J. Fluid Mech. (2019), *vol.* 880, R4, doi:10.1017/jfm.2019.776

JFM RAPIDS
journals.cambridge.org/rapids

Rotation of a superhydrophobic cylinder in a viscous liquid

Ehud Yariv[1,†] **and Michael Siegel**[2]

[1]Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

[2]Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102, USA

(Received 8 August 2019; revised 18 September 2019; accepted 19 September 2019)

The hydrodynamic quantification of superhydrophobic slipperiness has traditionally employed two canonical problems – namely, shear flow about a single surface and pressure-driven channel flow. We here advocate the use of a new class of canonical problems, defined by the motion of a superhydrophobic particle through an otherwise quiescent liquid. In these problems the superhydrophobic effect is naturally measured by the enhancement of the Stokes mobility relative to the corresponding mobility of a homogeneous particle. We focus upon what may be the simplest problem in that class – the rotation of an infinite circular cylinder whose boundary is periodically decorated by a finite number of infinite grooves – with the goal of calculating the rotational mobility (velocity-to-torque ratio). The associated two-dimensional flow problem is defined by two geometric parameters – namely, the number N of grooves and the solid fraction ϕ. Using matched asymptotic expansions we analyse the large-N limit, seeking the mobility enhancement from the respective homogeneous-cylinder mobility value. We thus find the two-term approximation,

$$1+\frac{2}{N}\ln\csc\frac{\pi\phi}{2},$$

for the ratio of the enhanced mobility to the homogeneous-cylinder mobility. Making use of conformal-mapping techniques and inductive arguments we prove that the preceding approximation is actually exact for $N=1,2,4,8,\ldots$. We conjecture that it is exact for all N.

Key words: drops and bubbles, low-Reynolds-number flows

1. Introduction

When a ‘rough’ hydrophobic solid is immersed in water, bubbles can get trapped within the vacancies of the roughness in a stable Cassie state. The resulting surface

† Email address for correspondence: udi@technion.ac.il

is known as superhydrophobic (Quéré 2008). As these surfaces tend to exhibit reduced resistance to liquid motion about them, they have been of interest to the fluid mechanics community (Bocquet & Lauga 2011). The analysis of flows about superhydrophobic surfaces is typically concerned with two classes of canonical problems: the first, rather idealized problem, concerns an externally imposed shear flow over a single superhydrophobic surface; the second, more representative of realistic configurations (Rothstein 2010), has to do with pressure-driven flow within 'superhydrophobic channels'. In both classes, interest lies not in the details of the flow, but rather in an appropriate lumped quantity that represents, in some averaged sense, the reduced friction due to superhydrophobicity. In the first class this quantity is provided by the intrinsic slip length (Davis & Lauga 2009; Crowdy 2010) – a pure geometric function of the surface morphology; in the second class it is provided by the effective slip length (Lauga & Stone 2003), which represents the excess volumetric flux in the channel.

Following recent experiments (Muralidhar *et al.* 2011; Castagna, Mazellier & Kourta 2018; Jetly, Vakarelski & Thoroddsen 2018) one can envision a third class of canonical problems, associated with rigid-body motion of a superhydrophobic 'particle'. To highlight that third class it is convenient to consider a small particle, where inertial effects are negligible. Given the linearity of the governing problem, the general properties of classical zero-Reynolds-number resistance problems (Happel & Brenner 1965; Hinch 1972) are expected to remain valid. These naturally suggest the particle mobility as the quantity of interest, which constitutes the pertinent counterpart of the slip-length concept in the first two problems.

We here consider a prototypic resistance problem, involving the rotation of a circular cylinder about its axis within a viscous liquid, where superhydrophobicity is brought about by the presence of a periodic array of infinite grooves, parallel to the cylinder axis, on its boundary. The same type of boundary was employed by Lauga & Stone (2003), who considered pressure-driven flow. With the resulting flow being two-dimensional, this may be the simplest resistance problem. Indeed, in the case of a homogeneous solid boundary, the flow is given by a two-dimensional rotlet. Following Lauga & Stone (2003), we assume that the curvature of the bubbles that are trapped in the grooves is the same as that of the circular groove boundaries. The compound boundary is accordingly circular.

Lauga & Stone (2003) solved an internal flow problem, driven by an imposed pressure gradient. We here solve an external flow problem, driven by the angular rotation of the cylinder relative to the otherwise quiescent liquid. Our goal is the angular mobility of the cylinder.

2. Problem formulation

An infinite solid cylinder of radius a is decorated with a finite number $N(\geqslant 1)$ of equally spaced grooves, infinitely extending parallel to the cylinder axis. Upon immersing the cylinder in a liquid (viscosity μ), an infinite bubble is trapped in each groove. The boundary of the cylinder accordingly consists of N liquid–gas interfaces, corresponding to the bubbles, and N solid–air interfaces, corresponding to the ridges which separate the bubbles. It is assumed that the curvature of the liquid–gas interfaces coincides with that of the solid ridges. The boundary of the cylinder cross-section is accordingly a circle. The geometry is portrayed in figure 1.

Our interest lies in the two-dimensional flow which is driven by the imposed rigid-body rotation of the cylinder about its axis, say with an angular velocity $\varOmega$.

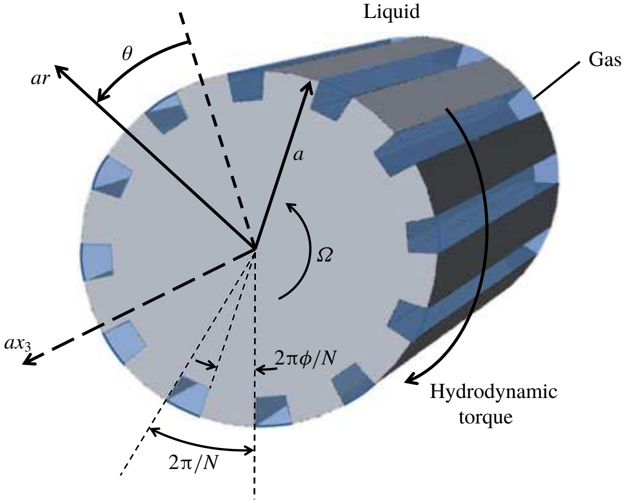

FIGURE 1. Schematic of the dimensional problem.

In particular, we seek to calculate the hydrodynamic torque (per unit length) about that axis. It is convenient to employ the ratio T of that torque to the corresponding torque about a groove-free cylinder, which is given by $4\pi\mu a^2\Omega$. Alternatively, we may employ its inverse $M = 1/T$, which represents the ratio of the angular mobility of the cylinder to the corresponding mobility of a groove-free cylinder.

We employ a dimensionless notation throughout, using a, $a\Omega$ and $\mu\Omega$ as the respective units of length, velocity and stress. We employ a non-rotating reference frame using the (x_1, x_2, x_3) Cartesian coordinates, with the x_3-axis coinciding with the cylinder axis. Writing $x = x_1$ and $y = x_2$, we also employ the (r, θ) polar coordinates in the xy-plane. With no loss of generality, the polar angle θ is defined such that, at the instantaneous moment considered, the angle of the centre of the nth ridge $(n = 0, 1, \ldots, N-1)$ is $\theta_n = 2\pi n/N$. In particular, $\theta_0 = 0$: the x-axis bisects the zeroth ridge.

Denoting the angle spanned by each bubble as $2\pi\phi/N$, the parameter ϕ (which satisfies $0 < \phi < 1$) represents the solid fraction of the compound boundary. Because of periodicity, it is sufficient to consider a 'unit sector' of angle $2\pi/N$ centred about $\theta = 0$, where the solid and free boundaries occupy the intervals $|\theta| < \phi\pi/N$ and $\phi\pi/N < |\theta| < \pi/N$, respectively.

The velocity field $\boldsymbol{u} = \hat{\boldsymbol{e}}_r u + \hat{\boldsymbol{e}}_\theta v$ satisfies the continuity and Stokes equations,

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \theta} = 0, \quad \frac{\partial p}{\partial r} = \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial p}{\partial \theta} = \nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}, \quad (2.1a\text{–}c)$$

wherein p is the dynamic pressure associated with the flow. In addition, it satisfies (i) kinematic impermeability,

$$u = 0 \quad \text{at } r = 1; \quad (2.2)$$

(ii) the periodicity conditions,

$$u = 0 \quad \text{at } \theta = \pm\frac{\pi}{N}; \quad (2.3)$$

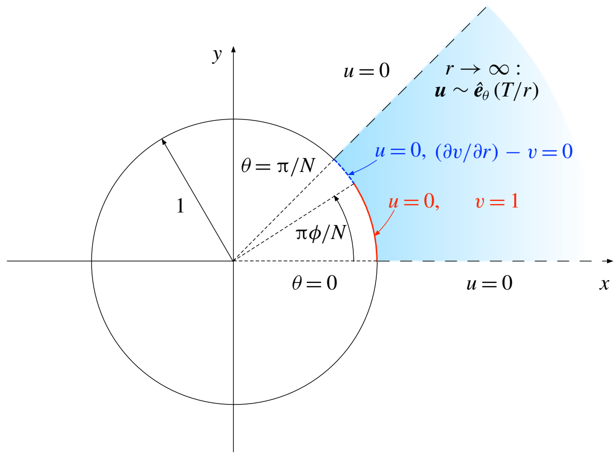

FIGURE 2. The hydrodynamic problem in the semi-sector $0<\theta<\pi/N$ (with $N=4$).

(iii) the decay requirement,

$$u,\, v\to 0 \quad \text{as } r\to\infty; \tag{2.4}$$

and (iv) the no-slip and shear-free conditions at $r=1$,

$$v=1 \quad \text{for } |\theta|<\frac{\phi\pi}{N}, \qquad \frac{\partial v}{\partial r}-v=0 \quad \text{for } \frac{\phi\pi}{N}<|\theta|<\frac{\pi}{N}. \tag{2.5a,b}$$

Two points are worth noticing. First, it is evident from the above formulation that p and v are even functions of θ, while u is an odd function of θ. It follows that one may actually solve the preceding problem in the 'semi-sector' $0<\theta<\pi/N$, with condition (2.3) becoming

$$u=0 \quad \text{at } \theta=0,\, \frac{\pi}{N}; \tag{2.6}$$

the associated problem is depicted in figure 2. Second, we note that in the absence of grooves the flow field is simply that of a two-dimensional rotlet (Pozrikidis 2011),

$$v=\frac{1}{r}, \qquad u\equiv 0, \qquad p\equiv 0. \tag{2.7a–c}$$

Our interest is in the hydrodynamic torque (per unit length in the x_3-direction) in the negative x_3-direction. Normalized by $4\pi\mu a^2\Omega$, it is given by

$$T=-\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\frac{\partial v}{\partial r}-v\right)_{r=1}\mathrm{d}\theta, \tag{2.8}$$

or, upon use of periodicity,

$$T=-\frac{N}{4\pi}\int_{-\pi\phi/N}^{\pi\phi/N}\left(\frac{\partial v}{\partial r}-v\right)_{r=1}\mathrm{d}\theta. \tag{2.9}$$

Note that in the Stokes regime, where the stress is divergence-free, the torque may be evaluated over any closed curve which encloses the circle $r = 1$. In particular, choosing a circle of large radius we obtain the following alternative to (2.8):

$$T = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\lim_{r\to\infty}\left[r^2\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\right]\,\mathrm{d}\theta. \tag{2.10}$$

The large-r velocity behaviour implied by (2.10) is

$$\boldsymbol{u} \sim \hat{\boldsymbol{e}}_\theta \frac{T}{r} + O(r^{-2}) \quad \text{for } r\to\infty. \tag{2.11}$$

In particular, the torque corresponding to (2.7) is $T = 1$.

A more convenient formulation is obtained by subtracting off the far-field rotlet (2.11)

$$\boldsymbol{u}' = \boldsymbol{u} - \hat{\boldsymbol{e}}_\theta \frac{T}{r}. \tag{2.12}$$

The excess velocity $\boldsymbol{u}' = \hat{\boldsymbol{e}}_r u' + \hat{\boldsymbol{e}}_\theta v'$, which vanishes in the case of a homogeneous surface (see (2.7)), satisfies (together with the original pressure field p) equations (2.1)–(2.3). In addition, it satisfies the dynamic conditions at $r = 1$ (cf. (2.5))

$$v' = 1 - T \quad \text{for } |\theta| < \frac{\phi\pi}{N}, \qquad \frac{\partial v'}{\partial r} - v' = 2T \quad \text{for } \frac{\phi\pi}{N} < |\theta| < \frac{\pi}{N}. \tag{2.13a,b}$$

The conceptual difference between the two formulations has to do with the calculation of the torque. In the original formulation, it is obtained by directly evaluating (2.9); in the excess-velocity formulation, it is obtained by imposing the refined condition on the far-field decay rate (cf. (2.4)),

$$\boldsymbol{u}' = O(r^{-2}) \quad \text{for } r \gg 1. \tag{2.14}$$

Note that this condition allows for the possibility of a faster decay rate than r^{-2}.

We have therefore obtained a well-posed linear problem which depends upon two parameters – namely, N and the solid fraction ϕ. It is not difficult to solve the preceding problem, for any numerical values of N and ϕ, using a Fourier-series expansion in the spirit of Lauga & Stone (2003). It turns out, however, that such a Fourier-series solution is inessential. To understand why, we consider now the asymptotic limit of large N.

3. Large-N limit

The large-N limit is addressed using matched asymptotic expansions. The inner region, of $O(1/N)$ extent, is described by the stretched coordinates

$$X = \frac{N(r-1)}{\pi}, \qquad Y = \frac{N\theta}{\pi}. \tag{3.1a,b}$$

In terms of these coordinates, the unit sector is bounded by $Y = \pm 1$. While X merely satisfies $X > 0$, we define the inner region by the requirement that X is $O(1)$ there.

The outer region corresponds to the limit $N\rightarrow\infty$ with r fixed. In that region, the fine details of the compound surface are not discerned. Since the excess velocity vanishes in the case of a homogeneous surface, we postulate that $\boldsymbol{u}'$ is exponentially small there (i.e. it vanishes at all asymptotic orders when expanded in inverse powers of N). This null solution trivially satisfies equations (2.1), conditions (2.3) and the strong version (2.14) of the decay condition.

Since the curvature is negligible in the inner region, it is evident that the Stokes equations (2.13) adopt there the approximate form (which resembles that in Cartesian coordinates),

$$\frac{\partial u'}{\partial X}+\frac{\partial v'}{\partial Y}\approx 0,\quad \frac{\pi}{N}\frac{\partial p}{\partial X}\approx \Delta u',\quad \frac{\pi}{N}\frac{\partial p}{\partial Y}\approx \Delta v', \tag{3.2a-c}$$

wherein $\Delta=\partial^2/\partial X^2+\partial^2/\partial Y^2$ denotes the inner-region Laplacian. At the same level of approximation, conditions (2.13) become

$$v'\approx 1-T \quad \text{for } |Y|<\phi,\qquad \frac{\partial v'}{\partial X}\approx 0 \quad \text{for } \phi<|Y|<1, \tag{3.3a,b}$$

at $X=0$. In addition, matching with the quiescent outer region requires large-X decay. By setting $T\approx 1$, we find that the solution is trivial. The nil inner solution at $O(1)$, which trivially satisfies (2.2)–(2.3), is consistent with our anticipation that the leading-order problem is identical to that about a homogeneous cylinder.

The problem structure suggests that the leading-order superhydrophobic effect is $O(1/N)$. We accordingly write

$$u'=\frac{\pi U}{N}+\cdots,\quad v'=\frac{\pi V}{N}+\cdots,\quad p=P+\cdots, \tag{3.4a-c}$$

where the fields U, V and P are presumably $O(1)$ functions of X and Y. Consistently, we write

$$T(N,\phi)=1+\frac{\pi\tau(\phi)}{N}+\cdots, \tag{3.5}$$

and seek to evaluate the leading-order correction $\tau(\phi)$.

The leading-order flow variables U, V and P are governed by (i) the continuity and Stokes equations (cf. (2.1)),

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0,\quad \frac{\partial P}{\partial X}=\Delta U,\quad \frac{\partial P}{\partial Y}=\Delta V; \tag{3.6a-c}$$

(ii) the kinematic condition at $X=0$,

$$U=0; \tag{3.7}$$

and (iii) the periodicity conditions (cf. (2.3)),

$$U=\frac{\partial V}{\partial Y}=0 \quad \text{at } Y=\pm 1. \tag{3.8}$$

In addition, they also satisfy the dynamic conditions at $X=0$ (cf. (2.5)),

$$V=-\tau \quad \text{for } |Y|<\phi,\qquad \frac{\partial V}{\partial X}=2 \quad \text{for } \phi<|Y|<1, \tag{3.9a,b}$$

and the decay condition

$$\lim_{X\to\infty} V = 0. \tag{3.10}$$

Note that the latter does not follow from the original decay condition (2.4), which does not apply in the inner region. Rather, it follows from the need to match the quiescent outer solution.

Defining $\tilde{V} = V + \tau - 2X$ we find that the fictitious flow field $\hat{\boldsymbol{e}}_r U + \hat{\boldsymbol{e}}_\theta \tilde{V}$, with the same pressure P, satisfies (3.6)–(3.8). In terms of that field, conditions (3.9) at $X = 0$ transform to the homogeneous conditions,

$$\tilde{V} = 0 \quad \text{for } |Y| < \phi, \qquad \frac{\partial \tilde{V}}{\partial X} = 0 \quad \text{for } \phi < |Y| < 1, \tag{3.11a,b}$$

while the far-field decay (3.10) is replaced by

$$\tilde{V} \sim -2X + \tau + o(1) \quad \text{for } X \gg 1. \tag{3.12}$$

The problem governing $(U, \tilde{V})$ is analogous to the problem of transverse shear over a periodic array of flat bubbles, which was solved by Philip (1972). Making use of the asymptotic behaviour of his solution at large distances (Lauga & Stone 2003), we find that

$$\tilde{V} \sim -2X + \frac{2}{\pi} \ln \sin \frac{\pi\phi}{2} + o(1) \quad \text{for } X \gg 1. \tag{3.13}$$

Comparing with (3.12) thus gives

$$\tau = \frac{2}{\pi} \ln \sin \frac{\pi\phi}{2}. \tag{3.14}$$

We conclude that the torque is

$$T = 1 + \frac{2}{N} \ln \sin \frac{\pi\phi}{2} + \cdots \quad \text{for } N \gg 1. \tag{3.15}$$

Inversion thus provides the requisite approximation for the enhanced mobility

$$M \sim 1 + \frac{2}{N} \ln \csc \frac{\pi\phi}{2} + \cdots \quad \text{for } N \gg 1. \tag{3.16}$$

4. Functional dependence upon N

We now consider the general case, where N is not necessarily large. Without trying to actually solve that formidable problem, we try to extract the structure by which the torque depends upon the number of grooves N. To that end we hereafter employ the subscript 'N' to denote quantities associated with N grooves. Thus, the excess liquid velocity is denoted by $\boldsymbol{u}'_N$, while the associated torque is denoted by T_N.

We define the streamfunction ψ_N associated with the excess velocity $\boldsymbol{u}'_N$ via the relation

$$\boldsymbol{u}'_N = \boldsymbol{\nabla} \psi_N \times \hat{\boldsymbol{e}}_z, \tag{4.1}$$

or, equivalently,

$$u_N' = \frac{1}{r}\frac{\partial \psi_N}{\partial \theta}, \quad v_N' = -\frac{\partial \psi_N}{\partial r}. \tag{4.2a,b}$$

The streamfunction ψ_N satisfies the biharmonic equation in the semi-sector $0 < \theta < \pi/N$. Since it is defined up to within an additive constant, the kinematic condition (2.2) may be simply written

$$\psi_N = 0 \quad \text{for } r = 1. \tag{4.3}$$

In addition to (4.3), ψ_N satisfies the periodicity conditions (cf. (2.6))

$$\frac{\partial \psi_N}{\partial \theta} = 0 \quad \text{at } \theta = 0, \frac{\pi}{N}; \tag{4.4}$$

the decay requirement (cf. (2.14))

$$\psi_N = O(1/r) \quad \text{for } r \to \infty; \tag{4.5}$$

and the dynamic conditions at $r = 1$ (cf. (2.13))

$$\frac{\partial \psi_N}{\partial r} = T_N - 1 \quad \text{for } 0 < \theta < \frac{\phi\pi}{N}, \tag{4.6}$$

$$\frac{\partial^2 \psi_N}{\partial r^2} - \frac{\partial \psi_N}{\partial r} = -2T_N \quad \text{for } \frac{\phi\pi}{N} < \theta < \frac{\pi}{N}. \tag{4.7}$$

We now employ the Goursat representation for biharmonic functions (Langlois & Deville 1964), writing $z = x + \mathrm{i}y$ and

$$\psi_N = \mathrm{Re}\{\bar{z} f(z) + g(z)\}, \tag{4.8}$$

where $f(z)$ and $g(z)$ are analytic functions in the semi-sector $0 < \arg z < \pi/N$ (with $|z| > 1$) and $\bar{z} = x - \mathrm{i}y$. Given (4.5), both f and $\mathrm{d}g/\mathrm{d}z$ are $O(z^{-2})$ at large $|z|$. Condition (4.3) suggests the following representation

$$\psi_N = \mathrm{Re}\left\{\left(\bar{z} - \frac{1}{z}\right) G(z)\right\}, \tag{4.9}$$

where $G(z)$ is analytic in the semi-sector. It corresponds to $f = G$ and $g = -G/z$ in (4.8). The function G is $O(z^{-2})$ at large $|z|$.

Since $G(z)/z$ is analytic in the semi-sector, its real part

$$\chi(r, \theta) = \mathrm{Re}\left\{\frac{G(z)}{z}\right\} \tag{4.10}$$

is harmonic there. In terms of χ, representation (4.9) reads

$$\psi_N = (r^2 - 1)\chi(r, \theta). \tag{4.11}$$

With that form, the impermeability condition (4.3) is indeed trivially satisfied. Upon differentiation with respect to r we find that conditions (4.6)–(4.7) become

$$2\chi(1, \theta) = T_N - 1 \quad \text{for } 0 < \theta < \frac{\phi\pi}{N}, \tag{4.12}$$

$$2\frac{\partial\chi}{\partial r}(1,\theta)=T_N \quad \text{for} \quad \frac{\phi\pi}{N}<\theta<\frac{\pi}{N}. \tag{4.13}$$

Also, upon differentiation with respect to θ we find that condition (4.4) becomes

$$\frac{\partial\chi}{\partial\theta}=0 \quad \text{at } \theta=0, \frac{\pi}{N} \quad \text{for } r>1. \tag{4.14}$$

Last, the decay condition (4.5) now reads

$$\chi=O(1/r^3) \quad \text{for } r\to\infty. \tag{4.15}$$

We have therefore transformed the flow problem into a boundary-value problem governing a harmonic function. Rather than trying to solve the above problem, we choose a different approach: we employ the structure (4.12)–(4.15) to relate the streamfunction appropriate for $2N$ grooves, ψ_{2N}, to that appropriate for N grooves, ψ_N. In doing so, it is convenient to denote the polar coordinates associated with the problem governing ψ_{2N} by, say, ρ and α, so as to distinguish them from the (r,θ) polar coordinates associated with the problem governing ψ_N. Adopting (4.3)–(4.7) to the excess-velocity problem for $2N$ grooves, as detailed in figure 3, the biharmonic function $\psi_{2N}(\rho,\alpha)$ must satisfy: (i) kinematic impermeability,

$$\psi_{2N}=0 \quad \text{for } \rho=1; \tag{4.16}$$

(ii) the periodicity conditions,

$$\frac{\partial\psi_{2N}}{\partial\alpha}=0 \quad \text{at } \alpha=0, \frac{\pi}{2N}; \tag{4.17}$$

(iii) the decay requirement,

$$\psi_{2N}=O(1/\rho) \quad \text{for } \rho\to\infty; \tag{4.18}$$

and (iv) the dynamic conditions at $\rho=1$,

$$\frac{\partial\psi_{2N}}{\partial r}=T_{2N}-1 \quad \text{for } 0<\alpha<\frac{\phi\pi}{2N}, \tag{4.19}$$

$$\frac{\partial^2\psi_{2N}}{\partial\rho^2}-\frac{\partial\psi_{2N}}{\partial\rho}=-2T_{2N} \quad \text{for } \frac{\phi\pi}{2N}<\alpha<\frac{\pi}{2N}. \tag{4.20}$$

Consider a conformal mapping $z=h(\zeta)$ from the complex ζ-plane, with $\zeta=\xi+\mathrm{i}\eta=\rho\mathrm{e}^{\mathrm{i}\alpha}$, to the z-plane. By choosing

$$h(\zeta)=\zeta^2, \tag{4.21}$$

the pre-image of the semi-sector in the z-plane is the semi-sector for the $2N$-grooves problem, where $0<\alpha<\pi/2N$: see figure 3. We claim that the relevant streamfunction is

$$\psi_{2N}=C_N\mathrm{Re}\left\{\left(\bar{\zeta}-\frac{1}{\zeta}\right)\frac{G(h(\zeta))}{\zeta}\right\}, \tag{4.22}$$

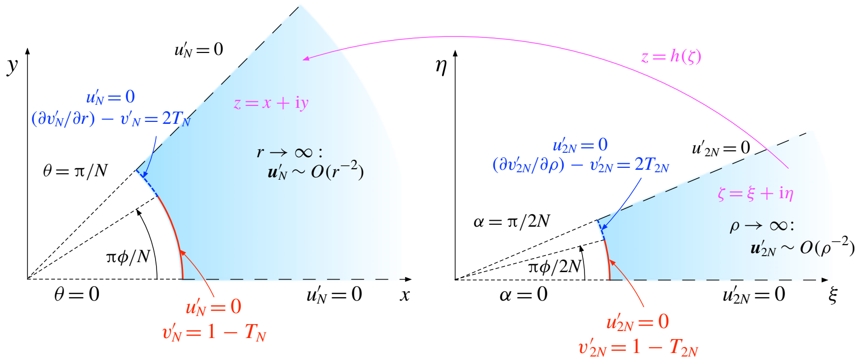

FIGURE 3. The excess-velocity boundary conditions for N and $2N$ grooves in the appropriate semi-sectors $0 < \theta < \pi/N$ and $0 < \alpha < \pi/2N$.

wherein C_N is a constant. Since representation (4.22) is of the Goursat form (4.8), it clearly satisfies the biharmonic equation. To prove our assertion, we accordingly need to show that conditions (i)–(iv) above are also satisfied.

To that end we make use of (4.10) and (4.21), which transform (4.22) to

$$\psi_{2N} = C_N(\rho^2 - 1)\chi(\rho^2, 2\alpha). \tag{4.23}$$

We now note that (i) representation (4.23) trivially satisfies the impermeability condition (4.16); (ii) differentiating it with respect to α and making use of (4.4) shows that the periodicity conditions (4.17) are satisfied; and (iii) substitution of (4.15) into (4.23) gives $\psi_{2N} = O(\rho^{-4})$ at large ρ, which accordingly satisfies the far-field requirement (4.18).

It remains to show that the dynamic conditions (4.19)–(4.20) are also satisfied. Differentiating (4.23) with respect to ρ shows that these conditions are equivalent to

$$2C_N\chi(1, 2\alpha) = T_{2N} - 1 \quad \text{for } 0 < \alpha < \frac{\phi\pi}{2N}, \tag{4.24}$$

$$4C_N\frac{\partial \chi}{\partial r}(1, 2\alpha) = T_{2N} \quad \text{for } \frac{\phi\pi}{2N} < \alpha < \frac{\pi}{2N}. \tag{4.25}$$

Comparing with (4.12)–(4.13) reveals that conditions (4.24)–(4.25) are satisfied if and only if

$$C_N(T_N - 1) = T_{2N} - 1, \quad 2C_N T_N = T_{2N}. \tag{4.26a,b}$$

Elimination of C_N provides the recursive equation

$$T_{2N} = \frac{2T_N}{1 + T_N}, \tag{4.27}$$

whose general solution is

$$T_N = \frac{N}{N + \mathcal{C}}. \tag{4.28}$$

The 'integration constant' $\mathcal{C}$ is independent of N and can only depend upon ϕ. We conclude that the mobility $M_N = 1/T_N$ is

$$M_N = \frac{N + \mathcal{C}(\phi)}{N}. \tag{4.29}$$

We have therefore obtained the general structure for the dependence of the cylinder mobility upon N. Since we did not solve the flow problem, we cannot obtain the explicit form of $\mathcal{C}(\phi)$ using that method. Nonetheless, that form may be obtained indirectly: making use of the large-N asymptotic sequence $1, 1/N, 1/N^2, \ldots$ we find that the exact expression (4.29) constitutes a large-N asymptotic expansion in this sequence which consists of only two terms, $1 + \mathcal{C}/N$. Comparing with (3.16) and making use of the uniqueness of the coefficients of any asymptotic expansion (Bender & Orszag 1978) we conclude that

$$\mathcal{C}(\phi) = 2 \ln \csc \frac{\pi\phi}{2}, \tag{4.30}$$

and that the asymptotic expansion (3.16) actually terminates after the $O(1/N)$ term. We have therefore obtained an exact expression for the mobility for the case where N is an integer power of two.

Incidentally, it is evident from our construction that for each such N the excess velocity $\boldsymbol{u}'_{2N}$ decays faster than $\boldsymbol{u}'_N$. To elucidate this mechanism, we write $N = 2^k$ with $k = 0, 1, 2, \ldots$ and postulate that $\psi_N = O(r^{-\lambda_k})$ for $r \to \infty$, with $\lambda_k \geqslant 1$. (This asymptotic relation is clearly more restrictive than (4.5).) From (4.11) and (4.23) it readily follows that $\lambda_{k+1} = 2\lambda_k + 2$. The solution of this recursive equation subject to $\lambda_0 = 1$ (which follows from (4.5)) is $\lambda_k = 3 \times 2^k - 2$. We conclude that the excess velocity $\boldsymbol{u}'_N$ is $O(r^{1-3N})$ at large r. This rapid increase in the decay rate is consistent with our postulate of exponentially small outer velocity in the large-N limit.

5. Concluding remarks

Motivated by the desire to understand rigid-body motion of a superhydrophobic particle in a viscous liquid, we have considered what may be the simplest problem in that class – namely, the rotation of an infinite cylinder which is decorated by longitudinal grooves. The dimensionless Stokes-flow problem involves two geometric parameters: the solid fraction ϕ and the number N of grooves. We have obtained a two-term asymptotic approximation in the limit where N is large. Making use of conformal-mapping techniques, we proved that this approximation is actually exact whenever N is an integer power of two. Comparison of the mobility expression (4.29)–(4.30) with that obtained from the Fourier-series solution of the Stokes-flow problem shows perfect agreement. We accordingly conjecture that (4.29)–(4.30) holds for all N.

The complex-variable approach has been used in this paper so as to extract the functional dependence of the rotational mobility upon the number of grooves. This approach can actually be used in order to transform the flow problem into a Hilbert problem, which results a semi-analytic expression for all the pertinent fluid dynamic quantities. While the semi-analytic Hilbert-transform solutions cannot provide the explicit mobility expression (4.29)–(4.30), they may be utilized for the construction of convenient numerical solutions in the more general case of a non-uniform (i.e. aperiodic) distribution of grooves. (Techniques introduced by

Crowdy 2013*b*, which avoid the difficulties associated with end point singularities at the points where the boundary conditions change type, may also be useful here.) The Hilbert-transform approach can be used, moreover, to recover analytical solutions in other two-dimensional Stokes-flow problems, such as the slip–stick Janus swimmers considered by Crowdy (2013*a*). Details will be given in a future paper.

Given the Stokes paradox, the rotational mobility obtained in the present contribution is the only existing component in the mobility tensor – no other mobility problem can be sensibly formulated in the present context of two-dimensional unbounded flow. On the other hand, there is no obstruction to finding the full resistance tensor for a two-dimensional particle near a plane wall (Crowdy 2013*c*) – a situation of interest in applications. The mobility of a cylinder near a superhydrophobic wall has been addressed recently by Schnitzer & Yariv (2019); a natural follow-up of both that work and the present analysis would involve the calculation of the mobility tensor of a superhydrophobic cylinder near a no-slip wall (Kaynan & Yariv 2017).

Acknowledgements

E.Y. acknowledges support from the Israel Science Foundation (grant no. 1081/16). M.S. acknowledges support from the National Science Foundation (grant no. DMS-1909407). We thank M. Hodes and D. Kane for useful discussions and for their help in preparing figure 1.

References

Bender, C. M. & Orszag, S. A. 1978 *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill.

Bocquet, L. & Lauga, E. 2011 A smooth future? *Nature* **10** (5), 334–337.

Castagna, M., Mazellier, N. & Kourta, A. 2018 Wake of super-hydrophobic falling spheres: influence of the air layer deformation. *J. Fluid Mech.* **850**, 646–673.

Crowdy, D. 2010 Slip length for longitudinal shear flow over a dilute periodic mattress of protruding bubbles. *Phys. Fluids* **22** (12), 121703.

Crowdy, D. G. 2013*a* Exact solutions for cylindrical slip–stick Janus swimmers in Stokes flow. *J. Fluid Mech.* **719**, R2.

Crowdy, D. G. 2013*b* Surfactant-induced stagnant zones in the Jeong–Moffatt free surface Stokes flow problem. *Phys. Fluids* **25** (9), 092104.

Crowdy, D. G. 2013*c* Wall effects on self-diffusiophoretic Janus particles: a theoretical study. *J. Fluid Mech.* **735**, 473–498.

Davis, A. M. J. & Lauga, E. 2009 Geometric transition in friction for flow over a bubble mattress. *Phys. Fluids* **21** (1), 011701.

Happel, J. & Brenner, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

Hinch, E. J. 1972 Note on the symmetries of certain material tensors for a particle in Stokes flow. *J. Fluid Mech.* **54** (03), 423–425.

Jetly, A., Vakarelski, I. U. & Thoroddsen, S. T. 2018 Drag crisis moderation by thin air layers sustained on superhydrophobic spheres falling in water. *Soft Matt.* **14**, 1608–1613.

Kaynan, U. & Yariv, E. 2017 Stokes resistance of a cylinder near a slippery wall. *Phys. Rev. Fluids* **2** (10), 104103.

Langlois, W. E. & Deville, M. O. 1964 *Slow Viscous Flow*. Springer.

Lauga, E. & Stone, H. A. 2003 Effective slip in pressure-driven Stokes flow. *J. Fluid Mech.* **489**, 55–77.

Muralidhar, P., Ferrer, N., Daniello, R. & Rothstein, J. P. 2011 Influence of slip on the flow past superhydrophobic circular cylinders. *J. Fluid Mech.* **680**, 459–476.

Philip, J. R. 1972 Flows satisfying mixed no-slip and no-shear conditions. *Z. Angew. Math. Phys.* **23** (3), 353–372.

POZRIKIDIS, C. 2011 *Introduction to Theoretical and Computational Fluid Dynamics*. Oxford University Press.

QUÉRÉ, D. 2008 Wetting and roughness. *Annu. Rev. Mater. Res.* **38** (1), 71–99.

ROTHSTEIN, J. P. 2010 Slip on superhydrophobic surfaces. *Annu. Rev. Fluid Mech.* **42** (1), 89–109.

SCHNITZER, O. & YARIV, E. 2019 Stokes resistance of a solid cylinder near a superhydrophobic surface. Part 1. Grooves perpendicular to cylinder axis. *J. Fluid Mech.* **868**, 212–243.