\Omega_3'',$$

and in the algebra $\Bbbk\langle a, b, c\rangle/(ca + ac,\ cb - bc,\ [a^2, b],\ [a, b^2],\ \Omega_2'',\ \Omega_3'')$ we get

$$\Omega_1'' = abc - bca \;=\tfrac{1}{2}a(a^2 - b^2) - \tfrac{1}{2}(a^2 - b^2)a \;=\tfrac{1}{2}[b^2, a] = 0,$$

as required. □

This brings us to our main result for $\mathcal{E}_3^!$.

Theorem 4.10.

(1) *$\mathcal{E}_3^!$ is a cci and $H_{\mathcal{E}_3^!}(t) = \frac{(1+t)(1+t+t^2)}{(1-t)}$.*
(2) *$\mathcal{E}_3^!$ is AS-Gorenstein and Auslander–Gorenstein.*
(3) *$\mathcal{E}_3^!$ is not AS-regular or Auslander-regular.*

Proof.

(1) By Lemmas 4.8 and 4.9, there exists a normalizing sequence $\{\Omega_1'', \Omega_2'', \Omega_3''\}$ of an Auslander-regular, CM algebra C'' with $\mathcal{E}_3^! \cong C''/(\Omega_1'', \Omega_2'', \Omega_3'')$. Moreover, $\operatorname{GKdim} C'' = 4$ by Lemma 4.8. Now consider the normal element $\Omega_4'' := \frac{1}{2}(a^2 + b^2) + c^2$ of $\mathcal{E}_3^!$. Then with variable ordering $a < b < c$, one can compute (via the GBNP package of GAP [11]) that the ideal of relations for $\mathcal{E}_3^!/(\Omega_4'')$ has Gröbner basis

$$\left\{\begin{array}{lllll} ba + 2ac - ab, & bc + \frac{1}{2}(b^2 - a^2), & ca + ac, & cb - bc, & c^2 + \frac{1}{2}(b^2 + a^2), \\ a^3, & b^3 + \frac{1}{3}(2a^2c + a^2b) & & & \end{array}\right\}.$$

Thus, $C''/(\Omega_1'', \Omega_2'', \Omega_3'', \Omega_4'') = \mathcal{E}_3^!/(\Omega_4'')$ has Hilbert series $1+3t+4t^2+3t^3+t^4$ and has GK-dimension 0. Since $\mathcal{E}_3^! = C''/(\Omega_1'', \Omega_2'', \Omega_3'')$ has GK-dimension 1 (Theorem 1.3(2)), we obtain $\{\Omega_1'', \Omega_2'', \Omega_3''\}$ as a regular normal sequence of C'' by Proposition 4.3(2). By definition, $\mathcal{E}_3^!$ is a cci.

Since $\{\Omega_1'', \Omega_2'', \Omega_3''\}$ is a regular normal sequence of C'' with degrees 3, 2, and 2, respectively, we get by Lemma 4.8

$$H_{\mathcal{E}_3^!}(t) = H_{C''}(t)(1 - t^2)^2(1 - t^3) = \frac{(1 - t^2)^2(1 - t^3)}{(1 - t)^3(1 - t^2)} = \frac{(1 + t)(1 + t + t^2)}{1 - t}.$$

(2) This follows from (1), along with Lemma 4.8 and [28, Theorems 3.6 and 6.3].
(3) We have by Theorem 1.3(1) and [42, Corollary 1.2] the result that if $\mathcal{E}_3^!$ had finite global dimension, then $\mathcal{E}_3^!$ would be a domain. But this contradicts Theorem 1.3(3). So, $\mathcal{E}_3^!$ is neither AS-regular nor Auslander-regular. □

Now we turn our attention to the AS-CM property of $\mathcal{E}_n^!$, and for this we need results on the depth of $\mathcal{E}_n^!$. We start with the following preliminary result.

Lemma 4.11. *For connected $\mathbb{N}$-graded algebras A and C, we have the statements below.*

1. *Suppose that A has a regular homogeneous element f of positive degree. Then* $\operatorname{depth} A > 0$.
2. *Let C be a Noetherian commutative algebra, and let M be a finitely generated module over C. If* $\operatorname{depth} M > 0$*, then there is a homogeneous element of positive degree $f \in C$ that is regular on M.*
3. *Let A be Noetherian and finitely generated over its affine center. Suppose that A has a homogeneous element g of positive degree such that* $\operatorname{GKdim} Ag \leq r$*. Then* $\operatorname{depth} A \leq r$.

Proof.

1. This is a connected $\mathbb{N}$-graded analogue of a standard homological result from the theory of commutative local rings; see, e.g., [9, Proposition 1.2.3]. Suppose, by way of contradiction, that $\operatorname{depth} A = 0$. Then, by definition, $\operatorname{Hom}_A(\Bbbk, A) \neq 0$, so there exists a one-dimensional nonzero ideal I of A. Let x be a generator of I. Since f has positive degree, we get $fx = 0$. This contradicts the regularity of f.
2. Let $\mathfrak{m} := C_{\geq 1}$ be the maximal graded ideal of C. If $\mathfrak{m}$ consists of nonregular elements of M, then $\mathfrak{m}$ is contained in the union of the associated primes of the C-module M. By the Noetherian property and prime avoidance, $\mathfrak{m}$ is actually contained in one associated prime $\mathfrak{p}$ of M. Thus, $\mathfrak{p} = \mathfrak{m}$. Now there exists a monomorphism $C/\mathfrak{p} \to M$. Composing this with the natural isomorphism $C/\mathfrak{m} \to C/\mathfrak{p}$, we get a nonzero C-module map $C/\mathfrak{m} \to M$. Thus, $\operatorname{Hom}_C(\Bbbk, M) \neq 0$ and $\operatorname{depth} M = 0$.
3. If $\operatorname{depth} A = 0$, then we are done. Now suppose that $\operatorname{depth} A > 0$. Let C be the center of A. Since A is finitely generated over C, $\operatorname{depth}_C A = \operatorname{depth}_A A > 0$. By part (2), there exists a homogeneous element $f \in C$ of positive degree that is regular on A. Replacing f by f^n for some $n \gg 0$, we may assume that $\deg f > \deg g$. Consider the sequence
$$0 \longrightarrow Ag \xrightarrow{\cdot f} Ag \longrightarrow Ag/Agf \longrightarrow 0.$$
Since $\operatorname{GKdim}(Ag) \leq r$ by assumption, we have $\operatorname{GKdim}(Ag/Agf) \leq r - 1$ [34, Proposition 8.3.5]. Define $\overline{A} := A/(f)$, and let $\overline{g}$ be the image of g in $\overline{A}$. Using the surjection $Ag/Agf \twoheadrightarrow Ag/(Ag \cap Af)$, we have
$$\operatorname{GKdim}(\overline{A}\overline{g}) = \operatorname{GKdim}(Ag/(Ag \cap Af)) \leq r - 1.$$
By induction, $\operatorname{depth}(\overline{A}) \leq r - 1$. Now the result holds by Rees's lemma [40, Theorem 8.34]. ☐

Theorem 4.12. *For every $n \geq 2$,* $\operatorname{depth}(\mathcal{E}_n^!) \leq 1$*. As a consequence, for $n \geq 4$, we get* $\operatorname{depth} \mathcal{E}_n^! < \operatorname{GKdim} \mathcal{E}_n^!$.

Proof. Let
$$g = \prod_{i<j} a_{i,j},$$
which is nonzero by Corollary 3.11. It is easy to check that $a_{i,j}g = a_{1,2}g$ by (E1.8.1). Hence, $\mathcal{D}_n g = \Bbbk[a_{1,2}]g$ or $\operatorname{GKdim} \mathcal{D}_n g = 1$. Since $\mathcal{E}_n^!$ is finitely generated over $\mathcal{D}_n$,

GKdim $\mathcal{E}_n^! g = 1$. The assertion follows by Lemma 4.11(3), and the consequence holds by Theorem 1.3(2). □

Proof of Theorem 1.6. Refer to Figure 1 throughout. By Theorem 1.3(2), we find that, to establish parts (1)–(3), it suffices to show that (a) $\mathcal{E}_2^!$ is AS-regular, (b) $\mathcal{E}_3^!$ is AS-Gorenstein and but not AS-regular, and (c) $\mathcal{E}_n^!$ is not AS-CM for $n \geq 4$. Now (a) holds as $\mathcal{E}_2^!$ is the commutative polynomial ring $\Bbbk[y_{1,2}]$, (b) holds by Theorem 4.10(2,3), and (c) follows from Theorema 4.12 and 1.3(1).

Moreover, (4) holds since $\mathcal{E}_2^! = \Bbbk[y_{1,2}]$ (AS-regular), since $\mathcal{E}_2 = \Bbbk[x_{1,2}]/(x_{1,2}^2)$ (quotient of an AS-regular algebra by a regular element), and by Theorems 4.10(1) and 4.6 for the algebras $\mathcal{E}_3^!$ and $\mathcal{E}_3$, respectively. □

Proof of Corollary 1.8. Refer to Figure 1 throughout. The result on Auslander-regularity holds for $n = 2$, as $\mathcal{E}_2^! = \Bbbk[y_{1,2}]$ is clearly Auslander-regular, for $n = 3$ by Theorem 4.10(3), and for $n \geq 4$ by Theorem 1.6(3). The result on Auslander–Gorenstein holds for $n = 2$ by Auslander-regularity, for $n = 3$ by Theorem 4.10(2), and for $n \geq 4$ by Theorem 1.6(3). The result on the CM condition holds for $n = 2, 3$ by the Auslander–Gorenstein condition with Theorem 1.3(1), and for $n \geq 4$ by Theorems 1.6(3) and 1.3(1). □

5. Further directions

First, we make the following remark about Question 1.4 and Conjecture 1.7 discussed in the Introduction.

Remark 5.1. If $\mathcal{E}_n^!$ is semiprime (i.e., if Question 1.4(1) is affirmative), then we obtain that depth $\mathcal{E}_n^! = 1$ (i.e., that Conjecture 1.7 holds). Namely, for any c.g. algebra $A \neq \Bbbk$, depth $A = 0$ implies that A is not semiprime as $\operatorname{Hom}_A(\Bbbk, A) \subseteq A$ is a nonzero nilpotent ideal of A. Therefore, depth $\mathcal{E}_n^! \geq 1$ when $\mathcal{E}_n^!$ is semiprime. Now Conjecture 1.7 follows from Theorem 4.12.

In addition to Question 1.4 and Conjecture 1.7, along with Question 3.7, we present here three other suggestions for further study of the quadratic dual $\mathcal{E}_n^!$ of Fomin–Kirillov algebras, yet there are numerous other directions that one could pursue motivated by Fomin–Kirillov's work [14] alone.

5.1. **On the center of $\mathcal{E}_n^!$.** In Section 2, we introduced the commutative subalgebras $\mathcal{C}_n$ and $\mathcal{D}_n$ of $\mathcal{E}_n^!$ in order to prove Theorem 1.3; recall $\mathcal{C}_n \cong \mathcal{D}_n$ by Proposition 3.10. But we ask the following.

Question 5.2. What is the presentation of the center $Z(\mathcal{E}_n^!)$ of $\mathcal{E}_n^!$?

On a related note, there is an important subalgebra of the Fomin–Kirillov algebra $\mathcal{E}_n$ constructed in [14] generated by its *Dunkl elements*. This subalgebra, which we denote by $\mathcal{F}_n$, is isomorphic to the cohomology of a flag manifold [14, Theorem 7.1]; the full presentation of $\mathcal{F}_n$ is also established in that result.

Question 5.3. What is the connection between $\mathcal{F}_n$ and the commutative algebras $Z(\mathcal{E}_n^!)$ and $\mathcal{C}_n$ discussed above?

5.2. **On the S_n-action on $\mathcal{E}_n^!$ and related algebras.** As discussed in Milinski and Schneider's study of *Nichols algebras* over Coxeter groups [36], the Fomin–Kirillov algebra $\mathcal{E}_n$ admits an action of the symmetric group S_n and, moreover, can be realized as a braided Hopf algebra in the category ${}^{S_n}_{S_n}\mathcal{YD}$ of Yetter–Drinfeld modules over S_n. Namely, $\mathcal{E}_n$ arises as a *pre-Nichols algebra* in ${}^{S_n}_{S_n}\mathcal{YD}$, and, in fact, it is conjectured that $\mathcal{E}_n$ is an honest Nichols algebra in ${}^{S_n}_{S_n}\mathcal{YD}$ (which has been verified for $n \leq 5$). Two algebras that are of interest in this context are the invariant subalgebra $\mathcal{E}_n^{S_n}$ and the skew group algebra $\mathcal{E}_n \rtimes S_n$. For instance, for a finite group G, there is a useful functor from ${}^{G}_{G}\mathcal{YD}$ to the category of $\Bbbk$-vector spaces sending a braided Hopf algebra $\mathcal{B}$ to the $\Bbbk$-Hopf algebra $\mathcal{B} \rtimes G$. Vital classes of finite-dimensional pointed Hopf algebras have been constructed in this fashion.

Now the quadratic dual $\mathcal{E}_n^!$ also admits an action of the symmetric group S_n, and an interesting direction for further research is to study the behavior of the resulting invariant ring and skew group algebra.

5.3. **On the Koszulity of $\mathcal{E}_n^!$.** As mentioned in the Introduction, the Fomin–Kirillov algebras $\mathcal{E}_n$ fail to be Koszul for $n \geq 3$ due to a result of Roos [39], so the same result holds for the quadratic dual $\mathcal{E}_n^!$. Toward understanding the cohomology rings $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ and $\mathrm{Ext}^*_{\mathcal{E}_n^!}(\Bbbk, \Bbbk)$ (for which $\mathcal{E}_n^!$ and $\mathcal{E}_n$, respectively, are the subalgebras generated in degree 1), the failure of Koszulity should be studied more carefully. Indeed, if $\mathcal{E}_n^!$ and $\mathcal{E}_n$ were Koszul, then they would equal $\mathrm{Ext}^*_{\mathcal{E}_n}(\Bbbk, \Bbbk)$ and $\mathrm{Ext}^*_{\mathcal{E}_n^!}(\Bbbk, \Bbbk)$, respectively.

As in [37, Section 2.4], we say that a graded algebra A is *p-Koszul* if

$$\mathrm{Ext}^{i,j}_A(\Bbbk, \Bbbk) = 0 \qquad \forall i < j \leq p.$$

For example, any graded algebra is 1-Koszul, any graded algebra generated in degree 1 is 2-Koszul, and any quadratic algebra is 3-Koszul. Moreover, a graded (quadratic) algebra is Koszul if and only if it is p-Koszul for all $p \geq 1$. By [37, Proposition 2.4.5], if A is a $(p-1)$-Koszul quadratic algebra, then for each $2 < i < p$ there is a natural perfect pairing

$$\mathrm{Ext}^{i,p}_A(\Bbbk, \Bbbk) \otimes \mathrm{Ext}^{p-i+2,p}_{A^!}(\Bbbk, \Bbbk) \longrightarrow \Bbbk.$$

So, we ask the following question.

Question 5.4. What is the maximum value of $p = p(n)$ for which $\mathcal{E}_n^!$ is p-Koszul?

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Department of Mathematics, The University of Illinois at Urbana–Champaign, Urbana, Illinois 61801

Email address: notlaw@illinois.edu

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195

Email address: zhang@math.washington.edu